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Qianfan Chen<sup>\*</sup>, Siddhant Jajodia<sup>\*\*</sup>, Tibor Jordán<sup>\*\*\*</sup>, and  
Kate Perkins<sup>‡</sup>

## Abstract

By mapping the vertices of a graph  $G$  to points in  $\mathbb{R}^3$ , and its edges to the corresponding line segments, we obtain a three-dimensional realization of  $G$ . A realization of  $G$  is said to be globally rigid if its edge lengths uniquely determine the realization, up to congruence. The graph  $G$  is called globally rigid if every generic three-dimensional realization of  $G$  is globally rigid.

We consider global rigidity properties of braced triangulations, which are graphs obtained from maximal planar graphs by adding extra edges, called bracing edges. We show that for every even integer  $n \geq 8$  there exist braced triangulations with  $3n - 4$  edges which remain globally rigid if an arbitrary edge is deleted from the graph. The bound is best possible. This result gives an affirmative answer to a recent conjecture. We also discuss the connections between our results and a related more general conjecture, due to S. Tanigawa and the third author.

## 1 Introduction

A  $d$ -dimensional *framework* (or geometric graph) is a pair  $(G, p)$ , where  $G$  is a simple graph and  $p : V(G) \rightarrow \mathbb{R}^d$  is a map. We also call  $(G, p)$  a *realization* of  $G$  in  $\mathbb{R}^d$ . The length of an edge  $uv$  in the framework is defined to be the distance between the points  $p(u)$  and  $p(v)$ . The framework is said to be *rigid* in  $\mathbb{R}^d$  if every continuous motion of its vertices in  $\mathbb{R}^d$  that preserves all edge lengths preserves all pairwise distances. It is *globally rigid* in  $\mathbb{R}^d$  if the edge lengths uniquely determine all pairwise distances. A realization  $(G, p)$  is *generic* if the set of the  $d|V(G)|$  coordinates of the vertices is algebraically independent over the rationals. It is known that for generic frameworks rigidity and global rigidity in  $\mathbb{R}^d$  depends only on the graph of the framework, for every  $d \geq 1$ . So we may call a graph  $G$  *rigid* (resp. *globally rigid*) in  $\mathbb{R}^d$  if every (or equivalently, if some)  $d$ -dimensional realization of  $G$  is rigid (resp. globally rigid).

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The characterization of rigid and globally rigid graphs is known for  $d = 1, 2$ . For  $d \geq 3$  these are major open problems. We refer the reader to [9, 11] for more details on the theory of rigid and globally rigid frameworks and graphs.

Rigid and globally rigid graphs occur in several applications, including sensor network localization [5], molecular conformation [4], formation control [15], and statics [10]. In some applications it is desirable to have a graph which remains rigid or globally rigid even if some vertices or edges are removed. In this paper we study graphs  $G$  for which  $G - e$  is globally rigid in  $\mathbb{R}^d$  for each edge  $e$  of  $G$ . They are called *redundantly globally rigid* in  $\mathbb{R}^d$ . In the rest of the paper we focus on the three-dimensional case, i.e.  $d = 3$ , and the following two conjectures concerning redundant global rigidity.

A *triangulation*  $T = (V, E)$  is a maximal planar graph on at least three vertices. A *braced triangulation*  $G = (V, E \cup B)$  is a graph obtained from a triangulation  $T = (V, E)$  by adding a set  $B$  of new edges, called the *bracing edges*. If  $|B| = 1$  (resp.  $|B| = 2$ ) then we say that  $G$  is a *uni-braced* (resp. *doubly braced*) triangulation. The characterization of globally rigid braced triangulations in  $\mathbb{R}^3$  is known, see Theorem 6 below. A conjectured sufficient condition for redundant global rigidity is as follows.

**Conjecture 1.** [8] *Every 5-connected braced triangulation  $G = (V, T \cup B)$  with  $|B| \geq 2$  is redundantly globally rigid in  $\mathbb{R}^3$ .*

A related extremal problem is to determine the smallest number of edges in a redundantly globally rigid graph in  $\mathbb{R}^3$  on  $n$  vertices, as a function of  $n$ , for all (sufficiently large)  $n$ . By a theorem of B. Hendrickson [4] every globally rigid graph on  $n \geq d + 2$  vertices is redundantly rigid<sup>1</sup> in  $\mathbb{R}^d$ . It is well-known that a rigid graph in  $\mathbb{R}^3$  on  $n \geq 3$  vertices has at least  $3n - 6$  edges. These facts imply that  $3n - 4$  is a lower bound for the extremal value, and  $n \geq 6$  must hold. It was conjectured in [7] that this lower bound is tight.

**Conjecture 2.** [7] *For every integer  $k$  there exists a redundantly globally rigid graph  $G$  in  $\mathbb{R}^3$  on  $n \geq k$  vertices with  $3n - 4$  edges.*

Note that the truth of Conjecture 1, combined with existence of arbitrarily large 5-connected triangulations, would imply Conjecture 2.

In the rest of the paper – after introducing the results from rigidity theory that we shall use – we consider doubly braced triangulations in which both bracing edges are dihedral (i.e. they connect non-adjacent vertices that belong to edge sharing faces). We shall prove sufficient conditions that guarantee that a specific edge can be removed from such a triangulation while preserving global rigidity.

Based on these results we can analyse special families of such triangulations which will lead to the proof of (a stronger form of) Conjecture 2. We shall prove that for every even integer  $n \geq 8$  there exist redundantly globally rigid graphs in  $\mathbb{R}^3$  on  $n$  vertices

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<sup>1</sup>The definitions of *redundantly rigid* graphs and *doubly redundantly rigid* graphs are similar to that of redundantly globally rigid graphs: these graphs remain rigid after removing any edge (resp. any pair of edges). A conjecture of W. Whiteley [13] states that 5-connected doubly braced triangulations are doubly redundantly rigid. The truth of Conjecture 1, together with Hendrickson's theorem, would imply an affirmative answer to Whiteley's conjecture.

with  $3n - 4$  edges<sup>2</sup>. In the last section we prove necessary conditions for the redundant global rigidity of braced triangulations and formulate a couple of conjectures.

## 2 Rigid and globally rigid graphs

We shall use the following results in order to verify the (global) rigidity of a graph. Let  $G = (V, E)$  be a graph. For a vertex  $v \in V$  we use  $N_G(v)$  (resp.  $d_G(v)$ ) to denote the set (resp. the number) of neighbours of  $v$  in  $G$ . For a set  $X \subseteq V$  the graph obtained from  $G$  by adding a complete graph on vertex set  $X$  (that is, by adding new edges connecting the vertex pairs  $x, y \in X$  which are not adjacent in  $G$ ) is denoted by  $G + K(X)$ .

**Theorem 1.** [12] *Let  $G = (V, E)$  be a graph,  $v \in V$ , and  $d \geq 1$ . If  $G - v$  is rigid and  $G - v + K(N_G(v))$  is globally rigid in  $\mathbb{R}^d$  then  $G$  is globally rigid in  $\mathbb{R}^d$ .*

A bracing edge  $uv$  is called *dihedral* if it connects the two non-adjacent vertices  $u, v$  of two edge sharing triangular faces on vertices  $uab$  and  $vab$ , respectively, of (some planar embedding of)  $G$ . In this case the resulting  $K_4$  subgraph forms a rigid *4-block*. In our proofs we shall consider braced triangulations with one or two such 4-blocks. By removing a single edge (or a vertex of degree five) from a (braced) triangulation, we may create a face whose boundary is a 4-cycle (5-cycle, resp.). We shall say that such a cycle is a *4-hole* (or *5-hole*) in (some planar embedding of) the graph. For simplicity we shall call a braced triangulation with dihedral bracing edges and a removed edge or degree-five vertex a *block and hole graph*. See [2, 13] for a more general definition and results on rigid block and hole graphs in three-space. We need the following corollaries of their results.

**Theorem 2.** [13] *Let  $G$  be a 4-connected block and hole graph which has a single 4-hole and a single 4-block. Then  $G$  is rigid in  $\mathbb{R}^3$ .*

**Theorem 3.** [2] *Let  $G'$  be a 5-connected block and hole graph with two 4-blocks and let  $G = G' - v$ , where  $v$  is a vertex of degree five in  $G'$  which is disjoint from the blocks. Then  $G$  is rigid in  $\mathbb{R}^3$ .*

Let  $G$  be a graph and let  $uv, vw$  be a pair of incident edges in  $G$ . Let  $E_{uv}^v$  be the set of the remaining edges incident with  $v$  and let  $E_{uv}^v = F \cup F'$  be a bipartition of  $E_{uv}^v$ . The (3-dimensional) *vertex splitting* operation (at  $v$ , on edges  $uv, vw$ ) adds a new vertex  $v'$  to the graph, adds the new edges  $uv', v'w, vv'$ , and then replaces every edge  $xv$  in  $F'$  by an edge  $xv'$ . The edges in  $F$  stay incident to  $v$ . See Figure 1. The vertex splitting is said to be *non-trivial* if  $F$  and  $F'$  are both non-empty.

An important conjecture in rigidity theory is that non-trivial vertex splitting preserves global rigidity in  $\mathbb{R}^d$ , for all  $d \geq 1$ , see [1]. The next result verifies a special case.

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<sup>2</sup>We can extend our result to odd values of  $n$  by using different techniques. We do not discuss this extension in this paper.

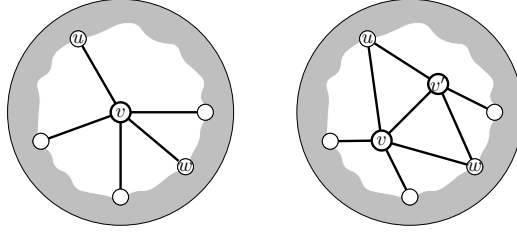


Figure 1: A non-trivial vertex splitting operation on edges  $uv$ ,  $vw$ .

**Theorem 4.** [8] *A graph is globally rigid if it can be obtained from  $K_5$  by a sequence of non-trivial vertex splitting operations.*

This theorem can be used in the analysis of globally rigid braced triangulations, due to the following combinatorial result.

**Theorem 5.** [8] *Every 4-connected uni-braced triangulation can be obtained from  $K_5$  by a sequence of non-trivial vertex splitting operations.*

Thus 4-connected uni-braced triangulations are globally rigid. A complete characterization, with no bounds on the number of bracing edges, is the following.

**Theorem 6.** [8] *A braced triangulation  $G = (V, E \cup B)$  with  $|V| \geq 5$  is globally rigid in  $\mathbb{R}^3$  if and only if  $G$  is 4-connected and  $|B| \geq 1$ .*

The inverse operation of vertex splitting is the contraction of an edge  $uv$  for which  $u$  and  $v$  have exactly two common neighbours. This operation takes a triangulation to a smaller triangulation. We shall also use the fact that an edge contraction decreases the vertex connectivity of a graph by at most one.

### 3 Redundant edges in braced triangulations

Every 5-connected braced triangulation with at least one bracing edge is globally rigid in  $\mathbb{R}^3$  by Theorem 6. In this section we describe several situations in which the removal of an edge from a 5-connected braced triangulation preserves global rigidity. The first lemma is an immediate corollary of Theorem 6.

**Lemma 1.** *Let  $G = (V, E \cup B)$  be a 5-connected braced triangulation with  $|B| \geq 2$ . Then  $G - e$  is globally rigid for every  $e \in B$ .*

In the rest of this section we shall assume that  $G$  is obtained from a triangulation by adding exactly two dihedral bracing edges that create two 4-blocks, with at most two vertices in common.

**Lemma 2.** *Let  $G = (V, E \cup B)$  be a 5-connected doubly braced triangulation with two 4-blocks. Suppose that  $e = uv \in E$  is an edge with  $d_G(v) = 5$  and  $v$  is disjoint from the 4-blocks. Then  $G - e$  is globally rigid.*

*Proof.* We shall prove that  $v$  satisfies the conditions of Theorem 1 in graph  $G-e$ . Since  $v$  is disjoint from the 4-blocks of  $G$ , the graph  $(G-e)-v (= G-v)$  is a block and hole graph with one 5-hole and two 4-blocks. The 5-connectivity of  $G$  and Theorem 3 imply that  $(G-e)-v$  is rigid. Next consider the graph  $H = (G-e)-v + K(N_{G-e}(v))$ . By 5-connectivity the four neighbours of  $v$  in  $G-e$  induce three edges in  $G-e$ . Thus three new edges are added to  $G-e$  to obtain  $H$ . Notice that  $H$  is a braced triangulation: two new edges can be used to triangulate the graph obtained from  $T = (V, E)$  by removing  $v$ , while the third one becomes a bracing edge. See Figure 2.

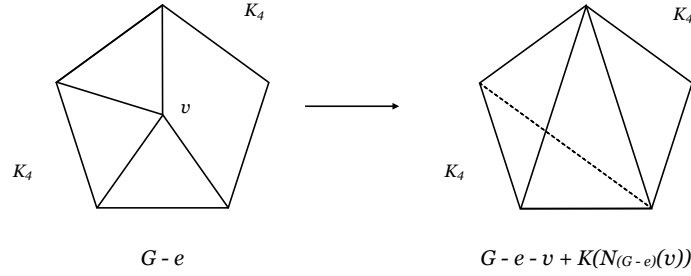


Figure 2: The neighbourhood of  $v$  in  $G-e$  and the edges they induce in  $H$ . The dashed edge is a bracing edge.

Since  $G$  is 5-connected,  $(G-e)-v = G-v$  is 4-connected. This implies that  $H$  is 4-connected. Hence  $H$  is globally rigid by Theorem 6. The lemma now follows from Theorem 1, applied to  $G-e$  and  $v$ .  $\square$

**Lemma 3.** *Let  $G = (V, E \cup B)$  be a 5-connected doubly braced triangulation with two 4-blocks. Suppose that  $e = uv \in E$  is an edge with  $d_G(v) = 5$  and  $v$  belongs to exactly one of the 4-blocks. Then  $G-e$  is globally rigid.*

*Proof.* Suppose that the 4-blocks are  $C_1$  and  $C_2$ , and  $v$  is part of  $C_1$ , say. Then the deletion of  $v$  from  $G-e$  creates a block and hole graph with a 4-block (namely,  $C_2$ ) and a 4-hole. Note that if  $v$  is not incident with the bracing edge  $f$  of  $C_1$  then  $f$  becomes an edge of the underlying (almost) triangulation of  $(G-e)-v$ . Since  $G$  is 5-connected,  $(G-e)-v = G-v$  is 4-connected. Thus  $(G-e)-v$  is rigid by Theorem 2. Furthermore, it follows that  $G-v + K(N_{G-e}(v))$  is a 4-connected braced triangulation with two bracing edges. Hence it is globally rigid by Theorem 6.

The lemma now follows from Theorem 1, applied to  $G-e$  and  $v$ .  $\square$

**Lemma 4.** *Let  $G = (V, E \cup B)$  be a 5-connected doubly braced triangulation with two 4-blocks  $C_1, C_2$  and let  $v \in V(C_1) - V(C_2)$ . Suppose that  $vw \in E \cap E(C_1)$  for which there is a triangular face  $uvw$  of  $T = (V, E)$  with  $u \notin V(C_1)$ . Let  $e = uv$ . Then  $G-e$  is globally rigid.*

*Proof.* We show that  $G-e$  can be obtained from  $K_5$  by a sequence of non-trivial vertex splitting operations. Observe that  $G-e$  has a 4-hole and two 4-blocks in

which  $v$  and  $w$  have exactly two common neighbours (the two other vertices of  $C_1$ ). Let  $H$  be the graph obtained from  $G - e$  by contracting the edge  $vw$ . It is easy to see that  $H$  is a 4-connected uni-braced triangulation. Thus  $H$  (and hence also  $G - e$ ) can be obtained from  $K_5$  by a sequence of non-trivial vertex splitting operations by Theorem 5. The lemma now follows from Theorem 4.  $\square$

The last lemma of this section is concerned with the case when the two 4-blocks share two vertices.

**Lemma 5.** *Let  $G = (V, E \cup B)$  be a 5-connected doubly braced triangulation with two 4-blocks  $C_1, C_2$  with  $V(C_1) \cap V(C_2) = \{a, b\}$ , where the crossing edges of  $C_1$  are  $bc$  and  $ad$ . Suppose that  $av$  is an edge for some vertex  $v$  disjoint from the 4-blocks. Then  $G - ab$ ,  $G - ac$ , and  $G - av$  are all globally rigid.*

*Proof.* We have  $V(C_1) - V(C_2) = \{c, d\}$ . Let us consider the removal of edge  $e = ab$ . Observe that in  $G - e$  the vertices  $c$  and  $a$  have exactly two common neighbours. Moreover,  $H$  is a 4-connected uni-braced triangulation, where  $H$  is the graph obtained from  $G - e$  by contracting the edge  $ca$ . Thus  $H$  (and hence also  $G - e$ ) can be obtained from  $K_5$  by a sequence of non-trivial vertex splitting operations by Theorem 5. Thus  $G - ab$  is globally rigid by Theorem 4.

The proof for edge  $ac$  is similar. In this case we delete the edge  $ac$ , contract the edge  $cd$ , and apply the same argument. Finally, to show that  $G - av$  is globally rigid, we use a similar proof again in which we delete  $av$  and then contract  $ac$ .  $\square$

## 4 Two families of graphs

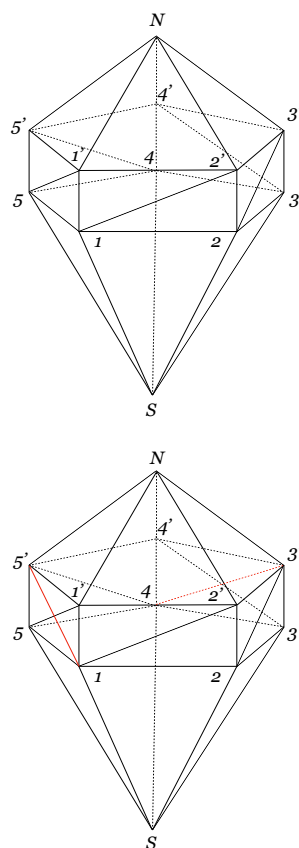
In this section, we define two infinite families of redundantly globally rigid doubly braced triangulations in  $\mathbb{R}^3$ .

**Definition 1** (Belted bipyramid). *For every  $n \geq 3$ , an  $n$ -gonal belted bipyramid, denoted by  $G(n)$ , is a graph on  $2n + 2$  vertices that is constructed as follows. Take two  $n$ -gonal pyramids with poles  $N$  and  $S$ , respectively, and label the vertices on the base of one pyramid 1 to  $n$  and on that of the other 1' to  $n'$  consecutively. Insert edges between the corresponding pairs of vertices (i.e. between 1 and 1', 2 and 2', and so on) and insert an edge between  $k$  and  $(k + 1)'$  for every  $1 \leq k \leq (n - 1)$ . Finally, insert an edge between  $n$  and 1'. See Figure 3.*

It is easy to see that  $G(n)$  is a triangulation. Let  $G(n, k)$  denote the graph obtained by inserting the edges  $1n'$  and  $k(k - 1)'$  to  $G(n)$ . Then  $G(n, k)$  is a doubly braced triangulation with two dihedral bracing edges. See Figure 3.

**Lemma 6.** *For every  $n \geq 5$ ,  $G(n)$  (and hence,  $G(n, k)$  for every  $2 \leq k \leq n$ ) is 5-connected.*

*Proof.* By using the structure and the symmetry of  $G(n)$  it is not hard to check that it is 5-connected. A simple argument is as follows: consider the base cycle  $C$  of one of the pyramids on vertex set  $(1, 2, \dots, n)$ . It is easy to verify that for every  $v \in V - V(C)$

Figure 3: The graphs  $G(5)$  and  $G(5, 4)$ .

there exist 5 paths from  $v$  to  $V(C)$  that are vertex-disjoint, apart from  $v$ . Furthermore, for every  $u, v \in V(C)$  there exist 5  $u$ - $v$ -paths that are vertex-disjoint apart from  $u, v$ . Since  $|V(C)| \geq 5$ , this implies that  $G(n)$  cannot have a vertex separator of size less than 5.  $\square$

**Theorem 7.** *For every  $n \geq 5$  and  $2 \leq k \leq n$  the graph  $G(n, k)$  is redundantly globally rigid in  $\mathbb{R}^3$ .*

*Proof.* Theorem 6 implies that  $G(n, k)$  is globally rigid in  $\mathbb{R}^3$ . It remains to show that the removal of any edge preserves global rigidity. First suppose that  $3 \leq k \leq n - 1$ , in which case the two 4-blocks are disjoint.

The bracing edges are redundant by Lemma 1. Note that each vertex has degree five in  $G(n, k)$ , except for the two poles and the end-vertices of the bracing edges. Thus we can use Lemmas 2 and 3 to show that most of the edges are redundant. The edges that do not satisfy the conditions of at least one of these two lemmas are the edges from the poles to the end-vertices of the bracing edges and, possibly, an edge that connects the end-vertices of different bracing edges. These edges are redundant by Lemma 4. So every edge is redundant and the graph is redundantly globally rigid, as required.



We can also show that  $G(n, 2)$  and  $G(n, n)$  are redundantly globally rigid by a similar argument. In these two special cases the two 4-blocks share two vertices, so we also need Lemma 5 in order to handle some of the edges incident with the intersection of the blocks.  $\square$

A slightly different construction is the following.

**Definition 2** (Flat belted bipyramid). *For every  $n \geq 4$ , an  $n$ -gonal flat belted bipyramid, denoted by  $F(n)$ , is a graph on  $2n$  vertices that is constructed as follows. Take  $G(n)$  and delete its two poles. Retaining the vertex labels described in Definition 1, for every  $1 \leq k \leq n$ , insert an edge between vertex 3 and vertex  $k$  (unless 3 is already adjacent to  $k$ ). Then, for every  $1 \leq k \leq n$ , insert an edge between vertex  $2'$  and vertex  $k'$  (unless  $2'$  is already adjacent to  $k'$ ). See Figure 4.*

It is easy to see that  $F(n)$  is a triangulation. Let  $H(n)$  be the graph obtained from  $F(n)$  by inserting edges  $1'2$  and  $3'4$ . See Figure 4. Thus  $H(n)$  is a doubly braced triangulation with two dihedral bracing edges that create two disjoint 4-blocks. Although  $F(n)$  is not 5-connected, a proof strategy similar to that of Lemma 6 can be used to show that  $H(n)$  is 5-connected.

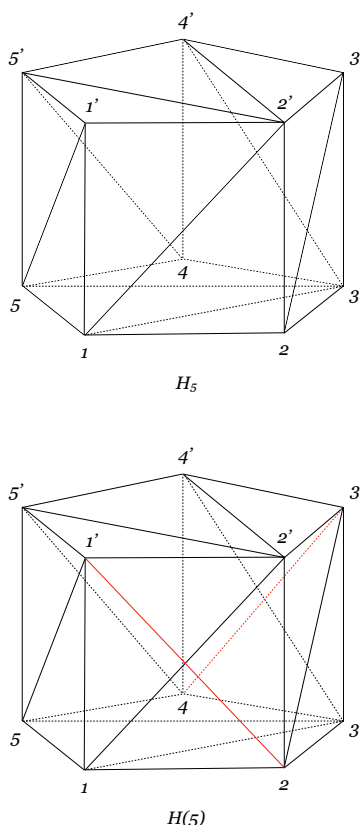


Figure 4: The graphs  $H_5$  and  $H(5)$ .

**Lemma 7.** *For every  $n \geq 4$  the graph  $H(n)$  is 5-connected.*

In fact we can show that  $H(4)$  is the smallest 5-connected doubly braced triangulation<sup>3</sup>.

**Theorem 8.**  *$H(n)$  is redundantly globally rigid in  $\mathbb{R}^3$  for  $n \geq 4$ .*

*Proof.* Theorem 6 implies that  $H(n)$  is globally rigid in  $\mathbb{R}^3$ . It remains to show that the removal of any edge preserves global rigidity. The rest of the proof is similar to that of Theorem 7, using the lemmas of the previous section. Note that in the case of  $H(n)$  the two 4-blocks are disjoint.  $\square$

The results of this section imply that Conjecture 2 is true.

**Theorem 9.** *For every even integer  $n \geq 8$  there exist redundantly globally rigid graphs in  $\mathbb{R}^3$  on  $n$  vertices with  $3n - 4$  edges.*

A simple degree count shows that there are no such graphs for  $n \leq 7$ .

As we noted earlier, redundantly globally rigid graphs are doubly redundantly rigid. Thus the graphs defined in this section are also examples of doubly redundantly rigid graphs with the smallest number of edges for every even  $n \geq 8$ . They are different from the ones constructed in [7], and easier to analyse.

## 5 Concluding remarks and conjectures

A natural question is whether the 5-connectivity condition in Conjecture 1 can be weakened. The next example shows that 5-connectivity is not necessary.

**Example** Consider the graph  $G$  in Figure 5. It is a 4-connected (but not 5-connected) doubly braced triangulation, and hence it is globally rigid by Theorem 6. We sketch a proof which shows that  $G - e$  is globally rigid for every edge  $e$ . By the symmetry of  $G$  we have four cases to consider: the deleted edge  $e$  is (i) a cross edge in the top  $K_4$ , (ii) a side in the top  $K_4$ , (iii) an edge from the  $K_4$  to the 4-cycle of the 4-separator, (iv) an edge of the 4-cycle of the separator.

In case (i)  $G - e$  is a 4-connected braced triangulation. In cases (ii) and (iii) we can apply (the proof of) Lemma 3 by noting that its proof works here by using the specific structure of  $G$  (rather than 5-connectivity). In case (iv) we perform two contractions and get to a 4-connected uni-braced triangulation as follows. Suppose, by symmetry, that  $e = cd$ . Then first contract an edge connecting  $c$  to the top  $K_4$ . Next contract one of the edges from  $c$  to the remainder of the top  $K_4$ . By contracting the appropriate edge we obtain a 4-connected uni-braced triangulation. Then global rigidity follows by Theorem 4.

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<sup>3</sup>The minimum degree condition implies that the number of vertices is at least eight, and equality holds only if the graph is 5-regular. Thus the complement of the graph is isomorphic to one of the following: (i) the disjoint union of a three-cycle and a five-cycle, (ii) the disjoint union of two four-cycles, (iii) a cycle on eight vertices. In the first two cases a simple analysis shows that the graph cannot be made planar by removing at most two edges. In the third case the graph is  $H(4)$ .

This leads us to the next question: is it possible to obtain a complete characterization of redundantly globally rigid braced triangulations, at least in some special cases (say, for doubly braced triangulations with two dihedral bracing edges)?

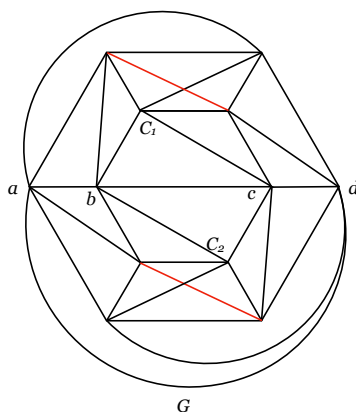


Figure 5: A redundantly globally rigid doubly braced triangulation  $G$  with a 4-separator  $S = \{a, b, c, d\}$ .

In this section we prove some necessary conditions and then formulate a conjecture. A  $k$ -separator  $S$  in a connected graph  $G = (V, E)$  is a set of vertices with  $|S| = k$  for which  $G - S$  is disconnected. For some  $X \subseteq V$  we use  $G[X]$  to denote the subgraph of  $G$  induced by vertex set  $X$ . It is known that for a minimal separator  $S$  in a triangulation  $G$  we have  $|S| \geq 3$ , the graph  $G - S$  has exactly two connected components, and  $G[S]$  is a cycle (see e.g. [8, Section 5]). For a separator  $S$  and connected component  $C$  of  $G - S$  we say that  $G[C \cup S]$  is an *extended component* of  $S$  in  $G$ .

**Lemma 8.** *Let  $G = (V, E \cup B)$  be a redundantly globally rigid braced triangulation and let  $S$  be a 4-separator in  $G$ . Suppose that  $S$  is a minimal separator in the underlying triangulation  $(V, T)$ . Then for every component  $C$  of  $G - S$  there exists a bracing edge incident with  $C$ .*

*Proof.* Let  $T = (V, E)$ . Since  $S$  is a minimal separator in  $T$ , the graph  $T - S$  (and hence also  $G - S$ ) has exactly two connected components  $C, D$ . For a contradiction suppose that there is no bracing edge incident with  $C$ . Since  $T[S]$  induces a 4-cycle the graph  $K$  obtained from the extended component  $G[C \cup S]$  of  $S$  by adding the edges that connect those vertex pairs of  $S$  which are not adjacent  $G$ , is a 4-connected uni-braced triangulation in which  $S$  induces a  $K_4$ . Let  $e$  be an edge of  $K$  incident with  $C$ . Then  $K - e$  is a minimally rigid graph on at least five vertices. By Hendrickson's theorem  $K - e$  is not globally rigid. The fact that  $G - e$  can be obtained from  $K - e$  by merging  $K - e$  and the other extended component  $G[D \cup S]$  along a complete graph (and, possibly, by deleting edges) implies that  $G - e$  is not globally rigid. This contradiction completes the proof.  $\square$

The proof shows that the lemma holds even if redundantly globally rigid is weakened to doubly redundantly rigid in the condition. If the underlying triangulation  $T$  is 4-connected, then every 4-separator of  $G$  is obviously a minimal separator in  $T$ , so the conditions of Lemma 8 are satisfied.

Let us consider the case when  $T$  is not 4-connected and  $G$  is doubly braced. Then for every 3-separator  $S$  of  $T$ , and corresponding components  $C, D$  of  $T - S$ , both bracing edges must connect  $C$  and  $D$  (for otherwise  $S$  is a 3-separator in  $G - e$  for some bracing edge  $e$ , contradicting redundant global rigidity). Call the components arising this way 3-separator components of  $T$ . It is not hard to see that this implies that  $T$  has exactly two minimal 3-separator components  $C_1$  and  $C_2$ , both bracing edges connect  $C_1$  and  $C_2$ , and that  $T$  can be made 4-connected by adding a single edge (from  $C_1$  to  $C_2$ ). We believe that in this rather special case  $G$  is redundantly globally rigid. Otherwise, when  $T$  is 4-connected, the necessary condition of Lemma 8, together with Hendrickson's connectivity condition, might be sufficient.

**Conjecture 3.** *Let  $G = (V, E \cup B)$  be a doubly braced triangulation. Then  $G$  is redundantly globally rigid in  $\mathbb{R}^3$  if and only if*

- (i)  $G - e$  is 4-connected for all  $e \in E \cup B$ , and
- (ii) either  $T = (V, E)$  has a 3-separator, or
- (iib) for every 4-separator  $S$  of  $G$  and component  $C$  of  $G - S$  there is a bracing edge incident with  $C$ .

Note that if  $G$  is doubly braced and the bracing edges induce two disjoint 4-blocks then  $T$  must be 4-connected. Thus in this case the conjecture can be simplified.

We close this section by noting that an interesting related open problem is to characterize globally rigid block and hole graphs (with no constraints on the size of the block and the number of holes - see [2] for the definition). It is possible that the global rigidity of these graphs can be characterized by Hendrickson's necessary conditions.

**Conjecture 4.** *A block and hole graph with a single block is globally rigid in  $\mathbb{R}^3$  if and only if it is 4-connected and redundantly rigid in  $\mathbb{R}^3$ .*

A characterization of redundantly rigid block and hole graphs with a single block can be obtained from a recent result in [6].

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