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Minimally globally rigid graphs with high minimum degree

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Abstract

Garamvölgyi and Jordán [4] recently showed that every minimally globally rigid graph in $\mathbb{R}^d$ contains a vertex of degree at most $2d+1$, and gave examples with minimum degree $d+2$ for all $d \geq 3$. We construct examples of minimally globally rigid graphs in $\mathbb{R}^d$ with minimum degree at least $2d-1$, for all $d \geq 1$.

1 Introduction

In this note we consider rigid and globally rigid graphs in $\mathbb{R}^d$. We refer the reader to [9, 12] for the basic definitions. We say that a globally rigid graph G in $\mathbb{R}^d$ is *minimally globally rigid* in $\mathbb{R}^d$ if $G - e$ is not globally rigid for all $e \in E(G)$. Garamvölgyi and Jordán [4] recently showed that every graph which is minimally globally rigid in $\mathbb{R}^d$ contains a vertex of degree at most $2d + 1$, and gave infinite families of examples with minimum degree $d + 2$ for all $d \geq 3$. We will construct families of examples of minimally globally rigid graphs in $\mathbb{R}^d$ with minimum degree at least $2d - 1$, for all $d \geq 1$. We conjecture that $2d - 1$ is the best possible upper bound for all $d \geq 2$.

The minimum degree of a globally rigid graph in $\mathbb{R}^d$ on at least $d + 2$ vertices is at least $d + 1$, for all $d \geq 1$. For $d = 1, 2$ it is known that the tight upper bound on the minimum degree of a minimally globally rigid graph in $\mathbb{R}^d$ is $d + 1$. A graph G on at least three vertices is globally rigid in $\mathbb{R}^1$ if and only if G is 2-connected. By applying a result of Dirac [2], which states that every minimally 2-connected graph contains a vertex of degree two, the tight bound for $d = 1$ follows. For $d = 2$ it was shown in [6] that every minimally globally rigid graph in $\mathbb{R}^2$ on at least four vertices contains a vertex of degree three.

The tight bounds for $d \geq 3$ have not been determined yet. We conjecture that the degree bound occuring in our construction is best possible.

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Conjecture 1. *Every minimally globally rigid graph in $\mathbb{R}^d$ contains a vertex of degree at most $2d-1$, for all $d \geq 2$.*

In our constructions we will use the vertex splitting and coning operations, which are fundamental tools in rigidity theory.

Given an integer d, a vertex v in a graph G, and two disjoint subsets U, W of the neighbours of v with $|W| = d-1$, the *d-dimensional vertex splitting operation along (U, W) at v* constructs a new graph by deleting all edges from v to U, adding an edge from each vertex of $U \cup W$ to a new vertex v' and an additional edge from v to v', see Figure 1. We say that the d-dimensional vertex splitting operation is *non-trivial* if the two split vertices v, v' both have degree at least $d+1$, or equivalently, U is nonempty and v has at least one neighbour which does not belong to $U \cup W$.

We will need the following result.

Theorem 2. *[10] Suppose $d \geq 1$ is an integer and G is a graph which can be obtained from K_{d+2} by a sequence of non-trivial d-dimensional vertex splitting operations. Then G is globally rigid in $\mathbb{R}^d$.*

Given an integer d, a vertex v in a graph G and two disjoint subsets U, W of the neighbours of v with $|W| = d$, the *d-dimensional non-adjacent vertex splitting operation along (U, W) at v* constructs a new graph by deleting all edges from v to U and adding an edge from each vertex of $U \cup W$ to a new vertex v'.

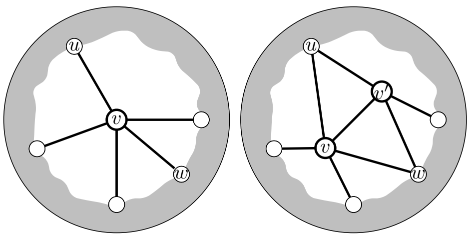

Figure 1: A 3-dimensional non-trivial vertex splitting operation, where $W = \{u, v\}$.

Theorem 3. *[14] Suppose G is a graph which is minimally rigid in $\mathbb{R}^d$ and G' is obtained from G by applying either the d-dimensional vertex splitting operation or the d-dimensional non-adjacent vertex splitting operation at a vertex v of G. Then G' is minimally rigid in $\mathbb{R}^d$.*

The *cone* of a graph G is obtained from G by adding a new vertex v adjacent to all vertices of G.

Theorem 4. *[13] Let $d \geq 1$ be an integer, let G be a graph and let G' be the cone of G. Then G is rigid in $\mathbb{R}^d$ if and only if G' is rigid in $\mathbb{R}^{d+1}$.*

2 The constructions

2.1 Rigidity circuits

An $\mathcal{R}_d$-*circuit* is a graph $G = (V, E)$ with the property that E is a circuit in the d-dimensional rigidity matroid, see [15]. It is well known that every $\mathcal{R}_d$-circuit satisfies $|V| \geq d + 2$ and $|E| \leq d|V| - \binom{d+1}{2} + 1$, so its minimum degree is at most $2d - 1$, for all $d \geq 2$. Thus the following result is best possible. Note that every graph on at least $d + 2$ vertices which is globally rigid in $\mathbb{R}^d$, is redundantly rigid by a result of Hendrickson [5]. This implies that every globally rigid $\mathcal{R}_d$-circuit is minimally globally rigid in $\mathbb{R}^d$.

Theorem 5. *For each $d \geq 2$, there are infinitely many $\mathcal{R}_d$-circuits which are globally rigid in $\mathbb{R}^d$ and have minimum degree $2d - 1$.*

Proof. For each $d \geq 1$, let $\mathcal{G}_d$ be the family of all $\mathcal{R}_d$-circuits which can be obtained from K_{d+2} by a sequence of non-trivial d-dimensional vertex splitting operations, have minimum degree at least $2d - 1$, and at most half their vertices have degree $2d - 1$. We will show that $\mathcal{G}_d$ is infinite by induction on d. The lemma will follow since all graphs in $\mathcal{G}_d$ are globally rigid in $\mathbb{R}^d$ by Theorem 2.

When $d = 1$, $\mathcal{G}_1$ is the family of all cycles with at least three vertices. Hence we may assume that $\mathcal{G}_{d-1}$ is infinite for some $d \geq 2$. Choose $G \in \mathcal{G}_{d-1}$ and put $n = |V(G)|$. Let H be the graph obtained from G by adding a new vertex v which is adjacent to all vertices of G. Since $G \in \mathcal{G}_{d-1}$, G can be obtained from K_{d+1} by a sequence of non-trivial $(d-1)$-dimensional vertex splittings. This implies that H can be obtained from K_{d+2} by a sequence of non-trivial d-dimensional vertex splittings: we construct the initial copy of K_{d+2} by joining the new vertex v to all vertices of the K_{d+1}, and then add v to the set of common neighbours in each vertex splitting in the sequence for G. Then H is globally rigid by Theorem 2. This implies, by [5], that H is redundantly rigid. Since G is an $\mathcal{R}_{d-1}$-circuit, this in turn implies, by counting edges, that H is an $\mathcal{R}_d$-circuit. In addition, H has minimum degree at least $2d - 2$ and contains at most $\frac{n}{2}$ vertices of degree $2d - 2$. We will show that, when n is sufficiently large, we can convert H to a graph in $\mathcal{G}_d$ by performing a sequence of non-trivial d-dimensional vertex splittings at v.

Let $V(H) = \{v, x_1, x_2, \ldots, x_n\}$ where $\deg_H(x_i) = 2d - 2$ for $1 \leq i \leq k$ and $\deg_H(x_i) \geq 2d - 1$ for $k < i \leq n$. Let $n = q(d - 1) + r$ for some integers $q \geq 0$ and $0 \leq r \leq d - 2$. When $n \geq 4(d - 1)$, we can choose a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_q\}$ of $V(H) - \{v\}$ such that $|V_i| = d - 1$ for $1 \leq i \leq q - 1$, $|V_q| = d - 1 + r$, and $x_i \in \bigcup_{j=2}^{q-1} V_j$ for all $1 \leq i \leq k$. Let $H_0, H_1, \ldots H_{q-1}$ be the sequence of graphs constructed recursively as follows. Let $H_0 = H$ and let H_1 be obtained from H_0 by splitting v along (V_1, V_2). For $2 \leq i \leq q - 1$ let H_i be obtained from H_{i-1} by splitting v along $(V_i \cup \{v_{i-1}\}, V_{i+1})$, where v_i is the new vertex created by the i'th vertex splitting. Then H_{q-1} can be obtained from K_{d+2} by a sequence of non-trivial d-dimensional vertex splits, $\deg_{H_{q-1}}(x) - \deg_G(x) = 2$ for all $x \in \bigcup_{j=2}^{q-1} V_j$ and $\deg_{H_{q-1}}(x) - \deg_G(x) = 1$ for all $x \in V_1 \cup V_q$. In addition, we have $\deg_{H_{q-1}}(v_1) =$

$2d-1$, $\deg_{H_{q-1}}(v_i) = 2d$ for all $2 \le i \le q-1$ and $\deg_{H_{q-1}}(v) = 2d-1+r$. It follows that H_{q-1} has minimum degree $2d-1$ and that the number of vertices of minimum degree in H_{q-1} is at most $|V_1| + |V_q| + 2 \le 3d-2$ more than the number of vertices of minimum degree in G. Since $|V(H_{q-1})| = |V(H)| + q - 1 = n + q = n + \lfloor \frac{n}{d-1} \rfloor$, we have $H_{q-1} \in \mathcal{G}_d$ when n is sufficiently large compared to d. It follows that we can construct infinitely many graphs in $\mathcal{G}_d$ by varying our choice of $G \in \mathcal{G}_{d-1}$. □

A more careful counting may give infinitely many globally rigid $\mathcal{R}_d$-circuits with minimum degree $2d-1$ and at most $3\binom{d+1}{2}$ vertices of degree $2d-1$. On the other hand, every $\mathcal{R}_d$-circuit of minimum degree $2d-1$ has at least $2\binom{d+1}{2} - 2$ vertices of degree $2d-1$. We next give examples of subfamilies of $\mathcal{G}_d$ which attain this lower bound when $1 \le d \le 3$.

Specific families of graphs

For $d = 1$, the cycles on at least three vertices are all in $\mathcal{G}_1$. They contain no vertices of degree 1.

For $d = 2$ consider the graphs that can be obtained from a maximal outerplanar graph of maximum degree four on at least 8 vertices with exactly two vertices a, b of degree two, by adding the edge ab. These graphs can be obtained from K_4 by a sequence of non-trivial vertex splitting operations by [3]. They contain exactly four vertices of degree 3.

For $d = 3$ consider the graphs that can be obtained from a 5-connected maximal planar graph on at least 20 vertices with maximum degree 6, by adding an edge connecting two non-adjacent vertices of degree five. It is known that we have infinitely many such graphs (see, e.g., [1]) and it was shown in [10] that these graphs can be obtained from K_5 by a sequence of non-trivial vertex splitting operations. They contain exactly ten vertices of degree 5.

2.2 Minimally rigid graphs

It is known that the minimum degree of a minimally rigid graph in $\mathbb{R}^d$ on at least $d+1$ vertices is at least d, and at most $2d-1$. We use a similar construction to construct graphs which are minimally rigid in $\mathbb{R}^d$ and all their vertices have degree $2d-1$ or $2d$.

Theorem 6. *Suppose $d \ge 1$ is an integer. Let $\mathcal{L}_d$ be the family of all graphs G which are minimally rigid in $\mathbb{R}^d$, have $d(d+1)$ vertices of degree $2d-1$ and all other vertices have degree $2d$. Then $\mathcal{L}_d$ is infinite.*

Proof. We proceed by induction on d. When $d = 1$, $\mathcal{L}_1$ is the family of all paths with at least two vertices. Hence we may assume that $\mathcal{L}_{d-1}$ is infinite for some $d \ge 2$. Choose $G \in \mathcal{L}_{d-1}$ and put $n = |V(G)|$. Let H be the graph obtained from G by adding a new vertex v which is adjacent to all vertices of G. Then H is the cone of G so is minimally rigid in $\mathbb{R}^d$ by Theorem 4. In addition, H has one vertex of degree n, $d(d-1)$ vertices of degree $2d-2$ and all other vertices have degree $2d-1$. We will show that, when n is sufficiently large, we can convert H to a graph in $\mathcal{L}_d$ by performing a sequence of vertex splits at v.

Let $V(H) = \{v, x_1, x_2, \ldots, x_n\}$ where $\deg_H(x_i) = 2d-2$ for $1 \leq i \leq d(d-1)$ and $\deg_H(x_i) = 2d-1$ for $d(d-1) < i \leq n$. Let $n = q(d-1)+r$ for some integers $q \geq 0$ and $0 \leq r \leq d-2$. When $n \geq d(d-1)+2(d-1)$, we can choose a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_q\}$ of $V(H)-\{v\}$ such that $q-r$ of the parts of $\mathcal{P}$ have cardinality $d-1$, the remaining r parts have cardinality d and $x_i \in \bigcup_{j=2}^{q-1} V_j$ for all $1 \leq i \leq d(d-1)$. Let $H_0, H_1, \ldots H_{q-1}$ be the sequence of graphs constructed recursively by putting $H_0 = H$ and then constructing H_i from H_{i-1} by d-dimensional (non-adjacent) vertex splitting operations at v by using the following rules: if v is incident with v_{i-1} in H_{i-1}, then the operation is made along $(V_i \cup \{v_{i-1}\}, V_{i+1})$, otherwise it is made along (V_i, V_{i+1}), where v_i is the new vertex created by the i'th vertex splitting. Furthermore, the non-adjacent version of the splitting operation is applied precisely when $|V_{i+1}| = d$ holds. Then H_{q-1} is minimally rigid in $\mathbb{R}^d$ by Theorem 3, $\deg_{H_{q-1}}(x) - \deg_G(x) = 2$ for all $x \in \bigcup_{j=2}^{q-1} V_j$ and $\deg_{H_{q-1}}(x) - \deg_G(x) = 1$ for all $x \in V_1 \cup V_q$. In addition, we have $\deg_{H_{q-1}}(v_i) \in \{2d-1, 2d\}$ for all $1 \leq i \leq q-1$ and $\deg_{H_{q-1}}(v) \in \{2d-1, 2d\}$. Hence $H_{q-1} \in \mathcal{L}_d$. Since $|V(H_{q-1})| = |V(H)| + q - 1 = n + q = n + \lfloor n/d \rfloor$, we can construct infinitely many graphs in $\mathcal{L}_d$ by varying our choice of $G \in \mathcal{L}_{d-1}$. □

3 Further remarks

One may also consider similar problems for minimally $\mathcal{R}_d$-connected graphs and minimally redundantly rigid graphs in $\mathbb{R}^d$. It was shown in [8] that the minimum degree of a minimally $\mathcal{R}_d$-connected graph is at most $4d-1$ for all $d \geq 3$, at most 5 for $d = 2$, and at most 2 for $d = 1$. Király [11] pointed out that the minimum degree of a minimally redundantly rigid graph in $\mathbb{R}^d$ is at most $2d+1$, for all $d \geq 1$. Since $\mathcal{R}_d$-circuits are minimally $\mathcal{R}_d$-connected and rigid $\mathcal{R}_d$-circuits are minimally redundantly rigid in $\mathbb{R}^d$, Theorem 5 also gives an infinte family of graphs which belong to both families and have minimum degree $2d-1$.

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