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Abstract

We provide a combinatorial study of split matroids, a class that was motivated by the study of matroid polytopes from a tropical geometry point of view. A nice feature of split matroids is that they generalize paving matroids, while being closed under duality and taking minors. Furthermore, these matroids proved to be useful in giving exact asymptotic bounds for the dimension of the Dressian, and also implied new results on the rays of the tropical Grassmannians.

In the present paper, we introduce the notion of elementary split matroids, a subclass of split matroids that contains all connected split matroids. We give a hypergraph characterization of elementary split matroids in terms of independent sets, and show that the proposed class is closed not only under duality and taking minors but also truncation. We further show that, in contrast to split matroids, the proposed class can be characterized by a single forbidden minor. As an application, we provide a complete list of binary split matroids.

Keywords: Binary matroids, Excluded minors, Split matroids, Paving matroids

1 Introduction

The class of split matroids was introduced by Joswig and Schröter [10] as an efficient tool in tropical geometry. Their definition was based on a polyhedral approach, by imposing conditions on the split hyperplanes of the matroid base polytope. In order to recall the definition, we first overview the polyhedral background.

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Given a polytope P , its intersection with a supporting hyperplane is called a **face** of P . The polytope itself is also considered to be a face. A face is a **facet** if it is properly contained in exactly one face, namely P . A **split** of P is a subdivision without new vertices which has exactly two maximal cells. The affine span of the intersection of the two cells is then called a **split hyperplane**. Two splits are **compatible** if the corresponding split hyperplanes do not meet in a relative interior point of P . Let $M = (S, r_M)$ be a matroid on ground set S with rank function r_M . We denote the rank of the matroid by r , that is, $r_M(S) = r$. The convex hull of the characteristic vectors of the bases of M is called the **matroid base polytope** of M and is denoted by $P(M)$. We denote by $\Delta(r, S)$ the $|S| - 1$ -dimensional hypersimplex representing the matroid base polytope of the rank- r uniform matroid on S , that is, the convex hull of all zero-one vectors over S with exactly r ones. For a flat F of M , the F -**hyperplane** is defined as $H(F) = \{x \in \mathbb{R}^S \mid x(F) = r_M(F)\}$, while the intersection $P(M) \cap H(F)$ is the **face of $P(M)$ defined by F** . Note that two flats might define the same face. A flat F is called a **facet** if it defines a facet and is inclusionwise minimal among flats defining $H(F)$. A facet F is a **split facet** if the corresponding F -hyperplane defines a split of $\Delta(r, S)$. Roughly, the split facets are the hyperplanes that are used to cut off parts of $\Delta(r, S)$ to obtain $P(M)$. Using this terminology, a matroid M is a **split matroid** if its split facets form a compatible system of splits of the affine hull of $P(M)$ intersected with the unit cube $[0, 1]^S$.

The goal of the present paper is to give a combinatorial understanding of split matroids.

Previous work. Joswig and Schröter [10] gave a thorough analysis of split matroids in terms of polyhedral geometry. They observed that it suffices to concentrate on the connected case, as a matroid is a split matroid if and only if at most one connected component is a non-uniform split matroid and all other components are uniform [10, Proposition 15]. For the connected case, they also gave a characterization that does not rely on polyhedral combinatorics, stating that a connected matroid is a split matroid if and only if for each split facet F the restriction $M|F$ and the contraction M/F both are uniform [10, Theorem 11].

Besides their applicability in tropical geometry, split matroids are also of combinatorial interest. In particular, the class of split matroids contains all paving matroids, a well-studied class with distinguished structural properties. A conjecture of Crapo and Rota [4], that was made precise by Mayhew, Newman, Welsh and Whittle [11], suggests that the asymptotic fraction of matroids on n elements that are paving tends to 1 as n tends to infinity. Therefore, an affirmative answer to the conjecture would imply that almost all matroids are split. A weakness of paving matroids is that their class is not closed under duality, a property that is desired in many cases. However, split matroids are closed both under duality and taking minors [10, Proposition 44], hence they form a large class with strong combinatorial properties.

In the light of minor-closedness, it is natural to ask what the excluded minors are for the class of split matroids. It is an easy exercise to show that uniform matroids are exactly the $U_{0,1} \oplus U_{1,1}$ minor-free matroids. The broader class of paving matroids coincides with the family of $U_{0,1} \oplus U_{2,2}$ -minor-free matroids [14]. For split

matroids, Joswig and Schröter [10, Question A] identified five forbidden minors, and Cameron and Mayhew [3] later verified that the list is complete. It is worth mentioning that $U_{0,\ell} \oplus U_{k,k}$ -minor-free matroids were studied for positive integers k and ℓ in general [5], while [18] gave excluded-minor characterizations for the class of so-called nearly-uniform and nearly-paving matroids.

Our results. In [10], split matroids were introduced via polyhedral geometry. The polyhedral point of view gives an insight into the geometry of the base polytope of split matroids, which in turn leads to a series of fundamental structural results. However, the polyhedral approach has two shortcomings when it comes to optimization. First, the definition is difficult to work with as it relies on the joint structure of split facets. For example, it is not immediate to see what the independent sets are, or how the rank of a set can be determined. Furthermore, as it was already observed in [10], the notion of split matroids is a bit subtle in the disconnected case. This is strengthened by the fact that while uniform or paving matroids can be characterized by a single excluded minor, the class of split matroids requires five of those.

The above observations suggest that there might be an intermediate matroid class that captures all the good characteristics of split matroids (i.e. closed under duality and taking minors) but is more convenient to work with in terms of optimization. We show that this is indeed true and introduce a class that we call elementary split matroids. The proposed class is a proper subclass of split matroids which includes all connected split matroids. The definition follows a combinatorial approach by setting the independent sets of the matroid to be the family of sets having bounded intersections with certain hyperedges. An analogous characterization was previously known for paving matroids, see [9, 19]: for a non-negative integer r , a ground set S of size at least r , and a (possibly empty) family $\mathcal{H} = \{H_1, \dots, H_q\}$ of proper subsets of S such that $|H_i \cap H_j| \leq r - 2$ for $1 \leq i < j \leq q$, the set system $\mathcal{B}_{\mathcal{H}} = \{X \subseteq S \mid |X| = r, X \not\subseteq H_i \text{ for } i = 1, \dots, q\}$ forms the set of bases of a paving matroid, and in fact every paving matroid can be obtained in this form. Elementary split matroids satisfy similar constraints; nevertheless, the underlying hypergraph might have more complex structure.

We show that the proposed class has various nice properties that partially follow from connected split matroids being special cases. However, elementary split matroids are closed not only under duality and taking minors but also truncation. Furthermore, the class can be characterized by the single forbidden minor $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$, therefore fitting in the list of earlier results. Based on the excluded-minor characterization, we give a new proof for the result of [3] on forbidden minors of split matroids and a complete description of binary split matroids.

The rest of the paper is organized as follows. Basic notation and definitions are introduced in Section 2. The hypergraph representation of elementary split matroids is presented in Section 3. Section 4 gives an excluded-minor characterization of the proposed class. Finally, Section 5 gives a new proof for the list of forbidden minors of split matroids, and further provides a complete list of binary split matroids.

2 Preliminaries

Let S be a ground set of size n . A **clutter** (or **Sperner family**) is a collection \mathcal{F} of subsets of S in which none of the sets is a subset of another. For subsets $X, Y \subseteq S$, the **difference** of X and Y is denoted by $X - Y$. If Y consists of a single element y , then $X - \{y\}$ and $X \cup \{y\}$ are abbreviated as $X - y$ and $X + y$, respectively.

A **matroid** is a pair $M = (S, \mathcal{I})$ where $\mathcal{I} \subseteq 2^S$ is the family of **independent sets** satisfying the so-called **independence axioms**:

- (I1) $\emptyset \in \mathcal{I}$,
- (I2) if $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$,
- (I3) for every subset $X \subseteq S$ the maximal subsets of X which are in \mathcal{I} have the same cardinality.

For a set $X \subseteq S$, the maximum size of an independent subset of X is the **rank** of X and is denoted by $r_M(X)$. The subscript M is dismissed when the matroid is clear from the context. The inclusionwise maximal members of \mathcal{I} are called **bases**. An inclusionwise minimal non-independent set forms a **circuit**, while a **loop** is a circuit consisting of a single element. The **dual** of M is the matroid $M^* = (S, \mathcal{I}^*)$ where $\mathcal{I}^* = \{X \subseteq S \mid S - X \text{ contains a basis of } M\}$. A **cocircuit** or **coloop** of M is a circuit or loop of M^* , respectively. The matroid is **connected** if for any two elements $e, f \in S$ there exists a circuit containing both. This can be shown to be equivalent to $r_M(X) + r_M(S - X) > r_M(S)$ for every $\emptyset \neq X \subsetneq S$. A set $X \subseteq S$ is **closed** or is a **flat** if $r_M(X + e) = r_M(X)$ for every $e \in S - X$. The **closure** of a set $X \subseteq S$, that is, the inclusionwise minimal closed set containing X is denoted by $\text{cl}_M(X)$. Two non-loop elements $e, f \in S$ are **parallel** if $r_M(\{e, f\}) = 1$. A flat of rank one is called a **parallel class**. A flat is **proper** if it has nonzero rank and it is not the ground set of the matroid. A subset $Z \subseteq S$ is **cyclic** if it is the (possibly empty) union of circuits, or equivalently, the matroid restricted to Z has no coloops. Bonin and de Mier [2] rediscovered the following axiom scheme for the cyclic flats of a matroid, first proved by Sims [15].

Proposition 1. *Let \mathcal{Z} be a collection of subsets of a ground set S and $r: \mathcal{Z} \rightarrow \mathbb{Z}_{\geq 0}$ a function. There is a matroid M on S for which \mathcal{Z} is the set of cyclic flats and r is the rank function of M restricted to \mathcal{Z} if and only if the following conditions are satisfied:*

- (Z0) \mathcal{Z} is a lattice under inclusion,
- (Z1) $r(0_{\mathcal{Z}}) = 0$ where $0_{\mathcal{Z}}$ is the zero of this lattice,
- (Z2) $0 < r(Y) - r(X) < |Y - X|$ for all $X, Y \in \mathcal{Z}$ with $X \subsetneq Y$,
- (Z3) $r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|$ for all $X, Y \in \mathcal{Z}$ with join $X \vee Y$ and meet $X \wedge Y$.

In this case, the independent sets of M are $\mathcal{I} = \{I \subseteq S \mid |I \cap Z| \leq r(Z) \text{ for each } Z \in \mathcal{Z}\}$.

For a non-negative integer $r \leq n$, the **uniform matroid** $U_{r,n}$ is defined on an n -element set by setting every subset of size at most r to be independent, that is, $\mathcal{I} = \{X \subseteq S \mid |X| \leq r\}$. When $r = n$, the matroid is called a **free matroid**, while the choice $r = 0$ results in a **rank-0 matroid**. For technical reasons, we allow the ground set of the matroid to be the empty set, i.e. $n = 0$, in which case the matroid is simply the **empty matroid** $M = (\emptyset, \{\emptyset\})$. A matroid of rank r is called **paving** if every set of size at most $r - 1$ is independent, or in other words, every circuit of the matroid has size at least r .

The **direct sum** $M_1 \oplus M_2$ of matroids $M_1 = (S_1, \mathcal{I}_1)$ and $M_2 = (S_2, \mathcal{I}_2)$ on disjoint ground sets is the matroid $M = (S_1 \cup S_2, \mathcal{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2 , that is, $\mathcal{I} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1 \text{ and } I_2 \in \mathcal{I}_2\}$. Given a non-negative integer k , the **k -truncation** of $M = (S, \mathcal{I})$ is the matroid $(M)_k = (S, \mathcal{I}_k)$ with $\mathcal{I}_k = \{X \in \mathcal{I} \mid |X| \leq k\}$. Given a subset $S' \subseteq S$, the **restriction** of M to S' is again a matroid $M|S' = (S', \mathcal{I}')$ with independence family $\mathcal{I}' = \{I \in \mathcal{I} \mid I \subseteq S'\}$. We also say that $M|S'$ is obtained by the **deletion** of $S - S'$, denoted by $M \setminus (S - S')$. The **contraction** of a subset $S'' \subseteq S$ results in a matroid $M/S'' = (S - S'', \mathcal{I}'')$ where $\mathcal{I}'' = \{I \in \mathcal{I} \mid I \subseteq S - S'', |I| = r_M(S'' \cup I) - r_M(S'')\}$. A matroid N that can be obtained from M by a sequence of deletions and contractions is called a **minor** of M . For uniform matroids, it is not difficult to see that $U_{k',\ell'}$ is a minor of $U_{k,\ell}$ if and only if $k' \leq k$ and $\ell - \ell' \geq k - k'$ hold. The following well-known result is [13, Theorem 4.3.1].

Proposition 2. *Let e be an element of a connected matroid M . Then M/e or $M \setminus e$ is connected.*

A class \mathcal{M} of matroids is **minor-closed** if for any member M of \mathcal{M} , each minor of M is also contained in \mathcal{M} . For a minor-closed class \mathcal{M} , a **nearly- \mathcal{M} matroid** is a matroid M such that $M/e \in \mathcal{M}$ or $M \setminus e \in \mathcal{M}$ for each element e . We will use the following observation of [16, 18].

Proposition 3. *The class of nearly- \mathcal{M} matroids is minor-closed for each minor-closed class \mathcal{M} of matroids.*

The **rank-2 wheel** $M(\mathcal{W}_2)$ is the matroid obtained from $U_{2,3}$ by adding a parallel copy of one of the elements of the ground set. The following is a consequence of a result of Gershkoff and Oxley [8].

Proposition 4. *Every connected non-uniform matroid contains $M(\mathcal{W}_2)$ as a minor.*

For connected matroids, the following proposition summarizes the relations between the different notions of flacets and characterizes compatibility, see [3, 6, 7, 10].

Proposition 5. *Let M be a connected matroid on ground set S with rank function r_M .*

- (a) *A subset $Z \subseteq S$ is a flacet of M if and only if it is a proper flat such that both $M|Z$ and M/Z are connected.*
- (b) *A flacet Z is a split flacet if and only if $|Z| \geq 2$, or equivalently, if Z is cyclic.*

(c) For distinct split facets F and G , the splits obtained from the F - and G -hyperplanes are compatible if and only if $|F \cap G| + r_M(S) \leq r_M(F) + r_M(G)$.

As we will show, the inequality of (c) motivates a matroid class slightly different from that of split matroids.

3 Hypergraph representation

In this section, we introduce the notion of elementary split matroids. Similarly to paving matroids, the definition is via hypergraphs, which will immediately imply that the proposed class is closed under duality, taking minors, and truncation.

Theorem 6. Let S be a ground set of size at least r , $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) collection of subsets of S , and r, r_1, \dots, r_q be non-negative integers satisfying

$$(H1) \quad |H_i \cap H_j| \leq r_i + r_j - r \text{ for } 1 \leq i < j \leq q.$$

Then $\mathcal{I} = \{X \subseteq S \mid |X| \leq r, |X \cap H_i| \leq r_i \text{ for } 1 \leq i \leq q\}$ forms the independent sets of a matroid with rank function $r_M(Z) = \min\{r, |Z|, \min_{1 \leq i \leq q} \{|Z - H_i| + r_i\}\}$. If furthermore

$$(H2) \quad |S - H_i| + r_i \geq r \text{ for } i = 1, \dots, q$$

holds, then the rank of the matroid is r .

Proof. The first two independence axioms clearly hold. A nice trick of the proof is that the third independence axiom (I3) and the rank formula is proved simultaneously. For any subset $Z \subseteq S$, let $I \subseteq Z$ be a maximal member of \mathcal{I} in the sense that I cannot be extended by an element of Z to a member of \mathcal{I} . If $|I| = \min\{r, |Z|\}$ then we are done, hence assume that strict inequality holds. Since I is maximal in Z , there is a hyperedge $H_z \in \mathcal{H}$ for every $z \in Z - I$ such that $|I \cap H_z| = r_z$ and $z \in H_z$. Furthermore, if $z', z'' \in Z - I$ are distinct elements, then the corresponding hyperedges $H_{z'}$ and $H_{z''}$ are identical as otherwise

$$\begin{aligned} |H_{z'} \cap H_{z''}| &\geq |I \cap H_{z'} \cap H_{z''}| \\ &= |I \cap H_{z'}| + |I \cap H_{z''}| - |I \cap (H_{z'} \cup H_{z''})| \\ &\geq r_{z'} + r_{z''} - |I| \\ &> r_{z'} + r_{z''} - r, \end{aligned}$$

contradicting (H1). Therefore there exists a hyperedge, say H_i , such that $Z - I \subseteq H_i$ and $|I \cap H_i| = r_i$. Thus we get $|I| = |I \cap H_i| + |I - H_i| = r_i + |Z - H_i|$, implying that the cardinality of I depends only on Z . Therefore the third independence axiom holds, and the rank formula is also verified. If (H2) holds, then the rank formula implies $r_M(S) = \min\{r, |S|, \min_{1 \leq i \leq q} \{|S - H_i| + r_i\}\} = r$, concluding the proof of the theorem. \square

We call the matroids that can be obtained in the form provided by Theorem 6 **elementary split matroids**. When (H2) fails for some $1 \leq i \leq q$, the rank of the matroid is less than r by the rank formula. In such a case, replacing r with $r' = \min_{1 \leq i \leq q} \{|S - H_i| + r_i\}$ does not violate (H1) while \mathcal{I} remains the same. Thus a rank- r elementary split matroid can be represented by a hypergraph $\mathcal{H} = \{H_1, \dots, H_q\}$ and values r, r_1, \dots, r_q satisfying both (H1) and (H2). It is not difficult to see that the underlying hypergraph can be chosen in such a way that

$$(H3) \quad r_i \leq r - 1 \text{ for } i = 1, \dots, q,$$

$$(H4) \quad |H_i| \geq r_i + 1 \text{ for } i = 1, \dots, q.$$

Indeed, if a pair (H_i, r_i) violates (H3) or (H4), then the corresponding constraint $|X \cap H_i| \leq r_i$ is redundant. Therefore, we call the representation **non-redundant** if all of (H1)–(H4) hold.

Elementary split matroids generalize paving matroids. Indeed, paving matroids correspond to the special case when $r_i = r - 1$ for $i = 1, \dots, q$. If, in addition, $|H_i| = r$ holds for $i = 1, \dots, q$, then we get back the class of sparse paving matroids.

Remark 7. The definition of elementary split matroids is closely related to the construction of matroids by cyclic flats, described in Proposition 1. Consider a non-redundant hypergraph representation $\mathcal{H} = \{H_1, \dots, H_q\}, r, r_1, \dots, r_q$ of a rank- r elementary split matroid. In order to exclude extreme cases, assume that $q \geq 1$, the r_i values are strictly positive, and (H2) holds with strict inequality for $1 \leq i \leq q$. We claim that the family $\mathcal{Z} = \{\emptyset, H_1, \dots, H_q, S\}$ satisfies the conditions of Proposition 1 with $r_M(\emptyset) = 0$, $r_M(H_i) = r_i$ and $r_M(S) = r$. Indeed, for different indices i and j we have $|H_i \cap H_j| \leq r_i + r_j - r \leq (|H_i| - 1) - 1 = |H_i| - 2$ by (H1), (H3) and (H4), hence $H_i \not\subseteq H_j$. Thus condition (Z0) is satisfied and $H_i \wedge H_j = \emptyset$, $H_i \vee H_j = S$ for each $i \neq j$. Condition (Z1) holds by $r_M(\emptyset) = 0$. Condition (Z2) for $X = \emptyset$ and $Y = H_i$ translates to $0 < r_i < |H_i|$, for $X = \emptyset$ and $Y = S$ it translates to $0 < r < |S|$, and for $X = H_i$ and $Y = S$ it translates to $0 < r - r_i < |S - H_i|$, all of which are satisfied by our assumptions. Conditions (Z0)–(Z2) imply that (Z3) is satisfied if either X or Y is $0_{\mathcal{Z}}$ or $1_{\mathcal{Z}}$, or if $X = Y$. If $X = H_i$ and $Y = H_j$ for $i \neq j$, then (Z3) is equivalent to $r_i + r_j \geq r + |H_i \cap H_j|$, that is, to (H1). Therefore, Proposition 1 provides another proof for M being a rank- r matroid whose system of cyclic flats is \mathcal{Z} . However, the addition of the missing extreme cases ensures that our class is minor-closed.

A nice feature of the class of split matroids is that it is closed under duality and taking minors. We show that the same holds for elementary split matroids. In addition, the class of elementary split matroids is closed for truncation, a property that split matroids do not satisfy in general. To see the latter, consider the matroid $M = (U_{1,2} \oplus U_{1,2} \oplus U_{1,2} \oplus U_{1,2})_3$, that is, the 3-truncation of the direct sum of four rank-1 uniform matroids on 2 elements. Then it is not difficult to check that M is connected and has a $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor, therefore it is not a split matroid, see Theorem 11 later.

Theorem 8. *The class of elementary split matroids is closed under duality, taking minors, and truncation.*

Proof. Let $M = (S, \mathcal{I})$ be a rank- r elementary split matroid and $\mathcal{H} = \{H_1, \dots, H_q\}$, r, r_1, \dots, r_q be a representation satisfying (H1) and (H2). For a non-negative integer $k < r$, the k -truncation of M is the matroid $(M)_k = (S, \mathcal{I}_k)$ where $\mathcal{I}_k = \{X \subseteq S \mid |X| \leq k, |X \cap H_i| \leq r_i \text{ for } 1 \leq i \leq q\}$. As $|H_i \cap H_j| \leq r_i + r_j - r \leq r_i + r_j - k$, $(M)_k$ is an elementary split matroid.

Now consider a set $Z \subseteq S$. The deletion of Z results in a matroid $M \setminus Z = (S - Z, \mathcal{I}_{S-Z})$ where $\mathcal{I}_{S-Z} = \{X \subseteq S - Z \mid |X| \leq r, |X \cap (H_i - Z)| \leq r_i \text{ for } 1 \leq i \leq q\}$. As $|(H_i \cap H_j) - Z| \leq |H_i \cap H_j| \leq r_i + r_j - r$, $M \setminus Z$ is an elementary split matroid. Note that (H2) might not hold for the restriction as the size of the ground set decreased, hence the rank of $M \setminus Z$ might be smaller than r .

Finally, define $\overline{H}_i := S - H_i$, $\overline{r} := |S| - r$, and $\overline{r}_i := |\overline{H}_i| - r + r_i$ for $i = 1, \dots, q$. Then $\overline{r} \leq |S|$ and $\overline{r}, \overline{r}_1, \dots, \overline{r}_q$ are non-negative by $r \leq |S|$ and (H2). By (H1), for $1 \leq i < j \leq q$, we obtain

$$\begin{aligned} |\overline{H}_i \cap \overline{H}_j| &= |S| - |H_i| - |H_j| + |H_i \cap H_j| \\ &\leq |S| - |H_i| - |H_j| + r_i + r_j - r \\ &= (|\overline{H}_i| - r + r_i) + (|\overline{H}_j| - r + r_j) - (|S| - r) \\ &= \overline{r}_i + \overline{r}_j - \overline{r}. \end{aligned}$$

By $r_i \geq 0$, for $i = 1, \dots, q$, we obtain

$$|S - \overline{H}_i| + \overline{r}_i = |H_i| + |S| - |H_i| - r + r_i \geq \overline{r}.$$

Therefore $\overline{\mathcal{H}} = \{\overline{H}_1, \dots, \overline{H}_q\}$, $\overline{r}, \overline{r}_1, \dots, \overline{r}_q$ satisfies all the conditions of Theorem 6, hence $\{X \subseteq S \mid |X| \leq \overline{r}, |X \cap \overline{H}_i| \leq \overline{r}_i \text{ for } i = 1, \dots, q\}$ forms the independent sets of a rank- \overline{r} elementary split matroid \overline{M} . For a set $X \subseteq S$ of size r , $|X \cap H_i| \leq r_i$ holds if and only if $|\overline{X} \cap \overline{H}_i| \leq \overline{r}_i$ holds, where $\overline{X} = S - X$. That is, the bases of M are exactly the complements of the bases of \overline{M} , thus \overline{M} coincides with the dual M^* of M .

As every minor of a matroid can be obtained by a series of deletions and contractions, and $M/Z = (M^* \setminus Z)^*$, the minor-closedness of the class of elementary split matroids follows. \square

Remark 9. Assume that the representation of M is non-redundant, that is, all of (H1)–(H4) are satisfied. By (H4), for $i = 1, \dots, q$, we obtain

$$\overline{r}_i = |\overline{H}_i| - r + r_i = |S| - |H_i| - r + r_i \leq \overline{r} - 1.$$

Furthermore, by (H3), for $i = 1, \dots, q$, we obtain

$$|\overline{H}_i| \geq |\overline{H}_i| - r + r_i + 1 = \overline{r}_i + 1.$$

That is, $\overline{\mathcal{H}} = \{\overline{H}_1, \dots, \overline{H}_q\}$, $\overline{r}, \overline{r}_1, \dots, \overline{r}_q$ satisfies (H3) and (H4) as well, hence then the representation of the dual provided by the proof of Theorem 8 is also non-redundant.

The following observation will be helpful when characterizing binary split matroids.

Theorem 10. *Consider a non-redundant representation $\mathcal{H} = \{H_1, \dots, H_q\}, r, r_1, \dots, r_q$ of an elementary split matroid M on ground set S . Then $M|H_i \cong U_{r_i, |H_i|}$ and $M/H_i \cong U_{r-r_i, |S-H_i|}$ for $i = 1, \dots, q$.*

Proof. Let $X \subseteq H_i$ be a subset of size r_i for some $1 \leq i \leq q$. Then $|X| = r_i < r$ by (H3) and $|X \cap H_j| \leq |H_i \cap H_j| \leq r_i + r_j - r < r_j$ by (H1) and (H3) for each index $j \neq i$, hence X is independent in M . As each independent subset of H_i has size at most r_i , we get that $M|H_i \cong U_{r_i, |H_i|}$. Considering the hypergraph representation of the dual matroid M^* constructed in the proof of Theorem 8, it follows that $M^*|\overline{H}_i \cong U_{\overline{r}_i, |\overline{H}_i|} = U_{|\overline{H}_i| - r + r_i, |\overline{H}_i|}$, hence $M/H_i = (M^*|\overline{H}_i)^* \cong U_{r-r_i, |H_i|}$. \square

4 Excluded-minor characterization

The aim of this section is to give an excluded-minor characterization of elementary split matroids. In contrast to split matroids where five forbidden minors are needed, elementary split matroids can be characterized by a single one. The next theorem determines the unique forbidden minor, and establishes a connection between elementary and connected split matroids.

Theorem 11. *The following are equivalent for a matroid M on ground set S .*

- (i) *M is an elementary split matroid.*
- (ii) *M has no $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor.*
- (iii) *M is a loopless and coloopless matroid whose proper cyclic flats form a clutter, or M is the direct sum of a uniform matroid with either a rank-0 matroid or a free matroid.*
- (iv) *M is a connected split matroid or the direct sum of two uniform matroids.*

Proof. (i) \Rightarrow (ii) The class of elementary split matroids is closed under taking minors by Theorem 8, hence it suffices to show that $M = U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ is not an elementary split matroid. Suppose to the contrary that there exists a hypergraph $\mathcal{H} = \{H_1, \dots, H_q\}$ and values r_1, \dots, r_q satisfying (H1)–(H4) with $r = 2$ which define M . As M has exactly one loop, there is an index i such that $|H_i| = 1$ and $r_i = 0$. We claim that H_i is the unique hyperedge in \mathcal{H} . Indeed, for an arbitrary index $j \neq i$, we have $0 \leq |H_i \cap H_j| \leq r_j - r < 0$ which is not possible. Hence $i = q = 1$ and $M \cong U_{0,1} \oplus U_{2,3}$, a contradiction.

(ii) \Rightarrow (iii) Suppose that M is $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor-free and has proper cyclic flats X and Y such that $X \subsetneq Y$. As X is a flat and Y is cyclic, $(M|Y)/X$ is loopless and not free, hence it has a $U_{1,2}$ -minor. Let $x \in X$ and $z \in S - Y$ and consider the matroid $M' = (M|(Y+z))/(X-x)$. As X is cyclic, x is a loop in $M/(X-x)$, and it is also a loop in M' . As Y is a flat, z is a coloop in $M|(Y+z)$, hence it is a coloop in M' as well. We get that $M' \cong U_{0,1} \oplus (M|Y)/X \oplus U_{1,1}$ where $(M|Y)/X$ has a $U_{1,2}$ -minor. Therefore M' has a $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor, hence so does M . This contradiction proves that the proper cyclic flats of M form a clutter.

It remains to consider the case when M is $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor-free and has a loop or a coloop. By duality, we may assume that M contains a loop. Let M' be the matroid obtained by deleting all the loops and coloops from M . If M' is empty, then M is the direct sum of a rank-0 and a (possibly empty) free matroid. Otherwise, as M' is loopless and coloopless, it has a $U_{1,2}$ -minor. This implies that M is coloopless, as otherwise it contains a $U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ -minor, contradicting the assumption. We also get that M' is connected, as otherwise it has a $U_{1,2} \oplus U_{1,2}$ -minor, meaning that M contains $U_{0,1} \oplus U_{1,2} \oplus U_{1,2}$ as a minor. By Proposition 4, each connected non-uniform matroid contains $M(\mathcal{W}_2)$ as a minor. However, $M(\mathcal{W}_2)$ has a $U_{1,2} \oplus U_{1,1}$ -minor, hence M' is necessarily a uniform matroid. Therefore M is the direct sum of a rank-0 and a uniform matroid.

(iii) \Rightarrow (iv) The implication is immediate if M is the direct sum of a uniform matroid with either a rank-0 matroid or a free matroid. Hence we may assume that M is loopless, coloopless and its cyclic flats form a clutter. If M is connected, we need to show that any two distinct split facets F and G are compatible, that is, $|F \cap G| + r_M(S) \leq r_M(F) + r_M(G)$ by Proposition 5. As proper cyclic flats form a clutter and each split facet is a proper cyclic flat, this inequality is equivalent to the cyclic flat axiom (Z3) for $X = F$ and $Y = G$.

Consider the case when M is disconnected and let Z_1, \dots, Z_t denote its connected components. As M is loopless and coloopless, Z_1, \dots, Z_t are proper cyclic flats. This implies $t = 2$, as otherwise Z_1 and $Z_1 \cup Z_2$ are both proper cyclic flats with $Z_1 \subsetneq Z_1 \cup Z_2$, contradicting the assumption that proper cyclic flats form a clutter. Assume that there is a proper cyclic flat $Z \notin \{Z_1, Z_2\}$. Then in the lattice of cyclic flats $Z \wedge Z_i = \emptyset$ and $Z \vee Z_i = S$, hence (Z3) implies that $r_M(Z) + r_M(Z_i) \geq r_M(S) + |Z \cap Z_i|$ holds for $i = 1, 2$. Thus we get

$$\begin{aligned} r_M(Z) + 1 &\leq |Z| \\ &= |Z \cap Z_1| + |Z \cap Z_2| \\ &\leq (r_M(Z) + r_M(Z_1) - r_M(S)) + (r_M(Z) + r_M(Z_2) - r_M(S)) \\ &= 2r_M(Z) - r_M(S) \\ &\leq r_M(Z) - 1, \end{aligned}$$

a contradiction. Therefore, the only proper cyclic flats of M are Z_1 and Z_2 , hence $M|_{Z_1}$ and $M|_{Z_2}$ are uniform matroids. We proved that M is the direct sum of two uniform matroids.

(iv) \Rightarrow (i) Assume first that M is the direct sum of a rank- r_1 uniform matroid on ground set H_1 and a rank- r_2 uniform matroid on ground set H_2 . Let $r := r_1 + r_2$. Then M is the elementary split matroid on ground set $S := H_1 \cup H_2$ corresponding to the hypergraph $\mathcal{H} = \{H_1, H_2\}$ and non-negative integers r, r_1, r_2 . Indeed, $r \leq |S|$ holds and (H1) is satisfied as $0 = |H_1 \cap H_2| \leq r_1 + r_2 - r = 0$.

Now consider the case when M is a connected split matroid. Let r denote the rank of M , $\mathcal{H} = \{H_1, \dots, H_q\}$ be the collection of split facets, and set the value of r_i to be the rank of H_i for $i = 1, \dots, q$. Then $r \leq |S|$ and the values r, r_1, \dots, r_q are non-negative. As M is a split matroid, any two split facets are compatible, therefore

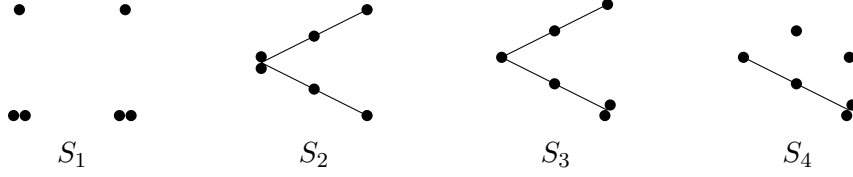


Figure 1: The connected excluded minors for split matroids [3].

(H1) is satisfied. By Theorem 6, $\mathcal{H} = \{H_1, \dots, H_q\}$, r, r_1, \dots, r_q define an elementary split matroid \hat{M} . We claim that \hat{M} is identical to M . Indeed, if a set $B \subseteq S$ is a basis in M then, by definition, it is also a basis in \hat{M} . If B is not a basis in M , then the characteristic vector of B is not contained in the base polytope of M . As the base polytope is completely determined by the facet inequalities (see e.g. [12, Theorem 2.6]), this means that $|B \cap F| > r_M(F)$ for some facet F of M . Since M is connected, necessarily $|F| \geq 2$. By Proposition 5(b), F is a split facet, hence B is not a basis of \hat{M} either. \square

5 Applications

As an application of our results, we give a new proof for the result of Cameron and Mayhew [3]. Furthermore, we further give a complete list of binary split matroids.

5.1 Split matroids

Based on the previous results, we give a different and shorter proof of the excluded minor characterization of split matroids originally proved by Cameron and Mayhew [3]. As already observed in [10], the only disconnected excluded minor for the class of split matroids is $M(\mathcal{W}_2) \oplus M(\mathcal{W}_2)$. This follows from Proposition 4 and the fact that a disconnected matroid is a split matroid if and only if it is the direct sum of a connected split matroid and uniform matroids (see [3, Proposition 2.7]). Joswig and Schröter [10] also identified four connected rank-3 excluded minors on 6 elements, these matroids S_1, S_2, S_3, S_4 are given by their geometric representations on Figure 1, see also [3].

Theorem 12. *The only connected excluded minors for split matroids are S_1, S_2, S_3 and S_4 .*

Proof. Let M be a connected matroid which is not a split matroid but each of its proper minors is. By Theorem 11, connected split matroids coincide with connected elementary split matroids, hence M is not elementary split while it is nearly-elementary-split by Proposition 2. As M is not elementary split, $(M/X) \setminus Y \cong U_{0,1} \oplus U_{1,2} \oplus U_{1,1}$ for some subsets $X, Y \subseteq S$ by Theorem 11. By M being loopless and coloopless, the sets X and Y are nonempty. Let $X' \subseteq X$ be a nonempty subset and consider the matroid $N = M/X'$. Notice that N is disconnected since it is split but not elementary split. Moreover, N is coloopless since M is coloopless.

We claim that N contains exactly one loop. If N is loopless, then each of its connected components has a $U_{1,2}$ -minor and each of its non-uniform components has an $M(\mathcal{W}_2)$ -minor by Proposition 4. As N is not the direct sum of two uniform matroids by Theorem 11, we get that it either contains $M(\mathcal{W}_2) \oplus U_{1,2}$ or $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$ as a minor. This contradicts Proposition 3 since neither of these two matroids is nearly-elementary-split. Similarly, if N has at least two loops, then it has a non-uniform connected component or at least two loopless connected components. Thus N contains either $U_{0,2} \oplus M(\mathcal{W}_2)$ or $U_{0,2} \oplus U_{1,2} \oplus U_{1,2}$ as a minor, both of which contradict Proposition 3. This proves that N has exactly one loop.

Suppose that $|X| \geq 2$ and choose distinct elements $x_1, x_2 \in X$. By our previous observation, M/x_1 contains exactly one loop l_1 and M/x_2 contains exactly one loop l_2 , that is, $\{x_1, l_1\}$ and $\{x_2, l_2\}$ are parallel classes of M . If $x_2 = l_1$, then $M/\{x_1, x_2\}$ is loopless. Otherwise, parallel classes $\{x_1, l_1\}$ and $\{x_2, l_2\}$ are disjoint and $M/\{x_1, x_2\}$ contains l_1 and l_2 as loops. Both of these cases contradict that $M/\{x_1, x_2\}$ contains exactly one loop. We conclude that $|X| = 1$. The class of split matroids is closed under duality, hence the dual M^* is also an excluded minor. Applying the previous argument to $(M^* \setminus X)/Y = ((M/X) \setminus Y)^* = U_{1,1} \oplus U_{1,2} \oplus U_{0,1}$, we get that $|Y| = 1$ holds as well.

We proved that M is a rank-3 matroid on 6 elements. Denote the element of X by a , the loop of M/a by b , the element of Y by e , the coloop of $M \setminus e$ by f , and the remaining two elements of the ground set by c and d . Then $\{e, f\}$ is a cocircuit of M , thus $M|\{a, b, c, d\}$ is a loopless rank-2 matroid containing the parallel class $\{a, b\}$, hence it is isomorphic to either $U_{1,2} \oplus U_{1,2}$ or $M(\mathcal{W}_2)$. The former case gives the matroid S_1 . In the latter case consider the lines $\{a, b, c, d\}$ and $\text{cl}_M(\{e, f\})$. If $\text{cl}_M(\{e, f\}) = \{e, f\}$, we get S_4 . Otherwise, the intersection of $\{a, b, c, d\}$ and $\text{cl}_M(\{e, f\})$ is a rank-1 flat, thus it is $\{a, b\}$, $\{c\}$ or $\{d\}$. The first case gives S_2 and the latter two cases give S_3 . \square

5.2 Binary split matroids

Acketa [1] gave a complete list of binary paving matroids: $U_{r,n}$ for $r \in \{0, 1, n-1, n\}$, loopless rank-2 matroids with at most three parallel classes, $M(K_4 - e)$, $M(K_4)$, $M(K_{2,3})$, F_7 , F_7^* and $AG(3, 2)$ (see also [13] for the definition of the latter three matroids). Based on this and our previous results, we extend this list to contain all binary split matroids. As each split matroid has at most one non-uniform connected component, we only consider the connected case. Recall that the only forbidden minor for binary matroids is $U_{2,4}$ by Tutte [17].

Theorem 13. *The following is a complete list of connected binary split matroids on at least two elements.*

- (a) *Matroids obtained by adding (possibly zero) parallel copies to an element of $U_{r-1,r}$ for any $r \geq 2$.*
- (b) *Loopless rank-2 matroids with exactly three parallel classes, and their duals.*
- (c) *Connected binary (sparse) paving matroids of rank and corank at least three: $M(K_4)$, F_7 , F_7^* , $AG(3, 2)$.*

Proof. It is not difficult to check that all the listed matroids are connected binary split matroids. It remains to prove that each binary split matroid M is included in the list. Connected matroids of rank or corank one are $U_{1,n}$ and $U_{n-1,n}$, and these are included in (a). Connected binary matroids of rank or corank two are exactly the matroids listed as (b). Thus we may assume that M has rank and corank at least three. As (c) contains binary paving matroids of rank and corank at least three from the list of Acketa [1], it only remains to consider the non-paving case.

Let M be given by a non-redundant hypergraph representation $\mathcal{H} = \{H_1, \dots, H_q\}$, r, r_1, \dots, r_q . As M is non-paving, $r_i \leq r - 2$ holds for some index i . By Theorem 10, $M/H_i \cong U_{r-r_i, |S-H_i|}$ where $2 \leq r - r_i$ by our assumption and $r - r_i + 1 \leq |S - H_i|$ as M is coloopless. As M contains no $U_{2,4}$ -minor, necessarily $|S - H_i| = r - r_i + 1$. Then $|H_i| = |S| - r + r_i - 1 \geq r_i + 2$ as M has corank at least 3. Furthermore, $M|H_i \cong U_{r_i, |H_i|}$ contains no $U_{2,4}$ -minor, implying $r_i = 1$ and $|S - H_i| = r$. Suppose that $q \geq 2$ and pick an index $j \neq i$. Applying (H1) and (H3), we get $1 + r_j - r \geq |H_i \cap H_j| \geq |H_j| - |S - H_i| \geq r_j + 1 - r$, hence $S - H_i \subseteq H_j$ and $|H_j| = r_j + 1$. Then $S - H_i \subseteq H_j$ implies that $r = |S - H_i| \leq |H_j| = r_j + 1 \leq r$, hence $S - H_j = H_j$ and $r_j = r - 1$. Therefore, S is the disjoint union of the rank-1 set H_i and the rank- $(r - 1)$ set H_j , contradicting the connectivity of M . This proves that $q = 1$, so $M \cong (U_{1, |H_i|} \oplus U_{r, r})_r$ as described in (a). \square

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References

- [1] D. M. Acketa. On binary paving matroids. *Discrete Mathematics*, 70(1):109–110, 1988.
- [2] J. E. Bonin and A. de Mier. The lattice of cyclic flats of a matroid. *Annals of Combinatorics*, 12(2):155–170, 2008.
- [3] A. Cameron and D. Mayhew. Excluded minors for the class of split matroids. *Australasian Journal of Combinatorics*, 79(2):195–204, 2021.
- [4] H. H. Crapo and G.-C. Rota. *On the Foundations of Combinatorial Theory: Combinatorial Geometries*. MIT press Cambridge, Mass., 1970.

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- [5] G. Drummond. A generalisation of uniform matroids. *Advances in Applied Mathematics*, 131:102248, 2021.
- [6] E. M. Feichtner and B. Sturmfels. Matroid polytopes, nested sets and Bergman fans. *Portugaliae Mathematica, Nova Série*, 62(4):437–468, 2005.
- [7] S. Fujishige. A characterization of faces of the base polyhedron associated with a submodular system. *Journal of the Operations Research Society of Japan*, 27(2):112–129, 1984.
- [8] Z. Gershkoff and J. Oxley. A notion of minor-based matroid connectivity. *Advances in Applied Mathematics*, 100:163–178, 2018.
- [9] J. Hartmanis. Lattice theory of generalized partitions. *Canadian Journal of Mathematics*, 11:97–106, 1959.
- [10] M. Joswig and B. Schröter. Matroids from hypersimplex splits. *Journal of Combinatorial Theory, Series A*, 151:254–284, 2017.
- [11] D. Mayhew, M. Newman, D. Welsh, and G. Whittle. On the asymptotic proportion of connected matroids. *European Journal of Combinatorics*, 32(6):882–890, 2011.
- [12] S. Oh and D. Xiang. The facets of the matroid polytope and the independent set polytope of a positroid. *arXiv preprint arXiv:1701.08483*, 2021.
- [13] J. Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
- [14] J. G. Oxley. Ternary paving matroids. *Discrete Mathematics*, 91(1):77–86, 1991.
- [15] J. A. Sims. *Some Problems in Matroid Theory*. PhD thesis, University of Oxford, 1980.
- [16] J. L. Tejada. Exploring matroid minors. Master’s thesis, California State University, San Bernardino, Electronic Theses, Projects, and Dissertations, 2020.
- [17] W. T. Tutte. A homotopy theorem for matroids, II. *Transactions of the American Mathematical Society*, 88(1):161–174, 1958.
- [18] V. N. Vega. Excluded minors for nearly-paving matroids. Master’s thesis, California State University, San Bernardino, Electronic Theses, Projects, and Dissertations, 2020.
- [19] D. J. A. Welsh. *Matroid Theory*. L. M. S. Monographs, No. 8. Academic Press, London-New York, 1976.