Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2021-03. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

Globally rigid graphs are fully reconstructible

Dániel Garamvölgyi, Steven J. Gortler, and Tibor Jordán

May 9, 2021

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Dániel Garamvölgyi⋆, Steven J. Gortler⋆⋆, and Tibor Jordán⋆⋆⋆

Abstract

A d-dimensional framework is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. The length of an edge $uv \in E$ in (G, p) is the distance between $p(u)$ and $p(v)$. The framework is said to be globally rigid in $\mathbb{R}^d$ if the graph G and its edge lengths uniquely determine (G, p), up to congruence. A graph G is called globally rigid in $\mathbb{R}^d$ if every d-dimensional generic framework (G, p) is globally rigid.

In this paper, we consider the problem of reconstructing a graph from the set of edge lengths arising from a generic framework. Roughly speaking, a graph G is strongly reconstructible in $\mathbb{C}^d$ if it is uniquely determined by the set of (unlabeled) edge lengths of any generic framework (G, p) in d-space, along with the number of its vertices. It is known that if G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then it is strongly reconstructible in $\mathbb{C}^d$. We strengthen this result and show that under the same conditions, G is in fact fully reconstructible in $\mathbb{C}^d$, which means that the set of edge lengths alone is sufficient to uniquely reconstruct G, without any constraint on the number of vertices.

We also prove that if G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then the d-dimensional generic rigidity matroid of G is connected. This result generalizes Hendrickson's necessary condition for global rigidity and gives rise to a new combinatorial necessary condition.

Finally, we provide new families of fully reconstructible graphs and use them to answer some questions regarding unlabeled reconstructibility posed in recent papers.

1 Introduction

A *(d-dimensional) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map that assigns a point in $\mathbb{R}^d$ to each vertex of G. The *length* of an

⋆Department of Operations Research, Eötvös University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `daniel.garamvolgyi@ttk.elte.hu`

⋆⋆Harvard School of Engineering and Applied Sciences, 33 Oxford St., Cambridge, MA 02138 USA. e-mail: `sjg@cs.harvard.edu`

⋆⋆⋆Department of Operations Research, Eötvös University, and the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `tibor.jordan@ttk.elte.hu`

edge uv in (G, p) is the Euclidean distance between $p(u)$ and $p(v)$. We also call (G, p) a *realization* of G in $\mathbb{R}^d$. The framework is *generic* if the set of the coordinates of its points is algebraically independent over $\mathbb{Q}$. We say that a d-dimensional framework (G, p) is *globally rigid* if every other realization (G, q) of G in $\mathbb{R}^d$ in which corresponding edges have the same length is congruent to (G, p). That is, the graph G and its edge lengths in (G, p) uniquely determine the pairwise distances of all vertices in (G, p). It is known that for generic d-dimensional frameworks, global rigidity depends only on the graph G, for all $d \geq 1$. We say that G is *(generically) globally rigid in* $\mathbb{R}^d$ if every (equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid. In the rest of this section we give a brief overview of our main results. Most of the definitions and more details are given in the next section.

Unlabeled rigidity is the study of what combinatorial and geometric information is determined by the (multi)-set of edge lengths arising from some d-dimensional framework (G, p). In [10], it was shown that if (G, p) is a generic globally rigid framework with n vertices in $\mathbb{R}^d$, where $n \geq d + 2$, then there can be no distinct realization (H, q) of any graph H with n vertices in $\mathbb{R}^d$ that produces the same edge lengths, up to trivialities. This result is essentially tight: if H is allowed to have more vertices or if G is not globally rigid, then p cannot be determined. To prove this result, it was sufficient to study the following, related graph reconstruction question.

The d-dimensional measurement variety $M_{d,G}$ of a graph G is the Zariski-closure of the set of all vectors arising as the squared edge lengths of d-dimensional realizations of G. It is natural to look for conditions under which the measurement variety itself determines the underlying graph. As this is dealing with varieties, it is simpler to do this analysis in the complex setting. Formally, we call G *strongly reconstructible in* $\mathbb{C}^d$ if for each pair of generic frameworks (G, p) and (H, q) in $\mathbb{C}^d$ with the same set of edge lengths and the same number of vertices, we have that G is isomorphic to H and the corresponding edges have the same length. This means, essentially, that the graph G, as well as the association between edges and coordinate axes, is uniquely determined by $M_{d,G}$ (see Theorem 2.18).

In [10], it was shown that for $d \geq 2$, globally rigid graphs in $\mathbb{R}^d$ on at least $d + 2$ vertices are strongly reconstructible in $\mathbb{C}^d$. Although this result was sufficient to answer the original question of [10], as a pure reconstruction question, the dependence on n remains unsatisfying. To this end, we call a graph G *fully reconstructible in* $\mathbb{C}^d$ if it is, roughly speaking, strongly reconstructible even when we do not require that G and H have the same number of vertices in the above definition. In [10], the question was posed whether global rigidity in $\mathbb{R}^d$ implies full reconstructibility in $\mathbb{C}^d$. For $d = 2$, the question was answered affirmatively in [6]. In this paper, as one of our main results, we substantially strengthen the previous results and show that for all $d \geq 2$,

if G is globally rigid in $\mathbb{R}^d$ on $n \geq d + 2$ vertices, then G is fully reconstructible in $\mathbb{C}^d$.

Results in [6] show that for $d = 2$, this is tight: if a graph is not globally rigid in $\mathbb{R}^2$, then it is not even strongly reconstructible in $\mathbb{C}^2$. The $d = 1$ case is slightly different but also fully characterizable using 3-connectivity. For $d \geq 3$, a characterization of full reconstructibility seems more elusive. In particular, we show that global rigidity in $\mathbb{R}^d$ is not necessary for full reconstructibility in $\mathbb{C}^d$. We also prove some positive and negative results regarding possible sufficient and possible necessary conditions for strong and full reconstructibility. These answer a number of questions that were posed in [10] and [6].

To prove our result on full reconstructibility we also prove a combinatorial theorem, interesting on its own right. We say that a graph is *M-connected in* $\mathbb{R}^d$ if its d-dimensional (generic) rigidity matroid is connected (see next section for detailed definitions). A combinatorial characterization of globally rigid graphs in $\mathbb{R}^d$ is known only for $d = 1, 2$, and is a major open problem for $d \geq 3$. In higher dimensions, Hendrickson's theorem (see Theorem 2.2) gives combinatorial necessary conditions that link global rigidity to connectivity and local rigidity properties of G. In this paper, as another main result, we strengthen one of Hendrickson's necessary conditions (redundant rigidity) by proving that for all $d \geq 1$,

if G is globally rigid in $\mathbb{R}^d$ on $n \geq d+2$ vertices, then it is M-connected in $\mathbb{R}^d$.

This result may lead to a better understanding of higher dimensional global rigidity. In particular, we use it to find new examples of so-called *H-graphs*, graphs that satisfy Hendrickson's conditions but are not globally rigid.

The rest of the paper is laid out as follows. In Section 2, we recall the definitions and results from rigidity theory and algebraic geometry that we shall use throughout the paper. Section 3 contains our main results: after making some structural observations about the measurement variety of graphs, we show that globally rigid graphs in $\mathbb{R}^d$ on at least $d + 2$ vertices are M-connected in $\mathbb{R}^d$ (Theorem 3.5) and, for $d \geq 2$, fully reconstructible in $\mathbb{C}^d$ (Theorem 3.6). In Section 4, we illustrate the results of the previous section with several examples. In particular, we use Theorem 3.5 to give new examples of H-graphs. We also answer questions from [6] and [10] related to the unlabeled reconstructibility problem, as well as pose new open questions. Finally, in Section 5 we prove some new results regarding M-connected graphs in $\mathbb{R}^d$.

2 Preliminaries

We start by fixing some conventions. In the following, graphs will be understood to be simple, that is, without parallel edges and loops. For a graph $G = (V, E)$, we shall use $\mathbb{R}^E$ and $\mathbb{C}^E$ to denote the $|E|$-dimensional real (complex, respectively) Euclidean space with axes labelled by the edges of G. We shall also often refer to the *configuration spaces* $\mathbb{R}^{nd}$ and $\mathbb{C}^{nd}$, where n denotes the number of vertices of G and $d \geq 1$ is some dimension. We shall really think of these spaces as $(\mathbb{R}^d)^V$ and $(\mathbb{C}^d)^V$, i.e. as n-tuples of d-dimensional vectors, indexed by the vertices of G. Nonetheless, as it is less cumbersome, we shall use the notation $\mathbb{R}^{nd}$ and $\mathbb{C}^{nd}$.

2.1 Real and complex frameworks

Let $G = (V, E)$ be a graph on n vertices and $d \geq 1$ some fixed integer. A *d-dimensional realization* of G is a pair (G, p) where $p : V \to \mathbb{R}^d$ maps the vertices of G into Euclidean space. We call such a point a *configuration* and we say that the pair

(G, p) is a *framework*. Two d-dimensional frameworks (G, p) and (G, q) are *equivalent* if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for every edge $uv \in E$, and *congruent* if the same holds for every pair of vertices $u, v \in V$. Here $\|\cdot\|$ denotes the Euclidean norm.

A framework is *(locally) rigid* if every continuous motion of the vertices which preserves the edge lengths takes it to a congruent framework, and *globally rigid* if every equivalent framework is congruent to it.

We say that a configuration $p \in \mathbb{R}^{nd}$ is *generic* if its $n \cdot d$ coordinates are algebraically independent over $\mathbb{Q}$. It is known that in any fixed dimension d, both local and global rigidity are generic properties of the underlying graph, in the sense that either every generic d-dimensional framework is locally/globally rigid or none of them are (see [4, 8]). Thus, we say that a graph is *rigid* (respectively *globally rigid*) in d dimensions if every (or equivalently, if some) generic d-dimensional realization of the graph is rigid (resp. globally rigid).

The function mapping the realizations of a graph to the sequence of its Euclidean squared edge lengths is called the *rigidity map* or *edge measurement map.* Let $G = (V, E)$ be a graph on n vertices. We denote the d-dimensional rigidity map of G by $m_{d,G} : \mathbb{R}^{nd} \to \mathbb{R}^E$, that is, for a d-dimensional realization (G, p) of G, the coordinate of $m_{d,G}(p)$ corresponding to the edge $uv \in E$ is $\|p(u) - p(v)\|^2$.

Analogously to the real case, we define a d-dimensional *complex framework* to be a pair (G, p), where $G = (V, E)$ is a graph and $p : V \to \mathbb{C}^d$ is a complex mapping. Given an edge $e = uv$ of G, its *complex squared length* in (G, p) is

$$m_{uv}(p) = (p(u) - p(v))^T \cdot (p(u) - p(v)) = \sum_{k=1}^{d} (p(u)_k - p(v)_k)^2,$$

where k indexes over the d dimension-coordinates. Note that in this definition we do not use conjugation. For real frameworks, this coincides with the usual (Euclidean) squared length, and it follows that we can extend $m_{d,G}$ to $\mathbb{C}^{nd} \to \mathbb{C}^E$ function by letting

$$m_{d,G}(p) = \big(m_{uv}(p)\big)_{uv \in E}.$$

We say, as in the real case, that two frameworks (G, p) and (G, q) are *equivalent* if $m_{d,G}(p) = m_{d,G}(q)$, and they are *congruent* if $m_{d,K_V}(p) = m_{d,K_V}(q)$, where K_V is the complete graph on the vertex set V. A configuration $p \in \mathbb{C}^{nd}$ is, again, generic, if the coordinates of p are algebraically independent over $\mathbb{Q}$. A point $p \in \mathbb{R}^{nd}$ is generic as a real configuration precisely if it is generic as a complex one.

Using these notions one can define the analogues of rigidity and global rigidity for complex frameworks. It turns out that, as in the real case, the (global) rigidity of generic complex frameworks only depends on the underlying graph, and the graph properties obtained in this way coincide with their real counterpart.

Theorem 2.1. [9, 10] Complex rigidity and global rigidity are generic properties and a graph G is rigid (respectively globally rigid) in $\mathbb{C}^d$ if and only if it is rigid (resp. globally rigid) in $\mathbb{R}^d$.

In light of Theorem 2.1, the terms "(globally) rigid in $\mathbb{R}^d$" and "(globally) rigid in $\mathbb{C}^d$" are interchargeable for a graph. We shall always use the former to emphasize that, although we often work in the complex setting, our results are related to the standard (real) notion of global rigidity.

It follows from the definitions that globally rigid graphs are rigid. The following much stronger necessary conditions of global rigidity are due to Hendrickson [11]. We say that a graph is *redundantly rigid* in a given dimension if it remains rigid after deleting any edge. A graph is *k-connected* for some $k \geq 2$ if it has at least $k+1$ vertices and it remains connected after deleting any set of less than k vertices.

Theorem 2.2. [11] Let G be a graph on at least $d+2$ vertices for some $d \geq 1$. Suppose that G is globally rigid in $\mathbb{R}^d$. Then G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.

In $d = 1, 2$ dimensions the conditions of Theorem 2.2 are, in fact, sufficient for global rigidity [12]. This fails in the $d \geq 3$ case and a combinatorial characterization of globally rigid graphs in these dimensions is a major open question.

2.2 The rigidity matrix and the rigidity matroid

The rigidity matroid of a graph G is a matroid defined on the edge set of G which reflects the rigidity properties of all generic realizations of G. For a general introduction to matroid theory we refer the reader to [17]. Let (G, p) be a realization of a graph $G = (V, E)$ in $\mathbb{R}^d$. The *rigidity matrix* of the framework (G, p) is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_iv_j \in E$, in the row corresponding to v_iv_j, the entries in the d columns corresponding to vertices v_i and v_j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros. In other words, it is $1/2$ times the Jacobian of the rigidity map $m_{d,G}$. The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of rows. It is known that any pair of generic frameworks (G, p) and (G, q) have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G) = (E, r_d)$ of the graph G. We can define the rigidity matrix $R(G, p)$ for complex frameworks in the same way as in the real case. This, again, allows us to define the rigidity matroid of the framework. It is not difficult to show that the rigidity matroid of a generic framework in $\mathbb{C}^d$ is, again, the d-dimensional rigidity matroid $\mathcal{R}_d(G)$.

We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$. A graph $G = (V, E)$ is *M-independent* in $\mathbb{R}^d$ if $r_d(G) = |E|$ and it is an *M-circuit* in $\mathbb{R}^d$ if it is not independent but every proper subgraph G' of G is independent. When the dimension d is clear from the context, we shall simply write M-independent and M-circuit, respectively. An edge e of G is an *M-bridge* in $\mathbb{R}^d$ if $r_d(G - e) = r_d(G) - 1$ holds.

Gluck characterized rigid graphs in terms of their rank.

Theorem 2.3. [7] Let $G = (V, E)$ be a graph with $|V| \geq d + 1$. Then G is rigid in $\mathbb{R}^d$ if and only if $r_d(G) = d|V| - \binom{d+1}{2}$.

Let $\mathcal{M}$ be a matroid on ground set E with rank function r. We can define a relation on the pairs of elements of E by saying that $e, f \in E$ are equivalent if $e = f$ or there is a circuit C of $\mathcal{M}$ with $\{e, f\} \subseteq C$. This defines an equivalence relation. The equivalence classes are the *connected components* of $\mathcal{M}$. The matroid is said to be *connected* if there is only one equivalence class, and *separable* otherwise. We shall use the fact that $\mathcal{M}$ is separable if and only if there is a partition $E = E_1 \cup E_2$ of E into two non-empty subsets for which

$$r(\mathcal{M}) = r(\mathcal{M}_1) + r(\mathcal{M}_2), \tag{1}$$

holds, where $\mathcal{M}_i$ denotes the restriction of $\mathcal{M}$ to E_i, $i = 1, 2$.

Given a graph $G = (V, E)$, the subgraphs induced by the edge sets of the connected components of $\mathcal{R}_d(G)$ are the *M-connected components* of G in $\mathbb{R}^d$. The graph is said to be *M-connected* in $\mathbb{R}^d$ if $\mathcal{R}_d(G)$ is connected, and *M-separable* in $\mathbb{R}^d$ otherwise. See Figure 1 for an example of an M-separable graph in $\mathbb{R}^3$.

Theorem 2.4. [12] Let G be a graph without isolated vertices and $d \in \{1, 2\}$. Then

(a) If G is globally rigid in $\mathbb{R}^d$ on at least $d + 2$ vertices, then it is M-connected in $\mathbb{R}^d$.

(b) If G is M-connected in $\mathbb{R}^d$, then it is redundantly rigid in $\mathbb{R}^d$.

Since an edge e is an M-bridge if and only if $\{e\}$ is the edge set of an M-connected component of G, and a rigid graph is redundantly rigid if and only if it has no M-bridges, part *(a)* of Theorem 2.4 is a strengthening of the second part of Theorem 2.2 in the $d \leq 2$ case. We shall show (Theorem 3.5) that this strengthening remains true in the $d \geq 3$ case. On the other hand, it is known that part *(b)* of Theorem 2.4 is not true in $d \geq 3$ dimensions.

Let $G = (V, E)$ be a graph on n vertices and (G, p) a framework in $\mathbb{C}^d$. The elements of $\ker(R(G, p)) \subseteq \mathbb{C}^{nd}$ are the *infinitesimal motions* of (G, p), while the elements of $\ker(R(G, p)^T) \subseteq \mathbb{C}^E$ are the *equilibrium stresses* of (G, p). We shall also use the notation $S(G, p) = \ker(R(G, p)^T)$. By basic linear algebra, $S(G, p)$ is the orthogonal complement of $\operatorname{span}(R(G, p))$ in $\mathbb{C}^E$. Since the elements of $S(G, p)$ capture the row dependences of $R(G, p)$, we have that G is M-independent in $\mathbb{R}^d$ if and only if for every generic realization (G, p) in $\mathbb{C}^d$, we have $S(G, p) = \{0\}$, and G is an M-circuit in $\mathbb{R}^d$ if and only if every generic realization (G, p) has a unique (up to scalar multiple) non-zero equilibrium stress ω and ω is non-zero on every edge of G.

Suppose that G is M-separable in $\mathbb{R}^d$ and let $E = E_1 \cup E_2$ be a partition of E into non-empty subsets such that for the graphs G_i induced by $E_i, i = 1, 2$, we have $r_d(G) = r_d(G_1) + r_d(G_2)$. It is not difficult to see that in this case for any generic realization (G, p) in $\mathbb{C}^d$, we have $\operatorname{span}(R(G, p)) = \operatorname{span}(R(G_1, p)) \oplus \operatorname{span}(R(G_2, p))$ as linear subspaces of $\mathbb{C}^E$, under the identification $\mathbb{C}^E = \mathbb{C}^{E_1} \times \mathbb{C}^{E_2}$. This also implies $S(G, p) = S(G_1, p) \oplus S(G_2, p)$ under the same identification.

2.3 Affine maps and conics at infinity

We say that a framework (G, p) in $\mathbb{C}^d$ has *full affine span* if the affine span of the image of the vertices under p is all of $\mathbb{C}^d$. A configuration $q \in \mathbb{C}^{nd}$, viewed as a point $q = (q_v)_{v \in V}$, is an *affine image* of p if $q_v = Ap_v + b, v \in V$ for some matrix $A \in \mathbb{C}^{d \times d}$

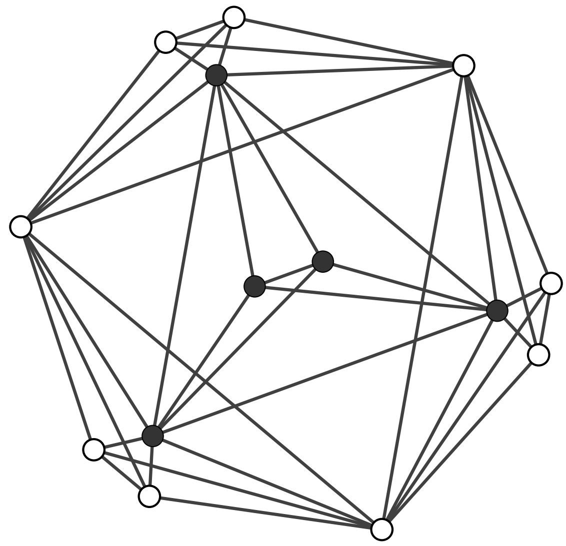


Figure 1: A 3-connected, redundantly rigid and M-separable graph in $\mathbb{R}^3$. This graph satisfies $r_3(G) = 36 = 27 + 9 = r_3(G^o) + r_3(K_5)$, where G^o is the outer ring of K_5's and K_5 is the subgraph induced by the black (filled) vertices.

and vector $b \in \mathbb{C}^d$. We say that p and q are *strongly congruent* if q can be obtained as the affine image of p under a rigid motion, i.e. an affine map $x \mapsto Ax + b$, such that $A^T A$ is the identity matrix.

Two frameworks (G, p) and (G, q) in $\mathbb{R}^d$ are strongly congruent if and only if they are congruent. This is not always the case for frameworks in $\mathbb{C}^d$. However, congruent frameworks that have full affine span are strongly congruent, see [9, Corollary 8].

Let (G, p) be a framework in $\mathbb{C}^d$. We say that the *edge directions of* (G, p) *lie on a conic at infinity* if there is a non-zero symmetric matrix Q such that for every edge uv of G, $(p(u) - p(v))^T Q(p(u) - p(v)) = 0$. The following lemma is implied by results of Connelly (see e.g. [4, Proposition 4.2]). For completeness, we give a proof.

Lemma 2.5. Let $G = (V, E)$ be a graph and (G, p) a framework in $\mathbb{C}^d$ such that its edge directions do not lie on a conic at infinity. Let (G, q) be a framework such that q is an affine image of p. Then $m_{d,G}(q) = m_{d,G}(p)$ if and only if q and p are congruent.

Proof. The "if" direction is immediate. In the other direction, suppose that $m_{d,G}(q) = m_{d,G}(p)$. Let $x \mapsto Ax + b$ be an affine transformation that sends p to q. It follows from the definitions that for any pair of vertices $u, v \in V$,

$$m_{uv}(q) = (p(u) - p(v))^T A^T A(p(u) - p(v)),$$

where the definition of m_{uv} is extended to all (possibly non-adjacent) vertex pairs u, v in a natural way. Therefore we have

$$m_{uv}(q) - m_{uv}(p) = (p(u) - p(v))^T (A^T A - I)(p(u) - p(v)). \tag{2}$$

By assumption, for every edge $uv \in E$, the left-hand side of (2) is zero. Since the edge lengths of (G, p) do not lie on a conic at infinity, this implies $A^T A - I = 0$, so that the left-hand side is zero for every pair of vertices $u, v \in V$, which is what we wanted to show. □

The following lemma is stated in [4] for frameworks in $\mathbb{R}^d$, but the same proof works for frameworks in $\mathbb{C}^d$.

Lemma 2.6. [4, Proposition 4.3] Let G be a graph in which each vertex has degree at least d. Then for every generic realization (G, p) in $\mathbb{C}^d$, the edge directions of (G, p) do not lie on a conic at infinity.

The following lemma is folklore.

Lemma 2.7. Let $(G, p), (G, q)$ be frameworks in $\mathbb{C}^d$ and suppose that q is an affine image of p. Then $S(G, p) \subseteq S(G, q)$. If both (G, p) and (G, q) have full affine span, then $S(G, p) = S(G, q)$.

Proof. Let $x \mapsto Ax + b$ be the affine transformation that maps p to q and let A' be the $nd \times nd$ block matrix with n copies of A in its diagonal and zeroes elsewhere, where n denotes the number of vertices of G. In other words, A' is the Kronecker product $I_n \otimes A$ of the $n \times n$ identity matrix and A.

Direct calculation shows that $R(G, q) = R(G, p)A'$, which immediately implies $S(G, p) = \ker(R(G, p)^T) \subseteq \ker(R(G, q)^T) = S(G, q)$. If (G, p) and (G, q) have full affine span, then the affine map sending p to q must necessarily be invertible, so that p is an affine image of q as well, implying $S(G, q) \subseteq S(G, p)$. □

Finally, we shall use the following property of globally rigid graphs which is easy to deduce from previous results on global rigidity and maximum rank stress matrices. We sketch the proof and refer the reader to [4, 8] for the definitions and key theorems.

Theorem 2.8. Let G be a globally rigid graph on $n \geq d + 2$ vertices in $\mathbb{C}^d$, for some $d \geq 1$ and (G, p) a generic realization of G in $\mathbb{C}^d$. For every realization (G, q) in $\mathbb{C}^d$ with $S(G, p) = S(G, q)$ we must have that q is an affine image of p.

Proof. Since G is globally rigid in $\mathbb{C}^d$, it is also globally rigid in $\mathbb{R}^d$. Let (G, p_0) be a generic realization of G in $\mathbb{R}^d$. It was shown in [8] that there exists an equilibrium stress ω_0 for (G, p_0) for which the associated stress matrix has rank $n - d - 1$. The complex version of [8, Lemma 5.8] then implies that (G, p) has an equilibrium stress ω such that the associated stress matrix has rank $n - d - 1$.

If $S(G, p) = S(G, q)$ for some realization (G, q), then ω is a stress for (G, q) as well. Then (the complex version of) [4, Proposition 1.2] implies that q is an affine image of p. □

2.4 Algebraic geometry background

We briefly recall the notions from algebraic geometry that we shall use. For a more detailed exposition, see [10, Appendix A] or [6, Section 2.2]. We say that a subset $X \subseteq \mathbb{C}^m$ is a *variety* if it is the set of simultaneous vanishing points of some polynomials $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_m]$. The varieties in $\mathbb{C}^m$ form the closed sets of the so-called

Zariski topology[1]. We denote by $I(X) \subseteq \mathbb{C}[x_1, \dots, x_m]$ the set of polynomials that vanish on X. We say that X is *irreducible* if it cannot be written as the proper union of a finite number of varieties, and it is *defined over* $\mathbb{Q}$ if $I(X)$ has a generating set consisting of polynomials with rational coefficients. The *dimension* of an irreducible variety X is the largest number k such that there exists a chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k = X$ of irreducible varieties.

Let $X \subseteq \mathbb{C}^m$ be an irreducible variety. At each point $x \in X$, we define the *Zariski tangent space* of X at x, denoted by T_xX, to be the kernel of the Jacobian matrix of a set of generating polynomials of $I(X)$, evaluated at x. Thus, T_xX is a linear subspace of $\mathbb{C}^m$. We say that x is *smooth* if $\dim(T_xX) = \dim(X)$. If X is *homogeneous* (i.e. it can be defined by homogenous polynomials, or equivalently, $tx \in X$ for every $x \in X$ and $t \in \mathbb{C}$) and $x \in X$ is a smooth point, we define the *Gauss fiber corresponding to* x to be the set $\{y \in X : y \text{ is smooth and } T_yX = T_xX\}$.[2]

Let $X \subseteq \mathbb{C}^m$ be a variety defined over $\mathbb{Q}$. We say that a point $x \in X$ is *generic in* X if the only polynomials with rational coefficients satisfied by x are those in $I(X)$. Note that a framework (G, p) in $\mathbb{C}^d$ is generic if and only if p is generic as a point of the variety $\mathbb{C}^{nd}$. It is known that if a point of X is generic, then it is smooth. We shall also need the following result.

Lemma 2.9. [10, Lemma A.6] Let $X \subseteq Y$ be irreducible varieties, with Y defined over $\mathbb{Q}$. Suppose that X has at least one point which is generic in Y. Then the points in X which are generic in Y are Zariski-dense in X.

2.5 The measurement variety

Recall that for a graph $G = (V, E)$, we denote its d-dimensional edge measurement map by $m_{d,G} : \mathbb{C}^{nd} \to \mathbb{C}^E$.

Definition 2.10. The *d-dimensional measurement variety* of a graph G (on n vertices), denoted by $M_{d,G}$, is the Zariski-closure of $m_{d,G}(\mathbb{C}^{nd})$.

We shall frequently use the following lemma on generic points. It follows by applying [10, Lemmas 4.4, A.7, A.8] to the varieties $\mathbb{C}^{nd}$, $M_{d,G}$ and the map $m_{d,G}$.

Lemma 2.11. Let $x \in M_{d,G}$ be a point in the measurement variety of G. Then x is generic in $M_{d,G}$ if and only if there is a generic point $p \in \mathbb{C}^{nd}$ for which $x = m_{d,G}(p)$.

It is known that the measurement variety, being the closure of the image of an irreducible variety defined over $\mathbb{Q}$, is also an irreducible variety defined over $\mathbb{Q}$. It follows from the definition and basic topological considerations that if $E' \subseteq E$ is

[1]Throughout the paper, unless otherwise noted, we shall work with the Zariski topology. In particular, given a subset $X \subseteq \mathbb{C}^d$ of some complex Euclidean space, we shall simply say that X is open (closed, respectively) to mean that it is Zariski-open (Zariski-closed, respectively), and we shall use $\overline{X}$ to denote the Zariski-closure of X in $\mathbb{C}^d$.

[2]In other words, the Gauss fiber corresponding to x is the fiber over T_xX of the rational function $X \dashrightarrow Gr(\dim(X), \mathbb{C}^m)$, defined by the mapping $x \mapsto T_xX$ on the smooth locus of X. Here, $Gr(\dim(X), \mathbb{C}^E)$ denotes the Grassmannian variety of $\dim(X)$-dimensional linear subspaces of $\mathbb{C}^n$.

a subset of edges inducing a subgraph G' of G, then $M_{d,G'} = \overline{\pi_{E'}(M_{d,G})}$, where $\pi_{E'} : \mathbb{C}^E \to \mathbb{C}^{E'}$ is the projection onto the coordinate axes corresponding to E', see [6, Lemma 3.8].

In what follows we shall frequently compare the measurement varieties of different graphs. In this case by writing $M_{d,G} = M_{d,H}$ ($M_{d,G} \subseteq M_{d,H}$, respectively) we mean that there is a bijection $\psi : E(G) \to E(H)$ between the edge sets of G and H such that $\Psi(M_{d,G}) = M_{d,H}$ ($\Psi(M_{d,G}) \subseteq M_{d,H}$, respectively), where $\Psi : \mathbb{C}^{E(G)} \to \mathbb{C}^{E(H)}$ is the mapping induced by ψ in the natural way. When we want to be more explicit about the underlying edge bijection, we shall write $M_{d,G} = M_{d,H}$ *under the edge bijection* ψ. Moreover, if we write both $M_{d,G} = M_{d,H}$ and $m_{d,G}(p) = m_{d,H}(q)$ in the same context, we shall mean that these equalities are satisfied under the same edge bijection.

The following results show that $\mathcal{R}_d(G)$ is "encoded" in the measurement variety in some sense. This has been observed before, see e.g. [6, 10, 18]. Using the terminology of the latter paper, the situation can be summarized by saying that the algebraic matroid corresponding to the variety $M_{d,G}$ is isomorphic to $\mathcal{R}_d(G)$.

Lemma 2.12. Let G be a graph on n vertices. Then

$$\dim(M_{d,G}) = r_d(G). \tag{3}$$

In particular, when $n \geq d+1$ we have $\dim(M_{d,G}) \leq nd - \binom{d+1}{2}$ and equality holds if and only if G is rigid in d dimensions. Moreover, G is M-independent in $\mathbb{R}^d$ if and only if $M_{d,G} = \mathbb{C}^E$.

Theorem 2.13. Let G and H be graphs with the same number of edges and suppose that $M_{d,G} = M_{d,H}$ under some edge bijection ψ. Then ψ defines an isomorphism between $\mathcal{R}_d(G)$ and $\mathcal{R}_d(H)$.

The following result shows that the measurement variety also encodes the space of stresses of generic frameworks. This follows from the fact that $S(G,p)$ is the orthogonal complement of $\mathrm{span}(R(G,p))$ in $\mathbb{C}^E$ using standard results in differential geometry, see [8, Lemma 2.21] or [10, Lemma 4.10].

Lemma 2.14. Let G be a graph and (G,p) a generic realization in $\mathbb{C}^d$ for some $d \geq 1$. Let $x = m_{d,G}(p) \in M_{d,G}$. Then the space of stresses $S(G,p)$ is the orthogonal complement of the tangent space $T_x(M_{d,G})$ in $\mathbb{C}^E$.

The lemma implies that if (G,p) and (G,q) are generic frameworks in $\mathbb{C}^d$, then $S(G,p) \neq S(G,q)$ if and only if the Gauss fibers corresponding to $m_{d,G}(p)$ and $m_{d,G}(q)$ are different. We shall use this corollary later.

2.6 Unlabeled reconstruction

In what follows it will be convenient to use the following notions. We say that two frameworks (G,p) and (H,q) are *length-equivalent* (*under the bijection* ψ) if there is a bijection ψ between the edge sets of G and H such that for every edge e of G, the length of e in (G,p) is equal to the length of $\psi(e)$ in (H,q).

Definition 2.15. Let (G, p) be a generic realization of the graph G in $\mathbb{C}^d$. We say that (G, p) is *strongly reconstructible* if for every generic framework (H, q) in $\mathbb{C}^d$ that is length-equivalent to (G, p) under some edge bijection ψ, where H has the same number of vertices as G, there is an isomorphism $\varphi : V(G) \to V(H)$ for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$.

In this paper we shall also consider the following stronger property, where the condition on the number of vertices of H is omitted.

Definition 2.16. Let G be a graph without isolated vertices and let (G, p) be a generic realization of G in $\mathbb{C}^d$. We say that (G, p) is *fully reconstructible* if for every generic framework (H, q) in $\mathbb{C}^d$ that is length-equivalent to (G, p) under some edge bijection ψ, where H has no isolated vertices, there is an isomorphism $\varphi : V(G) \to V(H)$ for which $\psi(uv) = \varphi(u)\varphi(v)$ for all $uv \in E$.

Note that, since we assume (G, p) to be generic, its edge lengths are pairwise distinct, and hence the bijection ψ is unique in the above definitions. Also note that in the definition of full reconstructibility it is essential to only consider *generic* length-equivalent frameworks (H, q), since for any (G, p) and any forest H with $|E|$ edges, we can find a (not necessarily generic) realization (H, q) that is length-equivalent to (G, p). In the case of strong reconstructibility, we can omit this genericity condition when G is rigid in $\mathbb{R}^d$, see [6, Theorem 3.6].

As Theorem 2.18 below shows, both strong and full reconstructibility of a generic framework can be characterized in terms of a certain uniqueness condition on the measurement variety $M_{d,G}$ of the underlying graph. This also implies that these notions are generic properties of a graph in the sense that if there is a generic framework (G, p) in $\mathbb{C}^d$ which is strongly (resp. fully) reconstructible, then every generic realization of G in $\mathbb{C}^d$ is strongly (resp. fully) reconstructible. This motivates the following definition.

Definition 2.17. A graph G is said to be *(generically) strongly reconstructible* (respectively *(generically) fully reconstructible*) in $\mathbb{C}^d$ if every generic realization (G, p) of G in $\mathbb{C}^d$ is strongly (respectively fully) reconstructible.

Theorem 2.18. Let G be a graph and $d \geq 1$ be fixed. The following are equivalent.

(i) G is generically strongly reconstructible (generically fully reconstructible, respectively) in $\mathbb{C}^d$.

(ii) There exists some generic framework (G, p) in $\mathbb{C}^d$ that is strongly reconstructible (fully reconstructible, respectively).

(iii) Whenever $M_{d,G} = M_{d,H}$ under an edge bijection ψ for some graph H, where H has the same number of vertices as G (where H has an arbitrary number of vertices, respectively), ψ is induced by a graph isomorphism.

The "strongly reconstructible" part of Theorem 2.18 is [6, Theorem 3.4]. The same proof works for the "fully reconstructible" version after omitting the condition on the number of vertices of H.

We close this section by recalling the main result of [10].

Theorem 2.19. [10, Theorem 3.4] Let G be a graph on at least $d+2$ vertices, where $d \geq 1$. Suppose that

- $d = 1$ and G is 3-connected, or
- $d \geq 2$ and G is globally rigid in $\mathbb{R}^d$.

Then G is strongly reconstructible in $\mathbb{C}^d$.

In the next section, we shall strengthen this result by proving that globally rigid graphs on at least $d+2$ vertices are, in fact, fully reconstructible in $\mathbb{C}^d$ for $d \geq 2$. The cases $d = 1, 2$ were already settled in [6] by verifying the following equivalence.

Theorem 2.20. [6, Theorem 5.19, Corollary 5.22, Theorem 5.1] Let G be a graph on at least $d+2$ vertices and without isolated vertices, where $d \in \{1, 2\}$. Then the following are equivalent.

- $d = 2$ and G is globally rigid in $\mathbb{R}^d$ (or $d = 1$ and G is 3-connected).
- G is strongly reconstructible in $\mathbb{C}^d$.
- G is fully reconstructible in $\mathbb{C}^d$.

In Section 4, we shall give examples showing that for $d \geq 3$, there are fully reconstructible graphs in $\mathbb{C}^d$ (on at least $d+2$ vertices) that are not globally rigid in $\mathbb{R}^d$.

3 Necessary conditions for global rigidity

In this section we prove our main results: globally rigid graphs in $\mathbb{R}^d$ on at least $d+2$ vertices are M-connected in $\mathbb{R}^d$ (Theorem 3.5) and fully reconstructible in $\mathbb{C}^d$ (Theorem 3.6). We start with some technical results about the structure of the measurement variety that we shall use in these proofs. Apart from Lemma 3.1, the lemmas in the next subsection are implicit in [10].

3.1 The structure of the measurement variety

The next lemma implies that the measurement variety of an M-separable graph G is the product of the measurement varieties of its M-connected components. A special case of this statement when G contains an M-bridge was proved in [6, Theorem 3.13].

Lemma 3.1. Let $d \geq 1$ and let $G = (V, E)$ be a graph. Suppose that there is a partition $E = E_1 \cup E_2$ of E into non-empty subsets such that $r_d(E) = r_d(E_1) + r_d(E_2)$. Let G_1 and G_2 be the subgraphs induced by E_1 and E_2, respectively. Then $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$ (under the identification $\mathbb{C}^E = \mathbb{C}^{E_1} \times \mathbb{C}^{E_2}$).

Proof. For $i = 1, 2$, M_{d,G_i} arises as the closure of the projection of $M_{d,G}$ onto the coordinate axes corresponding to E_i, so we have that $M_{d,G} \subseteq M_{d,G_1} \times M_{d,G_2}$. By [19, Chapter 3, Theorem 1.6], $M_{d,G_1} \times M_{d,G_2}$ is an irreducible variety of dimension $r_d(E_1) + r_d(E_2)$. Since this dimension equals the dimension $r_d(E)$ of the irreducible variety $M_{d,G}$, the two varieties must be equal. □

Lemma 3.2. Let G be a graph and (G, p) a generic framework in $\mathbb{C}^d$ with full affine span. Let $\mathcal{A} \subseteq \mathbb{C}^{nd}$ denote the set of affine images of p, and let $F \subseteq M_{d,G}$ be the Gauss fiber corresponding to $m_{d,G}(p)$. Then $\overline{m_{d,G}(\mathcal{A})} \subseteq \overline{F}$.

Proof. For clarity, we shall write m and M instead of $m_{d,G}$ and $M_{d,G}$ in the following. For any $q \in \mathcal{A}$, if q has full affine span, then by Lemma 2.7 we have $S(G, p) = S(G, q)$.

Let $\mathcal{A}^g$ denote the set of frameworks in $\mathcal{A}$ that are generic. Since (G, p) is generic, this set is non-empty, and since $\mathcal{A}$ is irreducible (being a linear space), Lemma 2.9 implies $\overline{\mathcal{A}^g} = \mathcal{A}$. Now for any $q \in \mathcal{A}^g$, we have $S(G, q) = S(G, p)$ by Lemma 2.7, and it follows by Lemma 2.14 that $T_{m(q)}M = T_{m(p)}M$, or in other words, $m(q) \in F$.

This shows that $m(\mathcal{A}^g) \subseteq F$. Taking closures and using the continuity of m, we have

$$\overline{F} \supseteq \overline{m(\mathcal{A}^g)} = \overline{m(\overline{\mathcal{A}^g})} = \overline{m(\mathcal{A})},$$

as desired. □

Although we shall not use this fact, we note that by [10, Lemma 4.6], $m_{d,G}(\mathcal{A})$ is a linear space, and in particular it is closed, so in the above lemma we could have written $m_{d,G}(\mathcal{A})$ instead of $\overline{m_{d,G}(\mathcal{A})}$.

Let G be a graph and $d \geq 1$. We say that a Gauss fiber F of $M_{d,G}$ is *generic* if it contains a point that is generic in $M_{d,G}$.

Lemma 3.3. Let G be a graph and $d \geq 2$.

a) If G is not M-independent in $\mathbb{R}^d$, then there exists a point $x \in M_{d-1,G} \subseteq M_{d,G}$ that is generic in $M_{d-1,G}$ and such that there are an infinite number of generic Gauss fibers F of $M_{d,G}$ with $x \in \overline{F}$.

b) If G is globally rigid in $\mathbb{R}^d$ on at least $d+2$ vertices and $x \in M_{d,G} \setminus M_{d-1,G}$, then there are at most a finite number of generic Gauss fibers F of $M_{d,G}$ with $x \in \overline{F}$.

Proof. *a)* [Following [10, Proposition 4.21]] For clarity, we shall write m instead of $m_{d,G}$ in the following. We note first that since G is not M-independent, it has at least $d + 2$ vertices and consequently any generic realization of G in $\mathbb{C}^d$ has full affine span.

Let (G, p) be a generic framework in $\mathbb{C}^d$ and let (G, q) be the framework in $\mathbb{C}^{d-1}$ obtained by projecting the image of each vertex in (G, p) onto the first $d-1$ coordinate axes. Then (G, q) is generic in $\mathbb{C}^{d-1}$, and consequently $x = m_{d,G}(q) \in M_{d-1,G}$ is generic in $M_{d-1,G}$ by Lemma 2.11. It is enough to find an infinite sequence of generic frameworks $(G, p_i), i \in \mathbb{N}$, with corresponding (generic) Gauss fibers $F_i, i \in \mathbb{N}$, such that $F_i \neq F_j$ for $i \neq j$ and such that q is an affine image of p_i, since by Lemma 3.2 this implies $x \in \overline{F}_i$.

We shall do this inductively. For the base case, let $p_1 = p$. Now suppose that for some $i > 1$, we have already found suitable frameworks $(G, p_j), j < i$. Since G is not

M-independent, each of these frameworks has a non-zero stress ω_j. By varying the last coordinate of the image of some vertices in (G, p), we can find a generic framework (G, p_i) that does not satisfy any of the stresses $\omega_j, j < i$ and whose projection onto the first $d-1$ coordinate axes is q.[3] This implies that $S(G, p_i) \neq S(G, p_j)$ for $j < i$. Let F_i denote the Gauss fiber corresponding to $m(p_i)$. By Lemma 2.14 we must have $T_{m(p_i)}M_{d,G} \neq T_{m(p_j)}M_{d,G}$, and hence $F_i \neq F_j$ for $j < i$.

b) This is an immediate consequence of Proposition 4.20 and Remark 4.8 of [10]. □

Corollary 3.4. Let G be a globally rigid graph in $\mathbb{R}^d$ on at least $d+2$ vertices for some $d \geq 2$ and suppose that $M_{d,G} = M_{d,H}$ under some edge bijection ψ for some graph H not necessarily on the same number of vertices as G. Then $M_{d-1,H} \subseteq M_{d-1,G}$ under ψ.

Proof. It follows from Theorem 2.2 that G has no M-bridges, which means it cannot be M-independent. Thus part *a)* of Lemma 3.3 implies that there is a generic point $x \in M_{d-1,H} \subseteq M_{d,H}$ such that there is an infinite number of generic Gauss fibers $F \subseteq M_{d,H}$ with $x \in \overline{F}$. Part *b)* of the same lemma then implies that x is in $M_{d-1,G}$. Now from the fact that x is generic in $M_{d-1,H}$ we have that every polynomial with rational coefficients satisfied by the points of $M_{d-1,G}$ must also be satisfied by the points of $M_{d-1,H}$. Since both $M_{d-1,H}$ and $M_{d-1,G}$ are defined over $\mathbb{Q}$, we must have $M_{d-1,H} \subseteq M_{d-1,G}$, as desired. □

3.2 Globally rigid graphs are M-connected

We are ready to prove the first main result of this section.

Theorem 3.5. Let $G = (V, E)$ be a globally rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices. Then G is M-connected in $\mathbb{R}^d$.

Proof. Let (G, p) be a generic framework in $\mathbb{C}^d$. We shall show that if G is not M-connected in $\mathbb{R}^d$, then there is a framework (G, q) in $\mathbb{C}^d$ such that $S(G, p) = S(G, q)$ but where q is not an affine image of p. Then, from Theorem 2.8, G cannot be globally rigid in $\mathbb{R}^d$, and we are done.

If G is not M-connected in $\mathbb{R}^d$, there must be a partition $E = E_1 \cup E_2$ of E into non-empty subsets with $r_d(E) = r_d(E_1) + r_d(E_2)$. Let G_1 and G_2 be the subgraphs consisting of full vertex set and the edge sets E_1 and E_2, respectively. Let $(G_1, p), (G_2, p)$ denote the respective sub-frameworks.

If G has an M-bridge e, then we may assume that $E_2 = \{e\}$ and $G_1 = G - e$. By Theorem 2.2, G is $(d+1)$-connected, so in this case the minimum degree of G_1 is at least d. Thus, by Lemma 2.6, the edge directions of (G_1, p) do not lie on a conic at infinity. If G contains no M-bridges, then G_1 contains an M-circuit, so in particular it has a subgraph of minimum degree at least d. Thus, by Lemma 2.6, applied to this subgraph, the edge directions of (G_1, p) do not lie on a conic at infinity.

[3] The existence of such a "generic lifting" follows from the basic fact that for any finite set $S \subseteq \mathbb{C}$ that is algebraically independent over $\mathbb{Q}$, the numbers $x \in \mathbb{C}$ for which $S \cup \{x\}$ is also algebraically independent form a dense subset of $\mathbb{C}$.

Now we find our promised (G, q). By Lemma 3.1, $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$. Let $m_{d,G}(p) = (x_1, x_2) \in M_{d,G}$. Now $x_2 \in M_{d,G_2}$ implies $4x_2 \in M_{d,G_2}$ and consequently $(x_1, 4x_2) \in M_{d,G}$. Since (G, p) was generic, (x_1, x_2) is generic in $M_{d,G}$ by Lemma 2.11, and this implies that $(x_1, 4x_2)$ is generic in $M_{d,G}$ as well. Using Lemma 2.11, it follows that there is a generic framework (G, q) in $\mathbb{C}^d$ with $m_{d,G}(q) = (x_1, 4x_2)$.

Let us consider the sub-frameworks (G_1, q) and (G_2, q). Both of these frameworks are generic. Also note that (G_1, q) is equivalent to (G_1, p) and (G_2, q) is equivalent to $(G_2, 2p)$. Since $S(G, p) = S(G, 2p)$, this implies $S(G, p) = S(G_1, p) \oplus S(G_2, p) = S(G_1, q) \oplus S(G_2, q) = S(G, q)$, as desired.

We have that (G_1, q) is equivalent to (G_1, p). As established above, the edge directions of (G_1, p) do not lie on a conic at infinity, Thus it follows from Lemma 2.5 that if q is an affine image of p, then q must be congruent to p. But q is clearly not congruent to p as (G_2, p) is not equivalent to (G_2, q). Thus p is not an affine image of q, as desired. □

Theorem 3.5 was known to hold in $\mathbb{R}^1$ (where global rigidity, 2-connectivity and M-connectivity are equivalent) and in $\mathbb{R}^2$, see [12] (c.f. Theorem 2.4 above). It is conjectured in [15] that globally rigid graphs in $\mathbb{R}^d$ are "non-degenerate" (for the definition see [15]). Since non-degenerate graphs are M-connected ([15, Lemma 3.2]), Theorem 3.5 gives an affirmative answer to a weaker, M-connected version of this conjecture.

Underlying the proof of Theorem 3.5 is the following structural observation on the measurement variety of G, which we give without details. Since G is globally rigid, [8, Theorem 4.4] and [10, Lemma 4.24, Remark 4.25] imply that for any generic Gauss fiber F of $M_{d,G}$ we have $\dim(\overline{F}) = \binom{d+1}{2}$. On the other hand, it follows from the definitions that if $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$, then $F = F_1 \times F_2$ where F_1 and F_2 are some generic Gauss fibers of M_{d,G_1} and M_{d,G_2}, respectively. Using Lemma 3.2 it can be shown that under the assumptions on G_1 made in the proof of Theorem 3.5, we have $\dim(\overline{F_1}) \geq \binom{d+1}{2}$. Since $\dim(\overline{F_2}) \geq 1$, this gives

$$\binom{d+1}{2} = \dim(\overline{F}) = \dim(\overline{F_1}) + \dim(\overline{F_2}) > \binom{d+1}{2},$$

a contradiction.

3.3 Globally rigid graphs are fully reconstructible

Our goal in this subsection is to prove the following result, which gives an affirmative answer to [10, Question 7.5].

Theorem 3.6. Let $d \geq 2$ and let G be a graph on $n \geq d + 2$ vertices that is globally rigid in $\mathbb{R}^d$. Then G is fully reconstructible in $\mathbb{C}^d$.

Theorem 4.7, below, will lead to examples that show that global rigidity is not necessary for full reconstructibility.

Our proof of Theorem 3.6 uses Theorem 3.5. In fact, as the following theorem shows, the former result is a strengthening of the latter.

Theorem 3.7. Let $G = (V, E)$ be a graph without isolated vertices and suppose that G is fully reconstructible in $\mathbb{C}^d$. Then G is M-connected in $\mathbb{R}^d$.

Proof. Suppose, for a contradiction, that there is a partition $E = E_1 \cup E_2$ of E into non-empty subsets with $r_d(E) = r_d(E_1) + r_d(E_2)$, and let G_1 and G_2 be the subgraphs induced by E_1 and E_2, respectively. By Lemma 3.1, this implies $M_{d,G} = M_{d,G_1} \times M_{d,G_2}$. If G_1 and G_2 have at least one vertex in common, then let H be the graph consisting of disjoint copies of G_1 and G_2. Otherwise, let H be the graph obtained from disjoint copies of G_1 and G_2 by identifying some vertex of G_1 and some vertex of G_2. In both cases, we have $m_{d,H}(\mathbb{C}^{n'd}) = m_{d,G_1}(\mathbb{C}^{n_1 d}) \times m_{d,G_2}(\mathbb{C}^{n_2 d})$, where n', n_1 and n_2 denote the number of vertices of H, G_1 and G_2, respectively. It follows that $M_{d,H} = M_{d,G_1} \times M_{d,G_2}$.[4] Since by construction G and H are not isomorphic, Theorem 2.18 then implies that G is not fully reconstructible, as desired. □

This theorem is similar in spirit to the following result from [6]: if G is strongly reconstructible in $\mathbb{C}^d$, then it cannot have an M-bridge in $\mathbb{R}^d$. Example 4.6 below shows that M-connectivity together with other natural conditions is not sufficient for strong reconstructibility.

We start by giving an outline of our proof of Theorem 3.6. By Theorem 2.18, to show that the globally rigid graph G is fully reconstructible, we need to show that whenever $M_{d,G} = M_{d,H}$ for some graph H without isolated vertices, we have that H is isomorphic to G (and the isomorphism induces the appropriate edge bijection). Let n and n' denote the number of vertices of G and H, respectively. If $n = n'$, then we are done by the strong reconstructibility of G (Theorem 2.19). If $n' < n$, then $M_{d,H}$ must necessarily be of lower dimension than $M_{d,G}$, which is impossible. The only remaining possibility to be ruled out is that $n' > n$; note that in this case $M_{d,G} = M_{d,H}$ (and in particular the equality of dimensions) implies that H is locally flexible in $\mathbb{C}^d$.

We rule this out as follows. From $M_{d,G} = M_{d,H}$ and $M_{d-1,H} \subseteq M_{d-1,G}$ (which follows from Corollary 3.4) we get lower bounds on the generic dimension of the space of infinitesimal motions of H in d and $d-1$ dimensions. We shall also show (Lemma 3.10) that as we decrease d, this generic dimension cannot decrease "too much", which gives us a lower bound on the number k_1 of one-dimensional infinitesimal motions of H. On the other hand, using Theorem 3.5, we can deduce that H is M-connected, and in particular connected, so that $k_1 = 1$, contradicting the lower bound we obtained previously.

The crux of our argument is a technical result (Lemma 3.9) which describes a particular infinitesimal rotation in d dimensions that, generically, cannot be decomposed into two $(d-1)$-dimensional infinitesimal rotations in a particular way. To prove this, we need the following lemma about the affine span of a certain variety.

Lemma 3.8. Let $G = (V, E)$ be a graph on n vertices and consider the mapping $f : \mathbb{C}^{2n} \to \mathbb{C}^E$ defined by

$$p = \big(p_v^{(1)}, p_v^{(2)}\big)_{v \in V} \longmapsto \big((p_v^{(1)} - p_u^{(1)})(p_v^{(2)} - p_u^{(2)})\big)_{uv \in E}.$$

[4] This follows from the basic fact that for any $U \subseteq \mathbb{C}^{E_1}, V \subseteq \mathbb{C}^{E_2}$ we have $\overline{U \times V} = \overline{U} \times \overline{V}$, where closures are meant in the respective Zariski topologies, see e.g. [6, Lemma 2.4].

Then $\overline{f(\mathbb{C}^{2n})}$ (and consequently $f(\mathbb{C}^{2n})$ itself) is not contained in any (linear) hyperplane in $\mathbb{C}^E$.

Proof. By direct calculation[5] we have that $f = m_{2,G} \circ \alpha$ where $\alpha : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is defined by

$$p = \left(p_v^{(1)}, p_v^{(2)}\right)_{v\in V} \longmapsto \left(\frac{p_v^{(1)} + p_v^{(2)}}{2}, \frac{p_v^{(1)} - p_v^{(2)}}{2\sqrt{-1}}\right)_{v\in V}.$$

Since α is a linear automorphism of $\mathbb{C}^{2n}$, this implies that $f(\mathbb{C}^{2n}) = m_{2,G}(\mathbb{C}^{2n})$, and thus $\overline{f(\mathbb{C}^{2n})} = M_{2,G}$. Let H be an arbitrary hyperplane in $\mathbb{C}^E$ whose orthogonal complement is generated by some non-zero $\omega \in \mathbb{C}^E$. We can find a generic framework (G, p) in $\mathbb{C}^2$ such that ω is not an equilibrium stress of (G, p). By Lemma 2.14, ω is not in the orthogonal complement of the tangent space of $M_{2,G}$ at $m_{2,G}(p)$. It follows that this tangent space is not contained in H, which implies that $M_{2,G}$ is not contained in H either, as desired. □

We introduce the following notation. Let $d \geq 3$ and let $G = (V, E)$ be a graph on n vertices. For a framework (G, p) in $\mathbb{C}^d$, let $W_1(G, p) \leq \mathbb{C}^{nd}$ denote the set of infinitesimal motions of (G, p) that are supported on the first $d-1$ coordinates, that is, infinitesimal motions of the form $(q_v, 0)_{v\in V}$, where $q_v \in \mathbb{C}^{d-1}$ for each vertex v. Similarly, let $W_2(G, p)$ denote the set of infinitesimal motions that are supported on the last $d-1$ coordinates.

Lemma 3.9. Let $d \geq 3$ and let $G = (V, E)$ be a graph on n vertices that is not M-independent in $\mathbb{R}^{d-2}$. Then there is a generic configuration $p = \left(p_v^{(1)}, \dots, p_v^{(d)}\right)_{v\in V} \in \mathbb{C}^{nd}$ such that the infinitesimal rotation $\varphi = \varphi(p)$ of (G, p) defined by

$$\varphi_v = (-p_v^{(d)}, 0, \dots, 0, p_v^{(1)}) \qquad \forall v \in V$$

is not in the subspace $W(G, p)$ of $\mathbb{C}^{nd}$ spanned by $W_1(G, p) \cup W_2(G, p)$. In particular, $W(G, p)$ is a proper subspace of the set of infinitesimal motions of (G, p).

Proof. Consider an arbitrary realization (G, p) in $\mathbb{C}^d$ and let $\widetilde{p} \in \mathbb{C}^{n(d-2)}$ denote the projection of p onto the middle $d-2$ coordinate axes. Suppose that $\varphi(p)$ can be written as $\varphi(p) = q + r$, where $q \in W_1(G, p)$ and $r \in W_2(G, p)$. Then $q_v = (-p_v^{(d)}, \widetilde{q}_v, 0)$ for every vertex $v \in V$, where $\widetilde{q} = (\widetilde{q}_v)_{v\in V} \in \mathbb{C}^{n(d-2)}$. The assumption that q is an infinitesimal motion of (G, p) means that

$$(p_v^{(1)} - p_u^{(1)})(p_v^{(d)} - p_u^{(d)}) = (\widetilde{p}_v - \widetilde{p}_u) \cdot (\widetilde{q}_v - \widetilde{q}_u), \qquad \forall uv \in E. \tag{4}$$

Now let $(G, \widetilde{p})$ be a fixed generic realization in $\mathbb{C}^{d-2}$. We shall show that we can choose $p^{(1)}, p^{(d)} \in \mathbb{C}^n$ in such a way that the framework (G, p) defined (coordinate-wise) by $p = (p^{(1)}, \widetilde{p}, p^{(d)})$ is generic in $\mathbb{C}^d$ and (4) does not hold for any choice of $\widetilde{q} \in \mathbb{C}^{n(d-2)}$. It follows that for such a p, $\varphi(p)$ cannot be in $W(G, p)$.

[5] This was already observed in [2].

Consider the linear subspace H of $\mathbb{C}^E$ defined as the image of $\mathbb{C}^{n(d-2)}$ under the linear mapping

$$\widetilde{q} = (\widetilde{q}_v)_{v\in V} \longmapsto R(G,\widetilde{p})\widetilde{q} = \big((\widetilde{p}_v - \widetilde{p}_u)\cdot(\widetilde{q}_v - \widetilde{q}_u)\big)_{uv\in E}.$$

Since G is not M-independent in $\mathbb{R}^{d-2}$, $(G,\widetilde{p})$ has a non-zero equilibrium stress $\omega \in \mathbb{C}^E$. Thus we have $\omega R(G,\widetilde{p}) = 0$, which implies that H is contained in the orthogonal complement of ω in $\mathbb{C}^E$, and in particular it is a proper subspace of $\mathbb{C}^E$. By Lemma 3.8, we can find $p^{(1)}, p^{(d)} \in \mathbb{C}^n$ such that

$$\big((p_v^{(1)} - p_u^{(1)})(p_v^{(d)} - p_u^{(d)})\big)_{uv\in E} \notin H.$$

Moreover, we can choose $p^{(1)}$ and $p^{(d)}$ in such a way that $p = (p^{(1)}, \widetilde{p}, p^{(d)})$ is generic in $\mathbb{C}^d$; this is because the set of pairs $(p^{(1)}, p^{(d)})$ for which the framework p obtained in this way is generic is a dense subset of $\mathbb{C}^{2n}$. By the above discussion, for such a choice of p we have that the infinitesimal rotation $\varphi = \varphi(p)$ is not contained in $W(G,p)$, as required. □

Lemma 3.10. Let $d \geq 3$ and let G be a graph on n vertices that is not M-independent in $\mathbb{R}^{d-2}$. For $i = 1,\dots,d$, let k_i denote the dimension of the set of infinitesimal motions of a generic realization of G in $\mathbb{C}^i$. Then for $i = 1,\dots,d-2$, we have $k_i \geq 2k_{i+1} - k_{i+2} + 1$.

Proof. The assumption that G is not M-independent in $\mathbb{R}^{d-2}$ implies that it is not M-independent in $\mathbb{R}^i$ for all $1 \leq i \leq d-2$. Thus, it is sufficient to prove for $i = d-2$. By Lemma 3.9, there is a generic realization (G,p) in $\mathbb{C}^d$ such that the subspace $W(G,p) \subseteq \mathbb{C}^{nd}$ generated by the subspaces $W_1(G,p)$ and $W_2(G,p)$ is a proper subset of the set of infinitesimal motions of (G,p). Note that $\dim(W_1(G,p)) = \dim(W_2(G,p)) = k_{d-1}$, and by the choice of (G,p), we have $\dim(W(G,p)) \leq k_d - 1$. Moreover, the subspace $W' = W_1(G,p) \cap W_2(G,p)$ consists of the infinitesimal motions of (G,p) that are supported on the middle $d-2$ coordinates. This implies $\dim(W') = k_{d-2}$. By basic linear algebra we have

$$\dim(W') + \dim(W(G,p)) = \dim(W_1(G,p)) + \dim(W_2(G,p)).$$

Substituting the above equalities and inequality and then rearranging gives

$$k_{d-2} \geq 2k_{d-1} - k_d + 1,$$

as desired. □

If G is M-independent in $\mathbb{R}^{d-2}$, then the conclusion of Lemma 3.10 does not hold. In this case G is M-independent in $\mathbb{R}^{d-1}$ and $\mathbb{R}^d$ as well, so we have $k_i = ni - |E|$ for $d-2 \leq i \leq d$, so that $k_{d-2} = 2k_{d-1} - k_d$.

The following combinatorial lemma lets us turn the recursive bound on k_i given in Lemma 3.10 into a lower bound that only depends on k_d and k_{d-1}.

Lemma 3.11. Let $d \geq 2$ be an integer and let $k_1, k_2, \dots, k_d \in \mathbb{Z}$ be a sequence of integers with $k_d = \binom{d+1}{2} + x$ and $k_{d-1} = \binom{d}{2} + y$ for some $x, y \in \mathbb{Z}$. Suppose that for $1 \leq i \leq d-2$ we have $k_i \geq 2k_{i+1} - k_{i+2} + 1$. Then for $1 \leq i \leq d-2$ we have $k_i \geq \binom{i+1}{2} + (d-i)(y-x) + x$.

Proof. Let $k_i = \binom{i+1}{2} + l_i$ for some numbers $l_i \in \mathbb{Z}, i = 1, \dots, d$; in particular, we have $l_d = x$ and $l_{d-1} = y$. For convenience, we also define $l_{d+1} = 2x - y$. We shall prove the following stronger statement: if the assumptions of the lemma hold, then we have $l_i \geq 2l_{i+1} - l_{i+2}$ and $l_i \geq (d-i)(y-x) + x$, for all $i = 1, \dots, d-1$. We proceed by induction on $j = d - i$. The $j = 1$ case immediately follows from the way we defined l_{d+1}. Now let $1 < j < d$. By using the lower bound on k_i we obtain

$$\begin{aligned} k_i &\geq 2k_{i+1} - k_{i+2} + 1 \\ &= (2\binom{i+2}{2} - \binom{i+3}{2} + 1) + (2l_{i+1} - l_{i+2}) \\ &= \binom{i+1}{2} + (2l_{i+1} - l_{i+2}), \end{aligned}$$

where we used the fact that for all $a \geq 1$, $\binom{a}{2} = 2\binom{a+1}{2} - \binom{a+2}{2} + 1$. This shows that $l_i \geq 2l_{i+1} - l_{i+2}$. By the induction hypothesis, this also gives

$$l_i - l_{i+1} \geq l_{i+1} - l_{i+2} \geq l_{i+2} - l_{i+3} \geq \dots \geq l_{d-1} - l_d = y - x,$$

so that $l_i \geq l_{i+1} + y - x \geq (d-i)(y-x) + x$, as required. □

Proof of Theorem 3.6. The $d = 2$ case follows from Theorem 2.20, so we only prove for $d \geq 3$. Let H be a graph on n' vertices, without isolated vertices and such that $M_{d,G} = M_{d,H}$ under some edge bijection ψ. Then by Corollary 3.4, we also have $M_{d-1,H} \subseteq M_{d-1,G}$ under ψ. Observe that since G is globally rigid, it is M-connected in $\mathbb{R}^d$ by Theorem 3.5, and thus so is H by Theorem 2.13 and the equality of the measurement varieties. In particular, H is a connected graph and it is not M-independent in $\mathbb{R}^d$ and thus not M-independent in $\mathbb{R}^{d-2}$.

Let $s = n' - n$ and let k_i denote the dimension of the set of infinitesimal motions of a generic realization of H in $\mathbb{C}^i$ for $1 = 1, \dots, d$. By considering the dimension of $M_{d,G} = M_{d,H}$ (using Lemma 2.12) and using the assumption that G is (globally) rigid, we get $n'd - k_d = nd - \binom{d+1}{2}$, implying $k_d = sd + \binom{d+1}{2}$. Similarly, from $M_{d-1,H} \subseteq M_{d-1,G}$ we have $n'(d-1) - k_{d-1} \leq n(d-1) - \binom{d}{2}$, so that $k_{d-1} \geq s(d-1) + \binom{d}{2}$. Note that here we used the fact that if G is (globally) rigid in $\mathbb{R}^d$, then it is rigid in $\mathbb{R}^{d-1}$, which follows from the coning theorem, discussed in Section 5.

Since $k_d \geq \binom{d+1}{2}$, we must have $s \geq 0$. This establishes that $n' \geq n$. On the other hand, from Lemmas 3.10 and 3.11 we get (substituting $x = sd$ and $y \geq s(d-1)$ into Lemma 3.11)

$$k_i \geq \binom{i+1}{2} - (d-i)s + sd = \binom{i+1}{2} + si$$

for $1 \leq i \leq d-2$, and in particular, $k_1 \geq s + 1$. But since H is connected, we know that generically the only 1-dimensional infinitesimal motions of H are translations, so that $k_1 = 1$. It follows that $s = 0$, so G and H have the same number of vertices. By Theorem 2.19, G is strongly reconstructible in $\mathbb{C}^d$, so ψ is induced by a graph isomorphism $\varphi : G \to H$, as desired. □

Applying Theorem 3.6 to a globally rigid subgraph of a graph, we obtain the following corollary.

Corollary 3.12. Let $d \geq 2$ and let (G, p) and (H, q) be generic frameworks in $\mathbb{C}^d$ that are length-equivalent under the edge bijection ψ. Let $G_0 = (V_0, E_0)$ be a globally rigid subgraph of $G = (V, E)$ and let H_0 denote the subgraph of H induced by $\psi(E_0)$. Then $\psi|_{E_0}$ is induced by an isomorphism $\varphi : V(G_0) \to V(H_0)$ and the frameworks $(G_0, p|_{V_0})$ and $(H_0, q|_{V(H_0)} \circ \varphi)$ are congruent.

The $d = 2$ case of Corollary 3.12 can be found in [6, Corollary 5.2].

4 Examples and open questions

In this section, we examine various examples related to M-connected and M-separable graphs, as well as the unlabeled reconstruction problem.

4.1 New examples of H-graphs

Following [13], we say that a graph G is an *H-graph* in $\mathbb{R}^d$ if it is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$ (i.e. it satisfies the necessary conditions of Theorem 2.2), but it is not globally rigid in $\mathbb{R}^d$. There are no H-graphs for $d = 1, 2$ but for $d \geq 3$ they exist and finding more examples may lead to a better understanding of higher dimensional global rigidity. For a long time, the complete bipartite graph $K_{5,5}$ was the only known H-graph in $\mathbb{R}^3$ (identified in [3]), until infinite families had been found in [13], see also [16].

Theorem 3.5 can be used to give new examples of H-graphs which are M-separable in $\mathbb{R}^3$. These also demonstrate that redundant rigidity and $(d+1)$-connectivity together do not imply M-connectivity in $d \geq 3$ dimensions.

Example 4.1. Consider the construction illustrated in Figure 2(a). It is easy to see that the graph G in the figure is 4-connected.

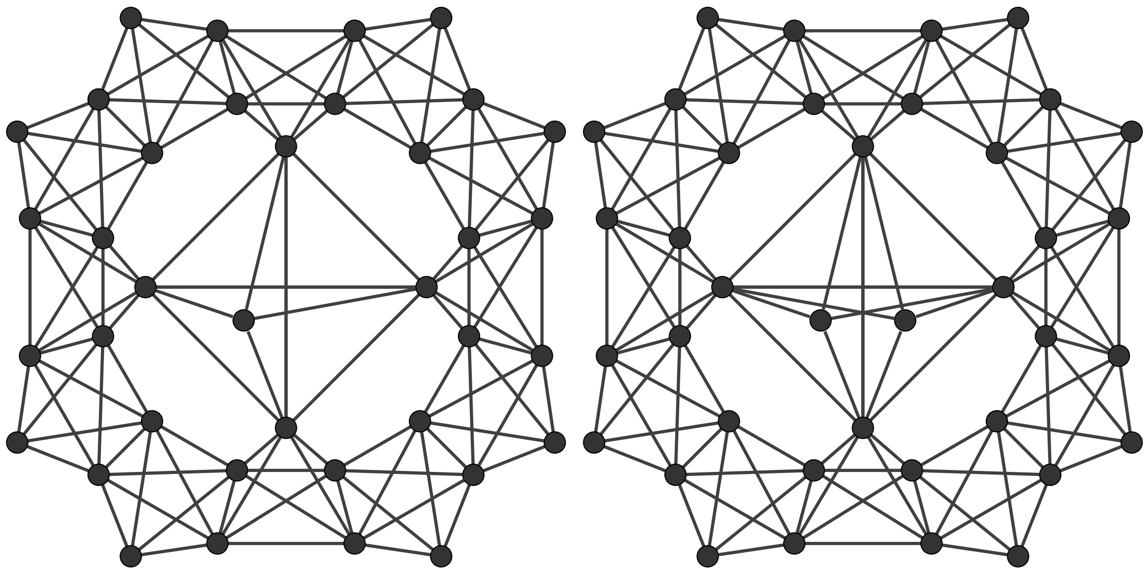
Claim. G is redundantly rigid and M-separable in $\mathbb{R}^3$.

Proof. We show that G is rigid by showing that a spanning subgraph of G can be reduced to K_4 by a sequence of the following operations: (i) deletion of a vertex of degree at least three, (ii) deletion of a vertex v of degree four and the addition of a new edge between two neighbours of v, (iii) the contraction of an edge uv for which u and v have exactly two common neighbours. It is well-known that the inverse operations (0- and 1-extension and vertex splitting) preserve rigidity in $\mathbb{R}^3$, see e.g. [21]. Thus, since K_4 is rigid, it will follow that G is also rigid.

First, delete the nine vertices of degree four from G and then delete one edge from each of the remaining copies of K_5. The resulting graph has 28 vertices and 78 edges. We shall reduce it to its internal K_4 subgraph. By the symmetry of the graph we can perform the reduction steps in groups of four in a symmetric way. First we contract four edges of the outer ring that do not belong to the four copies of $K_5 - e$, one from each "corner". These operations create a vertex of degree four in each corner, so we can apply operation (ii) at each of them to obtain a graph on 20 vertices and 54 edges, in which the four edges added form a four-cycle. After that we again apply operation (ii) in three rounds, decreasing the number of vertices by four in each round. If the added edges are chosen appropriately, we obtain a graph on 8 vertices, consisting of the internal K_4, a disjoint four-cycle C_4, and eight more edges that connect them (so that they span an 8-cycle). From here we apply operations (ii), (i), (ii), and then again (i) to get the K_4 subgraph.

Thus G is indeed rigid, that is, $r_3(G) = 3|V(G)| - 6 = 105$. Note that G is also redundantly rigid, because every edge of G belongs to a K_5 subgraph. To see that G is M-separable, first observe that if we remove one edge from each copy of K_5 in the outer ring G^o (say, one edge incident with each vertex of degree four) then we do not decrease its rank and obtain a spanning subgraph of G^o with 96 edges. Thus $r_3(G^o) \leq 96$. The inner K_5 has rank 9. Hence we must have $r_3(G) = r_3(G^o) + r_3(K_5)$, showing that G is indeed M-separable. □

Since M-separable graphs are not globally rigid by Theorem 3.5, G is indeed an H-graph in $\mathbb{R}^3$. We can obtain an infinite family of H-graphs in $\mathbb{R}^3$ from Example 4.1 by replacing the inner K_5 in Figure 2 with another 4-connected redundantly rigid graph K' in $\mathbb{R}^3$ on at least five vertices (as in Figure 2(b), where $K' = K_6 - e$).



(a) (b)

Figure 2: Graphs that are 4-connected, redundantly rigid and M-separable in $\mathbb{R}^3$. The graph shown in (a) satisfies $r_3(G) = 105 = 96 + 9 = r_3(G^o) + r_3(K_5)$, where G^o is the outer ring of K_5's.

We note that some of the H-graphs obtained in [13] (for example, the "6-ring" depicted in Figure 3) show that 4-connectivity, redundant rigidity, and M-connectivity together do not imply global rigidity in $\mathbb{R}^3$.

It is also interesting to note that every known H-graph in $\mathbb{R}^3$, except $K_{5,5}$, has a 4-separator.

Question 4.2. Is every 5-connected and redundantly rigid graph, other than $K_{5,5}$, globally rigid in $\mathbb{R}^3$?

The following related question also seems to be open.

Question 4.3. Is every 5-connected and redundantly rigid graph M-connected in $\mathbb{R}^3$?

Given a graph G, the *cone* of G is obtained by adding a new vertex v to G, along with new edges from v to every vertex of G. It is known that the cone of an H-graph in $\mathbb{R}^d$ is an H-graph in $\mathbb{R}^{d+1}$. We can use this fact to construct further families of H-graphs. However, these graphs will no longer be M-separable (see Theorem 5.3). Instead, we can use higher dimensional body-hinge graphs [13] to generalize the M-separable construction of Figure 2 to $d \geq 4$. We omit the details.

4.2 Unlabeled reconstructibility and small separators

In [10, Question 7.2] the authors asked whether every graph G that is 3-connected and redundantly rigid in $\mathbb{R}^d$ is determined by its measurement variety; that is, whether $M_{d,G} = M_{d,H}$ under some edge bijection ψ implies that G and H are isomorphic (note that here we do not require that the isomorphism induces ψ). Such a graph was called "weakly reconstructible in $\mathbb{C}^d$" in [6]. In the other direction, in [6, Section 7], the authors asked whether every graph on at least $d+2$ vertices that is strongly reconstructible in $\mathbb{C}^d$ for some $d \geq 3$ is globally rigid in $\mathbb{C}^d$, or (more weakly) whether it is $(d+1)$-connected.

In this subsection we provide negative answers to each of these questions for $d \geq 3$. Throughout this section we shall use the following (folklore) result which also appears in the proof of [6, Theorem 5.21].

Lemma 4.4. Let G and H be graphs on at least three vertices with G connected and let $\varphi_1, \varphi_2 : V(G) \to V(H)$ be injective graph homomorphisms. Suppose that φ_1 and φ_2 induce the same edge map $\psi : E(G) \to E(H)$. Then $\varphi_1(v) = \varphi_2(v)$ for all $v \in V(G)$.

Proof. Let $v \in V(G)$ be a vertex of degree at least two and let $vu, vu' \in E(G)$ be a pair of edges incident to v. Now $\varphi_1(v)$ is the unique vertex in H that is an end-vertex of both $\psi(vu)$ and $\psi(vu')$. Since $\varphi_2(v)$ can be described in the same way, we have $\varphi_1(v) = \varphi_2(v)$. Note that in a connected graph on at least three vertices, every edge has at least one end-vertex with degree at least two. This shows that φ_1 and φ_2 send at least one vertex of each edge in G to the same vertex in H. Since they also send each edge in G to the same edge in H, they must agree on every vertex of G. □

Example 4.5. Consider again the graph G shown in Figure 2(a). As we have seen in Example 4.1, G is 4-connected, redundantly rigid and M-separable in $\mathbb{R}^3$.

Claim. G is not strongly reconstructible in $\mathbb{C}^3$.

Proof. By the M-separability of G and Lemma 3.1, we have $M_{d,G} = M_{d,G^o} \times M_{d,K_5}$, where K_5 denotes the complete subgraph of G induced by the inner five vertices. Let ψ be a permutation of the edges of G that leaves the edges of G^o in place and permutes the edges of K_5 according to some permutation of its five vertices. Then $M_{d,G} = M_{d,G}$ under ψ, i.e. $M_{d,G}$ is invariant under the permutation of the coordinate axes in $\mathbb{C}^{E(G)}$ induced by ψ. However, ψ is not induced by a graph automorphism of G: since it leaves the edges of G^o in place, by Lemma 4.4 such an automorphism would have to leave the vertices of G^o in place and thus be the identity map on G, which does not induce ψ. By Theorem 2.18, this shows that G is not strongly reconstructible. □

It can be shown similarly that the graph G' shown in Figure 2(b) is not even weakly reconstructible in $\mathbb{C}^3$: the graph obtained by adding the missing edge to the inner K_6 and removing a different edge from it has the same measurement variety as G', even though the two graphs are not isomorphic. These examples can also be generalized to higher dimensions using the results on body-hinge graphs in [13].

The graphs considered in Example 4.5 are all M-separable. The next example is of an M-connected graph that is not strongly reconstructible.

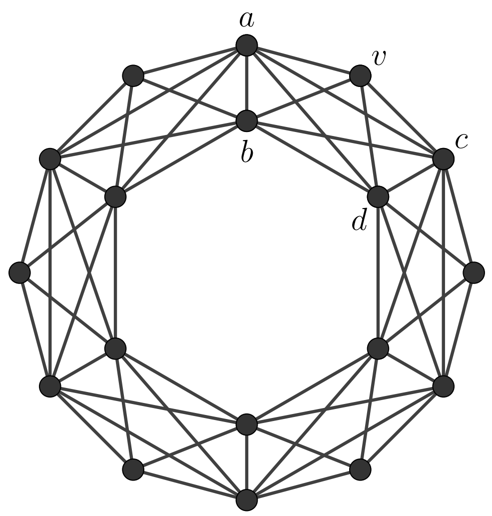


Figure 3: A graph that is 4-connected, redundantly rigid in $\mathbb{R}^3$ and M-connected in $\mathbb{R}^3$, but not globally rigid in $\mathbb{R}^3$.

Example 4.6. Let G be the 6-ring of K_5's shown in Figure 3. As noted before, it is 4-connected, redundantly rigid in $\mathbb{R}^3$ and M-connected in $\mathbb{R}^3$.

Claim. G is not strongly reconstructible in $\mathbb{C}^3$.

Proof. Let v be a vertex of degree four and let us denote the vertices of the K_5 subgraph that contains v by $\{v, a, b, c, d\}$, where the edges ab and cd are shared by neighbouring K_5's. Let $H = G - v$. It is easy to check that the edges ac, ad, bc, bd are all M-bridges in H. Thus by Lemma 3.1, we have $M_{d,H} = M_{d,H'} \oplus \mathbb{C}^4$, where $H' = H - \{ac, ad, bc, bd\}$. Let (G, p) be a generic realization of G and (by a slight abuse of notation) let (H, p) be its restriction H.

Consider the permutation ψ of the edges of H that leaves the edges of H' in place and maps ac, ad, bc, bd to bd, bc, ad, ac, respectively. Since $M_{d,H}$ is invariant under the permutation of coordinate axes induced by ψ, Lemma 2.11 implies that there exists a generic realization (H, q) of H in $\mathbb{C}^3$ such that $(H', q|_{V(H')})$ and $(H', p|_{V(H')})$ are equivalent and $m_{ac}(p) = m_{bd}(q)$, $m_{bd}(p) = m_{ac}(q)$, $m_{bc}(p) = m_{ad}(q)$, and $m_{ad}(p) = m_{bc}(q)$. This implies that the point configurations $(p(a), p(b), p(c), p(d))$ and $(q(b), q(a), q(d), q(c))$ are congruent, i.e. the points in them have the same pairwise squared distances. Since by genericity they have full affine span, this also implies that they are strongly congruent, in other words, there is a rigid motion of $\mathbb{C}^3$ that maps $p(a), p(b), p(c), p(d)$ to $q(b), q(a), q(d), q(c)$, respectively.

Let us extend (H, q) to a realization (G, q) by defining $q(v)$ to be the image of $p(v)$ under this rigid motion, and let us also extend ψ to all of $E(G)$ by mapping va, vb, vc, vd to vb, va, vd, vc, respectively. Then $m_{d,G}(p) = m_{d,G}(q)$ under the edge permutation ψ. On the other hand, ψ is not induced by a graph automorphism of G: since it leaves the edges of H' in place, by Lemma 4.4 such an automorphism would leave the vertices of H' in place and consequently it would have to be the identity map on G, which does not induce ψ. By Theorem 2.18, this shows that G is not strongly reconstructible in $\mathbb{C}^3$, as desired. □

Finally, we construct examples of fully reconstructible graphs with small separators. In the next proof, we shall use the following fact. Let G_1, G_2 be rigid graphs in $\mathbb{R}^d$ on at least $d+1$ vertices and let G be obtained from G_1 and G_2 by identifying k pairs of vertices. Then if $0 \leq k \leq d-1$, we have $r_d(G) = d|V(G)| - \binom{d+1}{2} - \binom{d-k+1}{2}$, and if $k \geq d$, then G is rigid. This follows e.g. from the "gluing lemma" [21, Lemma 11.1.9], or it can be seen directly by considering the infinitesimal motions of a generic realization of G in $\mathbb{R}^d$.

Theorem 4.7. Let $G = (V, E)$ be a graph with induced subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ for which $V_1 \cup V_2 = V$ and $V_1 \cap V_2$ induces a connected subgraph of G on at least three vertices. Let $d \geq 1$. If G_1 and G_2 are fully reconstructible rigid graphs on at least $d+1$ vertices in $\mathbb{C}^d$, then G is fully reconstructible in $\mathbb{C}^d$.

Proof. By Theorem 2.18, it suffices to show that if for some graph H we have $M_{d,G} = M_{d,H}$ under some edge bijection ψ, then ψ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$. Let H_1, H_2 be the subgraphs of H induced by $\psi(E_1)$ and $\psi(E_2)$, respectively. Now for $i = 1, 2$, $M_{d,G_i} = M_{d,H_i}$, so by the full reconstructibility of G_i, there is a graph isomorphism $\varphi_i : V_i \to V(H_i)$ that induces $\psi|_{E_i}$. Since $V_1 \cap V_2$ induces a connected subgraph of G, Lemma 4.4 applies to $\varphi_1|_{V_1 \cap V_2}$ and $\varphi_2|_{V_1 \cap V_2}$, giving $\varphi_1(v) = \varphi_2(v)$ for all $v \in V_1 \cap V_2$.

It follows that H is the union of two subgraphs H_1, H_2 which are isomorphic to G_1, G_2, respectively, and have $k \geq |V_1 \cap V_2|$ vertices in common. Let $\ell = |V_1 \cap V_2| \geq 3$. We first show that $k = \ell$. Note that $|V(G)| = |V(G_1)| + |V(G_2)| - \ell$ and $|V(H)| = |V(G_1)| + |V(G_2)| - k$, and hence $|V(H)| - |V(G)| = k - l$.

Let us first consider the $k \leq d-1$ case. Since by Theorem 2.13 the d-dimensional rigidity matroids of G and H are isomorphic, we obtain

$$d|V(G)| - \binom{d+1}{2} - \binom{d-\ell+1}{2} = r_d(G) = r_d(H) = d|V(H)| - \binom{d+1}{2} - \binom{d-k+1}{2}.$$

This gives

$$\binom{d+1}{2} + \binom{k}{2} = dk + \binom{d-k+1}{2} = d\ell + \binom{d-\ell+1}{2} = \binom{d+1}{2} + \binom{\ell}{2},$$

where the second equality comes from the previous equation and the first and third equalities from direct calculation. It follows that $k = \ell$. An analogous argument shows that $k = \ell$ holds in the $k \geq d$ case as well.

This implies that the only vertices of H that are in the image of both φ_1 and φ_2 are those in the image of $V(G_1) \cap V(G_2)$, where φ_1 and φ_2 agree. Hence we can "glue" φ_1 and φ_2, i.e. the mapping $\varphi : V \to V(H)$ defined by $\varphi|_{V_i} = \varphi_i, i = 1, 2$ is well-defined and is an isomorphism (in particular, it is injective). Then φ induces ψ, as required. □

We can slightly generalize Theorem 4.7 by only requiring that each connected component of the graph induced by $V_1 \cap V_2$ has at least three vertices.

Example 4.8. Let $d \geq 1$ and let G_d be the graph obtained by gluing two copies of the complete graph K_{d+2} along three pairs of vertices. Theorems 3.6 and 4.7 imply that G_d is fully reconstructible in $\mathbb{C}^d$. This example shows that fully (or strongly) reconstructible graphs need not be $(d+1)$-connected in the $d \geq 3$ case, which gives a negative answer to a question posed in [6, Section 7]. Also note that for $d \geq 4$, G_d is not even rigid in $\mathbb{R}^d$. It is unclear whether there exist non-rigid fully reconstructible graphs in $\mathbb{C}^3$.

4.3 Monotonicity of unlabeled reconstructibility

The graph G_d of Example 4.8 also shows that for $d \geq 4$, edge addition does not necessarily preserve strong (or full) reconstructibility in $\mathbb{C}^d$. Indeed, G_d is fully reconstructible in $\mathbb{C}^d$, but for any pair of non-neighbouring vertices $u, v \in V(G_d)$, uv is an M-bridge in $G_d + uv$. [6, Theorem 5.21] states that strongly reconstructible graphs (on at least $d + 2$ vertices and without isolated vertices) do not contain M-bridges, so that $G_d + uv$ is not strongly reconstructible in $\mathbb{C}^d$. It would be interesting to see whether this phenomenon can only happen if the newly added edge is an M-bridge.

Question 4.9. Let $d \geq 1$ and let $G = (V, E)$ be a graph on at least $d+2$ vertices that is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively). Is it true that if for some pair of vertices $u, v \in V$ we have $uv \notin E$ and $r_d(G) = r_d(G+uv)$, then $G + uv$ is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively)?

We can prove the following weaker result. We say that a pair $\{u, v\}$ of vertices in a graph G is *globally linked in G* in $\mathbb{C}^d$ if for every generic framework (G, p) in $\mathbb{C}^d$ and every equivalent realization (G, q), the squared distance between $p(u)$ and $p(v)$ is equal to the squared distance between $q(u)$ and $q(v)$.

Lemma 4.10. Let $G = (V, E)$ be a strongly reconstructible graph in $\mathbb{C}^d$ and suppose that a pair of vertices $u, v \in V$ is globally linked in G in $\mathbb{C}^d$. Then $G' = G + uv$ is strongly reconstructible in $\mathbb{C}^d$. Moreover, if G is fully reconstructible in $\mathbb{C}^d$, then so is G'.

Proof. By Theorem 2.18, it is sufficient to show that if $M_{d,G'} = M_{d,H'}$ under some edge bijection ψ, where H' is a graph on the same number of vertices as G', then ψ is induced by a graph isomorphism. Let H denote $H' - \psi(uv)$. Then $M_{d,G} = M_{d,H}$ under the edge bijection $\psi|_{E(G)}$, and thus (using the same theorem) the strong reconstructibility of G implies that $\psi|_{E(G)}$ is induced by a graph isomorphism $\varphi : V(G) \to V(H)$. It is sufficient to show that $\psi(uv) = \varphi(u)\varphi(v)$. After composing ψ with the edge bijection induced by φ^{-1}, this amounts to showing that if $M_{d,G+uv} = M_{d,G+u'v'}$ under the edge bijection that fixes the edges of G and sends uv to $u'v'$, then $\{u', v'\} = \{u, v\}$.

Let (G, p) be a generic realization of G in $\mathbb{C}^d$. By Lemma 2.11, there is a generic realization (G, q), equivalent to (G, p) and such that $\|p(u) - p(v)\| = \|q(u') - q(v')\|$. Since $\{u, v\}$ is globally linked in G, we must also have $\|p(u) - p(v)\| = \|q(u) - q(v)\|$. It follows from the genericity of q that $\{u, v\} = \{u', v'\}$, as required.

The same proof works when G is fully reconstructible in $\mathbb{C}^d$. $\square$

We may also consider the effect of edge deletion on unlabeled reconstructibility. The analogue of Question 4.9 for deleting edges is not true: it is not difficult to find an example of a graph G that is strongly (or fully) reconstructible in $\mathbb{C}^d$ but for which $G - uv$ is not strongly (or fully) reconstructible for some edge $uv \in E$, even though $r_d(G - uv) = r_d(G)$. This can happen e.g. if G is globally rigid and $G - uv$ contains an M-bridge. However, it is possible that the analogue of Lemma 4.10 for edge deletions is true.

Question 4.11. Let $d \geq 3$ and let $G = (V, E)$ be strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively). Is it true that if for some edge $uv \in E$ we have that $\{u, v\}$ is globally linked in $G - uv$, then $G - uv$ is strongly reconstructible in $\mathbb{C}^d$ (fully reconstructible in $\mathbb{C}^d$, respectively)?

The characterization of strong and full reconstructibility in $\mathbb{C}^1$ and $\mathbb{C}^2$ given by Theorem 2.20 shows that for $d = 2$, the answer to Question 4.11 is positive, while for $d = 1$, it is negative: let G be a 3-connected graph and suppose that $G - uv$ is not 3-connected for some edge $uv \in E(G)$. Then G is fully reconstructible in $\mathbb{C}^1$ and $\{u, v\}$ is globally linked in $G - uv$ (in fact, $G - uv$ is globally rigid in $\mathbb{C}^1$), but $G - uv$ is not strongly reconstructible in $\mathbb{C}^d$.

5 Graphs with nonseparable rigidity matroids

In light of Theorem 3.5, the combinatorial properties of M-connected graphs in $\mathbb{R}^d$ may be of interest in studying global rigidity. However, not much seems to be known about these graphs in the $d \geq 3$ case. In this section, we collect three results related to this notion.

Theorem 5.1. Let $G = (V, E)$ be an M-connected graph in $\mathbb{R}^d$. Then G is M-connected in $\mathbb{R}^{d'}$ for all $1 \leq d' \leq d$.

Proof. It suffices to consider the $d \geq 2$ case and show that G is M-connected in $\mathbb{R}^{d-1}$. We may also assume that G is an M-circuit in $\mathbb{R}^d$. Consider a generic realization (G, p)

of G in $\mathbb{R}^d$. Since G is an M-circuit, there exists a unique, (up to scalar multiplication) non-zero stress $\omega = (\omega_e)_{e\in E}$ of (G,p), which is non-zero on every edge $e \in E$.

For a contradiction, suppose that G is M-separable in $\mathbb{R}^{d-1}$ and let A, B be a separation, that is, $A \cup B = E$ and $r_{d-1}(A) + r_{d-1}(B) = r_{d-1}(E)$.

Let (G,p_i) be the $(d-1)$-dimensional realization of G obtained from (G,p) by a projection along the i-axis, for $1 \le i \le d$. These projected frameworks are also generic in $\mathbb{R}^{d-1}$ and ω is a stress on each (G,p_i). Let $R(G,p_i)$ be the matrix obtained from the rigidity matrix of (G,p) by replacing the $|V|$ columns corresponding to coordinate i by all-zero columns. Thus the rigidity matrix of (G,p_i) can be obtained from $R(G,p_i)$ by removing these zero columns.

Since A, B is a separation, we must have $\sum_{e\in A} \omega_e R_e(G,p_i) = 0$ for all $1 \le i \le d+1$, where $R_e(G,p_i)$ is the row of the edge e in $R(G,p_i)$. This gives

$$(d-1)\sum_{e\in A}\omega_e R_e(G,p) = \sum_{i=1}^{d}\sum_{e\in A}\omega_e R_e(G,p_i) = 0,$$

implying that the restriction of ω to A is a non-zero stress on a proper subframework of (G,p). Then extending this restricted stress to all of E by setting it zero on every $e \in B$ gives another stress of (G,p), contradicting the uniqueness of ω. □

Our next result is a characterization of M-connected cone graphs. Recall that the *cone* of a graph $G = (V,E)$, which we shall denote by G^v, is obtained by adding a new vertex v to G, along with the edges $uv, u \in V$. We shall use the following basic result on coning, due to W. Whiteley [20, 21], that we shall refer to as the coning theorem. Let (G^v,p) be a realization in $\mathbb{R}^{d+1}$ for some $d \ge 1$ and let (G,q) be a realization in $\mathbb{R}^d$ that is obtained from (G^v,p) by projecting $p(u), u \in V$ through $p(v)$ onto some hyperplane H not containing $p(v)$. Then (G^v,p) is independent (i.e. $\mathrm{rank}(R(G^v,p)) = |E(G^v)|$) in $\mathbb{R}^{d+1}$ if and only if (G,q) is independent in $\mathbb{R}^d$. This also implies that G is M-independent in $\mathbb{R}^d$ if and only if G^v is M-independent in $\mathbb{R}^{d+1}$.

We shall also use the fact that G is an M-circuit in $\mathbb{R}^d$ if and only if G^v is an M-circuit in $\mathbb{R}^{d+1}$. Although this result seems to be folklore, we could not find any proofs in the literature, so we provide one (due to W. Whiteley [22]) for completeness.

Lemma 5.2. Let $d \ge 1$ and let G be a graph and G^v its cone graph. Then G is an M-circuit in $\mathbb{R}^d$ if and only if G^v is an M-circuit in $\mathbb{R}^{d+1}$.

Proof. If G^v is an M-circuit in $\mathbb{R}^{d+1}$, then by the coning theorem G is M-dependent in $\mathbb{R}^d$. On the other hand, for every edge $uw \in E(G)$ we have that $(G-uw)^v = G^v - uw$ is M-independent in $\mathbb{R}^{d+1}$, so $G - uw$ is M-independent in $\mathbb{R}^d$. This shows that G is an M-circuit, as desired.

Now let G be an M-circuit in $\mathbb{R}^d$. Again by the coning theorem, $r_{d+1}(G^v) = |E(G^v)| - 1 = r_{d+1}(G^v - uw)$ for any edge $uw \in E(G)$. Thus, we only need to prove that $r_{d+1}(G^v - uv) = r_{d+1}(G^v)$ for any edge uv incident to the cone vertex.

Let $uv \in E(G^v)$ be such an edge and consider a framework (G^v,p) in which the vertices of G lie in the $x_{d+1} = 0$ hyperplane in a generic position and $p(v)$ lies outside of this hyperplane. Let w be a neighbour of u in G. The coning theorem implies that

$$\mathrm{rank}(R(G^v,p)) = r_{d+1}(G^v) = r_{d+1}(G^v - uw) = \mathrm{rank}(R(G^v - uw, p)).$$

Now consider a framework (G, q) obtained from (G, p) by changing the last coordinate of $p(w)$ by a sufficiently small amount, so that $\operatorname{rank}(R(G^v - uw, q)) = r_{d+1}(G^v) = \operatorname{rank}(R(G^v, q))$ still holds. Then there is an equilibrium stress ω of (G^v, q) that is non-zero on uw. This stress must be non-zero on uv as well, since in this framework, the rest of the edges incident to u all lie in the $x_{d+1} = 0$ hyperplane. This shows that

$$\operatorname{rank}(R(G^v - uv, q)) = \operatorname{rank}(R(G^v, q)) = r_{d+1}(G^v),$$

which implies $r_{d+1}(G^v - uv) = r_{d+1}(G^v)$, as desired. □

Theorem 5.3. Let G be a graph and let G^v denote its cone graph. Then G^v is M-connected in $\mathbb{R}^{d+1}$ if and only if G is connected and it has no M-bridges in $\mathbb{R}^d$.

Proof. For the "only if" direction, observe that coning takes an M-bridge in $\mathbb{R}^d$ to an M-bridge in $\mathbb{R}^{d+1}$,[6] and that an M-connected graph (on at least two edges) has no M-bridges. Moreover, M-connected graphs are 2-connected,[7] while the cone graph of a disconnected graph is not.

To prove the "if" direction, let us first observe that for any edge xy of G, xy is in the same M-connected component of G^v as vx and vy. Indeed, since xy is not an M-bridge in G, it is contained in some subgraph of G that is an M-circuit in $\mathbb{R}^d$, and by Lemma 5.2, the cone of this subgraph is an M-circuit in $\mathbb{R}^{d+1}$ which contains all three of these edges.

Thus it is sufficient to prove that any pair of cone edges vx, vy is in the same M-connected component of G^v. By assumption, there is a path $x = u_0, u_1, \ldots, u_k = y$ in G. Now by the previous observation vu_i and vu_{i+1} are in the same M-connected component of G^v, for all $0 \leq i < k$. By transitivity, we get that vx and vy are also in the same M-connected component, as desired. □

For a graph $G = (V, E)$ let $\operatorname{dof}_d(G) = d|V| - \binom{d+1}{2} - r_d(G)$ denote its "degrees of freedom" in the context of d-dimensional generic rigidity. The next theorem verifies a general combinatorial property of highly connected M-separable graphs and may be useful in the construction of further families of examples.

Theorem 5.4. Let $d \geq 1$ and let G be a $(d+1)$-connected and redundantly rigid graph in $\mathbb{R}^d$. Suppose that G is M-separable in $\mathbb{R}^d$ and let H_1, H_2,...,H_q be the M-connected components of G. Then

$$\sum_1^q \operatorname{dof}_d(H_i) \geq \binom{d+1}{2}.$$

Proof. Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices belonging to no other M-connected component than H_i, and let $Y_i = V(H_i) - X_i$ for $1 \leq i \leq q$. Let

[6] In order to see this first note that the coning theorem implies that G is rigid in $\mathbb{R}^d$ if and only if G^v is rigid in $\mathbb{R}^{d+1}$. This fact easily implies that if e is an M-bridge in G, and G is rigid in $\mathbb{R}^d$, then e is an M-bridge in G^v in $\mathbb{R}^{d+1}$. If G is not rigid, then make it rigid by adding a set of M-bridges and then apply the previous argument.

[7] This follows e.g. from Theorem 5.1 by recalling that in $\mathbb{R}^1$, a graph is M-connected if and only if it is 2-connected.

$n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$. Clearly, $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^q x_i + |\cup_{i=1}^q Y_i|$. Moreover, we have $\sum_{i=1}^q y_i \geq 2|\cup_{i=1}^q Y_i|$. Since G is redundantly rigid, every edge of G is in some M-circuit. Every M-circuit in $\mathbb{R}^d$ has at least $d+2$ vertices. Thus we have that $n_i \geq d+2$ for $1 \leq i \leq q$. Furthermore, since G is $(d+1)$-connected, $y_i \geq (d+1)$ holds for all M-components.

Let us choose a base B_i in each rigidity matroid $\mathcal{R}(H_i)$. Using the above inequalities we have

$$\begin{aligned} d|V| - \binom{d+1}{2} &= |\cup_{i=1}^q B_i| = \sum_{i=1}^q |B_i| = \sum_{i=1}^q \left(dn_i - \binom{d+1}{2} - \mathrm{dof}_d(H_i) \right) \\ &= d\sum_{i=1}^q n_i - \binom{d+1}{2} q - \sum_1^q \mathrm{dof}_d(H_i) \\ &= \left(d\sum_{i=1}^q x_i + \frac{d}{2}\sum_{i=1}^q y_i \right) + \left(\frac{d}{2}\sum_{i=1}^q y_i - \binom{d+1}{2} q \right) - \sum_1^q \mathrm{dof}_d(H_i) \\ &\geq d|V| - \sum_1^q \mathrm{dof}_d(H_i) \end{aligned}$$

Thus we must have $\sum_1^q \mathrm{dof}_d(H_i) \geq \binom{d+1}{2}$, as claimed. □

The graph of Figure 2(a) shows that Theorem 5.4 is, in some sense, tight: it has a unique non-rigid M-connected component in $\mathbb{R}^3$ with six degrees of freedom. Note that for $d = 1, 2$ the M-connected components of a graph are rigid. Thus the theorem implies that for $d \leq 2$ the $(d+1)$-connected redundantly rigid graphs are M-connected, which was shown in [12, Theorem 3.2] by a similar argument.

6 Acknowledgements

This work was supported by the Hungarian Scientific Research Fund grant no. K 135421, and the project Application Domain Specific Highly Reliable IT Solutions which has been implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the Thematic Excellence Programme TKP2020-NKA-06 (National Challenges Subprogramme) funding scheme.

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