Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2021-08. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Approximation by Lexicographically Maximal Solutions in Matching and Matroid Intersection Problems

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Abstract

We study how good a lexicographically maximal solution is in the weighted matching and matroid intersection problems. A solution is lexicographically maximal if it takes as many heaviest elements as possible, and subject to this, it takes as many second heaviest elements as possible, and so on. If the distinct weight values are sufficiently dispersed, e.g., the minimum ratio of two distinct weight values is at least the ground set size, then the lexicographical maximality and the usual weighted optimality are equivalent. We show that the threshold of the ratio for this equivalence to hold is exactly 2. Furthermore, we prove that if the ratio is less than 2, say α, then a lexicographically maximal solution achieves $(\alpha/2)$-approximation, and this bound is tight.

1 Introduction

Matching in bipartite graphs is one of the fundamental topics in combinatorics and optimization. Due to its diverse applications, various optimality criteria of matchings have been proposed based on the number of edges, the total weight of edges, etc. The concept of *rank maximality* is one of them, which is especially studied in the context of matching problems under preferences [2,7]. In this setting, we are given a partition

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$\{E_1, E_2, \ldots, E_k\}$ of the edge set E that represents a priority order, and a matching $M \subseteq E$ is *rank-maximal* if $|M \cap E_1|$ is maximized, and subject to this, $|M \cap E_2|$ is maximized, and so on. As pointed out in several papers [1, 2, 4, 7] (on generalized problems), the problem of finding a rank-maximal solution can be reduced to the usual weight-maximization setting by using sufficiently dispersed weights, e.g., by assigning $|E|^{k-i}$ for each element in E_i. (Clearly, it is not enough to assign arbitrary weights that are consistent with the priority order.)

In this paper, we explore the relation between rank-maximality and optimality in the weighted setting for two generalizations of bipartite matching: matching in general graphs and matroid intersection. To avoid confusion, we hereafter replace the term "rank" with "lex(icographical)" because we also deal with ranks in matroids. The main results of the paper (Theorems 2.1 and 2.2) state that, in both problems, the equivalence between lex-maximality and weighted optimality holds if the minimum ratio of two distinct weight values is larger than 2. This implies that we can choose the base of exponential weights as any constant larger than 2 (instead of $|E|$) in the aforementioned reduction. Furthermore, we show that if the minimum ratio is at most 2, say α, then a lex-maximal solution achieves $(\alpha/2)$-approximation of the maximum weight, and this bound is tight.

2 Results

We assume the basic notation and terminology on graphs and matroids (see, e.g., [8]).

Throughout the paper, let E be a finite ground set and $w\colon E \to \mathbb{R}_{>0}$ be a positive weight function on E. For a subset $X \subseteq E$, the weight $w(X)$ is defined as the sum of its elements. Let $w_1, w_2, \ldots, w_k$ be the distinct values of w in descending order. We assume $k \geq 2$ (otherwise, the results are trivial), and define

$$\alpha_w \coloneqq \min_{1 \leq i \leq k-1} \frac{w_i}{w_{i+1}}.$$

For a subset $X \subseteq E$ and $1 \leq i \leq k$, we denote by $X_i = \{\, e \in X \mid w(e) = w_i \,\}$ the restriction of X to the elements of weight w_i, and define $X_{\leq i} \coloneqq \bigcup_{j \leq i} X_j$ and accordingly $X_{\geq i}$, $X_{<i}$, and $X_{>i}$. For two subsets $X, Y \subseteq E$, we say that X is *lex-larger* than Y (or Y is *lex-smaller* than X) if there exists an index i such that $|X_j| = |Y_j|$ for any $j < i$ and $|X_i| > |Y_i|$. For a family $\mathcal{F} \subseteq 2^E$ of subsets of E, we say that $X \in \mathcal{F}$ is *lex-maximal* if no $Y \in \mathcal{F}$ is lex-larger than X. Note that a lex-maximal subset $X \in \mathcal{F}$ may not be unique but the sequence $|X_1|, |X_2|, \ldots, |X_k|$ is unique.

A *weighted matching instance* consists of an undirected graph $G = (V, E)$ and a weight function w on the edge set. We denote by $\mathrm{opt}(G, w)$ the optimal value, i.e., the maximum weight of a matching in G. In addition, we denote by $\mathrm{lexopt}(G, w)$ the weight of a lex-maximal matching in G (where $\mathcal{F}$ is the family of matchings in G), which takes as many edges of weight w_1 as possible, and subject to this, takes as many edges of weight w_2 as possible, and so on.

A *weighted matroid intersection instance* consists of two matroids $\mathbf{M}_1 = (E, \mathcal{I}_1)$ and $\mathbf{M}_2 = (E, \mathcal{I}_2)$ and a weight function w on the common ground set. We denote by

$\mathrm{opt}(\mathbf{M}_1, \mathbf{M}_2, w)$ the optimal value, i.e., the maximum weight of a common independent set in $\mathcal{I}_1 \cap \mathcal{I}_2$. In addition, we denote by $\mathrm{lexopt}(\mathbf{M}_1, \mathbf{M}_2, w)$ the weight of a lex-maximal common independent set (where $\mathcal{F} = \mathcal{I}_1 \cap \mathcal{I}_2$), which takes as many elements of weight w_1 as possible, and subject to this, takes as many elements of weight w_2 as possible, and so on.

The main theorems are stated as follows.

Theorem 2.1. *Let (G, w) be a weighted matching instance. If $\alpha_w \leq 2$, then*

$$\mathrm{lexopt}(G, w) \geq \frac{\alpha_w}{2} \cdot \mathrm{opt}(G, w).$$

Otherwise, a lex-maximal matching is a maximum-weight matching, and vice versa.

Theorem 2.2. *Let $(\mathbf{M}_1, \mathbf{M}_2, w)$ be a weighted matroid intersection instance. If $\alpha_w \leq 2$, then*

$$\mathrm{lexopt}(\mathbf{M}_1, \mathbf{M}_2, w) \geq \frac{\alpha_w}{2} \cdot \mathrm{opt}(\mathbf{M}_1, \mathbf{M}_2, w).$$

Otherwise, a lex-maximal common independent set is a maximum-weight common independent set, and vice versa.

We remark that the approximation ratio is tight as the weighted bipartite matching problem is included as a common special case. Let $G = (V, E)$ be a bipartite graph with $V = \{1, 2, 3, 4\}$ and $E = \{\, e_1 = \{1, 3\},\, e_2 = \{2, 3\},\, e_3 = \{2, 4\} \,\}$. Define $w(e_1) = 1$, $w(e_2) = x \in (1, 2]$, and $w(e_3) = 1$. We then have $\alpha_w = x$ (as $w_1 = x$ and $w_2 = 1$), $\mathrm{opt}(G, w) = 2$, and $\mathrm{lexopt}(G, w) = x$.

3 Proofs

We prove both theorems using the same strategy. The key definition and lemmas are as follows.

Definition 3.1. Let $X \subseteq E$ be a feasible solution that is not lex-maximal in a weighted matching or matroid intersection instance, and i be the smallest index such that $X_{\leq i}$ is not lex-maximal in the restricted instance whose ground set is $E_{\leq i}$. We say that a feasible solution $Y \subseteq E$ in the original instance is *eligible* if the following three conditions are satisfied:

- $|Y_j| = |X_j|$ for any $j < i$,
- $|Y_i| = |X_i| + 1$, and
- $|X_{>i} \setminus Y_{>i}| \leq 2$.

Intuitively, an eligible solution lexicographically improves the original solution at the most significant improvable class by sacrificing at most two lighter elements.

Lemma 3.2. *For any weighted matching instance (G, w) and any matching X that is not lex-maximal, there exists an eligible matching Y.*

Lemma 3.3. *For any weighted matroid intersection instance* $(\mathbf{M}_1, \mathbf{M}_2, w)$ *and any common independent set* X *that is not lex-maximal, there exists an eligible common independent set* Y*.*

We here prove Theorems 2.1 and 2.2 using Lemmas 3.2 and 3.3, whose proofs are given later.

Proof of Theorems 2.1 and 2.2. Let X^* be an optimal solution. Starting with $X = X^*$, we repeatedly update X to an eligible Y until it becomes lex-maximal. Let Y^* be the lex-maximal solution that is finally obtained.

In any update from X to Y, we have

$$w(Y) = w(Y_{\leq i}) + w(Y_{>i}) \geq w(X_{\leq i}) + w_i + w(X_{>i}) - 2w_{i+1} \geq w(X) - \frac{2 - \alpha_w}{\alpha_w} \cdot w_i.$$

If $\alpha_w > 2$, then we have $w(Y) > w(X)$, which cannot happen at the beginning when $X = X^*$. Thus, X^* is lex-maximal. As any lex-maximal solution has the same weight, we also see that any lex-maximal solution is optimal, and we are done for the second statements.

Suppose that $\alpha_w \leq 2$. Since we always have $|Y_j| = |X_j| = |Y_j^*|$ $(j < i)$ and $|Y_i| = |X_i| + 1 \leq |Y_i^*|$, we see that i is nondecreasing during the process and each w_i appears in the right-hand side at most $|Y_i^*|$ times in total. Thus, by repeating the above inequalities, we obtain

$$w(Y^*) \geq w(X^*) - \frac{2 - \alpha_w}{\alpha_w} \sum_{i=1}^{k} (w_i \cdot |Y_i^*|) = w(X^*) - \frac{2 - \alpha_w}{\alpha_w} \cdot w(Y^*),$$

which implies the first statements. □

3.1 Matching: Proof of Lemma 3.2

We prove Lemma 3.2 by contradiction.[1] Suppose to the contrary that there exists a counterexample, and take a minimal one in the following sense. First, the ground set E is minimized as the first priority, and subject to this, a counterexample matching X (that is not lex-maximal but admits no eligible matching Y) and a lex-maximal matching Z are taken so that the symmetric difference $X \triangle Z = (X \setminus Z) \cup (Z \setminus X)$ is minimized.

By the minimality, we have $E = X \cup Z = X \triangle Z$. Indeed, if there exists an edge $e \in E \setminus (X \cup Z)$, then we obtain a smaller counterexample by removing e (no eligible matching can newly appear), a contradiction. Similarly, if there exists $e \in X \cap Z$, then we obtain a smaller counterexample again by removing e (note that no edge is adjacent to e as X and Z are matchings), a contradiction.

Claim 3.4. *There are no adjacent edges of the heaviest weight.*

[1] We could give an algorithmic proof similar to the matroid intersection case in the next section, but we here describe an alternative proof because it is slightly simpler and may be extended to other problems.

Proof. Suppose to the contrary that $e_1 = \{v, u_1\} \in X$ and $e_2 = \{v, u_2\} \in Z$ are adjacent (at v) and $w(e_1) = w(e_2) = w_1$. If $u_1 = u_2$ (i.e., e_1 and e_2 are parallel), then $X' = (X \setminus \{e_1\}) \cup \{e_2\}$ is a smaller counterexample, a contradiction. Otherwise, consider the instance obtained by contracting the two edges e_1 and e_2, i.e., by merging u_1 and u_2 into a single vertex, removing the vertex v together with the incident edges, and restricting the weight function to the remaining set of edges.

As X is a matching that is not lex-maximal, this is true for $X' = X \setminus \{e_1\}$ after the contraction. By the minimality of the counterexample, there exists an eligible matching $Y' \subseteq E \setminus \{e_1, e_2\}$. Since at most one of u_1 and u_2 is matched by Y' in the original graph, we can add e_1 or e_2 to Y' to obtain a matching Y in the original instance. Moreover, the Y thus obtained is eligible by definition even if e_2 is added since $e_1 \in X_{\leq i}$ for any $i \geq 1$, contradicting our indirect assumption. □

Claim 3.5. $X_1 = \emptyset$.

Proof. Suppose to the contrary that there exists an edge $e \in X_1$. Then, by Claim 3.4, we have $w(f) < w(e) = w_1$ for each adjacent edge $f \in Z$. Thus, we can obtain a lex-larger matching Z' from Z by adding e and by removing all the (at most two) adjacent edges, a contradiction. □

By Claim 3.5, we have $|X_1| = 0 < |Z_1|$. Let $f \in Z$ be an edge with $w(f) = w_1$. Then, we can obtain an eligible matching Y from X by adding f and by removing all the (at most two) adjacent edges, a contradiction. This concludes the proof of the lemma.

3.2 Matroid Intersection: Proof of Lemma 3.3

The proof relies on (the correctness of) an augmenting path algorithm for the weighted matroid intersection problem, which was first described in [6]. We first review the basic facts based on [8, Sections 41.2 and 41.3].

Let $(\mathbf{M}_1, \mathbf{M}_2, w)$ be a weighted matroid intersection instance. For each matroid $\mathbf{M}_i$ ($i \in \{1, 2\}$), we denote the independent set family, the rank function, and the span function[2] by $\mathcal{I}_i \subseteq 2^E$, $r_i \colon 2^E \to \mathbb{Z}_{\geq 0}$, and $\mathrm{span}_i \colon 2^E \to 2^E$, respectively. For a set X and elements $x \in X$ and $y \notin X$, we write $X \setminus \{x\}$ and $X \cup \{y\}$ as $X - x$ and $X + y$, respectively.

Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ be a common independent set. The *exchangeability graph* with respect to I is a directed bipartite graph $D = (E \setminus I, I; A)$ defined as follows. Let $A \coloneqq A_1 \cup A_2$, where

$$A_1 \coloneqq \{\, (y, x) \mid x \in E \setminus I,\ y \in I,\ I + x - y \in \mathcal{I}_1 \,\},$$
$$A_2 \coloneqq \{\, (x, y) \mid x \in E \setminus I,\ y \in I,\ I + x - y \in \mathcal{I}_2 \,\}.$$

We also define

$$S \coloneqq \{\, s \in E \setminus I \mid I + s \in \mathcal{I}_1 \,\},$$
$$T \coloneqq \{\, t \in E \setminus I \mid I + t \in \mathcal{I}_2 \,\},$$

[2]This is also called the closure function; we follow the terminology of [8] for simplicity.

where elements in S and in T are called *sources* and *sinks*, respectively. Note that A_1 and S depend only on $\mathbf{M}_1$, and A_2 and T depend only on $\mathbf{M}_2$. Let $c\colon E \to \mathbb{R}$ be a cost function on the vertex set defined as follows:

$$c(e) \coloneqq \begin{cases} w(e) & (e \in I), \\ -w(e) & (e \in E \setminus I). \end{cases} \tag{3.1}$$

An S–T path P in D is *cheapest* if the total cost of its vertices is minimized. Subject to this, P is *shortest* if the number of its vertices is minimized.

For a nonnegative integer ℓ, a common independent set $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is said to be *ℓ-extreme* if $w(I)$ is maximized subject to $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|I| = \ell$. The following lemma leads to a simple augmenting path algorithm for the weighted matroid intersection problem. The key is that one can augment any ℓ-extreme solution to some $(\ell+1)$-extreme solution (if exists) by simply exchanging elements along a path.

Lemma 3.6 (cf. [8, Theorems 41.3, 41.5, and 41.6])**.** *Let $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ be an ℓ-extreme common independent set, and suppose that there exists a common independent set $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|J| > |I|$. Then, D contains an S–T path, which may consist of a single vertex in $S \cap T$. Let P be a shortest cheapest S–T path in D with respect to the cost function c defined as (3.1). Then, no inner vertex of P is a source or a sink, and $I \triangle P$ is an $(\ell+1)$-extreme common independent set.*

Now we start the proof of Lemma 3.3. Let $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ be a common independent set that is not lex-maximal, and let i be the smallest index such that $X_{\le i}$ is not lex-maximal in the restricted instance whose ground set is $E_{\le i}$.

Claim 3.7. *There exists a common independent set $Y' \subseteq E_{\le i}$ such that $|Y'_j| = |X_j|$ for any $j < i$, $|Y'_i| = |X_i| + 1$, $\mathrm{span}_1(X_{\le i}) \subseteq \mathrm{span}_1(Y')$, and $\mathrm{span}_2(X_{\le i}) \subseteq \mathrm{span}_2(Y')$.*

Proof. Define an auxiliary weight function $w'\colon E_{\le i} \to \mathbb{R}_{>0}$ by $w'(e) \coloneqq n^{i-j}$ for each $e \in E_j$ $(j = 1, 2, \ldots, i)$, where $n = |E_{\le i}|$. As remarked in the introduction, the lexicographical order coincides with the weighted order, i.e., for any two subsets $Z, Z' \subseteq E_{\le i}$, Z is lex-larger than Z' if and only if $w'(Z) > w'(Z')$.

Let $\ell \coloneqq |X_{\le i}|$. Then, $X_{\le i}$ is ℓ-extreme in the restricted instance $(\mathbf{M}_1|E_{\le i}, \mathbf{M}_2|E_{\le i}, w')$, which has a larger common independent set. By Lemma 3.6, one can obtain an $(\ell+1)$-extreme common independent set $Y' \subseteq E_{\le i}$ by flipping $X_{\le i}$ along a source-sink path P in the exchangeability graph. Note that $|Y'_j| = |X_j|$ for any $j < i$ and $|Y'_i| = |X_i|+1$ by the choice of i and the definition of w'.

If P consists of a single vertex (which is a source and a sink), then the claim immediately follows since $Y' = X_{\le i} + y$ for some element $y \in E_i$. Otherwise, let s and t be the first and last vertices of P, respectively. By the definitions of the exchangeability graph and the sources and sinks, $y \in \mathrm{span}_1(X_{\le i}) \subseteq \mathrm{span}_1(X_{\le i} + s)$ for every $y \in P - s$ and $y \in \mathrm{span}_2(X_{\le i}) \subseteq \mathrm{span}_2(X_{\le i} + t)$ for every $y \in P - t$ (recall that any $y \in P - s$ is not a source and any $y \in P - t$ is not a sink). Hence, we have

$\mathrm{span}_1(X_{\leq i} \cup P) = \mathrm{span}_1(X_{\leq i} + s)$ and $\mathrm{span}_2(X_{\leq i} \cup P) = \mathrm{span}_2(X_{\leq i} + t)$, and then

$$\begin{aligned}
\ell + 1 = |Y'| &= r_1(Y') \\
&\leq r_1(X_{\leq i} \cup P) = r_1(X_{\leq i} + s) \\
&\leq r_1(X_{\leq i}) + r_1(s) = |X_{\leq i}| + 1 = \ell + 1, \\
\ell + 1 = |Y'| &= r_2(Y') \\
&\leq r_2(X_{\leq i} \cup P) = r_2(X_{\leq i} + t) \\
&\leq r_2(X_{\leq i}) + r_2(t) = |X_{\leq i}| + 1 = \ell + 1,
\end{aligned}$$

which implies that the equality holds everywhere. Thus we have $\mathrm{span}_1(Y') = \mathrm{span}_1(X_{\leq i} + s) \supseteq \mathrm{span}_1(X_{\leq i})$ and $\mathrm{span}_2(Y') = \mathrm{span}_2(X_{\leq i} + t) \supseteq \mathrm{span}_2(X_{\leq i})$, which completes the proof. □

Take a subset $Y' \subseteq E_{\leq i}$ satisfying the conditions in Claim 3.7. We then obtain an eligible common independent set $Y \subseteq E$ from $Y' \cup X_{>i}$ by removing at most two elements in $X_{>i}$ as follows. If $r_1(Y' \cup X_{>i}) = r_2(Y' \cup X_{>i}) = |X| + 1$, then $Y' \cup X_{>i} \in \mathcal{I}_1 \cap \mathcal{I}_2$ and we do not need to remove any element. Suppose that $r_1(Y' \cup X_{>i}) < |X| + 1$. As $\mathrm{span}_1(X_{\leq i}) \subseteq \mathrm{span}_1(Y')$, we have $r_1(Y' \cup X_{>i}) \geq r_1(X) = |X|$, and hence $r_1(Y' \cup X_{>i}) = |X|$. This implies that $Y' \cup X_{>i}$ contains exactly one circuit of $\mathbf{M}_1$, which must intersect $X_{>i}$ since $Y' \in \mathcal{I}_1$. Hence, there exists an element $x \in X_{>i}$ such that $Y' \cup (X_{>i} - x) \in \mathcal{I}_1$. The same holds for $\mathbf{M}_2$. Thus we obtain a common independent set $Y \subseteq Y' \cup X_{>i}$ with $|X_{>i} \setminus Y_{>i}| \leq 2$, which is eligible.

4 Concluding Remarks

In this paper, we have analyzed how good a lex-maximal solution is in the weighted matching and matroid intersection problems based on how dispersed the distinct weight values are. It is well-known that, subject to a single matroid, a lex-maximal solution is always optimal, which can be found by a greedy algorithm. For more general independence systems, Jenkyns [3] and Korte and Hausmann [5] independently analyzed the worst-case approximation ratio of a greedy algorithm. In particular, the worst ratio is 2 in the weighted matching and matroid intersection problems, while a lex-maximal solution (which is a possible output of a greedy algorithm) always achieves the maximum weight if the distinct weight values are sufficiently dispersed. From this perspective, we have filled the gap between these two situations.

A natural question is as follows: how about a further (common) generalization, e.g., the weighted matroid parity problem? We just remark that it seems difficult to extend our algorithmic proof straightforwardly bacause no counterpart of augmentation from any ℓ-extreme solution to some $(\ell + 1)$-extreme solution along a path (Lemma 3.6) is known. A minimal counterexample proof (like the matching case) also seems nontrivial; in particular, we have found no counterpart of Claim 3.4.

Acknowledgments

We are grateful to Naoyuki Kamiyama for giving comments on a literature on rank-maximal matchings.

Kristóf Bérczi was supported by the Lendület Programme of the Hungarian Academy of Sciences – grant number LP2021-1/2021 and by the Hungarian National Research, Development and Innovation Office – NKFIH, grant number FK128673. Tamás Király was supported by the Hungarian National Research, Development and Innovation Office – NKFIH, grant number K120254. Yutaro Yamaguchi was supported by JSPS KAKENHI Grant Numbers JP20K19743 and JP20H00605 and by Overseas Research Program in Graduate School of Information Science and Technology, Osaka University. Yu Yokoi was supported by JSPS KAKENHI Grant Number JP18K18004.

"Application Domain Specific Highly Reliable IT Solutions" project has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme TKP2020-NKA-06 (National Challenges Subprogramme) funding scheme.

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