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Fair integral submodular flows

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Abstract

Integer-valued elements of an integral submodular flow polyhedron Q are investigated which are decreasingly minimal (dec-min) in the sense that their largest component is as small as possible, within this, the second largest component is as small as possible, and so on. As a main result, we prove that the set of dec-min integral elements of Q is the set of integral elements of another integral submodular flow polyhedron arising from Q by intersecting a face of Q with a box. Based on this description, we develop a strongly polynomial algorithm for computing not only a dec-min integer-valued submodular flow but even a cheapest one with respect to a linear cost-function. A special case is the problem of finding a strongly connected (or k -edge-connected) orientation of a mixed graph whose in-degree vector is decreasingly minimal.

1 Introduction

For an integral polyhedron Q , let $\overset{\dots}{Q}$ denote the set of integral elements of Q . We are interested in finding the most fair element z of $\overset{\dots}{Q}$ where fairness wants to reflect the intuitive feeling that the components of z are distributed as equitably as possible. There may be several ways to formally capture fairness. For example, if the difference of the largest and the smallest components of z is minimum, then z is felt rather fair. The square sum of the components is another, more global measure for fairness. In this work we are interested in a third natural possibility: decreasing minimality. Let us call an element $z \in \overset{\dots}{Q}$ decreasingly minimal (dec-min) if the largest component of z is as small as possible, within this, the second largest component (possibly with the same value as the largest component) is as small as possible, within this, the third largest component of z is small as possible, and so on. Actually, it is more convenient to work with the following slightly more general concept. Let F be a specified subset of the coordinates. We say that $z \in \overset{\dots}{Q}$ is **decreasingly minimal on F** (or **F -dec-min** for short) if the restriction of z to F is decreasingly minimal. For the trivial special case $F = \emptyset$, the set of F -decmin elements of $\overset{\dots}{Q}$ is $\overset{\dots}{Q}$ itself.

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In [14] and [15], we characterized dec-min elements of an M-convex set (which is, by an equivalent definition, the set $\overset{\dots}{B}$ of integral elements of an integral base-polyhedron B [21, 22]) and described a strongly polynomial algorithm to compute a cheapest dec-min element with respect to a linear cost function. In [16], an analogous investigation was carried out for network flows. In the present work, we consider decreasingly minimal integer-valued submodular flows, a common generalization of network flows and base-polyhedra.

Submodular flows were introduced by Edmonds and Giles [6] while the term itself was suggested by Zimmermann [25]. The notion became a standard tool in discrete optimization, see the books [10, 19, 22, 24]. Originally, Edmonds and Giles used submodular functions for the definition but supermodular functions could equally well be used, and in applications one often needs supermodular functions. Therefore, in the present paper, we replace the term submodular flow by **base-flow** (but emphasize that these are the same). From this point of view, the term base-polyhedron is adequate since it refers to neither submodular nor supermodular functions. Our newly suggested term base-flow is intended to capture this neutrality. In Section 1.2, we recall the formal definitions of base-polyhedra and base-flows, along with their basic properties.

1.1 Main results

One of our main goals is to provide a complete description of the set of F -dec-min elements of $\overset{\dots}{Q}$ for an integral base-flow polyhedron Q . Roughly, the theorem states that this set is the set of integral elements of a base-flow polyhedron obtained from Q by intersecting a face of Q with a box which is ‘narrow’ on F . In what follows, \mathbf{Z} denotes the set of integers while $\overline{\mathbf{Z}} := \mathbf{Z} \cup \{+\infty\}$ and $\underline{\mathbf{Z}} := \mathbf{Z} \cup \{-\infty\}$. Throughout we use \mathbf{R} to denote the set of reals.

THEOREM 1.1. *Let $D = (V, A)$ be a digraph endowed with integer-valued lower and upper bound functions $f : A \rightarrow \underline{\mathbf{Z}}$ and $g : A \rightarrow \overline{\mathbf{Z}}$ for which $f \leq g$. Let B be an integral base-polyhedron for which the base-flow polyhedron $Q = Q(f, g; B)$ is non-empty. Let $F \subseteq A$ be a specified subset of edges such that both f and g are finite-valued on F . Then there exists a face B^∇ of B and there exists a pair (f^*, g^*) of integer-valued bounding functions on A with $f \leq f^* \leq g^* \leq g$ such that an element $z \in \overset{\dots}{Q}$ is F -dec-min if and only if $z \in \overset{\dots}{Q}(f^*, g^*; B^\nabla)$. Moreover, $0 \leq g^*(e) - f^*(e) \leq 1$ for every $e \in F$.*

The proof will be prepared in Sections 2 and 3, and completed in Section 4. It should be noted that in the general case, when the finiteness of f and g on F is not assumed, it may be the case that no F -dec-min element exists at all. In Section 6 we describe a characterization for the existence of F -dec-min integral feasible base-flow, and show that Theorem 1.1 extends to this case, as well.

It is a known and easy property [8] that the intersection of two (integral) g -polymatroids is an (integral) base-flow polyhedron. Therefore Theorem 1.1 can be specialized to the following.

Corollary 1.2. *Let Q_1 and Q_2 be two integral g -polymatroids in \mathbf{R}^S for which their intersection Q is non-empty. Let $F \subseteq S$ be a specified subset of ground-set S . Then there exists a face Q^∇ of Q and there exists an integral box $T = T(f^*, g^*)$ in \mathbf{R}^S such that an element $z \in \overset{\dots}{Q}$ is F -dec-min if and only if $z \in \overset{\dots}{Q}^\nabla \cap T$. Moreover, $0 \leq g^*(s) - f^*(s) \leq 1$ for every element $s \in F$, and $f^*(s) = -\infty$, $g^*(s) = +\infty$ for every element $s \in S - F$. ■*

Recall that an M -convex set is the set of integral elements of an integral base-polyhedron. In Discrete convex analysis [21, 22], the set of integral elements of an integral g -polymatroid is called an M^\natural -convex set (pronounce M -nat-convex or M -natural-convex). As a base-polyhedron is a special g -polymatroid, an M -convex set is M^\natural -convex. Furthermore, the intersection of two M -convex (resp., M^\natural -convex) sets is called an M_2 -convex (resp., M_2^\natural -convex) set. It was proved by Frank [8] that the intersection of two integral g -polymatroids is an integral polyhedron, implying that an M_2^\natural -convex set is the set of integral elements of the intersection of two integral g -polymatroids. Therefore, Corollary 1.2 can be interpreted as a characterization of the set of F -dec-min elements of an M_2^\natural -convex set.

Our second main goal is to develop a strongly polynomial algorithm for computing the bounding functions f^* and g^* in the theorem, as well as the face B^∇ of B . Once these data are available, with the help of a standard base-flow algorithm [7, 10], one can compute an F -dec-min element of $\overset{\dots}{Q}$. Even more, with the help of a minimum cost base-flow algorithm [4, 20] a minimum cost F -dec-min element of $\overset{\dots}{Q}$ can also be computed in polynomial time, with respect to a linear cost-function $c : A \rightarrow \mathbf{R}$. Indeed, by Theorem 1.1, this latter problem is nothing but a minimum cost (f^*, g^*) -bounded base-flow problem, which can be solved in polynomial time. (See, e.g. [4, 20], and the book of Schrijver [24] (p. 1019)).

This approach gives rise to an algorithm in the special base-flow problem when the goal is to find a (minimum cost) k -edge-connected and in-degree-bounded orientation of a mixed graph for which the in-degree vector is decreasingly minimal.

1.2 Basic notions and notation

Let S be a finite ground-set. Two subsets X and Y of S are **intersecting** if $X \cap Y \neq \emptyset$. They are **properly intersecting** if none of $X \cap Y$, $X - Y$, $Y - X$ is empty. If, in addition, $S - (X \cup Y)$ is also non-empty, we speak of a **crossing** pair.

For a vector $x \in \mathbf{R}^S$ or a function $x : S \rightarrow \mathbf{R}$, the set-function $\tilde{x} : 2^S \rightarrow \mathbf{R}$ is defined by

$$\tilde{x}(Z) := \sum [x(s) : s \in Z] \quad (Z \subseteq S).$$

Such a function is modular in the sense that the modular equality

$$\tilde{x}(X) + \tilde{x}(Y) = \tilde{x}(X \cap Y) + \tilde{x}(X \cup Y) \quad (1.1)$$

holds for every pair $\{X, Y\}$ of subsets of S .

Unless stated otherwise, we assume throughout for a set-function that its value on the empty set is zero. Let $p : S \rightarrow \underline{\mathbf{Z}}$ be a set-function on S . When $p(S)$ is finite, the

set-function \bar{p} defined by

$$\bar{p}(X) := p(S) - p(S - X)$$

is called the **complementary set-function** or just the complement of p . Clearly, $\bar{p}(\emptyset) = 0$ and $\bar{p}(S) = p(S)$, and the complement of \bar{p} is p itself.

The supermodular inequality for $X, Y \subseteq S$ is as follows:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \quad (1.2)$$

The set-function p is called **fully supermodular** or just **supermodular** if (1.2) holds for every pair of subsets $X, Y \subseteq S$. We say the p is **intersecting (crossing) supermodular** if (1.2) holds for every intersecting (crossing) pair of subsets of S .

For a set-function $b : S \rightarrow \bar{\mathbf{Z}}$, the submodular inequality is as follows:

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \quad (1.3)$$

Function b is called fully (intersecting, crossing) submodular if (1.3) holds for every (intersecting, crossing) pair of subsets. Clearly, the complement of a fully (crossing) supermodular function is fully (crossing) submodular.

With a set function h with finite $h(S)$, we associate two polyhedra:

$$B'(h) := \{x \in \mathbf{R}^S : \tilde{x}(S) = h(S) \text{ and } \tilde{x}(Z) \geq h(Z) \text{ for all } Z \subset S\},$$

$$B(h) := \{x \in \mathbf{R}^S : \tilde{x}(S) = h(S) \text{ and } \tilde{x}(Z) \leq h(Z) \text{ for all } Z \subset S\}.$$

Obviously, $B(h) = B'(\bar{h})$ and $B'(h) = B(\bar{h})$.

For a (fully) supermodular function p with finite $p(S)$, the polyhedron $B := B'(p)$ is called a **base-polyhedron**. We say that B is a **0-base-polyhedron** if $p(S) = 0$, or equivalently, $\tilde{x}(S) = 0$ for each $x \in B$. A base-polyhedron can also be described with the help of a submodular function b , namely, $B = B(b)$ where b and p are complementary set-functions. The empty set, by convention, is also considered a base-polyhedron although it cannot be defined by a fully sub- (or supermodular) function (which is equivalent to saying that $B'(p)$ is never empty).

A basic property of non-empty base-polyhedra is that they uniquely determine their bounding fully supermodular (or submodular) function. Namely,

$$p(Z) = \min\{\tilde{x}(Z) : x \in B\} \quad (\text{or } b(Z) = \max\{\tilde{x}(Z) : x \in B\}). \quad (1.4)$$

This formula is particularly important from an algorithmic point of view since an algorithm developed for the case when the base-polyhedron B is defined by a fully supermodular p , the same algorithm can be used for an arbitrarily given base-polyhedron (defined, for example, by a crossing supermodular function) provided that a subroutine is available to compute $p(Z)$ in (1.4) for any input subset $Z \subseteq S$.

A rich overview of properties of base-polyhedra can be found in the books [10, 19]. For example, if p is integer-valued, then the base-polyhedron $B'(p)$ is an integral polyhedron. Furthermore, the face of a base-polyhedron, its translate by a vector, its intersection with a box are also base-polyhedra, as well as the direct sum and the Minkowski sum of base-polyhedra. For an arbitrary chain \mathcal{C} of subsets of S , the

polyhedron $\{x \in B'(p) : \tilde{x}(Z) = p(Z) \text{ for each } Z \in \mathcal{C}\}$ is a face of $B'(p)$ and every face of $B'(p)$ arises in this way.

In applications it is fundamental that crossing supermodular (or submodular) functions also define (possibly empty) base-polyhedra. The non-emptiness in this case was characterized by an elegant theorem of Fujishige [18]. For a general overview of base-polyhedra, see the book [10]. More recent results [1, 2] indicate that base-polyhedra defined by even weaker functions also show up in several applications. It is essential to see that formulating and proving results for base-polyhedra is significantly easier when the bounding set-function defining B is fully supermodular (or submodular) but in applications it is typical that B is described by a weaker function. For such situations it is a typical task to modify appropriately the algorithm developed for base-polyhedra given by fully supermodular functions.

Let $D = (V, A)$ be a loopless digraph endowed with integer-valued functions $f : A \rightarrow \underline{\mathbf{Z}}$ and $g : A \rightarrow \overline{\mathbf{Z}}$ for which $f \leq g$. Here f and g are serving as lower and upper bound functions, respectively. An edge e is called **tight** if $f(e) = g(e)$. The polyhedron $T(f, g) := \{x : f \leq x \leq g\}$ is called a **box**.

Let $\varrho_D(Z) = \varrho_A(Z)$ denote the number of edges of D entering $Z \subseteq V$ while $\delta_D(Z) = \delta_A(Z)$ is the number of edges of D leaving Z . For a function $x : A \rightarrow \mathbf{R}$ $\varrho_x(v) := \sum[x(uv) : uv \in A]$ and $\delta_x(v) := \sum[x(vu) : vu \in A]$. We call x **feasible** if $f \leq x \leq g$. It is a simple property that $f \leq g$ implies that $\varrho_g - \delta_f$ is a fully submodular function. Define function $\psi_x : V \rightarrow \mathbf{R}$ by $\psi_x(v) = \varrho_x(v) - \delta_x(v)$. The function ψ_x on V is sometime called the **net in-flow** of x . Define the set-function Ψ_x by $\Psi_x(Z) := \varrho_x(Z) - \delta_x(Z)$ for $Z \subseteq V$. Clearly, $\Psi(\emptyset) = \Psi_x(V) = 0$.

Let b be a crossing submodular function. A function $x : A \rightarrow \mathbf{R}$ is called a **submodular flow** if $\Psi_x \leq b$. When $f \leq x \leq g$, we speak of an (f, g) -**bounded** or **feasible** submodular flow. If there is a submodular flow x , then $\Psi_x(V) = 0$ implies that $b(V) \geq 0$. If $b(V) > 0$, then $b(V)$ can be reduced to 0 since this affects neither the (crossing) submodularity of b nor the requirement $\Psi_x \leq b$. Therefore we shall assume throughout that $b(V) = 0$, that is, the base-polyhedron $B(b)$ is a 0-base-polyhedron (with zero component-sum of each vector in B). The set

$$Q = \{x \in \mathbf{R}^A : f \leq x \leq g, \Psi_x \leq b\}$$

of (f, g) -feasible submodular flows is called a **submodular flow** polyhedron.

It is immediate from the definitions that x is a submodular flow precisely if its net in-flow vector ψ_x belongs to the 0-base-polyhedron B described by b . Since a base-polyhedron can also be defined with the help of a supermodular function, it follows that submodular flows can be defined with supermodular functions, as well, namely, if p is a crossing supermodular function, then the polyhedron

$$Q := \{x \in \mathbf{R}^A : f \leq x \leq g, \Psi_x \geq p\} \quad (1.5)$$

is also a submodular flow polyhedron. Therefore the role of submodular and supermodular functions in defining a submodular flow is completely symmetric, and hence a submodular flow could also be called as a supermodular flow. This is why we suggested in Section 1.1 the term base-flow rather than submodular flow. Actually, the

point is that in the definition of a base-flow it is the 0-base-polyhedron B that plays the essential role and not the way how B is given. Therefore a function (or vector) $x : A \rightarrow \mathbf{R}$ is a base-flow if $\psi_x \in B$ while a base-flow polyhedron

$$Q(f, g; B) := \{x \in \mathbf{R}^A : f \leq x \leq g, \psi_x \in B\} \quad (1.6)$$

is the set of feasible base-flows. When the 0-base-polyhedron $B = B'(p)$ is defined by a supermodular function p (with $p(V) = 0$), we speak of a p -base-flow.

A fundamental result of Edmonds and Giles [6] states for a crossing submodular function b that the linear system

$$\{f \leq x \leq g, \varrho_x(Z) - \delta_x(Z) \leq b(Z) \text{ for every } Z \subseteq V\}$$

describing a base-flow polyhedron is totally dual integral (TDI), implying that Q is an integral polyhedron whenever f, g, b are integer-valued.

An important feature of base-flows is that, given a subset $F \subseteq A$, the projection of a base-flow polyhedron Q to \mathbf{R}^F (that is, the restriction to F) is itself a base-flow polyhedron. The face of a base-flow polyhedron is also a base-flow polyhedron.

When B consists of a single element m (with $\tilde{m}(V) = 0$), a base-flow is a modular flow or mod-flow, which is a standard circulation when $m \equiv 0$. Decreasingly minimal mod-flows were investigated in [16]. Note that the projection of a mod-flow polyhedron is a base-flow polyhedron but typically not a mod-flow polyhedron.

Frank [7, 10] provided a necessary and sufficient condition for the non-emptiness of a base-flow polyhedron. For general crossing supermodular (or submodular) functions this condition is rather complicated but when p is fully supermodular, the formulation (and the proof of its necessity) is pretty straightforward.

THEOREM 1.3. *When p is fully supermodular, the p -base-flow polyhedron Q defined in (1.5) is non-empty if and only if $\varrho_g - \delta_f \geq p$, that is,*

$$\varrho_g(X) - \delta_f(X) \geq p(X) \text{ for every subset } X \subseteq V. \quad (1.7)$$

Note that in the special case of circulations (when $p \equiv 0$), we are back at Hoffman's circulation theorem.

2 L -upper-minimal base-flows

Let D, f, g, B , and $Q = Q(f, g; B)$ be the same as in Theorem 1.1, but in this section we do not use F . Let L be a subset of A for which f and g are finite-valued on L (that is, $f(e)$ may be $-\infty$ and $g(e)$ may be $+\infty$ only if $e \in A - L$.) We say that $z \in \overset{\dots}{Q}$ is **L -upper-minimal** or that z is an **L -upper-minimizer** if the number of g -saturated edges in L is as small as possible, where an edge $e \in L$ is called **g -saturated** if $z(e) = g(e)$. In this section, we are interested in characterizing the L -upper-minimizer elements of $\overset{\dots}{Q}$.

For a chain \mathcal{C} of subsets of V , we call an edge e **\mathcal{C} -entering** (respectively, **\mathcal{C} -leaving**) if e enters (resp., leaves) a member of \mathcal{C} , and e is **\mathcal{C} -neutral** when e neither enters nor leaves any member of \mathcal{C} . For a subset I of edges, let $\varrho_I(C)$ ($C \subseteq V$) denote the number of edges in I entering C , and let $\varrho_I(\mathcal{C})$ denote the number of \mathcal{C} -entering edges in I . When this number is positive, we say that I **enters** \mathcal{C} .

One of the goals of this section is to prove the following characterization of L -upper minimizer base-flows. This will serve as a main tool in proving Theorem 1.1.

THEOREM 2.1. *Let $Q = Q(f, g; B)$ be a non-empty integral base-flow polyhedron and L a subset of edges on which both f and g are finite-valued. There is a face B_L of B and there are bounds $f_L : A \rightarrow \underline{\mathbb{Z}}$ and $g_L : A \rightarrow \overline{\mathbb{Z}}$ with $f \leq f_L \leq g_L \leq g$ such that an element $z \in \overset{\circ}{\overset{\circ}{Q}}$ is an L -upper minimizer if and only if $z \in \overset{\circ}{\overset{\circ}{Q}}_L$ where $Q_L = Q(f_L, g_L; B_L)$.*

The proof will be prepared in this section and completed in Section 3.

If there is an edge $e \in L$ with $f(e) = g(e)$, then $x(e) = g(e)$ for each $x \in Q$. Therefore a base-flow x is g -saturated on L if and only if it is g -saturated on $L' := L - e$, and hence x is L -upper minimal precisely if it is L' -upper minimal. Therefore it suffices to prove the theorem for L' , and hence we can assume that L contains no tight edges, or in other words,

$$-\infty < f(e) < g(e) < +\infty \text{ for every edge } e \in L. \quad (2.1)$$

We shall also show how the chain determining the face B_L and the bounds (f_L, g_L) occurring in the theorem can be computed in polynomial time in the case when a subroutine is available to compute $p(Z)$ for any given subset $Z \subseteq S$, where p denotes the unique fully supermodular function p for which $B = B'(p)$.

2.1 Upper bound for the number of g -saturated edges

Our first goal is to show how a chain \mathcal{C} of subsets of V provides a lower bound for the number of g -saturated edges in L . Let p denote the unique fully supermodular function defining B , that is, $B = B'(p)$. Let $B_{\mathcal{C}}$ be the face of B defined by \mathcal{C} . Define the bounding pair $(f_{\mathcal{C}}(e), g_{\mathcal{C}}(e))$ for each edge $e \in A$, as follows. For $e \in L$, let

$$(f_{\mathcal{C}}(e), g_{\mathcal{C}}(e)) := \begin{cases} (g(e), g(e)) & \text{if } e \text{ enters at least two members of } \mathcal{C}, \\ (g(e) - 1, g(e)) & \text{if } e \text{ enters exactly one member of } \mathcal{C}, \\ (f(e), f(e)) & \text{if } e \text{ is } \mathcal{C}\text{-leaving,} \\ (f(e), g(e) - 1) & \text{if } e \text{ is } \mathcal{C}\text{-neutral.} \end{cases} \quad (2.2)$$

For $e \in A - L$, let

$$(f_{\mathcal{C}}(e), g_{\mathcal{C}}(e)) := \begin{cases} (g(e), g(e)) & \text{if } e \text{ is } \mathcal{C}\text{-entering,} \\ (f(e), f(e)) & \text{if } e \text{ is } \mathcal{C}\text{-leaving,} \\ (f(e), g(e)) & \text{if } e \text{ is } \mathcal{C}\text{-neutral.} \end{cases} \quad (2.3)$$

It follows from this definition that $f \leq f_{\mathcal{C}} \leq g_{\mathcal{C}} \leq g$. Note that these data define a base-flow polyhedron $Q(f_{\mathcal{C}}, g_{\mathcal{C}}; B_{\mathcal{C}})$ included in $Q(f, g; B)$.

Consider the following optimality criteria.

$$\left\{ \begin{array}{ll} \text{(O1)} & x(e) = f(e) \quad \text{if } e \in A \text{ is } \mathcal{C}\text{-leaving} \\ \text{(O2)} & x(e) = g(e) \quad \text{if } e \in A - L \text{ is } \mathcal{C}\text{-entering} \\ \text{(O3)} & g(e) - 1 \leq x(e) \leq g(e) \quad \text{if } e \in L \text{ enters exactly one member of } \mathcal{C} \\ \text{(O4)} & x(e) = g(e) \quad \text{if } e \in L \text{ enters at least two members of } \mathcal{C} \\ \text{(O5)} & f(e) \leq x(e) \leq g(e) - 1 \quad \text{if } e \in L \text{ is } \mathcal{C}\text{-neutral} \\ \text{(O6)} & \varrho_x(\mathcal{C}) - \delta_x(\mathcal{C}) = p(\mathcal{C}) \quad \text{if } \mathcal{C} \in \mathcal{C}. \end{array} \right. \quad (2.4)$$

An easy case-checking immediately shows the following.

Claim 2.2. *Let $x \in Q$. The union of the first five optimality criteria in (2.4) is equivalent to the requirement $f_{\mathcal{C}} \leq x \leq g_{\mathcal{C}}$. Criterion (O6) is equivalent to stating that $\psi_x \in B_{\mathcal{C}}$. ■*

Lemma 2.3. *Let $Q = Q(f, g; B)$ be a non-empty base-flow polyhedron and let p denote the unique supermodular function defining the 0-base-polyhedron B (that is, $B = B'(p)$). Let L be a subset of A meeting (2.1). Let x be an arbitrary element of Q and let*

$$X := \{e \in L : x(e) = g(e)\}. \quad (2.5)$$

Let \mathcal{C} be a chain of subsets of V . Then

$$|X| \geq \varrho_L(\mathcal{C}) - \sum [\varrho_g(\mathcal{C}) - \delta_f(\mathcal{C}) - p(\mathcal{C}) : \mathcal{C} \in \mathcal{C}]. \quad (2.6)$$

Moreover, (2.6) is met by equality if and only if the optimality criteria hold in (2.4), or, equivalently, $x \in Q(f_{\mathcal{C}}, g_{\mathcal{C}}; B_{\mathcal{C}})$.

Proof. If $\varrho_g(\mathcal{C}) - \delta_f(\mathcal{C}) - p(\mathcal{C}) = +\infty$, then (2.6) trivially holds, so we can assume that $\varrho_g(\mathcal{C}) < +\infty$, $\delta_f(\mathcal{C}) > -\infty$, and $p(\mathcal{C}) > -\infty$. Observe that, for any set I of edges,

$$\sum_{\mathcal{C} \in \mathcal{C}} \varrho_I(\mathcal{C}) \geq \varrho_I(\mathcal{C}), \quad (2.7)$$

(where $\varrho_I(\mathcal{C})$ denotes the number of \mathcal{C} -entering elements of I) and

$$\sum_{\mathcal{C} \in \mathcal{C}} \varrho_I(\mathcal{C}) = \varrho_I(\mathcal{C}) \Leftrightarrow \text{each edge in } I \text{ enters at most one member of } \mathcal{C}. \quad (2.8)$$

By applying (2.7) to $I := L - X$, the assumption $x \in Q$ implies

$$\sum_{\mathcal{C} \in \mathcal{C}} p(\mathcal{C}) \leq \sum_{\mathcal{C} \in \mathcal{C}} [\varrho_x(\mathcal{C}) - \delta_x(\mathcal{C})] \leq \sum_{\mathcal{C} \in \mathcal{C}} [\varrho_g(\mathcal{C}) - \delta_f(\mathcal{C}) - \varrho_{L-X}(\mathcal{C})] \leq \sum_{\mathcal{C} \in \mathcal{C}} [\varrho_g(\mathcal{C}) - \delta_f(\mathcal{C})] - \varrho_{L-X}(\mathcal{C}) \quad (2.9)$$

$$= \sum_{C \in \mathcal{C}} [\varrho_g(C) - \delta_f(C)] - \varrho_L(\mathcal{C}) + \varrho_X(\mathcal{C}),$$

from which

$$|X| \geq \varrho_X(\mathcal{C}) \geq \varrho_L(\mathcal{C}) + \sum_{C \in \mathcal{C}} [p(C) - \varrho_g(C) + \delta_f(C)], \quad (2.10)$$

and hence (2.6) follows.

To see the second part of the lemma, suppose first that equality holds in (2.6), and want to prove that the optimality criteria hold in (2.4). Equality in (2.6) implies that both inequalities in (2.10) are met with equalities. Therefore

$$|X| = \varrho_X(\mathcal{C}), \quad (2.11)$$

and each inequality in (2.9) is met with equalities, that is,

$$\sum_{C \in \mathcal{C}} p(C) = \sum_{C \in \mathcal{C}} [\varrho_x(C) - \delta_x(C)], \quad (2.12)$$

$$\sum_{C \in \mathcal{C}} [\varrho_x(C) - \delta_x(C)] = \sum_{C \in \mathcal{C}} [\varrho_g(C) - \varrho_{L-X}(C) - \delta_f(C)], \quad (2.13)$$

$$\sum [\varrho_{L-X}(C) : C \in \mathcal{C}] = \varrho_{L-X}(\mathcal{C}) \quad (2.14)$$

Here (2.12) holds precisely if $p(C) = \varrho_x(C) - \delta_x(C)$ for each $C \in \mathcal{C}$, which is exactly (O6).

Equality (2.13) implies that $\delta_x(C) = \delta_f(C)$ for each C and hence (O1) follows. (2.13) also implies for each \mathcal{C} -entering edge $e \in A - L$ that $x(e) = g(e)$, that is, (O2) holds. Furthermore, (2.13) implies for each \mathcal{C} -entering edge $e \in L$ that $g(e) - 1 \leq x(e) \leq g(e)$ (namely, $x(e) = g(e)$ when $e \in X$, and $x(e) = g(e) - 1$ when $e \in L - X$), that is, (O3) holds.

Equality (2.14) and (2.8) imply that $e \in L - X$ entering \mathcal{C} enters exactly one member of \mathcal{C} . Therefore, if an edge $e \in L$ enters at least two members of \mathcal{C} , then $e \in X$, that is, $x(e) = g(e)$, showing that (O4) holds.

Finally, (2.11) means that every g -saturated edge means that every g -saturated edge in L enters \mathcal{C} . Therefore, if an edge $e \in L$ does not enter \mathcal{C} , then $x(e) \leq g(e) - 1$, showing that (O5) holds.

To see the reverse implication, suppose that the element x of Q and the chain \mathcal{C} meet the six optimality criteria in the lemma. We have to show that (2.6) holds with equality. What we are going to prove is that both inequalities in (2.10) are met with equality.

Claim 2.4. $|X| = \varrho_X(\mathcal{C})$.

Proof. Suppose, indirectly, that $|X| > \varrho_X(\mathcal{C})$ which means that there is an edge $e \in X$ which is not \mathcal{C} -entering. Then $x(e) = g(e) > f(e)$ and (O1) imply that e is not \mathcal{C} -leaving either. That is, e is \mathcal{C} -neutral and hence (O5) implies that $x(e) \leq g(e) - 1$, a contradiction. ■

Therefore the first inequality in (2.10) is met indeed by equality. The second inequality in (2.10) holds with equality precisely if each of the three inequalities in (2.9) holds with equality. Consider these three inequalities separately.

Criterion (O6) states that $\varrho_x(C) - \delta_x(C) = p(C)$ for each $C \in \mathcal{C}$, implying (2.12), which shows that the first inequality in (2.9) holds with equality.

Claim 2.5. *The second inequality in (2.9) holds with equality, that is,*

$$\sum_{C \in \mathcal{C}} [\varrho_x(C) - \delta_x(C)] = \sum_{C \in \mathcal{C}} [\varrho_g(C) - \delta_f(C) - \varrho_{L-X}(C)]. \quad (2.15)$$

Proof. Observe first that (O1) implies

$$\sum_{C \in \mathcal{C}} \delta_x(C) = \sum_{C \in \mathcal{C}} \delta_f(C). \quad (2.16)$$

Let C be a member of \mathcal{C} . For an edge $e \in A - L$ entering C , (O2) implies that $x(e) = g(e)$. Furthermore, the definition of X shows that $x(e) = g(e)$ for $e \in X$. By integrating these observations, we get the following:

$$\begin{aligned} \varrho_x(C) &= \sum [x(e) : e \in A - (L - X), e \text{ enters } C] + \sum [x(e) : e \in L - X, e \text{ enters } C] \\ &= \sum [g(e) : e \in A - (L - X), e \text{ enters } C] + \sum [g(e) - 1 : e \in L - X, e \text{ enters } C] \\ &= \varrho_g(C) - \varrho_{L-X}(C). \end{aligned}$$

Therefore

$$\sum_{C \in \mathcal{C}} \varrho_x(C) = \sum_{C \in \mathcal{C}} [\varrho_g(C) - \varrho_{L-X}(C)].$$

This and (2.16) imply (2.15). ■

Claim 2.6. $\sum [\varrho_{L-X}(C) : C \in \mathcal{C} :] = \varrho_{L-X}(\mathcal{C})$.

Proof. The claim is equivalent to stating that every edge in $L - X$ entering \mathcal{C} enters exactly one member of \mathcal{C} . But this is true since if an edge $e \in L - X$, indirectly, enters at least two members of \mathcal{C} , then (O4) implies that $x(e) = g(e)$, that is, $e \in X$, a contradiction. ■

Claim 2.6 immediately implies that the third inequality in (2.9) also holds with equality. This and Claim 2.4 imply that both inequalities in (2.10) are met with equalities and hence we have equality in (2.6), as well. ■■

3 Min-max formula for L -upper-minimizers

In this section we provide a min-max theorem for the minimum number of g -saturated L -edges of a member of $\overset{\dots}{\mathcal{Q}}$. In Lemma 2.3, we did not need the unique supermodular function p defining the 0-base-polyhedron B in question. However, in formulating and proving the min-max theorem below, we shall rely on p . It will also be shown how the optima can be computed with the help of a standard algorithm to compute a cheapest integer-valued feasible base-flow.

THEOREM 3.1. *Let $Q = Q(f, g; B)$ be a non-empty base-flow polyhedron and let p denote the unique supermodular function defining 0-base-polyhedron B (that is, $B = B'(p)$). Let L be a subset of A meeting (2.1). The minimum number of g -saturated L -edges of an (f, g) -bounded integer-valued p -base-flow is equal to*

$$\max\{\varrho_L(\mathcal{C}) - \sum_{C \in \mathcal{C}} [\varrho_g(C) - \delta_f(C) - p(C)]\}, \quad (3.1)$$

where the maximum is taken over all chains \mathcal{C} of subsets C of V with $p(C) > -\infty$ and $\varrho_g(C) - \delta_f(C) < +\infty$. In particular, if the minimum is zero, the maximum is attained at the empty chain.

Proof. The first part of Lemma 2.3 implies that $\min \geq \max$.

For proving the reverse direction, consider the primal optimization problem which is to find an integral (f, g) -bounded base-flow which saturates (with respect to g) a minimum number of elements of L . We show that this is a minimum cost feasible base-flow problem on a modified digraph. To this end, we introduce a parallel copy e' of each $e \in L$. Let L' denote the set of new edges. Let

$$A_1 := A \cup L', \quad D' := (V, L'), \quad \text{and} \quad D_1 := (V, A_1).$$

Define g_L on A by $g_L := g - \chi_L$, that is, we reduce $g(e)$ by 1 for each $e \in L$. Since L contains no tight edges, $g_L \geq f$ follows. Let f_1 and g_1 be bounding functions and c_1 a cost function defined on A_1 as follows.

$$\begin{aligned} f_1(e) &:= f(e), & g_1(e) &:= g_L(e), & \text{and} & c_1(e) &:= 0 & \text{if } e \in A, \\ f_1(e') &:= 0, & g_1(e') &:= 1, & \text{and} & c_1(e') &:= 1 & \text{if } e' \in L'. \end{aligned}$$

Claim 3.2. *The problem of finding an (f, g) -bounded integer-valued p -base-flow on A admitting a minimum number of g -saturated L -edges is equivalent to finding a minimum c_1 -cost (f_1, g_1) -bounded integer-valued p -base-flow on A_1 .*

Proof. First, let $x \in \overset{\dots}{\mathcal{Q}}$ and let $X := \{e \in L : x(e) = g(e)\}$ denote the set of g -saturated elements of L . Let $X' := \{e' : e \in X\}$ denote the subset of L' corresponding to X . Define a p -base-flow x_1 on A_1 as follows.

$$\begin{aligned} x_1(e) &:= \begin{cases} x(e) & \text{if } e \in A - X, \\ g(e) - 1 & \text{if } e \in X, \end{cases} \\ x_1(e') &:= \begin{cases} 1 & \text{if } e' \in X', \\ 0 & \text{if } e' \in L' - X'. \end{cases} \end{aligned}$$

Then x_1 is an (f_1, g_1) -bounded p -base-flow on A_1 whose c_1 -cost is $|X|$.

Conversely, let x_1 be a minimum cost integer-valued (f_1, g_1) -bounded p -base-flow on A_1 . Observe that if $x_1(e') = 1$ for some $e' \in L'$, then $x_1(e) = g_1(e) = g(e) - 1$ where e is the edge in L corresponding to e' . Indeed, if we had $x_1(e) \leq g(e) - 2$, then the p -base-flow obtained from x_1 by adding 1 to $x_1(e)$ and subtracting 1 from $x_1(e')$ would be of smaller cost. It follows that the p -base-flow x on A defined by

$$x(e) := \begin{cases} x_1(e) + x_1(e') & \text{if } e \in L, \\ x_1(e) & \text{if } e \in A - L \end{cases} \quad (3.2)$$

is an (f, g) -bounded p -base-flow in D , for which the number of g -saturated L -edges is exactly the c_1 -cost of x_1 . \blacksquare

By Claim 3.2, we investigate the minimum of the c_1 -cost of integer-valued (f_1, g_1) -bounded p -base-flows on A_1 . In order to describe the dual optimization problem, let N be a $\{0, \pm 1\}$ -matrix whose columns correspond to the elements of A (the edge-set of D) while its rows correspond to the members of

$$\mathcal{P} := \{Z \subseteq V : p(Z) > -\infty\}.$$

The entry of N corresponding to $Z \in \mathcal{P}$ and $e \in A$ is $+1$ if e enters Z , -1 if e leaves Z , and 0 otherwise. Note that, for a vector $x \in A \rightarrow \mathbf{R}$, the requirement $\varrho_x(Z) - \delta_x(Z) \geq p(Z)$ for every $Z \subseteq V$ is equivalent to $Nx \geq p$. Let N' denote the matrix associated analogously with D' , and let $N_1 = [N, N']$.

The primal linear program is as follows:

$$\min\{c_1 x_1 : N_1 x_1 \geq p, x_1 \geq f_1, -x_1 \geq -g_1\}. \quad (3.3)$$

The dual linear program is as follows:

$$\max\{yp + z_1 f_1 - w_1 g_1 : yN_1 + z_1 - w_1 = c_1, y \geq 0, z_1 \geq 0, w_1 \geq 0\} \quad (3.4)$$

where $yp = \sum [y(Z)p(Z) : Z \in \mathcal{P}]$. Note that the components of $z_1 = (z, z')$ correspond to the edges in A and in L' , respectively, and the analogous statement holds for $w_1 = (w, w')$.

A fundamental theorem of Edmonds and Giles [6] states that the linear system in (3.3) is totally dual integral (TDI). In the present case, when each of f, g, p , and c_1 is integer-valued, the TDI-ness implies that both the primal and the dual program have an integer-valued optimal solution. In addition, since p is fully supermodular, the optimal y^* in (3.4) can be chosen in such a way that the sets C for which $y^*(C)$ is positive form a chain. Therefore we consider only those vectors y occurring in the constraint of (3.4) which are **chained** in the sense that $\mathcal{C} = \{C \subseteq V : y(C) > 0\}$ is a chain. We refer to \mathcal{C} as the **support chain** of y . The dual solution (y, z_1, w_1) is said to be **simple** if y is chained and $\min\{z_1(e), w_1(e)\} = 0$ for every edge $e \in A_1$.

Claim 3.3. *There exists an integer-valued optimal dual solution to (3.4) which is simple.*

Proof. We indicated already that there a dual integral solution (y, z_1, w_1) where y is chained. If both $z_1(e)$ and $w_1(e)$ are positive on an edge $e \in A_1$, then by reducing both $z_1(e)$ and $w_1(e)$ by $\min\{z_1(e), w_1(e)\}$ we obtain another dual solution whose dual cost is larger by $g_1(e) - f_1(e) \geq 0$ than the dual cost $yp + z_1f_1 - w_1g_1$ of (y, z_1, w_1) . ■

Let (y, z_1, w_1) be a simple (integer-valued) dual solution. Since z_1 and w_1 are non-negative vectors, y uniquely determines them. We describe explicitly how z_1 and w_1 can be expressed by y . Recall that chain \mathcal{C} consist of subsets $Z \subseteq V$ for which $y(Z)$ is positive. Define a function $\pi_y : V \rightarrow \mathbf{Z}$ as follows:

$$\pi_y(v) := \sum [y(C) : v \in C \in \mathcal{C}]. \quad (3.5)$$

Define $\Delta_y : A_1 \rightarrow \mathbf{Z}$ as follows. For $a = uv \in A_1$, let

$$\Delta_y(a) := \pi_y(v) - \pi_y(u).$$

Then $\Delta_y(a) > 0$ (respectively, $\Delta_y(a) < 0$) if and only if a is \mathcal{C} -entering (resp., \mathcal{C} -leaving) and hence $\Delta_y(a) = 0$ precisely when a is \mathcal{C} -neutral.

Claim 3.4. *Let $e = uv$ be an edge of D_1 . Then*

$$\Delta_y(a) + z_1(a) - w_1(a) = c_1(a). \quad (3.6)$$

Proof. It follows from the definition of Δ_y that

$$\Delta_y(a) = \sum [y(C) : C \in \mathcal{C} \text{ is entered by } a] \quad \text{when } a \text{ is } \mathcal{C}\text{-entering}, \quad (3.7)$$

$$\Delta_y(a) = - \sum [y(C) : C \in \mathcal{C} \text{ is left by } a] \quad \text{when } a \text{ is } \mathcal{C}\text{-leaving}. \quad (3.8)$$

$$\Delta_y(a) = 0 \quad \text{when } a \text{ is } \mathcal{C}\text{-neutral}. \quad (3.9)$$

These and the constraint $yN_1 + z_1 - w_1 = c_1$ in (3.4) imply (3.6). ■

Claim 3.5. *Let $e = uv \in A$.*

$$\text{If } e \text{ is } \mathcal{C}\text{-neutral, then} \quad \begin{cases} z_1(e) = 0 \\ w_1(e) = 0, \end{cases} \quad (3.10)$$

$$\text{if } e \text{ is } \mathcal{C}\text{-entering, then} \quad \begin{cases} z_1(e) = 0 \\ w_1(e) = \Delta_y(e), \end{cases} \quad (3.11)$$

$$\text{if } e \text{ is } \mathcal{C}\text{-leaving, then} \quad \begin{cases} z_1(e) = -\Delta_y(e) \\ w_1(e) = 0. \end{cases} \quad (3.12)$$

Proof. Since $c_1(e) = 0$ and (y, z_1, w_1) is simple, the claim follows by applying (3.6) to e in place of a . ■

Claim 3.6. Let $e' = uv \in L'$.

$$\text{If } e' \text{ is } \mathcal{C}\text{-neutral, then } \begin{cases} z_1(e') = 1 \\ w_1(e') = 0, \end{cases} \quad (3.13)$$

$$\text{if } e' \text{ is } \mathcal{C}\text{-entering, then } \begin{cases} z_1(e') = 0 \\ w_1(e') = \Delta_y(e') - 1, \end{cases} \quad (3.14)$$

$$\text{if } e' \text{ is } \mathcal{C}\text{-leaving, then } \begin{cases} z_1(e') = -\Delta_y(e') + 1 \\ w_1(e') = 0. \end{cases} \quad (3.15)$$

Proof. Since $c_1(e') = 1$ and (y, z_1, w_1) is simple, the claim follows by applying (3.6) to e' in place of a . ■

Claim 3.7. $z_1 f_1 = \sum [y(C) \delta_f(C) : C \in \mathcal{C}]$.

Proof. Since $f_1(e') = 0$ for $e' \in L'$, we have

$$\begin{aligned} z_1 f_1 &= \sum_{e \in A} z_1(e) f_1(e) + \sum_{e' \in L'} z_1(e') f_1(e') = \sum_{e \in A} z_1(e) f_1(e) \\ &= \sum [z_1(e) f(e) : e \in A, z_1(e) > 0] = \sum [-\Delta_y(e) f(e) : e \in A \text{ is } \mathcal{C}\text{-leaving}] \\ &= \sum [y(C) \delta_f(C) : C \in \mathcal{C}]. \end{aligned}$$

■

Claim 3.8. $w_1 g_1 = \sum [y(C) \varrho_g(C) : C \in \mathcal{C}] - \varrho_L(\mathcal{C})$.

Proof. Observe first that

$$\begin{aligned} &\sum [\Delta_y(e) g_1(e) : e \in A \text{ is } \mathcal{C}\text{-entering}] \\ &= \sum [\Delta_y(e) g(e) : e \in A - L \text{ is } \mathcal{C}\text{-entering}] + \sum [\Delta_y(e) (g(e) - 1) : e \in L \text{ is } \mathcal{C}\text{-entering}] \\ &= \sum [\Delta_y(e) g(e) : e \in A \text{ is } \mathcal{C}\text{-entering}] - \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}] \\ &= \sum_{C \in \mathcal{C}} y(C) \varrho_g(C) - \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}]. \end{aligned}$$

Second,

$$\begin{aligned} &\sum [\Delta_y(e') - 1 : e' \in L' \text{ is } \mathcal{C}\text{-entering}] = \sum [\Delta_y(e) - 1 : e \in L \text{ is } \mathcal{C}\text{-entering}] \\ &= \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}] - \sum [1 : e \in L \text{ is } \mathcal{C}\text{-entering}] \\ &= \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}] - \varrho_L(\mathcal{C}). \end{aligned}$$

Since $g_1(e') = 1$ for $e' \in L'$, we have

$$\begin{aligned}
& w_1 g_1 \\
&= \sum_{e \in A} w_1(e) g_1(e) + \sum_{e' \in L'} w_1(e') g_1(e') = \sum [w_1(e) g_1(e) : e \in A, w_1(e) > 0] + \sum_{e' \in L'} w_1(e') \\
&= \sum [\Delta_y(e) g_1(e) : e \in A \text{ is } \mathcal{C}\text{-entering}] + \sum [\Delta_y(e') - 1 : e' \in L' \text{ is } \mathcal{C}\text{-entering}] \\
&= \sum_{C \in \mathcal{C}} y(C) \varrho_g(C) - \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}] \\
&\quad + \sum [\Delta_y(e) : e \in L \text{ is } \mathcal{C}\text{-entering}] - \varrho_L(\mathcal{C}) \\
&= \sum_{C \in \mathcal{C}} y(C) \varrho_g(C) - \varrho_L(\mathcal{C}).
\end{aligned}$$

■

Claim 3.9. For a function $y : \mathcal{P} \rightarrow \mathbf{Z}_+$, let $z_1 : A_1 \rightarrow \mathbf{Z}_+$ and $w_1 : A_1 \rightarrow \mathbf{Z}_+$ be the vectors associated with y in Claims 3.5 and 3.6. Then the cost of the dual solution (y, z_1, w_1) in (3.4) is as follows.

$$yp + z_1 f_1 - w_1 g_1 = \varrho_L(\mathcal{C}) - \sum_{C \in \mathcal{C}} y(C) [\varrho_g(C) - \delta_f(C) - p(C)]. \quad (3.16)$$

Proof. By Claims 3.7 and 3.8, we have

$$\begin{aligned}
& yp + z_1 f_1 - w_1 g_1 \\
&= \sum_{C \in \mathcal{C}} y(C) p(C) + \sum_{C \in \mathcal{C}} y(C) \delta_f(C) - \sum_{C \in \mathcal{C}} y(C) \varrho_g(C) + \varrho_L(\mathcal{C}) \\
&= \varrho_L(\mathcal{C}) - \sum_{C \in \mathcal{C}} y(C) [\varrho_g(C) - \delta_f(C) - p(C)].
\end{aligned}$$

■

Claim 3.10. Let (y, z_1, w_1) be a simple integer-valued (possibly not optimal) solution to the dual problem (3.4). Let \mathcal{C} denote the support chain of y , and let z_1 and w_1 be the vectors associated with y in Claims 3.5 and 3.6. Assume that $y(C_1) \geq 2$ for a member C_1 of \mathcal{C} . Let y' be the function arising from y by reducing the value $y(C_1)$ by 1, and let z'_1 and w'_1 be the vectors associated with y' as described in Claims 3.5 and 3.6. Then

$$y'p + z'_1 f_1 - w'_1 g_1 \geq yp + z_1 f_1 - w_1 g_1.$$

Proof. By applying Claim 3.9 to (y, z_1, w_1) and to (y', z'_1, w'_1) , we get:

$$y'p + z'_1 f_1 - w'_1 g_1 = yp + z_1 f_1 - w_1 g_1 + [\varrho_g(C_1) - \delta_f(C_1) - p(C_1)] \geq yp + z_1 f_1 - w_1 g_1,$$

where the last inequality follows from (1.7). ■

Proposition 3.11. *Let (y, z_1, w_1) be a simple integer-valued optimal solution to the dual problem (3.4) with support chain \mathcal{C}_L of y , where z_1 and w_1 are the vectors associated with y in Claims 3.5 and 3.6. Define the function $y_L : \mathcal{P} \rightarrow \{0, 1\}$ to be 1 on the members of \mathcal{C}_L and 0 otherwise, and let z'_1 and w'_1 be the vectors associated with y_L as described in Claims 3.5 and 3.6. Then (y_L, z'_1, w'_1) is also a simple integer-valued optimal solution to the dual problem (3.4).*

Proof. By repeated applications of Claim 3.10, we obtain that $y_L p + z'_1 f_1 - w'_1 g_1 \geq y p + z_1 f_1 - w_1 g_1$, but here we must have equality due to the hypothesis that (y, z_1, w_1) is a dual optimum, and this equality shows, that (y_L, z'_1, w'_1) is also a dual optimum. ■

Let x_1^* be an integer-valued primal optimum in (3.3), that is, x_1^* is a minimum c_1 -cost (f_1, g_1) -bounded p -base-flow on the edge-set A_1 of D_1 . Let x^* be the (f, g) -bounded p -base-flow on D defined in (3.2). As noted in the proof of Claim 3.2, x^* is L -upper-minimizer, and the number of g -saturated L -edges of x is $c_1 x_1$.

Let (y_L, z'_1, w'_1) be a dual optimum in (3.4) ensured by Proposition 3.11, where y_L is $(0, 1)$ -valued with support chain \mathcal{C}_L . By Claim 3.9, the min-max theorem of Edmonds and Giles implies that the minimum number of g -saturated L -edges of an (f, g) -bounded integer-valued p -base-flow is equal to

$$c_1 x_1^* = y_L p + z'_1 f_1 - w'_1 g_1 = \varrho_L(\mathcal{C}_L) - \sum_{C \in \mathcal{C}} [\varrho_g(C) - \delta_f(C) - p(C)],$$

completing the proof of Theorem 3.1. ■ ■

Proof of Theorem 2.1. Let \mathcal{C}_L be an optimal chain in Theorem 3.1, and let (f_L, g_L) be the pair of bounding functions associated with \mathcal{C}_L in (2.2) and (2.3) (with \mathcal{C}_L in place of \mathcal{C}), that is, $f_L := f_{\mathcal{C}_L}$ and $g_L := g_{\mathcal{C}_L}$. Let B_L ($:= B_{\mathcal{C}_L}$) be the face of B determined by \mathcal{C}_L , and let $Q_L := Q(f_L, g_L; B_L)$. Theorem 3.1 and Lemma 2.3 imply that an element x of \ddot{Q} is L -upper minimal if and only if $x \in \ddot{Q}_L$, as required for Theorem 2.1. ■ ■ ■

Remark 3.12. The proofs in this section and Section 2 are a bit technical and lengthy but the underlying ideas are pretty standard. Our objective of this detailed description is to make easier to check the proofs. We note that the second part of Lemma 2.3 can also be proved on the basis of the complementary slackness condition for the dual linear programs (3.3) and (3.4) in the proof of Theorem 3.1. We also remark that the approach of introducing parallel edges in the proof of Theorem 3.1 is a well-known technique and is equivalent to considering a convex cost which is equal to 1 if $x(e) = g(e)$ and 0 if $f(e) \leq x(e) \leq g(e) - 1$. •

Algorithmic aspects. Since the cost-function c_1 is $\{0, 1\}$ -valued, the algorithm of Cunningham and Frank [4] computes in strongly polynomial time these primal and dual optima, provided a subroutine is available to minimize a submodular function. A simpler and more efficient submodular flow algorithm, based on a push-relabel approach, appeared in [11]. These algorithms actually work for the more general

setting when the bounding set-function is only crossing (sub- or) supermodular. Since in the present situation p is fully supermodular, as mentioned above, the sets on which y is positive form a chain \mathcal{C}_L of non-empty proper subsets of V . Observe that for determining the base-flow polyhedron $Q_L = Q(f_L, g_L; B_L)$ what we need is only a chained optimal dual solution y since this determines its support chain \mathcal{C}_L , which in turn gives rise to (f_L, g_L) as given in (2.2) and (2.3).

It should be emphasized that the same algorithmic approach works for an arbitrarily (or implicitly) given base-polyhedron B , for example, when B is defined by a crossing supermodular function. The only expectation is that a subroutine be available to compute

$$p(Z) = \min\{\tilde{x}(Z) : x \in B\} \quad (3.17)$$

for any input set $Z \subset V$. But a minimization problem over a base-polyhedron can be solved with an extension of Edmonds' greedy algorithm concerning polymatroids. See, for example, Section 14.5 in [10] along with the geometric view of the greedy algorithm on Page 488, which explains how the greedy algorithm can be implemented for base-polyhedra given implicitly. All what we need is an oracle for minimizing a submodular function.

We also remark that the submodular flow (= base-flow) algorithms in the literature work with crossing submodular functions. But these algorithms can be extended to the case when the 0-base-polyhedron B defining the base-flow polyhedron is given in an implicit form. What is really needed is a subroutine that is capable to find an integral element of B , and to decide for an element z of $\overset{\dots}{B}$ and a pair $\{s, t\}$ of elements of the ground-set whether $z' := z - \chi_s + \chi_t$ is in B .

4 Description of F -dec-min base-flows

Let $D = (V, A)$, F , (f, g) , and B be the same as in Theorem 1.1. In this section, we prove this theorem by describing the set of F -dec-min integral elements of a non-empty integral base-flow polyhedron $Q = Q(f, g; B)$. When $F = \emptyset$, each element of $\overset{\dots}{Q}$ is dec-min. In this case, the theorem is trivially true, so we assume henceforth that F is non-empty.

A natural reduction step consists of removing a tight edge e from F . This simply means that we replace F by $F' := F - e$ (but keep e in the digraph itself). Obviously, a member z of $\overset{\dots}{Q}$ is F -dec-min if and only if z is F' -dec-min. Therefore, we may always assume that F contains no tight edges, that is,

$$f(e) < g(e) \text{ for every } e \in F. \quad (4.1)$$

Removing tight edges from F will be used not only at the starting step of the proof (and the algorithm) but it is a basic tool in later phases, as well, when the bounds (f, g) are tightened and new tight edges arise in F . It is this reduction step that makes the current F smaller and smaller (see, Theorem 4.3).

Moreover, we assume that both f and g are finite-valued on F , that is,

$$-\infty < f(e) < g(e) < +\infty \text{ for every } e \in F, \quad (4.2)$$

which ensures the existence of an F -dec-min element of Q . When (4.2) is not assumed, it is possible that $\overset{\dots}{Q}$ has no F -dec-min element at all. We shall characterize this situation in Section 6.

4.1 Pre-decreasing minimality on F

Let p denote the unique fully supermodular function defining the 0-base-polyhedron B , that is, $B = B'(p)$. Let $\beta = \beta_F$ denote the smallest integer for which $\overset{\dots}{Q}$ has an element z that is **β -covered on F** (meaning that $z(e) \leq \beta$ for every edge $e \in F$), that is,

$$\beta_F = \min\{\max\{z(e) : e \in F\} : z \in \overset{\dots}{Q}(f, g; B)\}. \quad (4.3)$$

In the next section, we shall work out an algorithm to compute β_F in strongly polynomial time. Since we are interested in F -dec-min members of $\overset{\dots}{Q}$, we may assume that the largest g -value of the edges in F is this β . Let

$$L_\beta := \{e \in F : g(e) = \beta\}. \quad (4.4)$$

By the definition of β , L_β is non-empty. Now Condition (1.7) holds but, since F contains no tight edges and since β is minimal, after decreasing the g -value of the elements of L_β from β to $\beta - 1$, the resulting function $g^- := g - \chi_{L_\beta}$ violates (1.7), that is, $Q(f, g^-; B) = \emptyset$. Summing up, we shall rely on the following notation and assumptions.

$$\left\{ \begin{array}{l} F \text{ is non-empty and contains no } (f, g)\text{-tight edges,} \\ \beta := \max\{g(e) : e \in F\}, \\ L_\beta := \{e \in F : g(e) = \beta\}, \\ g^- := g - \chi_{L_\beta}, \\ \overset{\dots}{Q} = \overset{\dots}{Q}(f, g; B) \text{ is non-empty,} \\ \overset{\dots}{Q}(f, g^-; B) \text{ is empty.} \end{array} \right. \quad (4.5)$$

As a preparation for deriving the main result Theorem 1.1, we need the following relaxation of decreasing minimality. We call a member z of $\overset{\dots}{Q}$ **pre-decreasingly minimal (pre-dec-min, for short) on F** if the number μ of edges e in L_β with $z(e) = \beta$ is as small as possible. Obviously, z is pre-dec-min on F precisely if z is L_β -upper minimizer. It is also straightforward that if z is F -dec-min, then z is pre-dec-min on F .

Apply Theorem 2.1 to this $L := L_\beta$, and consider the base-flow polyhedron

$$Q_L = (f_L, g_L; B_L)$$

ensured by the theorem. Recall that the face B_L of B was defined by a chain \mathcal{C}_L while (f_L, g_L) was defined in (2.2) and (2.3). In the present special case of $L = L_\beta$, the definition of (f_L, g_L) in (2.2) and (2.3) specializes as follows. For $e \in L$, let

$$(f_L(e), g_L(e)) := \begin{cases} (\beta, \beta) & \text{if } e \text{ enters at least two members of } \mathcal{C}_L, \\ (\beta - 1, \beta) & \text{if } e \text{ enters exactly one member of } \mathcal{C}_L, \\ (f(e), f(e)) & \text{if } e \text{ is } \mathcal{C}_L\text{-leaving,} \\ (f(e), \beta - 1) & \text{if } e \text{ is } \mathcal{C}_L\text{-neutral.} \end{cases} \quad (4.6)$$

For $e \in A - L$, let

$$(f_L(e), g_L(e)) := \begin{cases} (g(e), g(e)) & \text{if } e \text{ is } \mathcal{C}_L\text{-entering,} \\ (f(e), f(e)) & \text{if } e \text{ is } \mathcal{C}_L\text{-leaving,} \\ (f(e), g(e)) & \text{if } e \text{ is } \mathcal{C}_L\text{-neutral.} \end{cases} \quad (4.7)$$

The optimality criteria (2.4), when applied to $z \in \overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$ in place of x , are as follows.

$$\begin{cases} \text{(O1)} & z(e) = f(e) & \text{if } e \in A \text{ is } \mathcal{C}_L\text{-leaving} \\ \text{(O2)} & z(e) = g(e) & \text{if } e \in A - L \text{ is } \mathcal{C}_L\text{-entering} \\ \text{(O3)} & \beta - 1 \leq z(e) \leq \beta & \text{if } e \in L \text{ enters exactly one member of } \mathcal{C}_L \\ \text{(O4)} & z(e) = \beta & \text{if } e \in L \text{ enters at least two members of } \mathcal{C}_L \\ \text{(O5)} & f(e) \leq z(e) \leq \beta - 1 & \text{if } e \in L \text{ is } \mathcal{C}_L\text{-neutral} \\ \text{(O6)} & \varrho_z(Z) - \delta_z(Z) = p(Z) & \text{if } Z \in \mathcal{C}_L. \end{cases} \quad (4.8)$$

By Lemma 2.3 and Theorem 2.1, we have the following.

Claim 4.1. For an element $z \in \overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$, the following properties are equivalent.

(A) z is pre-dec-min on F ,

(B) (4.8) holds,

(C) $z \in \overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}_L$. ■

Claim 4.2. An element z of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$ is F -dec-min if and only if z is an F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}_L$.

Proof. Suppose first that z is an F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$. Then z is surely F -pre-dec-min in $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$ and hence, by Claim 4.1, z is in $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}_L$. If, indirectly, $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}_L$ had an element z' which is decreasingly smaller on F than z , then z could not have been an F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$.

Conversely, let z' be an F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}_L$ and suppose indirectly that z' is not an F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$. Then any F -dec-min element z of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$ is decreasingly smaller on F than z' . But any F -dec-min element of $\overset{\circ}{\overset{\circ}{\overset{\circ}{\overset{\circ}{Q}}}}$ is pre-dec-min on F and

hence, by Claim 4.1, z is in $\overset{\dots}{Q}_L$, contradicting the assumption that z' was an F -dec-min element of $\overset{\dots}{Q}_L$. ■

For wider applicability, we must emphasize that, similarly to Theorem 1.1, the formulation of the next result relies on the 0-base-polyhedron B itself, but not on the unique fully supermodular function p defining B . (The proof, however, does refer to p .)

THEOREM 4.3. *Let D , (f, g) , F , and B be the same as in Theorem 1.1, and let $L := L_\beta$ (defined in (4.4)). Given (4.5), there is a set $F' \subset F$ for which an element z of $\overset{\dots}{Q}$ is an F -dec-min member of $\overset{\dots}{Q}$ if and only if z is an F' -dec-min member of $\overset{\dots}{Q}_L = \overset{\dots}{Q}(f_L, g_L; B_L)$. In addition, the box $T(f_L, g_L)$ is narrow on $F - F'$ in the sense that $0 \leq g_L(e) - f_L(e) \leq 1$ holds for every $e \in F - F'$.*

Proof. Apply Theorem 2.1 to $L = L_\beta$, and consider the face B_L of B ensured by the theorem. Let \mathcal{C}_L be the chain describing B_L (that is, $B_L = \{x \in B : \tilde{x}(C) = p(C) \text{ for each } C \in \mathcal{C}\}$ where p denotes the fully supermodular function p defining B). Let (f_L, g_L) be the pair of bounding functions defined in (4.6) and (4.7), and let $\overset{\dots}{Q}_L := \overset{\dots}{Q}(f_L, g_L; B_L)$.

Claim 4.4. *The subset $L' \subseteq L$ consisting of the \mathcal{C}_L -entering elements of L is non-empty.*

Proof. Let z be an element of $\overset{\dots}{Q}$ which is pre-dec-min on F . By Claim 4.1, $z \in \overset{\dots}{Q}_L$. By (4.5), there is an edge e in F , for which $z(e) = \beta = g(e)$, and hence $e \in L$. Since $g(e) = z(e) \leq g_L(e) \leq g(e)$ and F contains no (f, g) -tight edges, we have $f(e) < g(e) = g_L(e) = \beta$. This and definition (4.6) imply that e is \mathcal{C}_L -entering. ■

Since $L' \neq \emptyset$ by the claim, we have that

$$F' := F - L' \text{ is a proper subset of } F.$$

We are going to show that (f_L, g_L) and F' meet the requirements of the theorem. Call two vectors in \mathbf{Z}^A value-equivalent on L' if their restrictions to L' (that is, their projection to $\mathbf{Z}^{L'}$), when both arranged in a decreasing order, are equal.

Claim 4.5. *The members of $\overset{\dots}{Q}_L$ are value-equivalent on L' .*

Proof. By Claim 4.1, the members of $\overset{\dots}{Q}_L$ are exactly those elements of $\overset{\dots}{Q}$ which are pre-dec-min on F . Hence each member z of $\overset{\dots}{Q}_L$ has the same number μ of edges in L for which $z(e) = \beta$.

As F contains no (f, g) -tight edges, we have $z(e) \leq g_L(e) \leq \beta - 1$ for every edge $e \in L - L'$ and hence each element e of L with $z(e) = \beta$ belongs to L' , from which

$$|\{e \in L' : z(e) = \beta\}| = \mu.$$

Furthermore, we have $f_L(e) \geq \beta - 1$ for every element e of L' , from which L' has exactly $|L'| - \mu$ edges with $z(e) = \beta - 1$, implying that the members of $\overset{\dots}{Q}_L$ are indeed value-equivalent on L' . ■

Claim 4.2 implies that the F -dec-min elements of $\overset{\dots}{Q}$ are exactly the F -dec-min elements of $\overset{\dots}{Q}_L$, and hence it suffices to prove that an element z of $\overset{\dots}{Q}_L$ is an F -dec-min member of $\overset{\dots}{Q}_L$ if and only if z is an F' -dec-min member of $\overset{\dots}{Q}_L$. But this latter equivalence is an immediate consequence of Claim 4.5.

To prove the last part of Theorem 4.3, recall that $F - F' = L'$ and L' consisted of the \mathcal{C}_L -entering elements of L . But the definition of (f_L, g_L) in (4.6) implies that $\beta - 1 \leq f_L(e) \leq g_L(e) = \beta$ for every element e of L' , that is, the box $T(f_L, g_L)$ is indeed narrow on $F - F'$. ■ ■

4.2 Proof of the main result

After these preparations, we are in a position to prove our main result formulated in Section 1.

Proof of Theorem 1.1. We use induction on $|F|$. Since $f^* := f$, $g^* := g$, and $p^* := p$ clearly meet the requirements of the theorem when $F = \emptyset$, we can assume that F is non-empty. As before, we may assume that F contains no (f, g) -tight edges. By Theorem 4.3, it suffices to prove the theorem for $\overset{\dots}{Q}(f_L, g_L; B_L)$ and F' . But this follows by induction since F' is a proper subset of F . ■ ■ ■

4.3 Graph orientations

One of the starting points of the present investigation was the paper of Borradaile et al. [3], in which they considered (among others) strongly connected orientation of an undirected graph for which the indegree vector is decreasingly minimal (equitable in their term). They formulated a conjecture for characterizing dec-min strongly connected orientations. The conjecture has been proved in [14] in a more general framework concerning k -edge-connected in-degree constrained orientations. The characterization immediately gave rise to an algorithm for computing a dec-min orientation in question. The approach of [14], however, does not say anything about strongly connected dec-min orientations of a mixed graph.

It is known, however, (see, for example, [10]) that even the more general problem of finding a k -edge-connected in-degree constrained orientation of a mixed graph can be formulated as a special base-flow polyhedron problem (where the base-flow is defined by a crossing supermodular function). Therefore both Theorem 1.1 and the algorithm developed for computing minimum cost dec-min base-flow can be specialized to this dec-min orientation problem.

5 Algorithm for minimizing the largest base-flow value on F

Our next task is to describe a strongly polynomial algorithm to compute the bounding pair (f^*, g^*) and the face B^∇ of B in Theorem 1.1, the main result of the paper. In

Section 4 we derived Theorem 1.1 from Theorem 4.3. This derivation showed that if an algorithm is available to compute the face B_L of B and the bounds (f_L, g_L) occurring in Theorem 4.3, then at most $|F|$ applications of this algorithm results in the requested (f^*, g^*) and B^∇ . Therefore our main task is to show how the bounds (f_L, g_L) and face B_L in Theorem 4.3 can be computed.

Theorem 4.3 itself was derived by applying Theorem 2.1 in the special case $L := L_\beta$. Therefore, we can apply the algorithm outlined at the end of Section 3 once we are able to compute β_F defined in (4.3). Recall from Section 4.1 that β_F is nothing but the smallest integer for which $\overset{\dots}{Q}$ has an element z that is β_F -covered on F , or in other words, every component of z in F is at most β_F .

As a preparation to computing $\beta := \beta_F$, let S be a finite ground-set, $h : 2^S \rightarrow \underline{\mathbf{Z}}$ and $b : 2^S \rightarrow \overline{\mathbf{Z}}_+$ be set-functions. We call an integer μ **good** with respect to b and h if $\mu b(X) \geq h(X)$ for every $X \subseteq S$. An integer that is not good is called **bad**. The non-negativity of b implies that if μ is good, then so is every integer larger than μ . We assume that

$$h(X) \leq 0 \quad \text{whenever } b(X) = 0, \quad (5.1)$$

which is equivalent to requiring that there is a good μ . We also assume that

$$\text{there exists a subset } Y \subseteq S \text{ with } h(Y) > 0, \quad (5.2)$$

which is equivalent to requiring that the value $\mu = 0$ is bad. Let μ_{\min} denote the smallest good integer.

5.1 Computing the smallest good μ

We recall an algorithm from [16] to compute μ_{\min} . (In order to avoid confusion, here we use letter h for the function p used in [16].) The number μ_{\min} is nothing but the maximum of $\lceil h(X)/b(X) \rceil$ over the subsets X of S with $b(X) > 0$, and hence the algorithm may be viewed as a variant of the Newton-Dinkelbach algorithm.

The algorithm works if a subroutine is available to

$$\text{find a subset } X \subseteq S \text{ maximizing } h(X) - \mu b(X) \text{ for any fixed integer } \mu \geq 0. \quad (5.3)$$

This routine will actually be needed only for special values of μ when $\mu = \lceil h(X)/\ell \rceil \geq 0$ with $X \subseteq S$ and $1 \leq \ell \leq M$, where M denotes the largest finite value of b .

The algorithm starts with the bad $\mu_0 := 0$. Let

$$X_0 \in \arg \max \{h(X) - \mu_0 b(X) : X \subseteq S\},$$

that is, X_0 is a set maximizing the function $h(X) - \mu_0 b(X) = h(X)$. Note that the badness of μ_0 implies that $h(X_0) > 0$. Since, by assumption, there is a good μ , it follows that $\mu b(X_0) \geq h(X_0)$, and hence $b(X_0) > 0$.

The procedure determines one by one a series of pairs (μ_j, X_j) for subscripts $j = 1, 2, \dots$ where each integer μ_j is a tentative candidate for μ while X_j is a non-empty subset of S with $b(X_j) > 0$. Suppose that the pair (μ_{j-1}, X_{j-1}) has already been

determined for a subscript $j \geq 1$. Let μ_j be the smallest integer for which $\mu_j b(X_{j-1}) \geq h(X_{j-1})$, that is,

$$\mu_j := \left\lceil \frac{h(X_{j-1})}{b(X_{j-1})} \right\rceil.$$

If μ_j is bad, then let

$$X_j \in \arg \max \{h(X) - \mu_j b(X) : X \subseteq S\},$$

that is, X_j is a set maximizing the function $h(X) - \mu_j b(X)$. Since μ_j is bad, we have $h(X_j) - \mu_j b(X_j) > 0$, which implies $b(X_j) > 0$ by the assumption (5.1).

It was proved in [16] (Claim 7.1) that if μ_j is bad for some subscript $j \geq 0$, then $\mu_j < \mu_{j+1}$. This implies that there is a first subscript $\ell \geq 1$ during the run of the algorithm for which μ_ℓ is good. Theorem 7.2 in [16] states that $\mu_{\min} = \mu_\ell$, that is, μ_ℓ is the requested smallest good μ -value and $\ell \leq M$.

5.2 Computing β_F in strongly polynomial time

In the previous subroutine to compute the smallest good integer, S denoted the ground-set of the occurring set-functions h and b . Now we turn to the problem of computing β_F and apply the subroutine to set-functions defined on ground-set V in place of S .

As before, we suppose that there is an (f, g) -bounded base-flow (that is, $Q = Q(f, g; B)$ is non-empty), and also that F contains no (f, g) -tight edges. Based on the algorithm in Section 5.1, we describe first a strongly polynomial algorithm to compute β_F defined in (4.3), which is the smallest integer for which $\overset{\dots}{Q}$ has an element z satisfying $z(e) \leq \beta_F$ for every edge $e \in F$. Note that β_F can be interpreted as the smallest integer such that, by decreasing $g(e)$ to β_F for each edge $e \in F$ with $g(e) > \beta_F$, the resulting g^- and the unchanged f continue to meet the inequality $f \leq g^-$ and the inequality $\varrho_{g^-} - \delta_f \geq p$ in (1.7), which ensure that $Q(f, g^-; B)$ is non-empty.

The first requirement for β_F is that it should be at least the largest f -value on the edges in F , which is denoted by f_1 . Let $g_1 > g_2 > \dots > g_q$ denote the distinct g -values of the edges in F , and let $L := \{e \in F : g(e) = g_1\}$. Let $\beta_1 := \max\{f_1, g_2\}$.

It is known that there is a (purely combinatorial) strongly polynomial algorithm (see, for example [7]) to check whether a base-flow polyhedron is empty or not. We refer to such an algorithm as a base-flow feasibility subroutine. Such an algorithm either outputs an integral element of the base-flow polyhedron in question or else it outputs a set X violating (1.7).

With the help of this subroutine, we can check whether the g -value g_1 on the elements of L can uniformly be decreased to β_1 without destroying (1.7). If this is the case, then either $\beta_1 = f_1$, in which case a tight edge arises in F and we can remove this tight edge from F , or $\beta_1 = g_2$, in which case the number of distinct g -values becomes one smaller. Clearly, as the total number of distinct g -values in F is at most $|F|$, this kind of reduction may occur at most $|F|$ times.

Therefore, we are at a case when g_1 cannot be decreased to β_1 without violating (1.7), that is, $\beta_F > \beta_1$. We look for β_F in the form $\beta_F = \beta_1 + \mu$, that is, our goal is to compute the smallest positive integer μ such that \bar{Q} has an element which is $(\beta_1 + \mu)$ -covered on F . To this end, we show that computing such a μ is nothing but finding a smallest good μ with respect to set-functions $h := p'$ and b to be defined as follows.

Recall that $L = \{e \in F : g(e) = g_1\}$ and let $A_0 := A - L$ (that is, A_0 is the complement of L with respect to the whole edge-set A). Let g' denote the function arising from g by reducing $g(e)$ on the elements of L (where $g(e) = g_1$) to β_1 . Since $g' \geq f$ holds and hence $\varrho_{g'} - \delta_f$ is submodular, the set-function p' on V defined by

$$p'(Z) := p(Z) - \varrho_{g'}(Z) + \delta_f(Z) \quad (5.4)$$

is supermodular. Define a submodular function b on V by

$$b(Z) := \varrho_L(Z). \quad (5.5)$$

Recall that a non-negative integer μ is called good with respect to b and p' if

$$\mu b(Z) \geq p'(Z) \quad (5.6)$$

holds for every $Z \subseteq V$.

Claim 5.1. *For a non-negative integer μ , Condition (1.7) holds after increasing $g(e) = \beta_1$ uniformly by μ on the edges $e \in L$, or equivalently,*

$$\mu \varrho_L(Z) + \varrho_{g'}(Z) - \delta_f(Z) \geq p(Z) \quad (5.7)$$

holds for every $Z \subseteq V$ if and only if μ is good with respect to b and p' .

Proof. The claim follows immediately from the equivalence of (5.6) and (5.7). \blacksquare

Therefore our problem of computing β_F is reduced to computing the smallest good non-negative integer μ . As Q has no element which is g_1 -covered on F , Claim 5.1 shows that $\mu = 0$ is not good. Similarly, as Q has an element which g_1 -covered on F , Claim 5.1 shows that $\mu := g_1 - \beta_1$ is good.

Since b is submodular, p' is supermodular, and we have $\max\{b(Z) : Z \subseteq V\} \leq |L| \leq |A|$, we can apply the algorithm described in Section 5.1 to $h := p'$ and b . That algorithm needs the Subroutine (5.3) to compute a subset of V maximizing $p'(Z) - \mu b(Z)$ ($Z \subseteq V$) for any fixed integer $\mu \geq 0$. This subroutine is applied at most M times, where M denotes the largest value of b . Since the largest value of b is at most $|A|$, the subroutine (5.3) is applied at most $|A|$ times. Furthermore, by the definition of p' and b , the equivalent subroutine to minimize

$$\mu b(Z) - p'(Z) = \mu \varrho_L(Z) + \varrho_{g'}(Z) - \delta_f(Z) - p(Z)$$

can be realized with the help of a submodular function minimizing algorithm.

Therefore, the smallest good μ can be computed in strongly polynomial time, and hence the requested $\beta_F = \beta_1 + \mu$ may be assumed to be available.

6 Existence of an F -dec-min base-flow

In the previous sections, we assumed that the bounding functions f and g are finite-valued on F because this certainly ensured the existence of an F -dec-min element of $\overset{\dots}{Q}$ where $Q = (f, g; B)$ is a non-empty 0-base-flow polyhedron. The goal of the present section is to describe a characterization of the existence of F -dec-min elements when there are no a priori assumptions on the finiteness of f and g . Since every member of $\overset{\dots}{Q}$ is trivially F -dec-min when $F = \emptyset$, we assume that the specified subset F of A is non-empty.

First, we exhibit an easy reduction by which g can be made finite-valued on F without changing the set of F -dec-min elements of $\overset{\dots}{Q}$.

Lemma 6.1. *There is a function $g^- (\leq g)$ on A which is finite-valued on F such that the (possibly empty) set of F -dec-min elements of $\overset{\dots}{Q} := \overset{\dots}{Q}(f, g; B)$ is equal to the set of F -dec-min elements of $\overset{\dots}{Q}' := \overset{\dots}{Q}(f, g^-; B)$.*

Proof. Let z_1 be an element of $\overset{\dots}{Q}$ and let β denote the maximum value of its components. Define g^- as follows:

$$g^-(e) := \begin{cases} \min\{g(e), \beta\} & \text{if } e \in F, \\ g(e) & \text{if } e \in A - F. \end{cases} \quad (6.1)$$

As $g^- \leq g$, we have $\overset{\dots}{Q}' \subseteq \overset{\dots}{Q}$. In particular, an F -dec-min element z' of $\overset{\dots}{Q}'$ is in $\overset{\dots}{Q}$, and we claim that z' is actually F -dec-min in $\overset{\dots}{Q}$, as well. Indeed, if we had an element $z'' \in \overset{\dots}{Q}$ which is decreasingly smaller on F than z' , then z'' is not in $\overset{\dots}{Q}'$, that is, z'' is not (f, g^-) -bounded. Therefore there is an edge $a \in F$ for which $z''(a) > \beta$, implying that $\max\{z''(e) : e \in F\} > \beta \geq \max\{z'(e) : e \in F\}$. But this contradicts the assumption that z'' is decreasingly smaller on F than z' .

Conversely, suppose that z is an F -dec-min element of $\overset{\dots}{Q}$. Since the largest component of z_1 is β , the largest component of z is at most β , and hence $z \in \overset{\dots}{Q}'$. This and $\overset{\dots}{Q}' \subseteq \overset{\dots}{Q}$ imply that z is an F -dec-min element of $\overset{\dots}{Q}'$. \blacksquare

Let

$$\mathcal{P} := \{Z \subseteq V : \min\{\tilde{x}(Z) : x \in B\} > -\infty\}. \quad (6.2)$$

Note that if p denotes the unique fully supermodular function defining B , then

$$\mathcal{P} = \{Z \subseteq V : p(Z) > -\infty\}.$$

This implies that \mathcal{P} is closed under taking intersection and union. We also have $V \in \mathcal{P}$ since B is a 0-base-polyhedron. Therefore, each element $u \in V$ is contained in a unique smallest member $P(u)$ of \mathcal{P} (which is the intersection of all members of \mathcal{P} containing u).

To formulate the main result of this section, we introduce a set J of edges that encodes \mathcal{P} . Let

$$J := \{uv : v \in P(u) - u\}. \quad (6.3)$$

We refer to the elements of J as **jumping edges**. Clearly, $e = uv$ is a jumping edge precisely if e does not leave any member of \mathcal{P} , and a subset $X \subseteq V$ is a member of \mathcal{P} precisely if $\delta_J(X) = 0$.

Let

$$A_1 := \{e : e \in A, f(e) = -\infty\} \quad \text{and} \quad A_2 := \{vu : uv \in A - F, g(uv) = +\infty\},$$

and let

$$A^* := J \cup A_1 \cup A_2. \tag{6.4}$$

THEOREM 6.2. *Let $D = (V, A)$ be a digraph and $F \subseteq A$ a non-empty subset of edges, and let B be an integral 0-base-polyhedron. Let $f : A \rightarrow \underline{\mathbf{Z}}$ and $g : A \rightarrow \overline{\mathbf{Z}}$ be bounding functions on A with $f \leq g$ such that the base-flow polyhedron $\overset{\dots}{Q} := \overset{\dots}{Q}(f, g; B)$ is non-empty. Then there exists an F -dec-min element of $\overset{\dots}{Q}$ if and only if there is no di-circuit C with $C \cap F \neq \emptyset$ in the digraph $D^* = (V, A^*)$ defined by (6.4).*

Proof. Let $p : 2^V \rightarrow \underline{\mathbf{Z}}$ be the unique fully supermodular function on V determining B , that is, $B = B'(p)$. Since B is a 0-base-polyhedron, $p(V) = 0$. Suppose first that D^* includes a di-circuit C intersecting F , and assume, indirectly, that there exists an F -dec-min member z of $\overset{\dots}{Q}$. For $uv \in A$, define $z'(uv)$ as follows:

$$z'(uv) := \begin{cases} z(uv) - 1 & \text{if } uv \in C \cap A_1 \\ z(uv) + 1 & \text{if } vu \in C \cap A_2 \\ z(uv) & \text{otherwise.} \end{cases} \tag{6.5}$$

Let Z be any subset of V . Clearly,

$$\varrho_C(Z) = \delta_C(Z). \tag{6.6}$$

Let $C_1 := C \cap A_1$, $C_2 := C \cap A_2$, and $C_J := C \cap J$. Then

$$\varrho_{C_1}(Z) + \varrho_{C_2}(Z) + \varrho_{C_J}(Z) = \delta_{C_1}(Z) + \delta_{C_2}(Z) + \delta_{C_J}(Z) \tag{6.7}$$

and hence

$$\varrho_{C_J}(Z) - \delta_{C_J}(Z) = \delta_{C_1}(Z) + \delta_{C_2}(Z) - \varrho_{C_1}(Z) - \varrho_{C_2}(Z). \tag{6.8}$$

Furthermore,

$$\varrho_{z'}(Z) = \varrho_z(Z) - \varrho_{C_1}(Z) + \delta_{C_2}(Z), \tag{6.9}$$

$$\delta_{z'}(Z) = \delta_z(Z) - \delta_{C_1}(Z) + \varrho_{C_2}(Z), \tag{6.10}$$

from which, by recalling the notation $\Psi_z := \varrho_z - \delta_z$, we get

$$\Psi_{z'}(Z) = \Psi_z(Z) - \varrho_{C_1}(Z) + \delta_{C_2}(Z) + \delta_{C_1}(Z) - \varrho_{C_2}(Z) = \Psi_z(Z) + \varrho_{C_J}(Z) - \delta_{C_J}(Z). \tag{6.11}$$

Claim 6.3. *The vector z' defined in (6.5) is also in $\overset{\dots}{Q}$.*

Proof. It follows immediately from the definitions of D^* and z' that $f \leq z' \leq g$. To prove that $\Psi_{z'} := \varrho_{z'} - \delta_{z'} \geq p$, let Z be a member of \mathcal{P} . By the definition of J , no jumping edges leave Z , and hence $\delta_{C_J}(Z) = 0$. From (6.11) we get

$$\Psi_{z'}(Z) = \Psi_z(Z) + \varrho_{C_J}(Z) \geq p(Z) + 0 = p(Z),$$

as required. \blacksquare

By Claim 6.3, $z' \in \overset{\dots}{Q}$. Since $F \cap C \neq \emptyset$, z' is decreasingly smaller on F than z , a contradiction, showing that in the present case no dec-min element of $\overset{\dots}{Q}$ can exist.

To see the converse, suppose that there is no di-circuit of D^* intersecting F . We want to prove that there is an F -dec-min element of $\overset{\dots}{Q}$.

We claim that it suffices to prove this statement in the special case when g is finite-valued on F . Indeed, consider the function g^- introduced in (6.1). As $g^- \leq g$, there is no di-circuit described in the theorem with respect to (f, g^-) . Now g^- is finite-valued on F , and if there exists an F -dec-min (f, g^-) -bounded base-flow z , then it follows from Lemma 6.1 that z is an F -dec-min (f, g) -bounded base-flow.

Therefore, we can assume that g is finite-valued on F . In this case,

$$A_2 = \{vu : uv \in A, g(uv) = +\infty\}.$$

Claim 6.4. *Let $S \subset V$ be a set for which $\delta_{A^*}(S) = 0$, and let $e_0 \in F$ be an edge entering S . Then, for any $z \in \overset{\dots}{Q}$, we have*

$$z(e_0) \geq p(S) - [\varrho_g(S) - g(e_0)] + \delta_f(S), \quad (6.12)$$

and the right-hand side is finite.

Proof. Since $z \leq g$ and e_0 enters S , we have

$$\varrho_z(S) - z(e_0) \leq \varrho_g(S) - g(e_0),$$

from which

$$p(S) \leq \varrho_z(S) - \delta_z(S) = z(e_0) + [\varrho_z(S) - z(e_0)] - \delta_z(S) \leq z(e_0) + [\varrho_g(S) - g(e_0)] - \delta_f(S),$$

implying (6.12). Furthermore, $\delta_{A^*}(S) = 0$ implies that $f(e) > -\infty$ for every edge e of D leaving S and that $g(e) < +\infty$ for every edge e of D entering S . $\delta_{A^*}(S) = 0$ also implies that no jumping edge leaves S , which is equivalent to saying that $P(u) \subseteq S$ for each $u \in S$. But this latter property implies that $S \in \mathcal{P}$, that is, $p(S)$ is finite, from which the finiteness of the right-hand side of (6.12) follows. \blacksquare

Assume indirectly that $\overset{\dots}{Q}$ has no F -dec-min element, that is, for every element of $\overset{\dots}{Q}$ there exists another one which is decreasingly smaller on F . This implies that there is an edge $e_0 = ts$ in F for which $\overset{\dots}{Q}$ has an element z for an arbitrary small integer K such that $z(e_0) \leq K$.

Claim 6.5. *There exists an st -dipath P in D^* .*

Proof. Suppose, indirectly, that the set S of nodes reachable from s in D^* does not contain t . Since no edge of D^* leaves S and e_0 enters S , it follows from Claim 6.4 that there is a finite lower bound for $z(e_0)$, a contradiction. ■

The di-circuit formed by $e_0 = ts$ and the st -dipath P ensured by Claim 6.5 meets the requirement of the theorem. ■ ■

Extension of Theorem 1.1 With the help of Theorem 6.2 and Lemma 6.1, Theorem 1.1 can be extended to the case when (f, g) is not assumed to be finite-valued on F , but only the existence of a di-circuit in D^* intersecting F is excluded (which is equivalent, by Theorem 6.2, to the existence of an F -dec-min member of $\overset{\dots}{Q}$).

THEOREM 6.6. *Let $D = (V, A)$ be a digraph endowed with integer-valued lower and upper bound functions $f : A \rightarrow \underline{\mathbf{Z}}$ and $g : A \rightarrow \overline{\mathbf{Z}}$ for which $f \leq g$. Let B be an integral 0-base-polyhedron for which the base-flow polyhedron $Q = Q(f, g; B)$ is non-empty. Let $F \subseteq A$ be a specified subset of edges for which there exists an F -dec-min element of $\overset{\dots}{Q}$. Then there exists a face B^∇ of B and there exists a pair (f^*, g^*) of integer-valued bounding functions on A with $f \leq f^* \leq g^* \leq g$ such that an element $z \in \overset{\dots}{Q}$ is F -dec-min if and only if $z \in \overset{\dots}{Q}(f^*, g^*; B^\nabla)$. Moreover, $0 \leq g^*(e) - f^*(e) \leq 1$ for every $e \in F$.*

Proof. By Lemma 6.1, we can assume that g is finite-valued on F . By Theorem 6.2, there is no di-circuit C in D^* with $C \cap F \neq \emptyset$, implying that, for every edge $e = ts \in F$, the set S_e reachable in D^* from s meets the inequality (6.12) for any member z of $\overset{\dots}{Q}$. As the right-hand side of (6.12) is finite by Claim 6.4, there is a finite lower bound

$$f'(e) := p(S_e) - [\varrho_g(S_e) - g(e)] + \delta_f(S_e) \quad (6.13)$$

for $z(e)$. In this way, each $(-\infty)$ -valued lower bound on the edges in F can be made finite, and the original Theorem 1.1 applies. ■

We emphasize that for each $e \in F$ the set S_e occurring in the proof is easily computable and hence so is the finite lower bound $f'(e)$ given in (6.13). Therefore this reduction to the case when (f, g) is finite-valued on F is algorithmic.

7 Fractional dec-min elements

Throughout the paper we concentrated exclusively on integral dec-min elements but analogous questions concerning fractional dec-min elements of a base-flow polyhedron also make sense. In [16], we proved the following proposition.

Proposition 7.1. *If a convex subset P of \mathbf{R}^n admits a dec-min element x , then x is the unique dec-min element.*

This theorem can be applied to base-flow polyhedra, and since base-flow polyhedra are closed under projection, we even have the following.

Claim 7.2. *Let $Q = Q(f, g; B)$ be a non-empty base-flow polyhedron defined on the edge-set of digraph $D = (V, A)$, and let $F \subseteq A$ be a subset of edges. If Q admits a (fractional) element x which is F -dec-min, then the restriction $x'|_F$ of every F -dec-min element x' of Q to F is the same as the restriction $x|_F$ of x to F .*

Concerning the existence of an F -dec-min element of Q , we have the following theorem, which is the continuous counterpart of Theorem 6.2.

THEOREM 7.3. *There exists an F -dec-min element of Q if and only if there is no di-circuit C with $C \cap F \neq \emptyset$ in the digraph $D^* = (V, A^*)$ defined by (6.4).*

Proof. The proof is essentially the same as that of Theorem 6.2, except that the definition of $z'(uv) = z(uv) \pm 1$ in (6.5) should be changed to $z'(uv) := z(uv) \pm \delta$ using a sufficiently small $\delta > 0$ to meet the condition $z' \in Q$. ■

It remains a task for future research to construct an algorithm for computing an F -dec-min element of Q , when it exists.

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