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Rigid block and hole graphs with a single block

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Abstract

A block and hole graph is obtained from the graph of a plane triangulation by removing the interiors of some discs, defined by their boundary cycles, and then rigidifying the vertex sets of some of these cycles by adding new vertices and edges. These rigid parts form the so-called blocks, while the remaining cycles define the holes. Cruickshank, Kitson, and Power proved that a a block and hole graph G with a single block is generically minimally rigid in $\mathbb{R}^3$ if and only if G is $(3,6)$-sparse and has $3|V(G)|-6$ edges. This result implies that there is an efficient algorithm for testing whether such a graph is rigid, provided it has exactly $3|V(G)|-6$ edges.

In this paper we extend these results to block and hole graphs G with a single block and an arbitrary number of edges. The extension is based on a new formula for the degrees of freedom of such a graph. It also enables us to find a smallest set of new edges whose addition makes G rigid. We also point out that there is an underlying matroid which can be defined by $(3,6)$-sparsity.

1 Introduction

A d-dimensional *framework* (or geometric graph) is a pair (G,p), where G is a simple graph and $p : V(G) \to \mathbb{R}^d$ is a map. We also call (G,p) a *realization* of G in $\mathbb{R}^d$. The length of an edge uv in the framework is defined to be the distance between the points $p(u)$ and $p(v)$. The framework is said to be *rigid* in $\mathbb{R}^d$ if every continuous motion of its vertices in $\mathbb{R}^d$ that preserves all edge lengths preserves all pairwise distances. A relization (G,p) is *generic* if the set of the $d|V(G)|$ coordinates of the vertices is algebraically independent over the rationals. It is known that for generic frameworks rigidity in $\mathbb{R}^d$ depends only on the graph of the framework, for every $d \geq 1$. So we may call a graph G *(generically) rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) d-dimensional generic realization of G is rigid. We refer the reader to [9] for more details on the theory of rigid frameworks and graphs.

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Finding a combinatorial characterization of rigid graphs and determining the complexity of testing the rigidity of a graph in $\mathbb{R}^d$, for $d \geq 3$, are major open problems in rigidity theory. In this paper we consider the three-dimensional case $d = 3$. Our goal is to find further families of graphs for which we can test rigidity (and more generally, compute the degrees of freedom) in polynomial time.

The starting point of our investigations is the following result due to Gluck. We shall call a maximal planar graph with at least three vertices a *triangulation*. It is known that a rigid graph G in $\mathbb{R}^3$ on at least three vertices has at least $3|V(G)| - 6$ edges. If G is rigid and has exactly $3|V(G)| - 6$ edges then it is called *minimally rigid*.

Theorem 1.1. *[5] Every triangulation is minimally rigid in* $\mathbb{R}^3$.

Whiteley [10] initiated the study of the rigidity properties of modified triangulations that may contain blocks and holes. Extending the results of Finbow-Singh and Whiteley [4] on the single block and single hole case, Cruickshank, Kitson, and Power [3] characterized the minimally rigid triangulations with a single block and arbitrarily many holes.

To describe their result we need some definitions. Consider a planar embedding of a triangulation $G = (V, E)$. Note that G is 3-connected. A cycle C of G divides the plane into two parts and hence it determines two subgraphs of G that share the edges and vertices of C. Such a subgraph is called a *disc*. We say that it is *bounded* by C, or that C is its *boundary cycle*. The *interior* of a disc consists of the vertices and edges of the disc that do not belong to its boundary cycle. Two discs are *internally disjoint* if their common edges or vertices, if they exist, are part of their boundary cycles.

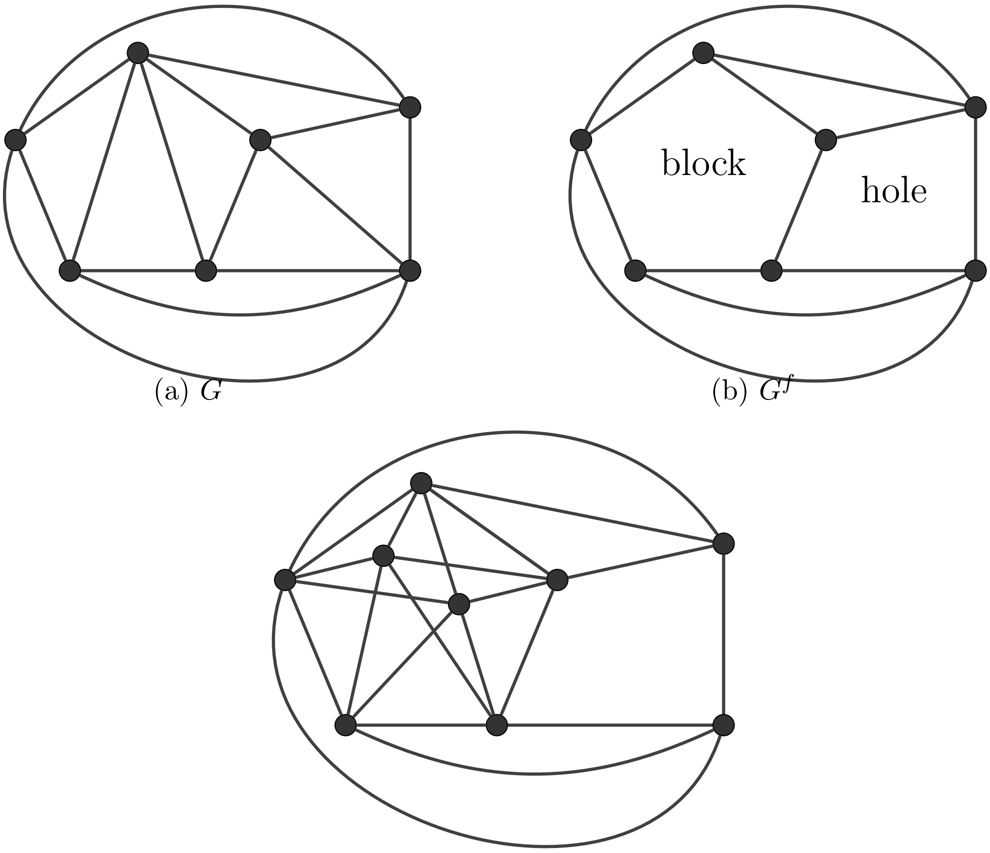
We say that a *face graph* G^f of G is obtained from (a planar embedding of) G by choosing a collection of pairwise internally disjoint discs, removing the interiors of these discs, and then labeling the non-triangular faces of the resulting (embedded) planar graph by either b (block) or h (hole)[1]. We may restrict ourselves to discs bounded by cycles of length at least four in G.

A *block-and-hole graph* $G^\diamond$ with face graph G^f is obtained from G^f by adding new vertices and edges that rigidify the vertex set of each block. This is achieved as follows. Let C be the boundary cycle of a block-labeled face. We add two new vertices x_C and y_C as well as edges that connect these new vertices to each vertex of C. Then the vertex set $V(C) \cup \{x_C, y_C\}$ induces a bipyramid, which is a minimally rigid graph (a triangulation). We denote this subgraph by B_C. The block and hole graph is the union of the face graph and these bipyramids, one for each block-labeled face[2]. See Figure 1.

We say that a graph $G = (V, E)$ is $(3, 6)$-*sparse* if $i_G(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 3$. Here $i_G(X)$ denotes the number of edges in the subgraph of G induced

[1] The definition of face graphs in [3, Definition 3] is slightly different. However, the authors (implicitely) use the definition given here.

[2] There are other ways to rigidify the block-labeled faces. Here we use this construction, which is called the *discus and hole graph* in [3].



(a) G (b) G^f

(c) $G^\diamond$

Figure 1: A triangulation G, a face graph G^f defined by two cycles of length five and four, respectively, and the block and hole graph $G^\diamond$. Graph $G^\diamond$ is rigid, but not minimally rigid.

by the vertex set X. If G is a $(3,6)$-sparse graph with $|E| = 3|V| - 6$ then G is called $(3,6)$-*tight*. Note that in a simple graph no subset $X \subseteq V$ with $|X| \leq 4$ can violate the sparsity count. Moreover, the subsets $X \subseteq V$ with $|X| = 2$ satisfy the weaker bound $i(X) \leq 3|X| - 5$. A subgraph $H = (V, F)$ of G is said to be a *maximal* $(3,6)$-*sparse* subgraph of G it is $(3,6)$-sparse but $H + e$ is not $(3,6)$-sparse for all $e \in E - F$.

Example 1. Let $G = (V, E)$ be the graph obtained from two disjoint triangles on vertex sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, respectively, by adding two vertices v_1, v_2 and then new edges from v_i to every other vertex, for $i = 1, 2$. See Figure 2. Graph G is not $(3,6)$-sparse, since $19 = |E| > 3|V| - 6 = 18$. The subgraphs $G_1 = G - v_1v_2$ and $G_2 = G - a_1a_2 - b_1b_2$ are both maximal $(3,6)$-sparse subgraphs of G. The graph G_1 is also $(3,6)$-tight.

It is known that minimally rigid graphs in $\mathbb{R}^3$ are $(3,6)$-tight, but there exist non-rigid $(3,6)$-tight graphs (e.g. the graph G_1 in Example 1.). The next result

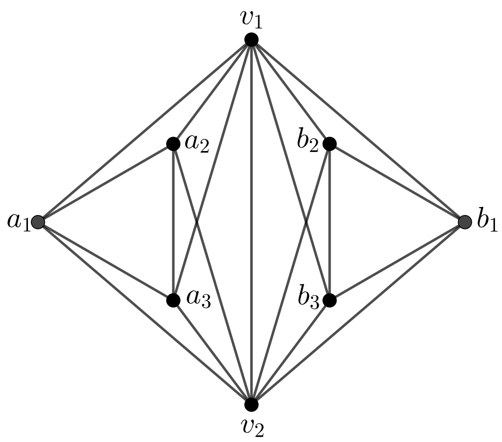


Figure 2: The graph of Example 1.

shows that for block and hole graphs with a single block this necessary condition is also sufficient.

Theorem 1.2. *[3, Theorem 36] Let $G^\diamond$ be a block and hole graph with a single block. Then $G^\diamond$ is minimally rigid in $\mathbb{R}^3$ if and only if $G^\diamond$ is $(3,6)$-tight.*

In order to extend Theorem 1.2 we recall the notion of the rigidity matroid of a graph and a recent result which provides a further link between $(3,6)$-sparsity and rigidity. The 3-dimensional *rigidity matroid*, denoted by $\mathcal{R}_3(G)$, of a graph $G = (V, E)$ is defined on the edge set E. The rank of this matroid, denoted by $r_3(G)$, determines whether G is rigid, and more generally, the so-called *degrees of freedom* of G. The graph G (with $|V| \geq 3$) is rigid if $r_3(G) = 3|V| - 6$ holds. See e.g. [8] for more details on the rigidity matroid. Thus we can test the rigidity of G if we can efficiently compute the rank of $\mathcal{R}_3(G)$.

The following result is due to J. Cheng and M. Sitharam [2]. See also [6] for a different proof and extensions to the higher dimensions.

Theorem 1.3. *[2] Let $H = (V, F)$ be a maximal $(3,6)$-sparse subgraph of graph $G = (V, E)$. Then $r_3(G) \leq |F|$.*

2 Rigid block and hole graphs with a single block

We start with three simple observations.

2.1 Preliminary observations

We say that a graph $G = (V, E)$ on at least three vertices is 2-connected if $G - v$ is connected for all $v \in V$.

Lemma 2.1. *Every face graph K is 2-connected.*

Proof. Since K is a face graph, it can be obtained from a(n embedded) triangulation G by removing the interiors of some discs. The graph G itself is a 2-connected (in fact, 3-connected) face graph, and the removals can be done independently, so it suffices to show that the removal of the vertices and edges of the interior of a disc D from a 2-connected face graph K' preserves 2-connectivity.

Let C be the boundary cycle of D and let K denote the graph obtained by removing the interior of D. The planarity and 2-connectivity of K' implies that for each vertex w which does not belong to D there exist two paths from w to $V(C)$ avoiding the interior of D which are vertex-disjoint apart from w. The existence of these paths and the cycle C in K show that it is impossible to disconnect K by removing a single vertex. Thus K is 2-connected, as required. $\square$

The statement of the lemma can be reversed in the following sense.

Lemma 2.2. *Let K be a 2-connected planar graph and let J be a non-triangular face in some planar embedding of K. Then there is a triangulation G for which K is a face graph of G in which J is a face.*

Proof. Consider a planar embedding of K in which J is a face. Since K is 2-connected, the boundary of each face is a cycle. Let G be a triangulation obtained from K by adding edges so that each face of length at least four becomes triangulated. Then K is the face graph of (this embedding of) G, where the discs that define K correspond to the faces in the embedding of K. In particular, J is a face. $\square$

Lemma 2.3. *Suppose that $H = (V, F)$ is a maximal $(3, 6)$-sparse subgraph of a 2-connected graph $G = (V, E)$. Then H is 2-connected.*

Proof. For a contradiction suppose that there is a partition $\{V_1, V_2, \{t\}\}$ of V, where $V_1, V_2 \subset V$ are non-empty vertex sets and $t \in V$, such that there is no edge in F between V_1 and V_2. Since G is 2-connected, there is an edge $f = u_1u_2$ with $f \in E - F$ and $u_i \in V_i$, $i = 1, 2$. By the maximality of H the addition of f to H destroys $(3, 6)$-sparsity. Hence there is a set $X \subseteq V$ with $|X| \geq 3$, $\{u_1, u_2\} \subseteq X$, and $i_H(X) = 3|X| - 6$. Let $X_i = (X \cap V_i) \cup \{t\}$ and $n_i = |X_i|$ for $i = 1, 2$. Thus $|X| = n_1 + n_2 - 1$. Since H is $(3, 6)$-sparse, we have

$$i_H(X) = i_H(X_1) + i_H(X_2) \leq 3n_1 - 5 + 3n_2 - 5 = 3(n_1 + n_2 - 1) - 7 = 3|X| - 7,$$

which contradicts the fact that $i_H(X) = 3|X| - 6$. $\square$

2.2 The rank of a block and hole graph with a single block

We are ready to state the rank formula of block and hole graphs with a single block, which is the main result of this section.

Theorem 2.4. *Let $G^\diamond$ be block and hole graph with a single block $B^\diamond$ and let $H = (V(G^\diamond), F)$ be a maximal $(3,6)$-sparse subgraph of $G^\diamond$ with $E(B^\diamond) \subseteq F$. Then*

$$r_3(G^\diamond) = |F| \tag{1}$$

In particular, $G^\diamond$ is rigid if and only if H is $(3,6)$-tight.

Proof. Since $B^\diamond$ is minimally rigid, it is $(3,6)$-sparse. Thus we can extend (the edge set of) the block to a maximal $(3,6)$-sparse subgraph. Therefore a maximal $(3,6)$-sparse subgraph $H = (V(G^\diamond), F)$ with $E(B^\diamond) \subseteq F$ indeed exists.

The inequality $r_3(G^\diamond) \leq |F|$ follows from Theorem 1.3. Let $V = V(G^\diamond)$ and define the number $k(G^\diamond, H) := 3|V| - 6 - |F|$. In order to show that equality holds, we shall prove that it is possible to add a set L of new edges to $G^\diamond$ with $|L| = k(G^\diamond, H)$, so that the resulting graph G^+ is a block and hole graph with a single block $B^\diamond$ in which $H + L$ is a minimally rigid spanning subgraph. If such a set L exists then the rigidity of $H + L$ implies that G^+ is rigid, and hence we can use the lower bound $r_3(G^\diamond) \geq r_3(G^+) - |L| = 3|V| - 6 - (3|V| - 6 - |F|) = |F|$ to deduce that equality holds in (1).

In the rest of the proof we shall fix a planar embedding of the face graph G^f (inherited from the planar embedding of the triangulation), which also fixes the planar embedding of its subgraph $H^- := H - \{x, y\}$, where x and y are the new vertices (i.e. the poles) of the bipyramid $B^\diamond$. We prove that the required set L exists by induction on $k(G^\diamond, H)$. First consider the base case $k(G^\diamond, H) = 0$. Then $L = \emptyset$ is a good choice, which can be seen as follows. Since we add no edges, we have $G^+ = G^\diamond$ and $H + L = H$. Thus H is $(3,6)$-tight. To show that H is minimally rigid we first use Lemmas 2.1 and 2.3 (and the fact that $B^\diamond$ is 3-connected) to deduce that H is 2-connected. It is easy to see that H^- is also 2-connected. Moreover, the facts that the edges of the block are all in H, H^- is a subgraph of the (embedded) face graph G^f, and Lemma 2.2 imply that H is a block and hole graph with a single block $B^\diamond$. We can now apply Theorem 1.2 to the $(3,6)$-tight graph H to deduce that it is minimally rigid. This completes the proof of the base case.

We next consider the general case. Let $k(G^\diamond, H) \geq 1$ and suppose that the statement holds for all block and hole graphs G' with a single block $B^\diamond$ and maximal $(3,6)$-sparse subgraph H' with $k(G', H') < k(G^\diamond, H)$. Our goal is to show that there is an edge $f = uv$ with $u, v \in V$, for which $G^\diamond + f$ is a block and hole graph with a single block $B^\diamond$, and $H + f$ is $(3,6)$-sparse. This will imply the theorem by induction, since $k(G^\diamond + f, H + f) = k(G^\diamond, H) - 1$.

Let us call a set $X \subseteq V$ with $|X| \geq 3$ *tight* if $i_H(X) = 3|X| - 6$ holds. Let $H[X]$ denote the subgraph of H induced by X. Note that for a tight set X we have that $H[X]$ is a triangle (if $|X| = 3$) or $H[X]$ has minimum degree at least three (if $|X| \geq 4$).

Claim 2.5. *Let X, Y be tight sets with $|X \cap Y| \geq 2$. Then $X \cup Y$ is tight, unless $|X \cap Y| = 2$ and $X \cap Y$ induces an edge.*

Proof. First suppose $|X \cap Y| \geq 3$. The $(3,6)$-sparsity of H and the supermodularity[3] of i_H imply that $3|X| - 6 + 3|Y| - 6 = i_H(X) + i_H(Y) \leq i_H(X \cap Y) + i_H(X \cup Y) \leq 3|X \cap Y| - 6 + 3|X \cup Y| - 6 = 3|X| - 6 + 3|Y| - 6$. Thus equality holds everywhere. In particular, $X \cup Y$ is tight. A similar argument shows that if $i_H(X \cup Y) \leq 3|X \cup Y| - 7$ then we must have $i_H(X \cap Y) \geq 3|X \cap Y| - 5$. Therefore $|X \cap Y| = 2$ and $i_H(X \cap Y) = 1$ must hold. It means that $X \cap Y$ induces an edge. □

Since the vertex set of block $B^\diamond$ is a tight set with more than two vertices, Claim 2.5 implies that the union T of all the tight sets that contain the (vertex set of the) block $B^\diamond$ is tight. Let G_T denote the subgraph of G^f induced by $T - \{x, y\}$. A proof similar to that of Lemma 2.3 shows that the subgraph of H induced by $T - \{x, y\}$, and hence also G_T, is 2-connected. Since $k(G^\diamond, H) \geq 1$, we must have $\bar{T} := V - T \neq \emptyset$. Therefore there is a face K of (the embedded) G_T, with boundary cycle C, for which the disc D bounded by C in (the embedded) G^f contains at least one vertex of $\bar{T}$. Since the block $B^\diamond$ is in T, K is different from the face that defines the block. Let V_{int} (resp. E_{int}) denote the set of the internal vertices (resp. edges) of D.

Claim 2.6. $E_{int} \subset F$.

Proof. Suppose that there is an edge $e = ab$ with $e \in E_{int} - F$. By the maximality of H the graph $H + e$ is not $(3,6)$-sparse. This implies that there is a tight set S with $\{a, b\} \subset S$. Note that at least one of a and b is not in T, and hence T is a proper subset of $T \cup S$. Then Claim 2.5, the maximality of T, and the fact that $H[S]$ has minimum degree at least two imply that S is disjoint from $\{x, y\}$. Hence $H[S] + e$ is, as an (embedded) subgraph of G^f, planar. This is a contradiction, since $H[S] + e$ cannot have more than $3|S| - 6$ edges by Euler's formula. □

Essentially the same proof leads to the following version, which shows that adding a diagonal of an internal face of D to H preserves $(3,6)$-sparsity.

Claim 2.7. *Let a, b be a pair of vertices on the boundary cycle C' of some face of D with $a \in V_{int}$. If $ab \notin E(G^f)$ then $H + ab$ is $(3,6)$-sparse.*

In the rest of the proof we consider two cases. In the first case we assume that each face in the interior of D is triangular. We claim that $|E_{int}| \geq 3|V_{int}|$. To see this observe that the planar embedding of D, which is a near triangulation, can be extended to a triangulation by adding a set E' of $|C| - 3$ edges, connecting pairs of vertices in the exterior of the disc. With these edges we have $3(|C| + |V_{int}|) - 6 = |E_{int}| + |C| + |E'| = |E_{int}| + |C| + |C| - 3$. Thus we have

$$|E_{int}| = 3|V_{int}| + |C| - 3, \tag{2}$$

and hence $|C| \geq 3$ implies $|E_{int}| \geq 3|V_{int}|$, as claimed. By using Claim 2.6 and $V(C) \subseteq T$ this yields $i_H(T \cup V_{int}) \geq 3|T| - 6 + 3|V_{int}| = 3|T \cup V_{int}| - 6$. Hence

[3] It is well-known and easy to check that for every graph $G = (V, E)$ and for every pair $X, Y \subseteq V$ we have $i_G(X) + i_G(Y) \leq i_G(X \cap Y) + i_G(X \cup Y)$.

equality must hold and $T \cup V_{int}$ is tight. This contradicts the maximality of T and shows that this case cannot occur.

Thus it remains to consider the case when there exists a non-triangular face in D, bounded by some cycle C' (of length at least four). The next claim shows that we can add a new diagonal to this face which is incident with a vertex of V_{int}.

Claim 2.8. *There is a vertex pair $\{u, v\} \in V(C')$ for which $u \in V_{int}$ and for which $G^f + f$ is simple and planar, where $f = uv$.*

Proof. Let us call an edge e of G^f which connects two non-consecutive vertices of C' an *outer diagonal*. Since $V_{int} \neq \emptyset$, we have $C \neq C'$, and hence at least one vertex of C', say u, must belong to V_{int}. If there is no outer diagonal incident with u then for the second neighbour $v \in V(C')$ of u on C' the pair $\{u, v\}$ satisfies the required properties. So it remains to consider the case when there is an outer diagonal uw. The cycle C' is divided into two internally disjoint subpaths P_1, P_2, connecting u to w. The key observation, which follows from the structure of the planar embedding, is that the set of the internal vertices of at least one of these paths, say P_1, is disjoint from C. In this case let us redefine u to be an internal vertex of P_1 and let v be an internal vertex of P_2. The existence of the outer diagonal and the choice of u imply that the pair $\{u, v\}$ satisfies the required properties. $\square$

Let f be the edge of Claim 2.8. Then $G^f + f$ is simple and planar. Furthermore, by Claim 2.7, $H + f$ is $(3, 6)$-sparse. Thus f is the desired edge: $G^\diamond + f$ is a block and hole graph with a single block $B^\diamond$ in which $H + f$ is $(3, 6)$-sparse. This completes the proof. $\square$

Figure 3: A triangulated dome with a single hole and single block.

We obtain the following characterization as a corollary.

Theorem 2.9. *Let $G^\diamond$ be a block and hole graph with a single block. Then $G^\diamond$ is rigid if and only if it has a minimally rigid spanning subgraph which is a block and hole graph with the same block.*

2.3 A matroid on the set of edges that triangulate the holes

We can deduce from Theorem 2.4 that the maximal $(3,6)$-sparse subgraphs of $G^\diamond$ that contain the block have the same size[4]. In fact with a simple extra argument we can show that the sparsity count defines a matroid on the missing edges of some triangulation in the following sense.

Let $G^\diamond$ be a block and hole graph on vertex set V with a single block $B^\diamond$. Consider the underlying (embedded) face graph G^f. Let J be a set of edges that triangulate the holes. Note that $G^f + J'$ is a face graph (and $G^\diamond + J'$ is a block and hole graph with a single block) for every $J' \subseteq J$. Let $\mathcal{H}$ be the family of maximal $(3,6)$-sparse subgraphs $H = (V, F)$ of $G^\diamond$ with $E(B^\diamond) \subseteq F$ and define

$$\mathcal{I} = \{I \subseteq J : (V, F \cup I) \text{ is } (3,6)\text{-sparse for some } H = (V,F) \in \mathcal{H}\}$$

We show that $\mathcal{I}$ corresponds to the independent sets of a matroid[5] on ground set J.

Theorem 2.10. *$(J, \mathcal{I})$ is a matroid.*

Proof. We shall verify that each of the axioms (M1)-(M3) is satisfied by $\mathcal{I}$. It is easy to see that (M1) and (M2) hold. To prove (M3) let $J' \subseteq J$ and let I', I'' be maximal subsets of J with $I', I'' \in \mathcal{I}$. By definition there exist maximal (3,6)-sparse subgraphs $H' = (V, F')$, $H'' = (V, F'')$ of $G^\diamond$ that contain the block and for which $H'_+ = (V, F' \cup I')$ and $H''_+ = (V, F'' \cup I'')$ are $(3,6)$-sparse. Observe that H'_+ and H''_+ are both maximal $(3,6)$-sparse subgraphs of $G^\diamond + J'$. Since $G^\diamond + J'$ is a block and hole graph with a single block, Theorem 2.4 implies that $|F' \cup I'| = |F'' \cup I''|$. Moreover, we have $|F'| = |F''|$, implying that $|I'| = |I''|$, as required. $\square$

Remark 1. It is well-known that, in general, the edge sets of the $(3,6)$-sparse subgraphs of a graph do not form the family of independent sets of a matroid on the edge set of a graph G. See Example 1. There are some rare exceptions: for example, when G has maximum degree at most five, these edge sets define a matroid [7]. Hence it may be interesting to note that, as it can be seen from Theorem 2.10, the $(3,6)$-sparsity count can be used to define a matroid on (the edge sets of the triangulations of) block and hole graphs with a single block.

Remark 2. We sketch a slightly a different approach that can be used to deduce the rank formula of Theorem 2.4, suggested by Csaba Király (private communication). We use the notation of Theorem 2.4. Let us triangulate the holes of $G^\diamond$ by adding a set of new edges. The resulting graph G^+ can be obtained from a triangulation by adding a vertex (i.e. one of the poles of the block). Thus it is rigid in $\mathbb{R}^3$ by Theorem 1.1 and the fact that adding a vertex of degree at least three to a rigid graph preserves rigidity. Hence a maximal $(3,6)$-sparse subgraph H^+ of G^+ that

[4]This is best possible: it is easy to construct a block and hole graph with two four-blocks and one four-hole from graph G of Example 1 for which this property fails for each of the two blocks.

[5]Recall the following matroid axioms: $(J, \mathcal{I})$ is a matroid if and only if (M1) $\emptyset \in \mathcal{I}$, (M2) if $I' \subseteq I$ and $I \in \mathcal{I}$ then $I' \in \mathcal{I}$, (M3) for every $J' \subseteq J$ and maximal subsets I', I'' of J' with $I', I'' \in \mathcal{I}$ we have $|I'| = |I''|$.

contains H must be $(3,6)$-tight by Theorem 1.3. It can be shown, as in the proof of Theorem 2.4, that H^+ is a block and hole graph with a single block. Hence it is rigid by Theorem 1.2. From here the argument of the first part of the proof of Theorem 2.4 can be used to obtain (1).

On one hand, this approach may lead to a somewhat shorter proof. On the other hand the proof of Theorem 2.4 given in Section 2.2 gives more structural information. We illustrate this by deducing the following single block and single hole theorem mentioned in the Introduction. The following proof deduces $(3,6)$-tightness from a connectivity condition without using the so-called girth inequalities of [3].

Theorem 2.11. *[4] Let $G^\diamond$ be a block and hole graph with a single block of size k and a single hole of size k. Then $G^\diamond$ is minimally rigid if and only if there exist k vertex disjoint paths connecting the vertex set of the block to the vertex set of the hole.*

Proof. First observe that $|E(G^\diamond)| = 3|V(G^\diamond)| - 6$. Necessity is easy to see: if the k paths do not exist, then - by Menger's theorem - there is a vertex set of size less than k that separates the block and the hole. It is easy to show that in this case $G^\diamond$ is not $(3,6)$-sparse, and hence it cannot be (minimally) rigid. To see sufficiency suppose that the k disjoint paths do exist but $G^\diamond$ is not rigid. Following the proof (and the notation) of Theorem 2.4 we obtain that $V - T$ is non-empty. Since there is a single hole, there is a unique non-triangular face in the interior of disc D, which must be equal to the single hole of $G^\diamond$. Moreover, it is separated from the block by the vertex set of the boundary cycle C of D. Due to the existence of the k disjoint paths, we must have $|C| \geq k$. By triangulating the hole by adding $k-3$ extra edges and using (2) we obtain $|E_{int}| \geq 3|V_{int}|$. As in the proof of the theorem, it leads to a contradiction. □

2.4 Algorithmic aspects

It is known that testing whether a given graph is $(3,6)$-sparse can be done in polynomial time. See e.g. [8] for the description of such an algorithm, which is based on matroidal methods. With this algorithm in hand we can construct a maximal $(3,6)$-sparse subgraph H of $G^\diamond$, starting from the single block $B^\diamond$, in a greedy manner. Theorem 2.4 implies that $r_3(G^\diamond)$ is equal to the number of edges in H. The proof of Theorem 2.4 also shows that we can determine, in polynomial time, a smallest set L of new edges for which $G^\diamond + L$ is rigid.

There exist other algorithmic approaches (e.g. the one based the degree constrained orientations [1]) which can be used in the single block setting to test $(3,6)$-sparsity and to obtain better running times. We omit the details.

3 Concluding remarks

Block and hole graphs with a single block can be used to model triangulated domes with doors and windows, where the gounded part corresponds to the block and the doors and windows correspond to the holes, see Figure 3. The results of this paper can be used to check whether such a structure is generically rigid (and more generally, to compute its degrees of freedom), irrespectively of the number of bars.

One may also consider block and hole graphs with a single hole. Interestingly, [3, Corollary 48] shows that a similar characterization applies: such a graph is minimally rigid in $\mathbb{R}^3$ if and only if it is $(3, 6)$-tight. It is not clear how this kind of duality can be used to obtain an efficient algorithm for testing whether a block and hole graph with a single hole is rigid. This question remains open.

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