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Vertex Splitting, Coincident Realisations and Global Rigidity of Braced Triangulations

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Abstract

We give a short proof of a result of Jordán and Tanigawa that a 4-connected graph which has a spanning plane triangulation as a proper subgraph is generically globally rigid in $\mathbb{R}^3$. Our proof is based on a new sufficient condition for the so called vertex splitting operation to preserve generic global rigidity in $\mathbb{R}^d$.

Keywords Bar-joint framework, global rigidity, vertex splitting, plane triangulation.

Mathematics Subject Classification 52C25, 05C10, 05C75

1 Introduction

We consider the problem of determining when a configuration consisting of a finite set of points in d-dimensional Euclidean space $\mathbb{R}^d$ is uniqely defined up to congruence by a given set of constraints which fix the distance between certain pairs of points. This problem was shown to be NP-hard for all $d \geq 1$ by Saxe [18], but becomes more tractable if we restrict our attention to generic configurations. Gortler, Healy and Thurston [9] showed that, for generic frameworks, uniqueness depends only on the underlying constraint graph. Graphs which give rise to uniquely realisable generic configurations in $\mathbb{R}^d$ are said to be *globally rigid in* $\mathbb{R}^d$. These graphs have been characterised for $d = 1, 2$, [13], but it is a major open problem in distance geometry to characterise globally rigid graphs when $d \geq 3$.

A recent result of Jordán and Tanigawa [17] characterises when graphs constructed from plane triangulations by adding some additional edges are globally rigid in $\mathbb{R}^3$.

Theorem 1. *Suppose that G is a graph which has a plane triangulation T as a spanning subgraph. Then G is globally rigid in $\mathbb{R}^3$ if and only if G is 4-connected and $G \neq T$.*

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We will give a short proof of this result. The main tool in our inductive proof is the (3-dimensional version of) the following result which gives a sufficient condition for the so called vertex splitting operation to preserve global rigidity in $\mathbb{R}^d$.

Theorem 2. *Let $G = (V, E)$ be a graph which is globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G' is obtained from G by a vertex splitting operation which splits v into two vertices v' and v'', and that G' has an infinitesimally rigid realisation in $\mathbb{R}^d$ in which v' and v'' are coincident. Then G' is generically globally rigid in $\mathbb{R}^d$.*

Theorem 2 may be of independent interest. It has aleady been used by Jordán, Kiraly and Tanigawa in [16] to repair a gap in the proof of their characterision of generic global rigidity for 'body-hinge frameworks' given in [15]. An analogous result to Theorem 2 was used in [12, 14] to obtain a characteriseation of generic global rigidity for 'cylindrical frameworks'. Theorem 2 is a special case of a conjecture of Whiteley, see [3, 4], that the vertex splitting operation preserves global rigidity in $\mathbb{R}^d$ if and only if both v' and v'' have degree at least $d + 1$ in G'.

2 Vertex splitting and coincident realisations

We will prove Theorem 2. We first define the terms appearing in the statement of this theorem. A *(d-dimensional) framework* is a pair (G, p) where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a point configuration. The *rigidity map* for G is the map $f_G : \mathbb{R}^{d|V|} \to \mathbb{R}^{|E|}$ which maps a configuration $p \in \mathbb{R}^{d|V|}$ to the sequence of squared edge lengths $(\|p(u) - p(v)\|^2)_{uv\in E}$. The framework (G, p) is *gloablly rigid* if, for every framework (G, q) with $f_G(p) = f_G(q)$, we have p is congruent to q. It is *rigid* if it is globally rigid within some open neighbourhood of p and is *infinitesimally rigid* if the Jacobean matrix of the rigidity map of G has rank $\min\{d|V| - \binom{d+1}{2}, \binom{d}{2}\}$ at p. Gluck [6] showed that every infinitesimally rigid framework is rigid and that the two properties are equivalent when p is *generic* i.e. the coordinates of p are algebraically independent over $\mathbb{Q}$. We say that the graph G is *rigid*, respectively *globally rigid*, in $\mathbb{R}^d$ if some, or equivalently every, generic framework (G, p) in $\mathbb{R}^d$ is rigid, respectively globally rigid. We refer the reader to the survey article [20] for more information on rigid frameworks.

We need the following result of Connelly and Whiteley [5] which shows that global rigidity is a stable property for infinitesimally rigid frameworks.

Lemma 3. *Suppose that (G, p) is an infinitesimally rigid, globally rigid framework on n vertices in $\mathbb{R}^d$. Then there exists an open neighbourhood N_p of p in $\mathbb{R}^{dn}$ such that (G, q) is infinitesimally rigid and globally rigid for all $q \in N_p$.*

Given a graph $G = (V, E)$ and $v \in V$ with neighbour set $N(v)$ the *(d-dimensional) vertex splitting operation* constructs a new graph G' by deleting v, adding two new vertices v' and v'' with $N(v') \cup N(v'') = N(v) \cup \{v', v''\}$ and $|N(v') \cap N(v'')| = d - 1$. Whiteley [19] showed that vertex splitting preserves generic rigidity in $\mathbb{R}^d$. More precisely he proved

Lemma 4. *Suppose that* (G, p) *is an infinitesimally rigid framework in* $\mathbb{R}^d$ *and that* G' *is obtained from* G *by a vertex split operation which splits a vertex* $v \in V(G)$ *into two vertices* v', v''*. Suppose further that the points in* $\{p(u) : u \in \{v\} \cup (N(v') \cap N(v''))\}$ *are in general position in* $\mathbb{R}^d$*. Then* (G', p') *is infinitesimally rigid for some* p' *with* $p'(v') = p(v)$ *and* $p'(x) = p(x)$ *for all* $x \in V(G) - v$*.*

Whiteley conjectured in [3, 4] that the vertex splitting operation will preserve generic global rigidity in $\mathbb{R}^d$ if and only if both v' and v'' have degree at least $d+1$ in G'. Theorem 2 verifies a special case of this conjecture.

Proof of Theorem 2: Let (G, p) be a generic realisation of G in $\mathbb{R}^d$ and let (G', p') be the $v'v''$-coincident realisation of G' obtained by putting $p'(u) = p(u)$ for all $u \in V - v$ and $p'(v') = p'(v'') = p(v)$. The genericity of p implies that the rank of the rigidity matrix of any $v'v''$-coincident realisation of G' will be maximised at (G', p') and hence (G', p') is infinitesimally rigid. The genericity of p also implies that (G, p) is globally rigid, and this in turn implies that (G', p') is globally rigid. We can now use Lemma 3 to deduce that (G', q) is globally rigid for any generic q sufficiently close to p'. Hence G' is globally rigid. •

3 Contractible edges in plane triangulations

A graph T is a *plane (near) triangulation* if it has a 2-cell embeding in the plane in which every (bounded) face has three edges on its boundary. We will need the following notation and elementary results for (a particular embedding of) a plane triangulation T. Every cycle C of T divides the plane into two open regions exactly one of which is bounded. We refer to the bounded region as the *inside of* C and the unbounded region as the *outside of* C. We say that C is a *separating cycle* of T if both regions contain vertices of T. If S is a minimal vertex cut-set of T then S induces a separating cycle C. It follows that every plane triangulation is 3-connected and that a plane triangulation is 4-connected if and only if it contains no separating 3-cycles. Given an edge e of T which belongs to no separating 3-cycle of T, we can obtain a new plane triangulation T/e by contacting the edge e and its end-vertices to a single vertex (which is located at the same point as one of the two end-vertices of e), and replacing the multiple edges created by this contraction by single edges.

Hama and Nakamoto [10], see also Brinkman et al [1], showed that every 4-connected plane triangulation T other than the octahedron has an edge e such that T/e is a 4-connected plane triangulation. We will obtain more detailed information on the distribution of such contractible edges in this section. We will frequently use the facts that T/e is 4-connected if and only if e belongs to no separating 4-cycle of T, that no separating 4-cycle in a 4-connected triangulation can have a chord, and that no proper subgraph of a 4-connected triangulation can be a plane triangulation. Our first lemma is statement (b) in the proof of [1, Theorem 0.1]. We include a proof for the sake of completeness.

Lemma 5. *Let T be a 4-connected plane triangulation with at least 7 vertices, u be a vertex of T of degree 4 and $e_1 = uv_1, e_2 = uv_2$ be two cofacial edges of T. Then T/e_i is 4-connected for some $i = 1, 2$.*

Proof. Suppose, for a contradiction, that T/e_i is not 4-connected for both $i = 1, 2$. Let $C_1 = v_1v_2v_3v_4v_1$ be the separating 4-cycle of T which contains the neighbours of u. Since T/e_1 is not 4-connected, T has a separating 4-cycle C_2 containing e_1. Since no separating 4-cycle of T can have a chord, $C_2 = wv_1uv_3w$ for some vertex $w \in V(T)\backslash(V(C_1) \cup \{u\})$. Similary, since T/e_2 is not 4-connected, T has a separating 4-cycle $C_3 = w'v_2uv_2w'$ for some $w' \in V(T)\backslash(V(C_1) \cup \{u\})$. If $w' \neq w$ then $T[V(C_1) \cup \{u, w, w'\}]$ contains a subgraph homeomorphic to K_5 contradicting the planarity of T. On the other hand, if $w = w'$, then $T[V(C_1) \cup \{u, w\}]$ is a proper subtriangulation of T and this contradicits the hypothesis that T is a 4-connected triangulation. □

Lemma 6. *Suppose that T is a 4-connected plane triangulation with at least 7 vertices and F is a face of T. Then T/e is 4-connected for some edge e of $T - V(F)$.*

Proof. Suppose that the lemma is false and that (T, F) is a counterexample. Fix a plane embedding of T with F as the unbounded face. Let $C = v_1v_2v_3v_4v_1v$ be a separating 4-cycle such that the set of vertices inside C is minimal with respect to inclusion. Since T is 4-connected, C has no chords and hence, relabelling $V(C)$ if necessary, we may assume that $v_1, v_2 \notin V(F)$. Let uv_1 be an edge from a vertex u in the interior of C to v_1. Since T/v_1u is not 4-connected, v_1u belongs to a separating 4-cycle C_2 of T. The minimality of C_1 implies that $C_2 = wv_1uv_3w$ for some vertex w outside C_1, and hence that u is the only vertex inside C_1 (otherwise $C_3 = v_1uv_3v_2v_1$ would contradict the minimality of C_1). This in turn implies that u has degree 4 in T, and we can now use Lemma 5 to deduce that T/uv_2 is 4-connected. □

Lemma 7. *Let T be a 4-connected plane triangulation on at least seven vertices, $uv \in E$ and F, F' be the faces of T which contain uv. Let x, y be two non-adjacent vertices of T and let S be the set of all edges of T which lie on an xy-path in T of length two. Then T/e is a 4-connected plane triangulation for at least one edge $e \in E(T) \setminus (E(F) \cup E(F') \cup S)$.*

Proof: It suffices to show that we can find an edge $e \in E(T)\backslash(E(F) \cup E(F') \cup S)$ with the property that e is in no separating 4-cycle of T.

We may assume without loss of generality that F is the unbounded face of T. Choose a 4-cycle C_1 in T as follows. If T has a separating 4-cycle then choose C_1 to be a separating 4-cycle of T such that the set of vertices inside C_1 is minimal with respect to inclusion. If T has no separating 4-cycles then put $E(C_1) = (E(F) \cup E(F')) - uv$. Let $C_1 = v_1v_2v_3v_4v_1$ and let T_1 be the plane near triangulation induced in T by $V(C_1)$ and the vertices inside C_1. The choice of C_1 implies that T_1 is a wheel on five vertices or T_1 is 4-connected.

We first consider the case when T_1 is 4-connected. If T/e is 4-connected for all $e \in E(T_1)\backslash E(C_1)$ then the lemma will hold for any edge $e \in E(T_1)\backslash(E(C_1) \cup S)$. Hence we may assume that T/e is not 4-connected for some edge e of $E(T_1)\backslash E(C_1)$. Then e is contained in a separating 4-cycle C_2 of T. The minimality of C_1 implies that $C_2 \not\subseteq T_1$ and the fact that $|V(T_1)\backslash V(C_1)| \geq 2$ imply that either C_2 or C_1 has a chord, contradicting the 4-connectivity of T.

It remains to consider the case when T_1 is a wheel on five vertices. Then the unique vertex u of $T_1 - C_1$ has degree four in T and we can apply Lemma 5 to deduce that, after a possible relabelling of $V(C)$, both T/uv_1 and T/uv_3 are 4-connected. If $uv_1, uv_3 \notin S$ then we are done. Hence we may assume that $\{x, y\} = \{v_1, v_3\}$, and that neither T/uv_2 nor T/uv_4 is 4-connected. Then uv_2, uv_4 belong to a separating 4-cycle of T so some vertex $w \in V(T)\backslash V(T_1)$ is adjacent to both v_2, v_4.

Relabelling v_1, v_3 if necessary. we may assume that v_1 lies in the interior of the 4-cycle $C_2 = v_2uv_4wv_2$. If v_1 is the only vertex in the interior of C_2 then w has degree 4 in T and we can apply Lemma 5 to deduce that T/wv_1 is 4-connected. Hence we may assume that there are at least two vertices in the interior of C_2. This in turn implies that $C_3 = v_2v_1v_4wv_2$ is a separating 4-cycle of T.

Let T_3 be the near triangulation induced in T by $V(C_3)$ and the vertices inside of C_3. Let C_3' be a separating 4-cycle of T with $V(C_3') \subseteq V(T_3)$ and such that the set of vertices inside C_3' is minimal with respect to inclusion and T_3' be the near triangulation induced in T by $V(C_3')$ and the vertices inside of C_3'. We can repeat the above argument with C_1 replaced by C_3' to deduce that there exists an edge $e \in E(T_3')\backslash E(C_3')$ such that T/e is 4-connected. Then e is the required edge of T. •

4 Braced triangulations

A *braced plane triangulation* is a graph $G = (V, E \cup B)$ which is the union of a plane triangulation $T = (V, E)$ and a (possibly empty) set of additional edges B, which we refer to as the *bracing edges of* G. We say that G is a *braced plane triangulation* when G is given with a particular 2-cell embedding of T in the plane. Given a braced plane triangulation $G = (T, B)$ and an edge e of T which belongs to no separating 3-cycle of T, we denote the braced plane triangulation obtained by contacting the edge e by $G/e = (T/e, B_e)$ where the set of bracing edges B_e is obtained from B by replacing any multiple edges in G/e by single edges (in particular any edge of B which becomes parallel to an edge of T/e is deleted).

We can use Lemma 7 to obtain a result on infinitesimally rigid realisations of braced 4-connected triangultions in $\mathbb{R}^3$ in which two adjacent vertices are coincident.

Theorem 8. *Let G be a braced plane triangulation which is obtained from a 4-connected plane triangulation T by adding a brace $b = xy$ and let $uv \in E(T)$. Then G has an infinitesimally rigid uv-coincident realisation in $\mathbb{R}^3$.*

Proof. We use induction on $|V(T)|$. Let C and C' be the faces of T which contain uv and let S be the set of edges of T which lie on an xy-path of length two. Since T is 4-connected, we have $|V(T)| \geq 6$ with equality only if T is the octahedron.

Suppose T is the octahedron. Then $T - (C \cup C') \cong K_2$. Let e be the unique edge in $T - (C \cup C')$. If $e \in S$ then b is incident with an end vertex of both uv and e and, up to symmetry, there is a unique choice for uv and b. We can now use a direct computation to find a uv-coincident realisation of G in $\mathbb{R}^3$. Hence we may assume that $e \notin S$. Then $G/e \cong K_5$ and it is easy to see that every generic uv-coincident framework $(G/e, p)$ is infinitesimally rigid. We can now use Lemma 4 to construct an infinitesimally rigid uv-coincident framework (G, p').

Hence we may assume that $|V(T)| \geq 7$. Lemma 7 implies that there exists an edge $e \in E(T) \setminus (E(C) \cup E(C') \cup S)$ such that T/e is 4-connected. We can now apply induction to deduce that any generic uv-coincident framework $(G/e, p)$ is infinitesimally rigid and then use Lemma 4 to construct an infinitesimally rigid uv-coincident framework (G, p'). □

We can combine Theorems 2 and 8 with the following 'gluing lemma' to prove Theorem 1.

Lemma 9. *Let G_1, G_2 be rigid graphs, $x \in V(G_1) \backslash V(G_2)$, $y \in V(G_2) \backslash V(G_1)$, $z \in V(G_1) \cap V(G_2)$, $xz \in E(G_1)$ and $|(V(G_1) \cap V(G_2))| \geq 3$. Put $G = (G_1 \cup G_2) - xz + xy$. Suppose that (G_1, p_1) is an infinitesimally rigid realisation of G_1 and that p_1 is generic on $(V(G_1) \cap V(G_2)) \cup \{x\}$. Then (G, p) is infinitesimally rigid for some p with $p|_{G_1} = p_1$.*

Proof. Let (G_1', p_1') be obtained from $(G_1 - xz, p_1)$ by adding the vertex y at a point $p_1'(y)$ which is algebraically independent from $p_1(V(G_1))$, and then adding an edge from y to x and all vertices in $(V(G_1) \cap V(G_2))$. Then (G_1', p_1') is infinitesimally rigid since it can be obtained from (G_1, p_1) by a 1-extension[1] and a possibly empty sequence of edge additions. Since G can be obtained from G_1' by replacing the subgraph induced by the edges from y to $V(G_1) \cap V(G_2)$ with the rigid graph G_2, (G, p) will be infinitesimally rigid for any generic extension p of p_1'. □

Proof of Theorem 1

Let $G = (T, B)$ where B is the set of braces of G. Necessity follows from the fact that every globally rigid graph on at least five vertices is 4-connected and redundantly rigid by [11] (and the fact that if $B = \emptyset$ then G would not have enough edges to be redundantly rigid). We prove sufficiency by induction on $|V(T)|$. If $|V(T)| = 5$ then $G \cong K_5$ and we are done since K_5 is globally rigid. Hence we may assume that $|V(T)| \geq 6$.

Suppose T is 4-connected. Choose $b = xy \in B$ and let S be the set of edges of T which lie on an xy-path of length two. If $|V(T)| = 6$ then T is the octahedron and $G/e \cong K_5$ for all $e \in E(T) \backslash S$, so G/e is globally rigid. We can now apply Theorems 2 and 8 to deduce that G is globally rigid. Hence we may assume that $|V(T)| \geq 7$. Lemma 7 now implies that there exists an edge $e \in E(T) \setminus S$ such that T/e is 4-connected. Then $T/e + b$ is globally rigid by induction, and we can again use Theorems 2 and 8 to deduce that G is globally rigid.

[1]The 1-extension operation constructs a graph G from a graph H by deleting an edge v_1v_2 and then adding a new vertex v and four new edges vv_1, vv_2, vv_3, vv_4 to H. It can be seen that if (H, p) is an infinitesimally rigid framework and the points $p(v_i)$, $1 \leq i \leq 4$, are in general position then (G, p') will be inifinitesimally rigid for any generic extension p' of p, see [20].

Hence we may assume that T is not 4-connected. Choose a fixed embedding of T in the plane and let C_1 be a separating 3-cycle in T such that the set W of vertices inside C_1 is minimal with respect to inclusion. Let T_1 be the subgraph of T induced by $V(C_1) \cup W$. Since G is 4-connected there is a brace $xy \in B$ with $x \in W$ and $y \in V(T) \backslash V(T_1)$. The minimality of C_1 implies that T_1 is 4-connected or is isomorphic to K_4.

Suppose $T_1 \cong K_4$. We first consider the case when there exists a vertex $z \in V(C_1)$ which is not adjacent to y in T. Then G/xz is a 4-connected braced triangulation with at least one brace so is globally rigid by induction. In addition, $T - x$ is a plane triangulation so is rigid. This allows us to construct an xz-coincident infinitesimally rigid realisation (G, p) from a generic infinitesimally rigid realisation $(G - x, p')$ by putting $p(x) = p'(z)$ and using the fact that x has at least three neighbours other than z in G. Theorem 2 now implies that G is globally rigid. It remains to consider the case when, for every brace $b = xy$ incident to x in G, y is adjacent to every vertex of C_1 in T. Planarity now implies that xy is the unique brace incident to x and $V(C_1) \cup \{y\}$ induces a copy of K_4 in T. The fact that $|V(T)| \geq 6$ now implies that $T-x$ is not 4-connected. In addition, $G-x = (T-x, B-xy)$ is a 4-connected braced plane triangulation, and has at least at least one brace since $T-x$ is not 4-connected. Then $G - x$ is globally rigid, by induction, and the fact that x has degree four in G now implies that G is globally rigid.

Hence we may assume that T_1 is 4-connected. Planarity now implies that some vertex $z \in V(C_1)$ is not adjacent to x. Then $G_1 = T_1 + xz$ is a braced 4-connected plane triangulation with exactly one brace. By Theorem 8, G_1 has an infinitesimally rigid uv-coincident realisation for all $e = uv \in E(T_1)$. We can now use Lemma 9 to deduce:

(∗) G has an infinitesimally rigid uv-coincident realisation for all edges $e = uv$ of T_1 which are not induced by $V(C_1) \cup \{x\}$.

Suppose T_1 is isomorphic to the octahedron. Let $e = uv$ be the unique edge of T_1 which is not incident to a vertex in $V(C_1) \cup \{x\}$. Then $G/e = T/e + xy$ is a 4-connected braced triangulation with at least one brace so is globally rigid by induction. We can now use Theorem 2 and (∗) to deduce that G is globally rigid.

It remains to consider the case when $|V(T_1)| \geq 7$. By Lemma 6, there is an edge $e = uv \in E(T_1)$ such that T_1/uv is 4-connected and $u, v \notin V(C_1)$. Then G/e is a 4-connected braced triangulation with at least one brace which, by induction, is globally rigid. Theorem 2 and (∗) now imply that G is globally rigid. □

5 Closing Remarks

1. It follows from a result of Cauchy [2], that every graph which triangulates the plane is generically rigid in $\mathbb{R}^3$. Fogelsanger [8] extended this result to triangulations of an arbitrary surface. It is natural to conjecture that Theorem 1 can be extended in the same way.

Conjecture 10. *Let G be a graph which has a triangulation T of some surface S as a spanning subgraph. Then G is globally rigid if and only if G is 4-connected and, when S has genus zero, $G \neq T$.*

This conjecture appeared as a question in [17] and was verified when S is the sphere, projective plane or torus.

2. Let $G = (V, E)$ be a graph and $vv' \in E$. Fekete, Jordán and Kaszanitzky [7] showed that G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^2$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^2$ (where $G - vv'$ and G/vv' are obtained from G by, respectively, deleting and contracting the edge vv'). We conjecture that the same result holds in $\mathbb{R}^d$.

Conjecture 11. *Let $G = (V, E)$ be a graph and $vv' \in E$. Then G can be realised as an infinitesimally rigid bar-joint framework (G, p) in $\mathbb{R}^d$ with $p(v) = p(v')$ if and only if $G - vv'$ and G/vv' are both generically rigid in $\mathbb{R}^d$.*

The proof in [7] is based on a characterisation of independence in the '2-dimensional generic vv'-coincident rigidity matroid'. It is unlikely that a similar approach will work in $\mathbb{R}^d$ since it is notoriously difficult to characterise independence in the d-dimensional generic rigidity matroid for $d \geq 3$. But it is conceivable that there may be a geometric argument which uses the generic rigidity of $G - vv'$ and G/vv' to construct an infinitesimally rigid vv'-coincident realisation of G.

3. We can use the proof technique of Theorem 2 to show that Conjecture 11 would imply the following weak version of Whiteley's conjecture on vertex splitting.

Conjecture 12. *Let $H = (V, E)$ be a graph which is generically globally rigid in $\mathbb{R}^d$ and $v \in V$. Suppose that G is obtained from H by a d-dimensional vertex splitting operation which splits v into two new vertices v' and v''. If $G - v'v''$ is generically rigid in $\mathbb{R}^d$, then G is generically globally rigid in $\mathbb{R}^d$.*

Jordán, Király and Tanigawa [15, Theorem 4.3] state Conjecture 12 as a result of Connelly [4, Theorem 29] but this is not true - they are misquoting Connelly's theorem.

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