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Decreasing Minimization on M-convex Sets

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Abstract

The present work is the first member of a pair of papers concerning decreasingly-minimal (dec-min) elements of a set of integral vectors, where a vector is dec-min if its largest component is as small as possible, within this, the next largest component is as small as possible, and so on. This notion showed up earlier under various names at resource allocation, network flow, matroid, and graph orientation problems. Its fractional counterpart was also seriously investigated.

The domain we consider is an M-convex set, that is, the set of integral elements of an integral base-polyhedron. A fundamental difference between the fractional and the discrete case is that a base-polyhedron has always a unique dec-min element, while the set of dec-min elements of an M-convex set admits a rich structure, described here with the help of a 'canonical chain'. As a consequence, we prove that this set arises from a matroid by translating the characteristic vectors of its bases with an integral vector.

By relying on these characterizations, we prove that an element is dec-min if and only if the square-sum of its components is minimum, a property resulting in a new type of min-max theorems. The characterizations also give rise, as shown in the companion paper, to a strongly polynomial algorithm and to a proof of a conjecture on dec-min orientations.

Keywords: Submodular optimization, Matroid, Base-polyhedron, M-convex set, Lexicographic minimization.
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1 Introduction

We investigate a problem which we call 'discrete decreasing minimization'. An element of a set of vectors is called decreasingly minimal (dec-min) if its largest component is as small as possible, within this, its second largest component is as small as possible, and so

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on. The term discrete decreasing minimization refers to the problem of finding a dec-min element (or even a cheapest dec-min element with respect to a given weighting) of a set of integral vectors. In the present work, this set is an M-convex set, which is nothing but the set of integral elements of an integral base-polyhedron. Note that one may consider the analogous term 'increasing maximization' (inc-max), as well.

The goal of this paper is to develop structural characterizations of the set of dec-min elements of an M-convex set. These form the bases, in [12], for developing a strongly polynomial algorithm, as well as for exploring and exhibiting various applications. Actually, earlier special cases played a major motivating role for our investigations, and this is why we first exhibit these catalyzing initial results briefly in Section 1.1 below. The main results will be described in Section 1.2. The research was strongly motivated by the theory of Discrete Convex Analysis (DCA), but the paper is self-contained and does not rely on any prerequisite from DCA.

1.1 Background problems

There are several independent sources of the topic we study. The first one is about graph orientation. Borradaile et al. [3] solved the problem of finding an orientation of a graph with decreasingly minimal (or egalitarian in their terms) in-degree vector. Our general approach provides an extension concerning in-degree constrained k-edge-connected dec-min orientation. The second source comes from resource allocation [6, 22, 23, 24, 25, 26]. For example, Harvey et al. [20] described an algorithm which computes a subset F of edges of a bipartite graph $G = (S, T; E)$ for which $d_F(t) = 1$ holds for every $t \in T$ and the degree-vector $\{d_F(s) : s \in S\}$ is decreasingly minimal. Our present framework makes it possible to manage significantly more general problems, for example, the one where F is requested to be degree-constrained and to have γ edges. Our work is strongly related to a classic paper of Megiddo [31] on 'source optimal' network flows, which is equivalent to finding a feasible st-flow that is increasingly maximal on the set of source edges. A fundamental difference is that Megiddo considers fractional flows while we characterize integer-valued flows which are inc-max on the set of source edges. Finally, we mention the 'shifted matroid optimization' problem due to Levin and Onn [27] which seeks for k bases $Z_1, \dots, Z_k$ of a matroid for which the vector $\sum_i \chi_{Z_i}$ is, in our terms, decreasingly minimal. Our approach permits to extend their results to k distinct matroids.

The present paper works out the theoretical background for dealing with these problems in a uniform framework. In the second part of this work, we describe these applications in detail, and provide strongly polynomial algorithms for their solution.

1.2 Main goals

Each of the four problems in Section 1.1 may be viewed as a special case of a single discrete optimization problem: characterize decreasingly minimal elements of an M-convex set [32, 33] (or, in other words, dec-min integral elements of a base-polyhedron). By one of its equivalent definitions, an M-convex set is nothing but the set of integral elements of an integral base-polyhedron.

We characterize dec-min elements of an M-convex set as those admitting no local improvement, and prove that the set of dec-min elements is itself an M-convex set arising by translating a matroid base-polyhedron with an integral vector. This result implies that decreasing minimality and increasing maximality coincide for M-convex sets. We shall also show that an element of an M-convex set is dec-min precisely if it is a square-sum minimizer. Using the characterization of dec-min elements, we shall derive a novel min-max theorem for the minimum square-sum of elements of an integral member of a base-polyhedron.

The structural description of the set of dec-min elements of an M-convex set (namely, that this set is a matroidal M-convex set) makes it possible to solve the algorithmic problem of finding a minimum cost dec-min element. (In the continuous case this problem simply did not exist due to the uniqueness of the fractional dec-min element of a base-polyhedron.) In the companion paper [12], we shall also describe a polynomial algorithm for finding a minimum cost (in-degree constrained) dec-min orientation. Furthermore, we shall outline an algorithm to solve the minimum cost version of the resource allocation problem of Harvey et al. [20] mentioned in Section 1.1. Furthermore, as an essential extension of the algorithm of Harada et al. [19], we describe a strongly polynomial algorithm to solve a minimum cost version of the decreasingly minimal degree-bounded subgraph problem in a bipartite graph $G = (S, T, E)$. We may consider two versions here. In the simpler one, we have a cost-function on the node-set of G, that is, on the ground-set of the corresponding M-convex set. Due to the matroidal description of the set of dec-min elements of an M-convex set, this min-cost version becomes rather easy since the matroid greedy algorithm can be applied. Significantly more complex, however, is the other min-cost version when there is a cost-function on the set of edges of G.

The topic of our investigations may be interpreted as a discrete counterpart of the work by Fujishige [15] from 1980 on the lexicographically optimal base of a base-polyhedron B, where lexicographically optimal is essentially the same as decreasingly minimal. He proved that there is a unique lexicographically optimal member x_0 of B, and x_0 is the unique minimum norm (that is, the minimum square-sum) element of B. This uniqueness result reflects a characteristic difference between the behaviour of the fractional and the discrete versions of decreasing minimization since in the latter case the set of dec-min elements (of an M-convex set) is typically not a singleton, and it actually has, as indicated above, a matroidal structure. While the present paper focuses on the unweighted case, the lexicographically optimal base of a base-polyhedron is defined and analyzed with respect to a weight vector in [15].

Fujishige also introduced the concept of principal partitions concerning the dual structure of the minimum norm point of a base-polyhedron. Actually, he introduced a special chain of the subsets of ground-set S and his principal partition arises by taking the difference sets of this chain. We will prove that there is an analogous concept in the discrete case, as well. As an extension of the above-mentioned elegant result of Borradaile et al. [4] concerning graphs, we show that there is a canonical chain describing the structure of dec-min elements of an M-convex set. The relation between our canonical partition and Fujishige's principal partition is clarified in [11], showing that the canonical partition is an intrinsic structure of an M-convex set consistent with the principal partition of a base-polyhedron.

1.3 Notation

Throughout the paper, S denotes a finite non-empty ground-set. For elements $s, t \in S$, we say that $X \subset S$ is an $s\overline{t}$**-set** if $s \in X \subseteq S - t$. For a vector $m \in \mathbf{R}^S$ (or function $m : S \to \mathbf{R}$), the restriction of m to $X \subseteq S$ is denoted by $m|X$. We also use the notation $\widetilde{m}(X) = \sum[m(s) : s \in X]$. With a small abuse of notation, we do not distinguish between a one-element set $\{s\}$ called a singleton and its only element s. When we work with a chain C of non-empty sets $C_1 \subset C_2 \subset \cdots \subset C_q$, we sometimes use C_0 to denote the empty set without assuming that C_0 is a member of C. The characteristic (or incidence) vector of a subset Z is denoted by χ_Z, that is, $\chi_Z(s) = 1$ if $s \in Z$ and $\chi_Z(s) = 0$ otherwise. For a polyhedron B, $\ddot{B}$ (pronounce: dotted B) denotes the set of integral members (elements, vectors, points) of B, that is,

$$\ddot{B} := B \cap \mathbf{Z}^S. \tag{1.1}$$

For a set-function h, we allow it to have value $+\infty$ or $-\infty$, while $h(\emptyset) = 0$ is assumed throughout. Where $h(S)$ is finite, the **complementary function** $\overline{h}$ is defined by $\overline{h}(X) = h(S) - h(S - X)$. For functions $f : S \to \mathbf{Z} \cup \{-\infty\}$ and $g : S \to \mathbf{Z} \cup \{+\infty\}$ with $f \leq g$, the polyhedron $T(f, g) = \{x \in \mathbf{R}^S : f \leq x \leq g\}$ is called a **box**. If $g(s) \leq f(s) + 1$ holds for every $s \in S$, we speak of a **small box**. For example, the $(0, 1)$-box is small, and so is any set consisting of a single integral vector.

2 Base-polyhedra and M-convex sets

Let S be a finite non-empty ground-set. Let b be a set-function for which $b(X) = +\infty$ is allowed but $b(X) = -\infty$ is not. The submodular inequality for subsets $X, Y \subseteq S$ is defined by

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$

We say that b is (fully) submodular if the submodular inequality holds for every pair of subsets $X, Y \subseteq S$ with finite b-values. A set-function p is supermodular if $-p$ is submodular.

For a (fully) submodular integer-valued set-function b on S for which $b(\emptyset) = 0$ and $b(S)$ is finite, the **base-polyhedron** B in $\mathbf{R}^S$ is defined by

$$B = B(b) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S),\ \widetilde{x}(Z) \leq b(Z) \text{ for every } Z \subset S\}, \tag{2.1}$$

which is possibly unbounded.

A special base-polyhedron is the one of matroids. Given a matroid M, Edmonds proved that the polytope (that is, the convex hull) of the incidence (or characteristic) vectors of the bases of M is the base-polyhedron $B(r)$ defined by the rank function r of M, that is, $B(r) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = r(S) \text{ and } \widetilde{x}(Z) \leq r(Z) \text{ for every subset } Z \subset S\}$. It can be proved that a kind of converse also holds, namely, every (integral) base-polyhedron in the unit $(0, 1)$-cube is a matroid base-polyhedron. We call the translation of a matroid base-polyhedron a **translated matroid base-polyhedron**. It follows that the intersection of a base-polyhedron with a small box is a translated matroid base-polyhedron.

A base-polyhedron $B(b)$ is never empty, and $B(b)$ is known to be an integral polyhedron. (A rational polyhedron is **integral** if each of its faces contains an integral element. In particular, a pointed rational polyhedron is integral if all of its vertices are integral.) By convention, the empty set is also considered a base-polyhedron. Note that a real-valued submodular function b also defines a base-polyhedron $B(b)$ but in the present work we are interested only in integer-valued submodular functions and integral base-polyhedra.

We call the set $\dddot{B}$ of integral elements of an integral base-polyhedron B an **M-convex set**. Originally, this basic notion of DCA introduced by Murota [32] (see, also the book [33]), was defined as a set of integral points in $\mathbf{R}^S$ satisfying certain exchange axioms, and it is known that the two properties are equivalent ([33, Theorem 4.15]). The set of integral elements of a translated matroid base-polyhedron will be called a **matroidal M-convex set**.

A non-empty base-polyhedron B can also be defined by a supermodular function p for which $p(\emptyset) = 0$ and $p(S)$ is finite as follows:

$$B = B'(p) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = p(S), \widetilde{x}(Z) \geq p(Z) \text{ for every } Z \subset S\}. \tag{2.2}$$

It is known that B uniquely determines both p and b, namely, $b(Z) = \max\{\widetilde{x}(Z) : x \in B\}$ and $p(Z) = \min\{\widetilde{x}(Z) : x \in B\}$. The functions p and b are complementary functions, that is, $b(X) = p(S) - p(S - X)$ or $p(X) = b(S) - b(S - X)$ (where $b(S) = p(S)$).

For a set $Z \subset S$, $p|Z$ denotes the restriction of p to Z, while $p' = p/Z$ is the set-function on $S - Z$ obtained from p by contracting Z, which is defined for $X \subseteq S - Z$ by $p'(X) = p(X \cup Z) - p(Z)$. Note that p/Z and $\overline{p}|(S - Z)$ are complementary set-functions. It is also known for disjoint subsets Z_1 and Z_2 of S that

$$(p/Z_1)/Z_2 = p/(Z_1 \cup Z_2), \tag{2.3}$$

When $p(Z)$ is finite, the base-polyhedron $B'(p|Z)$ is called the **restriction** of $B'(p)$ to Z.

Let $\{S_1, \dots, S_q\}$ be a partition of S and let p_i be a supermodular function on S_i. Let p denote the supermodular function on S defined by $p(X) := \sum[p_i(S_i \cap X) : i = 1, \dots, q]$ for $X \subseteq S$. The base-polyhedron $B'(p)$ is called the **direct sum** of the q base-polyhedra $B'(p_i)$. Obviously, a vector $x \in \mathbf{R}^S$ is in $B'(p)$ if and only if each x_i is in $B'(p_i)$ $(i = 1, \dots, q)$, where x_i denotes the restriction $x|S_i$ of x to S_i.

It is known that a face F of a non-empty base-polyhedron B is also a base-polyhedron. The (special) face of $B'(p)$ defined by the single equality $\widetilde{x}(Z) = p(Z)$ is the direct sum of the base polyhedra $B'(p|Z)$ and $B'(p/Z)$. More generally, any face F of B can be described with the help of a chain $(\emptyset \subset)$ $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ of subsets by $F := \{z : z \in B,\ p(C_i) = \widetilde{z}(C_i)$ for $i = 1, \dots, \ell\}$. (In particular, when $\ell = 1$, the face F is B itself.) Let $S_1 := C_1$ and $S_i := C_i - C_{i-1}$ for $i = 2, \dots, \ell$. Then F is the direct sum of the base-polyhedra $B'(p_i)$, where p_i is a supermodular function on S_i defined by $p_i(X) := p(X \cup C_{i-1}) - p(C_{i-1})$ for $X \subseteq S_i$. In other words, p_i is a set-function on S_i obtained from p by deleting C_{i-1} and contracting $S - C_i$. The unique supermodular function p_F defining the face F is given by $\sum[p_i(S_i \cap X) : i = 1, \dots, \ell]$. A face F is the set of elements x of B minimizing cx whenever $c : S \to \mathbf{R}$ is a linear cost function such that $c(s) = c(t)$ if $s, t \in S_i$ for some i and $c(s) > c(t)$ if $s \in S_i$ and $t \in S_j$ for some subscripts $i < j$.

The intersection of an integral base-polyhedron $B = B'(p)$ $(= B(\overline{p}))$ and an integral box $T(f, g)$ is an integral base-polyhedron. The intersection is non-empty if and only if

$$p \leq \widetilde{g} \quad \text{and} \quad \widetilde{f} \leq \overline{p}. \tag{2.4}$$

For an element m of a base-polyhedron $B = B(b)$ defined by a (fully) submodular function b, we call a subset $X \subseteq S$ **m-tight** (with respect to b) if $\widetilde{m}(X) = b(X)$. Clearly, the empty set and S are m-tight, and m-tight sets are closed under taking union and intersection. Therefore, for each subset $Z \subseteq S$, there is a unique smallest m-tight set $T_m(Z; b)$ including Z. When $Z = \{s\}$ is a singleton, we simply write $T_m(s; b)$ to denote the smallest m-tight set containing s. When the submodular function b is understood from the context, we abbreviate $T_m(Z; b)$ to $T_m(Z)$. Analogously, when $B = B'(p)$ is given by a supermodular function p, we call $X \subseteq S$ **m-tight** (with respect to p) if $\widetilde{m}(X) = p(X)$. In this case, we also use the analogous notation $T_m(Z) = T_m(Z; p)$ and $T_m(s) = T_m(s; p)$. Observe that for complementary functions b and p, X is m-tight with respect to b precisely if $S - X$ is m-tight with respect to p.

3 Decreasingly minimal elements of M-convex sets

3.1 Decreasing minimality

For a vector x, let $x{\downarrow}$ denote the vector obtained from x by rearranging its components in a decreasing order. For example, We call two vectors x and y (of same dimension) **value-equivalent** if $x{\downarrow} = y{\downarrow}$.

A vector x is **decreasingly smaller** than vector y, in notation $x <_{\rm dec} y$ if $x{\downarrow}$ is lexicographically smaller than $y{\downarrow}$ in the sense that they are not value-equivalent and $x{\downarrow}(j) < y{\downarrow}(j)$ for the smallest subscript j for which $x{\downarrow}(j)$ and $y{\downarrow}(j)$ differ. For example, $x = (2, 5, 5, 1, 4)$ is decreasingly smaller than $y = (1, 5, 5, 5, 1)$ since $x{\downarrow} = (5, 5, 4, 2, 1)$ is lexicographically smaller than $y{\downarrow} = (5, 5, 5, 1, 1)$. We write $x \leq_{\rm dec} y$ to mean that x is decreasingly smaller than or value-equivalent to y.

For a set Q of vectors, $x \in Q$ is **globally decreasingly minimal** or simply **decreasingly minimal** (**dec-min**, for short) if $x \leq_{\rm dec} y$ for every $y \in Q$. Note that the dec-min elements of Q are value-equivalent. Therefore an element m of Q is dec-min if its largest component is as small as possible, within this, its second largest component (with the same or smaller value than the largest one) is as small as possible, and so on. An element x of Q is said to be a **max-minimized** element (a **max-minimizer**, for short) if its largest component is as small as possible. A max-minimizer element x is **pre-decreasingly minimal** (**pre-dec-min**, for short) in Q if the number of its largest components is as small as possible. Obviously, a dec-min element is pre-dec-min, and a pre-dec-min element is max-minimized.

In an analogous way, for a vector x, we let $x{\uparrow}$ denote the vector obtained from x by rearranging its components in an increasing order. A vector y is **increasingly larger** than vector x, in notation $y >_{\rm inc} x$, if they are not value-equivalent and $y{\uparrow}(j) > x{\uparrow}(j)$ holds for the smallest subscript j for which $y{\uparrow}(j)$ and $x{\uparrow}(j)$ differ. We write $y \geq_{\rm inc} x$ if either $y >_{\rm inc} x$ or x and y are value-equivalent. Furthermore, we call an element m of Q **(globally) increasingly maximal** (**inc-max** for short) if its smallest component is as large as possible over the elements of Q, within this its second smallest component is as large as possible, and so on. Similarly, we can use the analogous terms **min-maximized** and **pre-increasingly maximal** (**pre-inc-max**).

It should be emphasized that a dec-min element of a base-polyhedron B is not necessarily integer-valued. For example, if $B = \{(x_1, x_2) : x_1 + x_2 = 1\}$, then $x^* = (1/2, 1/2)$ is a dec-min element of B. In this case, the dec-min members of $\overset{....}{B}$ are $(0, 1)$ and $(1, 0)$.

Therefore, finding a dec-min element of B and finding a dec-min element of $\overset{....}{B}$ (the set of integral points of B) are two distinct problems, and we shall concentrate only on the second, discrete problem. In what follows, the slightly sloppy term integral dec-min element of B will always mean a dec-min element of $\overset{....}{B}$. (The term is sloppy in the sense that an integral dec-min element of B is not necessarily a dec-min element of B).

We call an integral vector $x \in \mathbf{Z}^S$ **uniform** if all of its components are the same integer ℓ, and **near-uniform** if its largest and smallest components differ by at most 1, that is, if $x(s) \in \{\ell, \ell + 1\}$ for some integer ℓ for every $s \in S$. Note that if Q consists of integral vectors and the component-sum is the same for each member of Q, then any near-uniform member of Q is obviously both decreasingly minimal and increasingly maximal integral vector.

3.2 Characterizing dec-min elements

Let $B = B(b) = B'(p)$ be a base-polyhedron defined by an integer-valued submodular function b or supermodular function p (where b and p are complementary set-functions). Let m be an integral member of B, that is, $m \in \overset{....}{B}$. A set $X \subseteq S$ is m-tight with respect to b precisely if its complement $S - X$ is m-tight with respect to p. Recall that $T_m(s; b)$ denoted the unique smallest m-tight set (with respect to b) containing s. In other words, $T_m(s; b)$ is the intersection of all m-tight sets containing s. The easy equivalences in the next claim will be used throughout.

Claim 3.1. *Let $m \in \overset{....}{B}$, and let s and t be elements of S, and $m' := m + \chi_s - \chi_t$. The following properties are pairwise equivalent.*

(A) $m' \in \overset{....}{B}$.

(P1) *There is no $t\overline{s}$-set which is m-tight with respect to p.*

(P2) $s \in T_m(t; p)$.

(B1) *There is no $s\overline{t}$-set which is m-tight with respect to b.*

(B2) $t \in T_m(s; b)$. ■

A **1-tightening step** for $m \in \overset{....}{B}$ is an operation that replaces m by $m' := m + \chi_s - \chi_t$ where s and t are elements of S for which $m(t) \geq m(s) + 2$ and m' belongs to $\overset{....}{B}$. Note that m' is both decreasingly smaller and increasingly larger than m.

Since the mean of the components of m does not change at a 1-tightening step while the square-sum of the components of m strictly drops, consecutive 1-tightening steps may occur only a finite number of times (even if B is unbounded).

A member m of $\overset{....}{B}$ is **locally decreasingly minimal** in $\overset{....}{B}$ if there are no two elements s and t of S such that $m' := m + \chi_s - \chi_t$ is an element of $\overset{....}{B}$ and m' is decreasingly smaller than m. Note that m' is decreasingly smaller than m precisely if $m(t) \geq m(s) + 2$. Obviously, m is locally decreasingly minimal if and only if there is no 1-tightening step for m. Note that in this case, m' is also increasingly larger than m. Analogously, m is **locally increasingly maximal** if there are no two elements s and t of S such that $m' := m + \chi_s - \chi_t$ is an element of $\overset{....}{B}$ and m' is decreasingly larger than m.

The equivalence of the properties in the next claim is immediate from the definitions.

Claim 3.2. *For an integral element m of the integral base-polyhedron $B = B(b) = B'(p)$, the following conditions are pairwise equivalent.*

(A1) *There is no 1-tightening step for m.*

(A2) *m is locally decreasingly minimal.*

(A3) *m is locally increasingly maximal.*

(P1) *$m(s) \geq m(t) - 1$ holds whenever $t \in S$ and $s \in T_m(t; p)$.*

(P2) *Whenever $m(t) \geq m(s) + 2$, there is a $t\overline{s}$-set X which is m-tight with respect to p.*

(B1) *$m(s) \geq m(t) - 1$ holds whenever $s \in S$ and $t \in T_m(s; b)$.*

(B2) *Whenever $m(t) \geq m(s) + 2$, there is an $s\overline{t}$-set Y which is m-tight with respect to b.* ■

For a given vector m in $\mathbf{R}^S$, we call a set $X \subseteq S$ an **m-top set** (or a top-set with respect to m) if $m(u) \geq m(v)$ holds whenever $u \in X$ and $v \in S - X$. Both the empty set and the ground-set S are m-top sets, and m-top sets are closed under taking union and intersection. If $m(u) > m(v)$ holds whenever $u \in X$ and $v \in S - X$, we speak of a **strict m-top set**. Note that the number of strict non-empty m-top sets is at most n for every $m \in \ddot{B}$ while $m \equiv 0$ exemplifies that even all of the non-empty subsets of S can be m-top sets.

Theorem 3.3. *Let b be an integer-valued submodular function and let $p := \overline{b}$ be its complementary (supermodular) function. For an integral element m of the integral base-polyhedron $B = B(b) = B'(p)$, the following four conditions are pairwise equivalent.*

(A) *There is no 1-tightening step for m (or any one of the six other equivalent properties holds in Claim 3.2).*

(B) *There is a chain C of m-top sets ($\emptyset \subset$) $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ which are m-tight with respect to p (or equivalently, whose complements are m-tight with respect to b) such that the restriction $m_i = m|S_i$ of m to S_i is near-uniform for each member S_i of the S-partition $\{S_1, \dots, S_\ell\}$, where $S_1 = C_1$ and $S_i := C_i - C_{i-1}$ $(i = 2, \dots, \ell)$.*

(C1) *m is (globally) decreasingly minimal in $\ddot{B}$.*

(C2) *m is (globally) increasingly maximal in $\ddot{B}$.*

Proof. (B)→(A): If $m(t) \geq m(s) + 2$, then there is an m-tight set C_i containing t and not containing s, from which Property (A) follows from Claim 3.2.

(A)→(B): Let C be a longest chain consisting of non-empty m-tight and m-top sets $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$. For notational convenience, let $C_0 = \emptyset$ (but C_0 is not a member of of C). We claim that C meets the requirement of (B). If, indirectly, this is not the case, then there is a subscript $i \in \{1, \dots, \ell\}$ for which m is not near-uniform within $S_i := C_i - C_{i-1}$. This means that the max m-value β_i in S_i is at least 2 larger than the min m-value α_i in S_i, that is, $\beta_i \geq \alpha_i + 2$. Let $Z := \cup[T_m(t; p) : t \in S_i,\ m(t) = \beta_i]$. Then Z is m-tight. Since C_i is m-tight, $T_m(t; p) \subseteq C_i$ holds for $t \in S_i$ and hence $Z \subseteq C_i$. Furthermore, (A) implies that $m(v) \geq \beta_i - 1$ for every $v \in Z \cap S_i$.

Consider the set $C' := C_{i-1} \cup Z$. Then C' is m-tight, and $C_{i-1} \subset C' \subset C_i$. Moreover, we claim that C' is an m-top set. Indeed, if, indirectly, there is an element $u \in C'$ and an element $v \in S - C'$ for which $m(u) < m(v)$, then $u \in Z \cap S_i$ and $v \in C_i - Z$ since both C_{i-1}

and C_i are m-top sets. But this is impossible since the m-value of each element of $Z \cap S_i$ is β_i or $\beta_i - 1$ while the m-value of each element of $C_i - Z$ is at most $\beta_i - 1$.

The existence of C' contradicts the assumption that C was a longest chain of m-tight and m-top sets, and therefore m must be near-uniform within each S_i, that is, C meets indeed the requirements in (B).

(C1)→(A) and (C2)→(A): Property (A) must indeed hold since a 1-tightening step for m results in an element m' of $\ddddot{B}$ which is both decreasingly smaller and increasingly larger than m.

(B)→(C1): We may assume that the elements of S are arranged in an m-decreasing order $s_1, \dots, s_n$ (that is, $m(s_1) \geq m(s_2) \geq \cdots \geq m(s_n)$) in such a way that each C_i in (B) is a starting segment. Let m' be an element of $\ddddot{B}$ which is decreasingly smaller than or value-equivalent to m. Recall that $m|X$ denoted the vector m restricted to a subset $X \subseteq S$.

Lemma 3.4. *For each $i = 0, 1, \dots, \ell$, vector $m'|C_i$ is value-equivalent to vector $m|C_i$.*

Proof. Induction on i. For $i = 0$, the statement is void so we assume that $1 \leq i \leq \ell$. By induction, we may assume that the statement holds for $j \leq i - 1$ and we want to prove it for i. Since $m'|C_{i-1}$ is value-equivalent to $m|C_{i-1}$ and C_{i-1} is m-tight, it follows that C_{i-1} is m'-tight, too.

Let β_i denote the max m-value of the elements of $S_i = C_i - C_{i-1}$. By the hypothesis in (B), the maximum and the minimum of the m-values in S_i differ by at most 1. Hence we can assume that there are $r_i > 0$ elements in S_i with m-value β_i and $|S_i| - r_i \geq 0$ elements with m-value $\beta_i - 1$.

As $m|C_{i-1}$ is value-equivalent to $m'|C_{i-1}$ and m' was assumed to be decreasingly smaller than or value-equivalent to m, we can conclude that $m'|(S - C_{i-1})$ is decreasingly smaller than or value-equivalent to $m|(S - C_{i-1})$. Therefore, S_i contains at most r_i elements of m'-value β_i and hence

$$\begin{aligned} p(C_i) &\leq \widetilde{m}'(C_i) = \widetilde{m}'(C_{i-1}) + \widetilde{m}'(S_i) \\ &\leq \widetilde{m}'(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1) \\ &= \widetilde{m}(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1) = p(C_i), \end{aligned}$$

from which equality follows everywhere. In particular, S_i contains exactly r_i elements of m'-value β_i and $|S_i| - r_i$ elements of m'-value $\beta_i - 1$, proving the lemma. ■

By the lemma, m' is value-equivalent to m, and hence m is a decreasingly minimal element of $\ddddot{B}$, that is, (C1) follows.

(B)→(C2): The property in (C1) that m is globally decreasing minimal in $\ddddot{B}$ is equivalent to the statement that $-m$ is globally increasing maximal in $-\ddddot{B}$, that is, (C2) holds with respect to $-m$ and $-\ddddot{B}$. As we have already proved the implications (C2)→(A)→(B)→(C1), it follows that (C1) holds for $-m$ and $-\ddddot{B}$. But (C1) for $-m$ and $-\ddddot{B}$ is just the same as (C2) for m and $\ddddot{B}$. ■■

Remark 3.1. The equivalence of (C1) and (C2) in Theorem 3.3 shows that an element of an M-convex set is decreasingly minimal if and only if it is increasingly maximal. In the intersection of two M-convex sets (called an M_2-convex set in [33]), however, decreasing minimality and increasing maximality do not coincide. For example, consider two M-convex sets

$$\begin{aligned} \overset{....}{B_1} &= \{(2,0,0,0),\ (1,-1,1,1),\ (2,-1,1,0),\ (1,0,0,1)\}, \\ \overset{....}{B_2} &= \{(2,0,0,0),(1,-1,1,1),(2,1,0,1),(1,0,1,0)\}. \end{aligned}$$

In their intersection $\overset{....}{B_1} \cap \overset{....}{B_2} = \{(2,0,0,0),\ (1,-1,1,1)\}$, the element $x = (2,0,0,0)$ is increasingly maximal while $y = (1,-1,1,1)$ is decreasingly minimal. ■

3.3 Minimizing the sum of the k largest components

A decreasingly minimal element of $\overset{....}{B}$ has the starting property that its largest component is as small as possible. As a natural extension, one may be interested in finding a member of $\overset{....}{B}$ for which the sum of the k largest components is as small as possible. We refer to this problem as **min k-largest-sum**.

Theorem 3.5. *Let B be an integral base-polyhedron and k an integer with $1 \leq k \leq n$. Then any dec-min element m of $\overset{....}{B}$ is a solution to the min k-largest-sum problem.*

Proof. Observe first that if z_1 and z_2 are dec-min elements of $\overset{....}{B}$, then it follows from the very definition of decreasing minimality that the sum of the first j largest components of z_1 and of z_2 are the same for each $j = 1, \ldots, n$.

Let K denote the sum of the first k largest components of any dec-min element, and assume indirectly that there is a member $y \in \overset{....}{B}$ for which the sum of its first k largest components is smaller than K. Assume that the componentwise square-sum of y is as small as possible. By the previous observation, y is not a dec-min element. Theorem 3.3 implies that there are elements s and t of S for which $y(t) \geq y(s) + 2$ and $y' := y - \chi_t + \chi_s$ is in $\overset{....}{B}$. The sum of the first k largest components of y' is at most the sum of the first k largest components of y, and hence this sum is also smaller than K. But this contradicts the choice of y since the componentwise square-sum of y' is strictly smaller than that of y. ■

This theorem shows that M-convex sets have a striking property. Namely, any dec-min element of an M-convex set $\overset{....}{B}$ is simultaneously a solution to the min k-largest-sum problem for *each* $k = 1, 2, \ldots, n$. We say that such an element m is a **simultaneous k-largest-sum minimizer**. This notion has been investigated in the literature of majorization [2, 29, 35] under the name of 'least majorized' element. In particular, Tamir [35] proved the existence of a least majorized integral element for integral base-polyhedra. (Actually, he proved this even for g-polymatroids, but this more general result is an easy consequence of the special case concerning base-polyhedra).

The following result shows that this property actually characterizes dec-min elements of an M-convex set.

Theorem 3.6. *Let B be an integral base-polyhedron. An element m of $\overset{....}{B}$ is dec-min if and only if m is a simultaneous k-largest-sum minimizer.*

Proof. The content of Theorem 3.5 is that a dec-min element is a simultaneous k-largest-sum minimizer. To see the converse, let $m \in \overset{....}{B}$ be a simultaneous k-largest-sum minimizer. Suppose indirectly that m is not dec-min. By Theorem 3.3, there is a 1-tightening step for m, that is, there are elements s and t of S with $m(s) \geq m(t) + 2$ such that $m' := m - \chi_s + \chi_t$ is in $\overset{....}{B}$. Let k' denote the number of components of m with value at least $m(s)$. Then the sum of the k' largest components of m' is one less than the sum of the k' largest components of m, contradicting the assumption that m is k-sum-minimizer for each $k = 1, 2, \ldots, n$. ■

4 Characterizing the set of pre-decreasingly minimal elements

We continue to assume that p is an integer-valued (with possible $-\infty$ values but with finite $p(S)$) supermodular function, which implies that $B = B'(p)$ is a non-empty integral base-polyhedron. We have already proved that an integral element m of B (that is, an element of $\overset{....}{B}$) is decreasingly minimal (= dec-min) precisely if m is increasingly maximal (= inc-max).

One of our main goals is to prove that the set $\mathrm{dm}(\overset{....}{B})$ of all dec-min elements of $\overset{....}{B}$ is an M-convex set, meaning that there exists an integral base-polyhedron $B^\bullet \subseteq B$ such that $\mathrm{dm}(\overset{....}{B})$ is the set of integral elements of $B^\bullet$. In addition, we shall show that $\mathrm{dm}(\overset{....}{B})$ is actually a matroidal M-convex set, that is, $B^\bullet$ is a special base-polyhedron which is obtained from a matroid base-polyhedron by translating it with an integral vector.

The base-polyhedron $B^\bullet$ will be obtained with the help of a decomposition of B along a certain 'canonical' partition $\{S_1, S_2, \ldots, S_q\}$ of S into non-empty sets. To this end, we start by introducing the first member S_1 of this partition along with a matroid on S_1. The set S_1, depending only on B, will be called the peak-set of S.

4.1 Max-minimizers and pre-dec-min elements

Recall that an element of $\overset{....}{B}$ was called a max-minimizer if its largest component was as small as possible, while a max-minimizer was called a pre-dec-min element of $\overset{....}{B}$ if the number of its maximum components was as small as possible. As a dec-min element of $\overset{....}{B}$ is automatically pre-dec-min (in particular, a max-minimizer), we start our investigations by studying max-minimizers and pre-dec-min elements of $\overset{....}{B}$. For a number β, we say that a vector is **β-covered** if each of its components is at most β. Throughout our discussions,

$$\beta_1 := \beta(B) \tag{4.1}$$

denotes the smallest integer for which $\overset{....}{B}$ has a β_1-covered element. In other words, β_1 is the largest component of a max-minimizer of $\overset{....}{B}$. Therefore β_1 is the largest component of any pre-dec-min (and hence any dec-min) element of $\overset{....}{B}$. Note that an element m of $\overset{....}{B}$ is β_1-covered precisely if m is a max-minimizer. For any real number $\alpha \in \mathbf{R}$, let $\lceil \alpha \rceil$ denote the smallest integer not smaller than α.

Theorem 4.1. *For the largest component β_1 of a max-minimizer of $\ddot{\ddot{B}}$, one has*

$$\beta_1 = \max\{\left\lceil \frac{p(X)}{|X|} \right\rceil : \emptyset \neq X \subseteq S\}. \tag{4.2}$$

Proof. Formula (2.4), when applied to the special case with $f \equiv -\infty$ and $g \equiv \beta$, implies that B has a β-covered element if and only if

$$\beta|X| \geq p(X) \quad \text{whenever} \;\; X \subseteq S. \tag{4.3}$$

Moreover, if β is an integer and (4.3) holds, then B has an integral β-covered element. As $\beta|X| \geq p(X)$ holds for an arbitrary β when $X = \emptyset$, it follows that the smallest integer β meeting this (4.3) is indeed $\max\{\lceil p(X)/|X| \rceil : \emptyset \neq X \subseteq S\}$. ■

For a β_1-covered element m of $\ddot{\ddot{B}}$, let $r_1(m)$ denote the number of β_1-valued components of m. Recall that for an element $s \in S$ we denoted the unique smallest m-tight set containing s by $T_m(s) = T_m(s; p)$ (that is, $T_m(s)$ is the intersection of all m-tight sets containing s). Furthermore, let

$$S_1(m) := \cup\{T_m(t) : m(t) = \beta_1\}. \tag{4.4}$$

Then $S_1(m)$ is m-tight and $S_1(m)$ is actually the unique smallest m-tight set containing all the β_1-valued elements of m.

Theorem 4.2. *A β_1-covered element m of $\ddot{\ddot{B}}$ is pre-dec-min if and only if $m(s) \geq \beta_1 - 1$ for each $s \in S_1(m)$.*

Proof. Necessity. Let m be a pre-dec-min element of $\ddot{\ddot{B}}$. For any β_1-valued element $t \in S$ and any element $s \in T_m(t)$, we claim that $m(s) \geq \beta_1 - 1$. Indeed, if we had $m(s) \leq \beta_1 - 2$, then the vector m' arising from m by decreasing $m(t)$ by 1 and increasing $m(s)$ by 1 belongs to B (since $T_m(t)$ is the smallest m-tight set containing t) and has one less β_1-valued components than m has, contradicting the assumption that m is pre-dec-min.

Sufficiency. Let m' be an arbitrary β_1-covered integral element of B. Abbreviate $S_1(m)$ by Z and let h' denote the number of elements $z \in Z$ for which $m'(z) = \beta_1$. Then

$$\begin{aligned} |Z|(\beta_1 - 1) + r_1(m) &= \widetilde{m}(Z) = p(Z) \leq \widetilde{m}'(Z) \\ &\leq h'\beta_1 + (|Z| - h')(\beta_1 - 1) = |Z|(\beta_1 - 1) + h' \\ &\leq |Z|(\beta_1 - 1) + r_1(m'), \end{aligned}$$

from which $r_1(m) \leq r_1(m')$, as required. ■

Define the set-function h_1 on S as follows.

$$h_1(X) := p(X) - (\beta_1 - 1)|X| \;\; \text{for} \;\; X \subseteq S. \tag{4.5}$$

Theorem 4.3. *For the minimum number r_1 of β_1-valued components of a β_1-covered member of $\ddot{\ddot{B}}$, one has*

$$r_1 = \max\{h_1(X) : X \subseteq S\}. \tag{4.6}$$

Proof. Let m be an element of $\ddot{\ddot{B}}$ for which the maximum of its components is β_1, and let X be an arbitrary subset of S. Suppose that X has ℓ β_1-valued components. Then

$$p(X) \leq \widetilde{m}(X) \leq \ell\beta_1 + (|X| - \ell)(\beta_1 - 1) = |X|(\beta_1 - 1) + \ell \leq |X|(\beta_1 - 1) + r_1(m), \quad (4.7)$$

from which $r_1(m) \geq p(X) - (\beta_1 - 1)|X| = h_1(X)$, implying that

$$r_1 = \min\{r_1(m) : m \in \ddot{\ddot{B}},\ m \text{ is } \beta_1\text{-covered}\} \geq \max\{h_1(X) : X \subseteq S\}.$$

In order to prove the reverse inequality, we have to find a β_1-covered integral element m of B and a subset X of S for which $r_1(m) = h_1(X)$, which is equivalent to requiring that each of the three inequalities in (4.7) holds with equality. That is, the following three **optimality criteria** hold: (a) X is m-tight, (b) X contains all β_1-valued components of m, and (c) $m(s) \geq \beta_1 - 1$ for each $s \in X$.

Let m be a pre-dec-min element of B. Then $S_1(m)$ is m-tight, $S_1(m)$ contains all β_1-valued elements and, by Theorem 4.2, $m(s) \geq \beta_1 - 1$ for all $s \in S_1(m)$, therefore m and $S_1(m)$ satisfy the three optimality criteria. ∎

Note that r_1 is the number of β_1-valued components of any pre-dec-min element (and in particular, any dec-min element) of $\ddot{\ddot{B}}$.

4.2 The peak-set S_1

Since the set-function h_1 introduced in (4.5) is supermodular, the maximizers of h_1 are closed under taking intersection and union. Let S_1 denote the unique smallest subset of S maximizing h_1. In other words, S_1 is the intersection of all sets maximizing h_1. We call this set S_1 the **peak-set** of B (and of $\ddot{\ddot{B}}$).

Theorem 4.4. *For every pre-dec-min (and in particular, for every dec-min) element m of $\ddot{\ddot{B}}$, the set $S_1(m)$ introduced in* (4.4) *is independent of the choice of m and $S_1(m) = S_1$, where S_1 is the peak-set of B.*

Proof. It follows from Theorem 4.3 that, given a pre-dec-min element m of B, a subset X is maximizing h_1 precisely if the three optimality criteria mentioned in the proof hold. Since $S_1(m)$ meets the optimality criteria, it follows that $S_1 \subseteq S_1(m)$. If, indirectly, there is an element $s \in S_1(m) - S_1$, then $m(s) = \beta_1 - 1$ since S_1 contains all the β_1-valued elements. By the definition of $S_1(m)$, there is a β_1-valued element $t \in S_1(m)$ for which the smallest m-tight set $T_m(t)$ contains s, but this is impossible since S_1 is an m-tight set containing t but not s. ∎

Since $S_1 = S_1(m)$ is m-tight and near-uniform, we obtain that

$$\beta_1 = \left\lceil \frac{\widetilde{m}(S_1)}{|S_1|} \right\rceil = \left\lceil \frac{p(S_1)}{|S_1|} \right\rceil,$$

and the definitions of S_1 and r_1 imply that

$$r_1 = p(S_1) - (\beta_1 - 1)|S_1|. \quad (4.8)$$

Proposition 4.5. $S_1 = \{s \in S$: there is a pre-dec-min element $m \in \ddot{\ddot{B}}$ with $m(s) = \beta_1\}$. *For every pre-dec-min element m of* $\ddot{\ddot{B}}$, $m(s) \geq \beta_1 - 1$ *holds for every* $s \in S_1$, *and* $m(s) \leq \beta_1 - 1$ *holds for every* $s \in S - S_1$.

Proof. If $m(s) = \beta_1$ for some pre-dec-min m, then $s \in S_1(m) = S_1$. Conversely, let $s \in S_1$ and let m be a pre-dec-min element. We are done if $m(s) = \beta_1$. If this is not the case, then $m(s) = \beta_1 - 1$ by Theorem 4.2. By the definition of $S_1(m)$, there is an element $t \in S_1(m)$ for which $m(t) = \beta_1$ and $s \in T_m(t)$. But then $m' := m + \chi_s - \chi_t$ is in $\ddot{\ddot{B}}$, $m'(s) = \beta_1$ and m' is also pre-dec-min as it is value-equivalent to m. ■

4.3 Separating along S_1

Let S_1 be the peak-set occurring in Theorem 4.4 and let $S'_1 := S - S_1$. Let $p_1 = p|S_1$ denote the restriction of p to S_1, and let $B_1 \subseteq \mathbf{R}^{S_1}$ denote the base-polyhedron defined by p_1, that is, $B_1 := B'(p_1)$. Suppose that $S'_1 \neq \emptyset$ and let $p'_1 := p/S_1$, that is, p'_1 is the set-function on S'_1 obtained from p by contracting S_1 ($p'_1(X) = p(S_1 \cup X) - p(S_1)$ for $X \subseteq S'_1$).

Consider the face F of B determined by S_1, that is, F is the direct sum of the base-polyhedra $B_1 = B'(p_1)$ and $B'_1 = B'(p'_1)$. Then the dec-min elements of $\ddot{\ddot{B}}_1$ are exactly the integral elements of the intersection of B_1 and the box given by $\{x : \beta_1 - 1 \leq x(s) \leq \beta_1$ for every $s\}$. Hence the dec-min elements of $\ddot{\ddot{B}}_1$ are near-uniform.

Theorem 4.6. *An integral vector* $m = (m_1, m'_1)$ *is a dec-min element of* $\ddot{\ddot{B}}$ *if and only if* m_1 *is a dec-min element of* $\ddot{\ddot{B}}_1$ *and* m'_1 *is a dec-min element of* $\ddot{\ddot{B}}'_1$.

Proof. Suppose first that m is a dec-min element of $\ddot{\ddot{B}}$. Then $S_1 = S_1(m)$ by Theorem 4.4 and m is a max-minimizer, implying that every component of m in S_1 is of value $\beta_1 - 1$ or value β_1, and m has exactly r_1 components of value β_1. Therefore each of the components of m_1 is $\beta_1 - 1$ or β_1, that is, m_1 is near-uniform. Since m_1 is obviously in $\ddot{\ddot{B}}_1$, m_1 is indeed dec-min in $\ddot{\ddot{B}}_1$.

Since $\widetilde{m}(S_1) = p(S_1)$, for a set $X \subseteq S'_1$, we have

$$\widetilde{m}'_1(X) = \widetilde{m}(X) = \widetilde{m}(S_1 \cup X) - \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup X) - p(S_1) \geq p(S_1 \cup X) - p(S_1) = p'_1(X).$$

Furthermore

$$\widetilde{m}'_1(S'_1) = \widetilde{m}(S'_1) = \widetilde{m}(S_1 \cup S'_1) - \widetilde{m}(S_1) = p(S_1 \cup S'_1) - p(S_1) = p'_1(S'_1),$$

that is, m'_1 is in $\ddot{\ddot{B}}'_1$. If, indirectly, m'_1 is not dec-min, then, by applying Theorem 3.3 to S'_1, m'_1, and p'_1, we obtain that there are elements t and s of S'_1 for which $m'_1(t) \geq m'_1(s) + 2$ and ($*$) no $t\overline{s}$-set exists which is m'_1-tight with respect to p'_1. On the other hand, m is a dec-min element of $\ddot{\ddot{B}}$ for which

$$m(t) = m'_1(t) \geq m'_1(s) + 2 = m(s) + 2,$$

and hence there must be a $t\overline{s}$-set Y which is m-tight with respect to p.

Since S_1 is m-tight with respect to p, the set $S_1 \cup Y$ is also m-tight with respect to p. Let $X := S'_1 \cap Y$. Then

$$\widetilde{m}(X) + \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y) = p(S_1 \cup X),$$

and hence

$$\widetilde{m}'_1(X) = \widetilde{m}(X) = p(S_1 \cup X) - \widetilde{m}(S_1) = p(S_1 \cup X) - p(S_1) = p'_1(X),$$

that is, X is a $t\overline{s}$-set which is m'_1-tight with respect to p'_1, in contradiction with statement $(*)$ above that no such set exists.

To see the converse, assume that m_1 is a dec-min element of $\overset{....}{B}_1$ and m'_1 is a dec-min element of $\overset{....}{B}'_1$. This immediately implies that m is in the face F of B determined by S_1. Suppose, indirectly, that m is not a dec-min element of $\overset{....}{B}$. By Theorem 3.3, there are elements t and s of S for which $m(t) \geq m(s) + 2$ and $(**)$ no $t\overline{s}$-set exists which is m-tight with respect to p. If $t \in S_1$, then s cannot be in S_1 since the m-value of each element of S_1 is β_1 or $\beta_1 - 1$. But S_1 is m_1-tight with respect to p and hence it is m-tight with respect to p, contradicting property $(**)$. Therefore t must be in S'_1, implying, by Proposition 4.5, that s is also in S'_1.

Since m'_1 is a dec-min element of $\overset{....}{B}'_1$, there must be a $t\overline{s}$-set $Y \subset S'_1$ which is m'_1-tight with respect to p'_1. It follows that

$$\widetilde{m}(Y) = \widetilde{m}'_1(Y) = p'_1(Y) = p(S_1 \cup Y) - p(S_1) \leq \widetilde{m}(S_1 \cup Y) - \widetilde{m}(S_1) = \widetilde{m}(Y),$$

from which $\widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y)$, contradicting property $(**)$ that no $t\overline{s}$-set exists which is m-tight with respect to p. ■

An important consequence of Theorem 4.6 is that, in order to find a dec-min element of $\overset{....}{B}$, it will suffice to find separately a dec-min element of $\overset{....}{B}_1$ (which was shown above to be a near-uniform vector) and a dec-min element of $\overset{....}{B}'_1$. The algorithmic details is discussed in [12].

Theorem 4.7. *Let S_1 be the peak-set of $\overset{....}{B}$. For an element m_1 of $\overset{....}{B}_1$, the following properties are pairwise equivalent.*

(A1) *m_1 has r_1 $(= p(S_1) - (\beta_1 - 1)|S_1| > 0)$ components of value β_1 and $|S_1| - r_1$ (≥ 0) components of value $\beta_1 - 1$.*

(A2) *m_1 is near-uniform.*

(A3) *m_1 is dec-min in $\overset{....}{B}_1$.*

(B1) *m_1 is the restriction of a dec-min element m of $\overset{....}{B}$ to S_1.*

(B2) *m_1 is the restriction of a pre-dec-min element m of $\overset{....}{B}$ to S_1.*

Proof. The implications (A1)→(A2)→(A3) and (B1)→(B2) are immediate from the definitions.

(A3)→(B1): Let m'_1 be an arbitrary dec-min element of $\overset{....}{B}'_1$. By Theorem 4.6, $m := (m_1, m'_1)$ is a dec-min element of $\overset{....}{B}$ and hence m_1 is indeed the restriction of a dec-min element of $\overset{....}{B}$ to S_1.

(B2)→(A1): By Theorems 4.2 and 4.4, we have $m_1(s) \geq \beta_1 - 1$ for each $s \in S_1(m) = S_1$, that is, $\beta_1 - 1 \leq m_1(s) \leq \beta_1$. By letting r' denote the number of β_1-valued components of m_1, we obtain by (4.8) that

$$r_1 + (\beta_1 - 1)|S_1| = p_1(S_1) = \widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + r'$$

and hence $r' = r_1$. ∎

Theorem 4.6 implies that, in order to characterize the set of dec-min elements of $\overset{....}{B}$, it suffices to characterize the set of dec-min elements of $\overset{....}{B'_1}$.

Theorem 4.8. *Let β_2 denote the smallest integer for which $\overset{....}{B'_1}$ has a β_2-covered element, that is, $\beta_2 = \beta(B'_1)$. Then*

$$\beta_2 = \max\{\left\lceil \frac{p'_1(X)}{|X|} \right\rceil : \emptyset \neq X \subseteq S - S_1\}, \tag{4.9}$$

where $p'_1(X) = p(X \cup S_1) - p(S_1)$. Furthermore, β_2 is the largest component in $S - S_1$ of every dec-min element of $\overset{....}{B}$, and $\beta_2 < \beta_1$.

Proof. Formula (4.9) follows by applying Theorem 4.1 to base-polyhedron B'_1 ($= B'(p'_1)$) in place of B. By Theorem 4.6, the largest component in $S - S_1$ of any dec-min element m of $\overset{....}{B}$ is β_2. By Theorem 4.4, $S_1(m) = S_1$, and the definition of $S_1(m)$ shows that $m(s) \leq \beta_1 - 1$ holds for every $s \in S - S_1$, from which $\beta_2 < \beta_1$ follows. ∎

4.4 The matroid M_1 on S_1

It is known from the theory of base-polyhedra that the intersection of an integral base-polyhedron with an integral box is a (possibly empty) integral base-polyhedron. Moreover, if the box in question is small, then the intersection is actually a translated matroid base-polyhedron (meaning that the intersection arises from a matroid base-polyhedron by translating it with an integral vector). This result is a consequence of the theorem that (∗) any integral base-polyhedron in the unit (0, 1)-cube is the convex hull of (incidence vectors of) the bases of a matroid.

Consider the special small integral box $T_1 \subseteq \mathbf{Z}^{S_1}$ defined by

$$T_1 := \{x : \beta_1 - 1 \leq x(s) \leq \beta_1\}$$

and its intersection $B^\bullet_1 := B_1 \cap T_1$ with the base-polyhedron B_1 investigated above. Therefore $B^\bullet_1$ is a translated matroid base-polyhedron and Theorem 4.7 implies the following.

Corollary 4.9. *The dec-min elements of $\overset{....}{B}_1$ are exactly the integral elements of the translated matroid base-polyhedron $B^\bullet_1$.* ∎

Our next goal is to reprove Corollary 4.9 by concretely describing the matroid in question and not relying on the background theorem (∗) mentioned above. For a dec-min element m_1 of $\overset{....}{B}_1$, let

$$L_1(m_1) := \{s \in S_1 : m_1(s) = \beta_1\}.$$

We know from Theorem 4.7 that $|L_1(m_1)| = r_1$. Define a set-system $\mathcal{B}_1$ as follows:

$$\mathcal{B}_1 := \{L \subseteq S_1 : L = L_1(m_1) \text{ for some dec-min element } m_1 \text{ of } \overset{....}{B}_1\}. \tag{4.10}$$

We need the following characterization of $\mathcal{B}_1$.

Proposition 4.10. *An r_1-element subset L of S_1 is in $\mathcal{B}_1$ if and only if*

$$|L \cap X| \geq p_1'(X) := p_1(X) - (\beta_1 - 1)|X| \text{ whenever } X \subseteq S_1. \tag{4.11}$$

Proof. Suppose first that $L \in \mathcal{B}_1$, that is, there is a dec-min element m_1 of $\overset{....}{B}_1$ for which $L = L_1(m_1)$. Then

$$(\beta_1 - 1)|X| + |X \cap L| = \widetilde{m}_1(X) \geq p_1(X),$$

for every subset $X \subseteq S_1$ from which (4.11) follows.

To see the converse, let $L \subseteq S_1$ be an r_1-element set meeting (4.11). Let

$$m_1(s) := \begin{cases} \beta_1 & \text{if } s \in L \\ \beta_1 - 1 & \text{if } s \in S - L. \end{cases} \tag{4.12}$$

Then obviously $L = L_1(m_1)$. Furthermore,

$$\widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + |L| = (\beta_1 - 1)|S_1| + r_1 = p(S_1)$$

and

$$\widetilde{m}_1(X) = (\beta_1 - 1)|X| + |L \cap X| \geq p_1(X) \text{ whenever } X \subset S_1,$$

showing that $m_1 \in B_1$. Since $m_1 \in T_1$, we conclude that m_1 is a dec-min element of $\overset{....}{B}_1$. ■

Theorem 4.11. *The set-system $\mathcal{B}_1$ defined in* (4.10) *forms the set of bases of a matroid M_1 on ground-set S_1.*

Proof. The set-system $\mathcal{B}_1$ is clearly non-empty and all of its members are of cardinality r_1. It is widely known [5] that for an integral submodular function b on a ground-set S_1 the set-system

$$\{ L \subseteq S_1 : |L \cap X| \leq b(X) \text{ whenever } X \subset S_1, |L| = b(S_1) \},$$

if non-empty, satisfies the matroid basis axioms. This implies for the supermodular function p_1' that the set-system $\{L : |L \cap X| \geq p_1'(X) \text{ whenever } X \subset S_1, |L| = p_1'(S_1)\}$, if non-empty, forms the set of bases of a matroid. By applying this fact to the supermodular function p_1' defined by $p_1'(X) := p_1(X) - (\beta_1 - 1)|X|$, one obtains that $\mathcal{B}_1$ is non-empty and forms the set of bases of a matroid. ■

In this way, we proved the following more explicit form of Corollary 4.9.

Corollary 4.12. *Let $\Delta_1 : S_1 \to \mathbf{Z}$ denote the integral vector defined by $\Delta_1(s) := \beta_1 - 1$ for $s \in S_1$. A member m_1 of $\overset{....}{B}_1$ is decreasingly minimal if and only if there is a basis B_1 of M_1 such that $m_1 = \chi_{B_1} + \Delta_1$.* ■

4.5 Value-fixed elements of S_1

We say that an element $s \in S$ is **value-fixed** with respect to $\ddot{\ddot{B}}$ if $m(s)$ is the same for every dec-min element m of $\ddot{\ddot{B}}$. In Section 6.3, we will show a description of value-fixed elements of $\ddot{\ddot{B}}$. In the present section, we consider the value-fixed elements with respect to B_1, that is, $s \in S_1$ is value-fixed if $m_1(s)$ is the same for every dec-min element $m_1 \in \ddot{\ddot{B}}_1$. Recall that $m_1 \in \ddot{\ddot{B}}_1$ was shown to be dec-min precisely if $\beta_1 - 1 \leq m_1(s) \leq \beta_1$ for each $s \in S_1$.

A **loop** of a matroid is an element $s \in S_1$ not belonging to any basis. (Often the singleton $\{s\}$ is called a loop, that is, $\{s\}$ is a one-element circuit). A **co-loop** (or cut-element or isthmus) of a matroid is an element s belonging to all bases.

Proposition 4.13. *M_1 has no loops.*

Proof. By Proposition 4.5, for every $s \in S_1$ there is a pre-dec-min element m of $\ddot{\ddot{B}}$ for which $m(s) = \beta_1$. Then $m_1 := m|S_1$ is a pre-dec-min element of $\ddot{\ddot{B}}_1$ by Theorem 4.7 from which s_1 belongs to a basis of M_1 by Corollary 4.12. ■

The proposition implies that:

Proposition 4.14. *If $s \in S_1$ is value-fixed (with respect to B_1), then $m_1(s) = \beta_1$ for every dec-min element m_1 of $\ddot{\ddot{B}}_1$.* ■

By Corollary 4.12, an element $s \in S_1$ is a co-loop of M_1 if and only if $m_1(s) = \beta_1$ holds for every dec-min element m_1 of $\ddot{\ddot{B}}_1$. This and Theorem 4.6 imply the following.

Theorem 4.15. *For an element $s \in S_1$, the following properties are pairwise equivalent.*

(A) *s is a co-loop of M_1.*

(B) *s is value-fixed.*

(C) *$m(s) = \beta_1$ holds for every dec-min element m of $\ddot{\ddot{B}}$.* ■

Our next goal is to characterize the set of value-fixed elements of S_1. Consider the family of subsets S_1 defined by

$$\mathcal{F}_1 := \{X \subseteq S_1 : \beta_1|X| = p_1(X)\}. \tag{4.13}$$

The empty set belongs to $\mathcal{F}_1$ and it is possible that $\mathcal{F}_1$ has no other members. By standard submodularity arguments, $\mathcal{F}_1$ is closed under taking union and intersection. Let F_1 denote the unique largest member of $\mathcal{F}_1$. It is possible that $F_1 = S_1$ in which case we call S_1 **degenerate**.

Theorem 4.16. *An element $s \in S_1$ is value-fixed if and only if $s \in F_1$.*

Proof. Let m_1 be a dec-min member of $\ddot{\ddot{B}}_1$. Then

$$\beta_1|F_1| \geq \widetilde{m}_1(F_1) \geq p_1(F_1) = \beta_1|F_1|$$

and hence we must have $\beta_1 = m_1(s)$ for every $s \in F_1$, that is, the elements of F_1 are indeed value-fixed.

Conversely, let s be value-fixed, that is, $m_1(s) = \beta_1$ for each dec-min element m_1 of $\overset{....}{B}_1$. Let m_1 be a dec-min member of $\overset{....}{B}_1$. Let Z denote the unique smallest set containing s for which $\widetilde{m}_1(Z) = p_1(Z)$. (That is, $Z = T_{m_1}(s; p_1)$.) We claim that $m_1(t) = \beta_1$ for every element $t \in Z$. For if $m_1(t) = \beta_1 - 1$ for some t, then $m_1' := m_1 - \chi_s + \chi_t$ would also be a dec-min member of $\overset{....}{B}_1$, contradicting the assumption that s is value-fixed. Therefore $p_1(Z) = \widetilde{m}_1(Z) = \beta_1|Z|$ from which the definition of F_1 implies that $Z \subseteq F_1$ and hence $s \in F_1$. ■

5 The set of dec-min elements of an M-convex set

Let $B = B'(p)$ denote again an integral base-polyhedron defined by the (integer-valued) supermodular function p. As in the previous section, $\overset{....}{B}$ continues to denote the M-convex set consisting of the integral vectors (points, elements) of B. Our present goal is to provide a complete description of the set of decreasingly-minimal (= egalitarian) elements of $\overset{....}{B}$ by identifying a partition of the ground-set, to be named the canonical partition, inherent in this problem. As a consequence, we show that the set of dec-min elements has a matroidal structure and this feature makes it possible to solve the minimum cost dec-min problem.

5.1 Canonical partition and canonical chain

In Section 4 we introduced the integer β_1 as the minimum of the largest component of the elements of $\overset{....}{B}$ as well as the notion of peak-set S_1 of S. We considered the face of B defined by S_1 that was the direct sum of base-polyhedra $B_1 = B'(p_1)$ and $B_1' = B'(p_1')$, where p_1 denoted the restriction of p to S_1 while p_1' arose from p by contracting S_1 (that is, $p_1'(X) = p(S_1 \cup X) - p(S_1)$).

A consequence of Theorem 4.6 is that, in order to characterize the set of dec-min elements of $\overset{....}{B}$, it suffices to characterize separately the dec-min elements of $\overset{....}{B}_1$ and the dec-min elements of $\overset{....}{B}_1'$. In Theorem 4.7, we characterized the dec-min elements of $\overset{....}{B}_1$ as those belonging to the small box $T_1 := \{x \in \mathbf{R}^{S_1} : \beta_1 - 1 \leq x(s) \leq \beta_1 \text{ for } s \in S_1\}$. We also proved that the set $\overset{....}{B}{}_1^{\bullet}$ of dec-min elements of $\overset{....}{B}_1$ can be described with the help of matroid M_1. If the peak-set S_1 happens to be the whole ground-set S, then the characterization of the set of dec-min elements of $\overset{....}{B}$ is complete. If $S_1 \subset S$, then our remaining task is to characterize the set of dec-min elements of $\overset{....}{B}_1'$. This can be done by repeating iteratively the separation procedure to the base-polyhedron $B_1' = B'(p_1') \subseteq \mathbf{R}^{S-S_1}$ described in Section 4 for B.

In this iterative way, we are going to define a partition $\mathcal{P}^* = \{S_1, S_2, \dots, S_q\}$ of S which determines a chain $C^* = \{C_1, C_2, \dots, C_q\}$ where $C_i := S_1 \cup S_2 \cup \cdots \cup S_i$ (in particular $C_q = S$), and the supermodular function

$$p_i' := p/C_i \quad \text{on set } \overline{C_i} := S - C_i$$

which defines the base-polyhedron $B_i' = B'(p_i')$ in $\mathbf{R}^{\overline{C_i}}$. Moreover, we define iteratively a decreasing sequence $\beta_1 > \beta_2 > \cdots > \beta_q$ of integers, a small box

$$T_i := \{x \in \mathbf{R}^{S_i} : \beta_i - 1 \leq x(s) \leq \beta_i \text{ for } s \in S_i\}, \tag{5.1}$$

and the supermodular function p_i on S_i, where

$$p_i := p'_{i-1}|S_i \quad (= (p/C_{i-1})|S_i), \tag{5.2}$$

that is,

$$p_i(X) = p(X \cup C_{i-1}) - p(C_{i-1}) \text{ for } X \subseteq S_i.$$

Let $B_i := B'(p_i) \subseteq \mathbf{R}^{S_i}$ be the base-polyhedron defined by p_i.

In the general step, suppose that the pairwise disjoint non-empty sets $S_1, S_2, \dots, S_{j-1}$ have already been defined, along with the decreasing sequence $\beta_1 > \beta_2 > \cdots > \beta_{j-1}$ of integers. If $S = S_1 \cup \cdots \cup S_{j-1}$, then by taking $q := j-1$, the iterative procedure terminates. So suppose that this is not the case, that is, $C_{j-1} \subset S$. We assume that p_{j-1} on S_{j-1} has been defined as well as p'_{j-1} on $\overline{C_{j-1}}$.

Let

$$\beta_j = \max\{\left\lceil \frac{p'_{j-1}(X)}{|X|} \right\rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\}, \tag{5.3}$$

that is,

$$\beta_j = \max\{\left\lceil \frac{p(X \cup C_{j-1}) - p(C_{j-1})}{|X|} \right\rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\}. \tag{5.4}$$

Note that, by the iterative feature of these definitions, Theorem 4.8 implies that

$$\beta_j < \beta_{j-1}.$$

Furthermore, let h_j be a set-function on $\overline{C_{j-1}}$ defined as follows:

$$h_j(X) := p'_{j-1}(X) - (\beta_j - 1)|X| \text{ for } X \subseteq \overline{C_{j-1}}, \tag{5.5}$$

and let $S_j \subseteq \overline{C_{j-1}}$ be the peak-set of $\overline{C_{j-1}}$ assigned to $B'_{j-1} := B'(p'_{j-1})$, that is, S_j is the smallest subset of $\overline{C_{j-1}}$ maximizing h_j. Finally, let $p_j := p'_{j-1}|S_j$ and let $p'_j := p'_{j-1}/S_j$. Observe by (2.3) that $p'_j = p/C_j$. Therefore p_j is a set-function on S_j while p'_j is defined on $\overline{C_j}$.

We shall refer to the partition $\mathcal{P}^*$ and the chain C^* defined above as the **canonical partition** and **canonical chain** of S, respectively, assigned to B, while the sequence $\beta_1 > \cdots > \beta_q$ will be called the **essential value-sequence** of $\overset{....}{B}$. Let $B^\oplus$ denote the face of B defined by the canonical chain C^*, that is, $B^\oplus$ is the direct sum of the q base-polyhedra $B'(p_i)$ $(i = 1, \dots, q)$. Finally, let T^* be the direct sum of the small boxes T_i $(i = 1, \dots, q)$, that is, T^* is the integral box defined by the essential value-sequence as follows:

$$T^* := \{x \in \mathbf{R}^S : \beta_i - 1 \leq x(s) \leq \beta_i \text{ whenever } s \in S_i \ (i = 1, \dots, q)\}, \tag{5.6}$$

and let

$$B^\bullet := B^\oplus \cap T^*.$$

is always an integral base-polyhedron and hence $B^\bullet$ is an integral base-polyhedron. Furthermore, $B^\bullet$ is the direct sum of the q base-polyhedra $B_i \cap T_i$ $(i = 1, \dots, q)$, where $B_i = B'(p_i)$, implying that a vector m is in $\overset{....}{B}{}^\bullet$ if and only if each m_i is in $\overset{....}{B_i} \cap T_i$, where $m_i = m|S_i$.

Theorem 5.1. *Let $B = B'(p)$ be an integral base-polyhedron on ground-set S. The set of decreasingly-minimal elements of $\overset{....}{B}$ is (the* M*-convex set) $\overset{....}{B}{}^{\bullet}$. Equivalently, an element $m \in \overset{....}{B}$ is decreasingly minimal if and only if its restriction $m_i := m|S_i$ to S_i belongs to $B_i \cap T_i$ for each $i = 1, \dots, q$, where $\{S_1, \dots, S_q\}$ is the canonical partition of S belonging to B, T_i is the small box defined in* (5.1)*, and B_i is the base-polyhedron $B'(p_i)$ belonging to the supermodular set-function p_i defined in* (5.2).

Proof. We use induction on q. Suppose first that $q = 1$, that is, $S_1 = S$ and $B_1 = B$. If m is a dec-min element of B, then the equivalence of Properties (A1) and (A3) in Theorem 4.7 implies that m is in $\overset{....}{B}{}^{\bullet}$. If, conversely, $m \in \overset{....}{B}{}^{\bullet}$, then m is near-uniform and, by the equivalence of Properties (A1) and (A3) in Theorem 4.7 again, m is dec-min.

Suppose now that $q \geq 2$ and consider the base-polyhedron $B'_1 = B'(p'_1)$ appearing in Theorem 4.6. The iterative definition of the canonical partition $\mathcal{P}^*$ implies that the canonical partition of $S - S_1$ assigned to B'_1 is $\{S_2, \dots, S_q\}$ and the essential value-sequence belonging to B'_1 is $\beta_2 > \beta_3 > \cdots > \beta_q$. Also, the canonical chain $C' := \{C'_2, \dots, C'_q\}$ of B'_1 consists of the sets $C'_i = S_2 \cup \cdots \cup S_i = C_i - S_1$ $(i = 2, \dots, q)$.

By applying the inductive hypothesis to B'_1, we obtain that an integral element m'_1 of B'_1 is dec-min if and only if m'_1 is in the face of B'_1 defined by chain C' and m'_1 belongs to the box $T' := \{x \in \mathbf{R}^{S-S_1} : \beta_i - 1 \leq x(s) \leq \beta_i$ whenever $s \in S_i$ $(i = 2, \dots, q)\}$. By applying Theorem 4.6, we are done in this case as well. ∎

Corollary 5.2. *Let $B = B'(p)$ be an integral base-polyhedron on ground-set S. Let $\{C_1, \dots, C_q\}$ be the canonical chain, $\{S_1, \dots, S_q\}$ the canonical partition of S, and $\beta_1 > \beta_2 > \cdots > \beta_q$ the essential value-sequence belonging to $\overset{....}{B}$. Then an element $m \in \overset{....}{B}$ is decreasingly minimal if and only if each C_i is m-tight (that is, $\widetilde{m}(C_i) = p(C_i)$) and $\beta_i - 1 \leq m(s) \leq \beta_i$ holds for each $s \in S_i$ $(i = 1, \dots, q)$.* ∎

5.2 Obtaining the canonical chain and value-sequence from a dec-min element

The main goal of this section is to show that the canonical chain and value-sequence can be rather easily obtained from an arbitrary dec-min element of $\overset{....}{B}$. This approach will be crucial in developing a polynomial algorithm in [12] for computing the essential value-sequence along with the canonical chain and partition.

Let m be an element of $\overset{....}{B}$. We called a set $X \subseteq S$ m-tight if $\widetilde{m}(X) = p(X)$. Recall from Section 2 that, for a subset $Z \subseteq S$, $T_m(Z) = T_m(Z; p)$ denoted the unique smallest m-tight set including Z, that is, $T_m(Z)$ is the intersection of all the m-tight sets including Z. Obviously,

$$T_m(Z) = \cup(T_m(z) : z \in Z). \tag{5.7}$$

Let m be an arbitrary dec-min element of $\overset{....}{B}$. We proved that m is in the face $B^{\oplus}$ of B defined by the canonical chain $C^* = \{C_1, \dots, C_q\}$ belonging to B. Therefore each C_i is m-tight with respect to p. Furthermore $m_i := m|S_i$ belongs to the box T_i defined in (5.1). This implies that $m(s) \geq \beta_i - 1$ for every $s \in C_i$ and $m(s') \leq \beta_{i+1}$ for every $s' \in \overline{C_i}$. (The last inequality holds indeed since $s' \in \overline{C_i}$ implies that $s' \in S_j$ for some $j \geq i+1$ from which $m(s') \leq \beta_j \leq \beta_{i+1}$.) Since $\beta_{i+1} \leq \beta_i - 1$, we obtain that each C_i is an m-top set.

Since m_i is near-uniform on S_i with values β_i and possibly $\beta_i - 1$, we obtain

$$\beta_i = \left\lceil \frac{\widetilde{m}_i(S_i)}{|S_i|} \right\rceil = \left\lceil \frac{p_i(S_i)}{|S_i|} \right\rceil = \left\lceil \frac{p(C_i) - p(C_{i-1})}{|S_i|} \right\rceil.$$

Let $L_i := \{s \in S - C_{i-1} : m(s) = \beta_i\}$ and let $r_i := |L_i|$. Then $p_i(S_i) = \widetilde{m}_i(S_i) = (\beta_i - 1)|S_i| + r_i$ and hence

$$r_i = p(C_i) - p(C_{i-1}) - (\beta_i - 1)|S_i|. \tag{5.8}$$

The content of the next lemma is that, once C_{i-1} is given, the next member C_i of the canonical chain (and hence S_i, as well) can be expressed with the help of m. Recall that $T_m(L_i) = T_m(L_i; p)$ denoted the smallest m-tight set including L_i.

Lemma 5.3. $C_i = C_{i-1} \cup T_m(L_i; p)$.

Proof. Recall the definition of function h_i given in (5.5). We have

$$h_i(S_i) = r_i \tag{5.9}$$

since $h_i(S_i) = p'_{i-1}(S_i) - (\beta_i - 1)|S_i| = p(S_i \cup C_{i-1}) - p(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(C_i) - \widetilde{m}(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(S_i) - (\beta_i - 1)|S_i| = r_i$.

Since $L_i \subseteq C_i$ and each of C_{i-1}, C_i, and $T_m(L_i)$ are m-tight, we have $C_{i-1} \cup T_m(L_i; p) \subseteq C_i$. For $X' := T_m(L_i) \cap \overline{C_{i-1}}$ we have

$$\begin{aligned} h_i(X') &= p(C_{i-1} \cup T_m(L_i)) - p(C_{i-1}) - (\beta_i - 1)|X'_i| \\ &= \widetilde{m}(C_{i-1} \cup T_m(L_i)) - \widetilde{m}(C_{i-1}) - (\beta_i - 1)|X'_i| \\ &= \widetilde{m}(X') - (\beta_i - 1)|X'_i| = |L_i| = r_i = h_i(S_i), \end{aligned}$$

that is, X' is also a maximizer of $h_i(X)$. Since S_i was the smallest maximizer of h_i, we conclude that $C_{i-1} \cup T_m(L_i; p) \supseteq C_i$. ■

The lemma implies that both the essential value-sequence $\beta_1 > \cdots > \beta_q$ and the canonical chain C^* belonging to $\ddot{\ddot{B}}$ can be directly obtained from m.

Corollary 5.4. *Let m be an arbitrary dec-min element of $\ddot{\ddot{B}}$. The essential value-sequence and the canonical chain belonging to $\ddot{\ddot{B}}$ can be described as follows. Value β_1 is the largest m-value and C_1 is the smallest m-tight set containing all β_1-valued elements. Moreover, for $i = 2, \ldots, q$, β_i is the largest value of $m|\overline{C_{i-1}}$ and C_i is the smallest m-tight set (with respect to p) containing each element of m-value at least β_i.* ■

A detailed algorithm based on this corollary will be described in [12]. Note that a dec-min element m of $\ddot{\ddot{B}}$ may have more than q distinct values. For example, if $q = 1$ and $L_1 \subset C_1 = S$, then m has two distinct values, namely β_1 on the elements of L_1 and $\beta_1 - 1$ on the elements of $S - L_1$, while its essential value-sequence consists of the single member β_1.

A direct proof Corollary 5.4 implies that the chain of subsets and value-sequence assigned to a dec-min element m of $\ddddot{B}$ in the corollary do not depend on the choice of m. Here we describe an alternative, direct proof of this consequence.

Theorem 5.5. *Let m be an arbitrary dec-min element of $\ddddot{B}$. Let β_1 denote the largest value of m and let C_1 denote the smallest m-tight set (with respect to p) containing all β_1-valued elements. Moreover, for $i = 2, 3, \dots, q$, let β_i denote the largest value of $m|\overline{C_{i-1}}$ and let C_i denote the smallest m-tight set containing each element of m-value at least β_i. Then the chain $C_1 \subset C_2 \subset \cdots \subset C_q$ and the sequence $\beta_1 > \beta_2 > \cdots > \beta_q$ do not depend on the choice of m.*

Proof. Let z be dec-min element of $\ddddot{B}$. We use induction on the number of elements t of S for which $m(t) > z(t)$. If no such an element t exists, then $m = z$ and there is nothing to prove. So assume that $z \neq m$.

Let $L_i := \{t \in S_i : m(t) = \beta_i\}$. As m is dec-min, the definition of C_i implies that $m(s) = \beta_i - 1$ holds for every element $s \in S_i - L_i$. Let $t \in L_i$ and let $s \in T_m(t) - L_i$. Then $m' := m + \chi_s - \chi_t$ is also a dec-min element of $\ddddot{B}$, and we say that m' is obtained from m by an elementary step. Observe that $T_m(t) = T_{m'}(s)$ and hence the chain and the value-sequence assigned to m' is the same as those assigned to m.

Let i denote the smallest subscript for which $m|S_i$ and $z|S_i$ differ. Since z is dec-min, $z(s) \leq \beta_i$ holds for every $s \in S_i$. Let $L'_i := \{t \in S_i : z(t) = \beta_i\}$. Then $z(v) \leq \beta_i - 1$ for every $v \in S_i - L'_i$, and $|L'_i| \leq |L_i|$ as z is dec-min. Therefore

$$\widetilde{z}(S_i) \;\leq\; \beta_i|L'_i| + (\beta_i - 1)(|S_i - L'_i|) \;=\; (\beta_i - 1)|S_i| + |L'_i| \;\leq\; (\beta_i - 1)|S_i| + |L_i|.$$

On the other hand,

$$\begin{aligned}\widetilde{z}(S_i) &= \widetilde{z}(C_i) - \widetilde{z}(C_{i-1}) = \widetilde{z}(C_i) - \widetilde{m}(C_{i-1})\\ &\geq p(C_i) - \widetilde{m}(C_{i-1}) = \widetilde{m}(C_i) - \widetilde{m}(C_{i-1}) = \widetilde{m}(S_i) = (\beta_i - 1)|S_i| + |L_i|.\end{aligned}$$

Therefore we have equality throughout, in particular, $\widetilde{z}(C_i) = p(C_i)$, $|L'_i| = |L_i|$, and $z(v) = \beta_i - 1$ for every $v \in S_i - L'_i$.

Let $t \in L_i$ be an element for which $m(t) > z(t)$. Then $m(t) = \beta_i$ and $z(t) = \beta_i - 1$. It follows that $T_m(t)$ contains an element s for which $z(s) > m(s)$, implying that $m(s) = \beta_i - 1$ and $z(s) = \beta_i$. Now $m(t) > m'(t) = z(t)$ holds for the dec-min element $m' := m + \chi_s - \chi_t$ obtained from m by an elementary step, and therefore we are done by induction. ∎

5.3 Matroidal description of the set of dec-min elements

In Section 4.4, we introduced a matroid M_1 on S_1 and proved in Corollary 4.9 that the dec-min elements of $\ddddot{B}_1$ are exactly the integral elements of the translated base-polyhedron of M_1, where the translation means the addition of the constant vector $(\beta_1 - 1, \dots, \beta_1 - 1)$ of dimension $|S_1|$. The same notions and results can be applied to each subscript $i = 2, \dots, q$. Furthermore, by formulating Lemma 4.11 for subscript i in place of 1, we obtain the following.

Proposition 5.6. *The set-system* $\mathcal{B}_i := \{L \subseteq S_i : L = L_i(m_i)$ for some dec-min element m_i of $\overset{....}{B}_i\}$ *forms the set of bases of a matroid* M_i *on ground-set* S_i. *An* r_i*-element subset* L *of* S_i *is a basis of* M_i *if and only if*

$$|L \cap X| \geq p'_i(X) := p_i(X) - (\beta_i - 1)|X| \tag{5.10}$$

holds for every $X \subseteq S_i$. ■

It follows that a vector m_i on S_i is a dec-min element of $\overset{....}{B}_i$ if and only if $\beta_i - 1 \leq m_i(s) \leq \beta_i$ for each $s \in S_i$ and the set $L_i := \{s \in S_i : m_i(s) = \beta_i\}$ is a basis of M_i. Let M^* denote the direct sum of matroids $M_1, \dots, M_q$ and let $\Delta^* \in \mathbf{Z}^S$ denote the translation vector defined by

$$\Delta^*(s) := \beta_i - 1 \ \text{ whenever } \ s \in S_i,\ i = 1, \dots, q.$$

By integrating these results, we obtain the following characterization.

Theorem 5.7. *Let* B *be an integral base-polyhedron. An element* m *of (the M-convex set)* $\overset{....}{B}$ *is decreasingly minimal if and only if* m *can be obtained in the form* $m = \chi_L + \Delta^*$ *where* L *is a basis of the matroid* M^*. *The base-polyhedron* $B^\bullet$ *arises from the base-polyhedron of* M^* *by adding the translation vector* Δ^*. *Concisely, the set of dec-min elements of* $\overset{....}{B}$ *is a matroidal M-convex set.* ■

Cheapest dec-min element An important algorithmic consequence of Theorems 5.1 and 5.7 is that they help solve the **cheapest dec-min element** problem, which is as follows. Let $c : S \to \mathbf{R}$ be a cost function and consider the problem of computing a dec-min element m of an M-convex set $\overset{....}{B}$ for which cm is as small as possible.

By Theorem 5.7 the set $\overset{....}{B}{}^\bullet$ of dec-min elements of $\overset{....}{B}$ can be obtained from a matroid M^* by translation. Namely, there is a vector $\Delta^* \in \mathbf{Z}^S$ such that m is in $\overset{....}{B}{}^\bullet$ if and only if there is a basis L of M^* for which $m = \chi_L + \Delta^*$. Note that the matroid M^* arises as the direct sum of matroids M_i defined on the members S_i of the canonical partition. M_1 is described in Proposition 4.10 and the other matroids M_i may be determined analogously in an iterative way. To realize this algorithmically, we must have a strongly polynomial algorithm to compute the canonical partition as well as the essential value-sequence. Such an algorithm will be described in [12].

Therefore, in order to find a minimum c-cost dec-min element of $\overset{....}{B}$, it suffices to find a minimum c-cost basis of M^*. Note that, in applying the greedy algorithm to the matroids M_i in question, we need a rank oracle, which can be realized with the help of a submodular function minimization oracle by relying on the definition of bases in (4.11).

Recall that for integral bounds $f \leq g$, the intersection B_1 of a base-polyhedron B and the box $T(f, g)$, if non-empty, is itself a base-polyhedron. Therefore the algorithm above can be applied to the M-convex set $\overset{....}{B}_1$, that is, we can compute a cheapest dec-min element of the intersection $\overset{....}{B}_1 = \overset{....}{B} \cap T(f, g)$.

6 Integral square-sum and difference-sum minimization

For a vector $z \in \mathbf{Z}^S$, we can conceive several natural functions to measure the uniformity of its component values $z(s)$ for $s \in S$. Here are two examples:

$$\textbf{square-sum}: \quad W(z) := \sum [z(s)^2 : s \in S], \tag{6.1}$$

$$\textbf{difference-sum}: \quad \Delta(z) := \sum [|z(s) - z(t)| : s \neq t,\ s,t \in S]. \tag{6.2}$$

For vectors z_1 and z_2 with $\widetilde{z_1}(S) = \widetilde{z_2}(S)$, z_1 may be felt more uniform than z_2 if $W(z_1) < W(z_2)$, and z_1 may also be felt more uniform if $\Delta(z_1) < \Delta(z_2)$. The first goal of this section is to show, by establishing a fairly general theorem, that a dec-min element of an M-convex set $\dddot{B}$ is simultaneously a minimizer of these two functions. The second goal of this section is to derive a min-max formula for the minimum integral square-sum of an element of an M-convex set $\dddot{B}$, along with characterizations of (integral) square-sum minimizers and dual optimal solutions.

6.1 Symmetric convex minimization

Let S be a non-empty ground-set of n elements: $S = \{1, 2, \ldots, n\}$. We say that function $\Phi : \mathbf{Z}^S \to \mathbf{R}$ is **symmetric** if

$$\Phi(z(1), z(2), \ldots, z(n)) = \Phi(z(\sigma(1)), z(\sigma(2)), \ldots, z(\sigma(n))) \tag{6.3}$$

for all permutations σ of $(1, 2, \ldots, n)$. We call a function $\Phi : \mathbf{Z}^S \to \mathbf{R}$ **convex** if

$$\lambda\Phi(x) + (1-\lambda)\Phi(y) \geq \Phi(\lambda x + (1-\lambda)y) \tag{6.4}$$

whenever $x, y \in \mathbf{Z}^S$, $0 < \lambda < 1$, and $\lambda x + (1-\lambda)y$ is an integral vector; and **strictly convex** if

$$\lambda\Phi(x) + (1-\lambda)\Phi(y) > \Phi(\lambda x + (1-\lambda)y) \tag{6.5}$$

whenever $x, y \in \mathbf{Z}^S$, $0 < \lambda < 1$, and $\lambda x + (1-\lambda)y$ is an integral vector.

In the special case where φ is a function in one variable, it can easily be shown that the convexity of φ is equivalent to the weaker requirement that the inequality

$$2\varphi(k) \leq \varphi(k-1) + \varphi(k+1) \tag{6.6}$$

holds for every integer k. It is strictly convex in the sense of (6.5) if and only if $2\varphi(k) < \varphi(k-1) + \varphi(k+1)$ holds for every integer k. For example, $\varphi(k) = k^2$ is strictly convex while $\varphi(k) = |k|$ is convex but not strictly. Given a function φ in one variable, define Φ by

$$\Phi(z) := \sum [\varphi(z(s)) : s \in S] \tag{6.7}$$

for $z \in \mathbf{Z}^S$. Such a function Φ is called a **symmetric separable convex function**; note that Φ is indeed convex in the sense of (6.4). When φ is strictly convex, Φ is also called strictly convex.

Example 6.1. The square-sum $W(z)$ in (6.1) is a symmetric convex function which is separable and strictly convex. ∎

Example 6.2. The difference-sum $\Delta(z)$ in (6.2) is a symmetric convex function which is neither separable nor strictly convex. More generally, for a nonnegative integer K, the function defined by

$$\Delta_K(z) := \sum [(|z(s) - z(t)| - K)^+ : s \neq t,\ s, t \in S]$$

is a symmetric convex function, where $(x)^+ = \max\{x, 0\}$. ∎

The following statements show a close relationship between decreasing minimality and the minimization of symmetric convex Φ over an M-convex set $\overset{....}{B}$.

Proposition 6.1. *Let B be an integral base-polyhedron and Φ a symmetric convex function. Then each dec-min element of $\overset{....}{B}$ is a minimizer of Φ over $\overset{....}{B}$.*

Proof. Since the dec-min elements of $\overset{....}{B}$ are value-equivalent and Φ is symmetric, the Φ-value of each dec-min element is the same value μ. We claim that $\Phi(m) \geq \mu$ for each $m \in \overset{....}{B}$. Suppose indirectly that there is an element m of $\overset{....}{B}$ for which $\Phi(m) < \mu$. Then m is not dec-min in $\overset{....}{B}$ and Property (A) in Theorem 3.3 implies that there is a 1-tightening step for m resulting in decreasingly smaller member of $\overset{....}{B}$, that is, there exist $s, t \in S$ such that $m(t) \geq m(s) + 2$ and $m' := m + \chi_s - \chi_t \in \overset{....}{B}$.

Let $\alpha = m(t) - m(s)$, where $\alpha \geq 2$, and define $z = m + \alpha(\chi_s - \chi_t)$. Since z is obtained from m by interchanging the components at s and t, $\Phi(m) = \Phi(z)$ by symmetry (6.3). Note that the vector z may not be a member of $\overset{....}{B}$. For $\lambda = 1 - 1/\alpha$ we have

$$\lambda m + (1 - \lambda) z = \left(1 - \frac{1}{\alpha}\right) m + \frac{1}{\alpha}\,(m + \alpha(\chi_s - \chi_t)) = m + \chi_s - \chi_t = m' \in \overset{....}{B}\ (\subseteq \mathbf{Z}^S), \quad (6.8)$$

from which $\lambda\Phi(m) + (1 - \lambda)\Phi(z) \geq \Phi(m')$ by convexity (6.4). Since $\Phi(m) = \Phi(z)$, this implies $\Phi(m) \geq \Phi(m')$. After a finite number of such 1-tightening steps, we arrive at a dec-min element m_0 of $\overset{....}{B}$, for which $\mu = \Phi(m_0) \leq \Phi(m) < \mu$, a contradiction. ∎

Note that if Φ is convex but not strictly convex, then Φ may have minimizers that are not dec-min elements. This is exemplified by the identically zero function Φ for which every member of $\overset{....}{B}$ is a minimizer. However, for strictly convex functions we have the following characterization.

Theorem 6.2. *Given an integral base-polyhedron B and a symmetric strictly convex function Φ, an element m of $\overset{....}{B}$ is a minimizer of Φ if and only if m is a dec-min element of $\overset{....}{B}$.*

Proof. If m is a dec-min element, then m is a Φ-minimizer by Proposition 6.1. To see the converse, let m be a Φ-minimizer of $\overset{....}{B}$. If, indirectly, m is not a dec-min element, then Property (A) in Theorem 3.3 implies that there is a 1-tightening step for m, that is, there exist $s, t \in S$ such that $m(t) \geq m(s)+2$ and $m' := m+\chi_s-\chi_t \in \overset{....}{B}$. For $\alpha = m(t)-m(s)$, $\lambda = 1-1/\alpha$, and $z = m + \alpha(\chi_s - \chi_t)$, we have (6.8), from which we obtain $\lambda\Phi(m) + (1-\lambda)\Phi(z) > \Phi(m')$ by strict convexity (6.5), and hence $\Phi(m) > \Phi(m')$, a contradiction to the assumption that m is a Φ-minimizer. ■

We obtain the following as corollaries of this theorem.

Corollary 6.3. *Let B be an integral base-polyhedron and Φ a symmetric separable convex function. Then each dec-min element of $\overset{....}{B}$ is a minimizer of Φ over $\overset{....}{B}$, and the converse is also true if, in addition, Φ is strictly convex.* ■

Corollary 6.4. *For an M-convex set $\overset{....}{B}$, an element m of $\overset{....}{B}$ is a square-sum minimizer if and only if m is a dec-min element of $\overset{....}{B}$.* ■

An immediate consequence of Corollary 6.3 is that a square-sum minimizer of $\overset{....}{B}$ minimizes an arbitrary symmetric separable discrete convex function. Note, however, that this consequence immediately follows from a much earlier result of Groenevelt [18] below, which deals with the minimization of a (not-necessarily symmetric) separable convex function.

Theorem 6.5 (Groenevelt [18]; cf. [16, Theorem 8.1])**.** *Let B be an integral base-polyhedron, $\overset{....}{B}$ be the set of its integral elements, and $\Phi(z) = \sum[\varphi_s(z(s)) : s \in S]$ for $z \in \mathbf{Z}^S$, where $\varphi_s : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ is a discrete convex function for each $s \in S$. An element m of $\overset{....}{B}$ is a minimizer of $\Phi(z)$ if and only if $\varphi_s(m(s)+1) + \varphi_t(m(t)-1) \geq \varphi_s(m(s)) + \varphi_t(m(t))$ whenever $m + \chi_s - \chi_t \in \overset{....}{B}$.* ■

A dec-min element is also characterized as a difference-sum minimizer.

Theorem 6.6. *For an M-convex set $\overset{....}{B}$, an element m of $\overset{....}{B}$ is a difference-sum minimizer if and only if m is a dec-min element of $\overset{....}{B}$.*

Proof. By Proposition 6.1 every dec-min element is a difference-sum minimizer. To show the converse, suppose indirectly that there is difference-sum minimizer m that is not dec-min in $\overset{....}{B}$. Property (A) in Theorem 3.3 implies that there is a 1-tightening step for m, that is, there exist $s, t \in S$ such that $m(t) \geq m(s) + 2$ and $m' := m + \chi_s - \chi_t \in \overset{....}{B}$. Here we observe that $|m'(s) - m'(t)| = |m(s) - m(t)| - 2$ and

$$(|m'(v)-m'(s)|+|m'(v)-m'(t)|)-(|m(v)-m(s)|+|m(v)-m(t)|) = \begin{cases} -2 & \text{if } m(s) < m(v) < m(t) \\ 0 & \text{otherwise.} \end{cases}$$

This shows $\Delta(m') \leq \Delta(m) - 2$, a contradiction. ■

Remark 6.1. We emphasize that there is a fundamental difference between the problems of finding a minimum square-sum element over a base-polyhedron B and over the M-convex set $\overset{....}{B}$ (the set of integral elements of B). In the first case (investigated by Fujishige [15, 16]), there alway exists a single, unique solution, while in the second case, the square-sum minimizer elements of $\overset{....}{B}$ have an elegant matroidal structure. Namely, Corollary 6.4 shows that the square-sum minimizers are exactly the dec-min elements of $\overset{....}{B}$ and hence, by Theorem 5.7, the set of square-sum minimizers of an M-convex set arises from the bases of a matroid by translating their incidence vectors with a vector. ■

Remark 6.2. Corollary 6.4 says that an element m of an M-convex set $\overset{....}{B}$ is dec-min precisely if m is a square-sum minimizer. One may feel that it would have been a more natural approach to derive this equivalence by showing that $x \leq_{\rm dec} y$ holds precisely if $W(x) \leq W(y)$. Perhaps surprisingly, however, this equivalence fails to hold, that is, the square-sum is not order-preserving with respect to the quasi-order $\leq_{\rm dec}$. To see this, consider the following four vectors in increasing order:

$$m_1 = (2,3,3,1) <_{\rm dec} m_2 = (3,3,3,0) <_{\rm dec} m_3 = (2,2,4,1) <_{\rm dec} m_4 = (3,2,4,0).$$

Their square-sums admit a different order:

$$W(m_1) = 23, \quad W(m_2) = 27, \quad W(m_3) = 25, \quad W(m_4) = 29.$$

The four vectors m_i $(i = 1,2,3,4)$ form an M-convex set. Among these four elements, m_1 is the unique dec-min element and the unique square-sum minimizer but the decreasing-order and the square-sum order of the other three elements are different. We remark that if φ in (6.7) is not only strictly convex but 'rapidly' increasing as well, then $x <_{\rm dec} y$ can be proved to be equivalent to $\Phi(x) < \Phi(y)$. This intuitive notion of rapid increase is formalized in [11]. ■

Remark 6.3. For the intersection of two M-convex sets, dec-min elements and square-sum minimizers may not coincide. Here is an example. Let

$$\overset{....}{B_1} = \{(3,3,3,0),\ (2,2,4,1),\ (2,3,3,1),\ (3,2,4,0)\},$$
$$\overset{....}{B_2} = \{(3,3,3,0),\ (2,2,4,1),\ (3,2,3,1),\ (2,3,4,0)\},$$

which are both M-convex. In their intersection $\overset{....}{B_1} \cap \overset{....}{B_2} = \{(3,3,3,0),(2,2,4,1)\}$, the vector $(3,3,3,0)$ is the unique dec-min element while $(2,2,4,1)$ is the unique square-sum minimizer. This demonstrates that the two notions of optima may differ for the intersection of two M-convex sets. ■

Remark 6.4. For $a,b,c \geq 0$, the function defined by

$$\Phi(z) = a \sum_{s \in S} |z(s)| + b \sum_{s \neq t} |z(s) - z(t)| + c \sum_{s \neq t} |z(s) + z(t)|$$

is a symmetric convex function. More generally, a function of the form

$$\Phi(z) = \sum_{s \in S} \varphi_1(z(s)) + \sum_{s \neq t} \varphi_2(|z(s) - z(t)|) + \sum_{s \neq t} \varphi_3(z(s) + z(t)),$$

where $\varphi_1, \varphi_2, \varphi_3 : \mathbf{Z} \to \mathbf{R}$ are (discrete) convex functions, is a symmetric convex function which is not separable. Such a function is an example of the so-called 2-separable convex functions. By Theorem 6.2, a dec-min element of $\overset{....}{B}$ is a minimizer of function Φ over $\overset{....}{B}$. The minimization of 2-separable convex functions is investigated in depth by Hochbaum and others [1, 21, 22] using network flow techniques. ■

Remark 6.5. Theorem 6.2 is a discrete counterpart of a result of Maruyama [30] for the continuous case. See also Nagano [34, Corollary 13]. Symmetric convex function minimization is studied, mainly for the continuous case, in the literature of majorization [2, 29]. ■

Remark 6.6. A min-max formula can be derived for the square-sum (see Section 6.2) and, more generally, for separable convex functions from the Fenchel-type duality theorem in DCA [32, 33]. However, we cannot use the Fenchel-type duality theorem to obtain a min-max formula for non-separable symmetric convex functions, since non-separable symmetric convex functions are not necessarily M-convex. ■

6.2 Min-max theorem for integral square-sum

Recall the notation $W(z) = \sum[z(s)^2 : s \in S]$ for the square-sum of $z \in \mathbf{Z}^S$. Given a polyhedron B, we say that an element $m \in \overset{\cdots}{B}$ is a **square-sum minimizer** (over $\overset{\cdots}{B}$) or that m is an **integral square-sum minimizer** of B if $W(m) \leq W(z)$ holds for each $z \in \overset{\cdots}{B}$. The main goal of this section is to derive a min-max formula for the minimum integral square-sum of an element of an M-convex set $\overset{\cdots}{B}$, along with a characterization of (integral) square-sum minimizers.

A set-function p on S can be considered as a function defined on (0, 1)-vectors. It is known that p can be extended in a natural way to every vector π in $\mathbf{R}^S$, as follows. For the sake of this definition, we may assume that the elements of S are indexed in a decreasing order of the components of π, that is, $\pi(s_1) \geq \cdots \geq \pi(s_n)$ (where the order of the components of π with the same value is arbitrary). For $j = 1, \ldots, n$, let $I_j := \{s_1, \ldots, s_j\}$ and let

$$\hat{p}(\pi) := p(I_n)\pi(s_n) + \sum_{j=1}^{n-1} p(I_j)[\pi(s_j) - \pi(s_{j+1})]. \tag{6.9}$$

Obviously, $p(Z) = \hat{p}(\chi_Z)$. The function $\hat{p}$ is called the **linear extension** of p.

Remark 6.7. The linear extension was first considered by Edmonds [5] who proved for a polymatroid $P = P(b)$ defined by a monotone, non-decreasing submodular function b that $\max\{\pi x : x \in \overset{\cdots}{P}\} = \hat{b}(\pi)$ when π is non-negative. The same approach shows for a base-polyhedron $B = B'(p)$ defined by a supermodular function p that $\min\{\pi x : x \in \overset{\cdots}{B}\} = \hat{p}(\pi)$. Another basic result is due to Lovász [28] who proved that p is submodular if and only if $\hat{p}$ is concave. We do not, however, explicitly need these results, and only remark that in the literature the linear extension is often called Lovász extension. ■

Our approach is as follows. First, we consider an arbitrary set-function p on S (supermodular or not) along with the polyhedron

$$B = B'(p) := \{x : x \in \mathbf{R}^S,\ \widetilde{x}(Z) \geq p(Z) \text{ for every } Z \subset S \text{ and } \widetilde{x}(S) = p(S)\},$$

and develop an easily checkable lower bound for the minimum square-sum over the integral elements of B. If this lower bound is attained by an element m of $\overset{\cdots}{B}$, then m is certainly a square-sum minimizer independently of any particular property of p. For general p, the lower bound (not surprisingly) is not always attainable. We shall prove, however, that it is attainable when p is (fully) supermodular. That is, we will have a min-max theorem for the minimum square-sum over an M-convex set $\ddddot{B}$, or in other words, we will have an easily checkable certificate for an element m of $\ddddot{B}$ to be a minimizer of the square-sum.

We shall need the following two claims. For any real number $\alpha \in \mathbf{R}$, let $\lfloor \alpha \rfloor$ denote the largest integer not larger than α, and $\lceil \alpha \rceil$ the smallest integer not smaller than α.

Claim 6.7. *For $m, \pi \in \mathbf{Z}^S$, one has*

$$\sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil \geq \sum_{s \in S} m(s)[\pi(s) - m(s)]. \tag{6.10}$$

Moreover, equality holds if and only if

$$m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\} \quad \text{for every } s \in S. \tag{6.11}$$

Proof. The claim follows by observing that $\lfloor a/2 \rfloor \lceil a/2 \rceil \geq b(a - b)$ holds for any pair of integers a and b, where equality holds precisely if $b \in \{\lfloor a/2 \rfloor, \lceil a/2 \rceil\}$. ∎

Let p be an arbitrary set-function on S with $p(\emptyset) = 0$ and consider an integral element m of the polyhedron $B = B'(p)$. Recall that a non-empty subset $X \subseteq S$ was called a strict π-top set if $\pi(u) > \pi(v)$ held whenever $u \in X$ and $v \in S - X$. In what follows, for an $m \in \ddddot{B}$, m-tightness of a subset $Z \subseteq S$ means $\widetilde{m}(Z) = p(Z)$.

Claim 6.8. *For $m \in \ddddot{B}$ and $\pi \in \mathbf{Z}^S$, one has*

$$\hat{p}(\pi) \leq \sum_{s \in S} m(s)\pi(s). \tag{6.12}$$

Moreover, equality holds if and only if each (of the at most n) strict π-top set is m-tight.

Proof. Suppose that the elements of S are indexed in such a way that $\pi(s_1) \geq \pi(s_2) \geq \cdots \geq \pi(s_n)$. For $j = 1, \dots, n$, let $I_j := \{s_1, \dots, s_j\}$. Then

$$\begin{aligned}
\hat{p}(\pi) &= p(I_n)\pi(s_n) + \sum_{j=1}^{n-1} p(I_j)[\pi(s_j) - \pi(s_{j+1})] \\
&\leq \widetilde{m}(I_n)\pi(s_n) + \sum_{j=1}^{n-1} \widetilde{m}(I_j)[\pi(s_j) - \pi(s_{j+1})] \\
&= \sum_{1 \leq i \leq j \leq n} m(s_i)\pi(s_j) - \sum_{1 \leq i \leq j \leq n-1} m(s_i)\pi(s_{j+1}) \\
&= \sum_{1 \leq i \leq j \leq n} m(s_i)\pi(s_j) - \sum_{1 \leq i < j' \leq n} m(s_i)\pi(s_{j'}) \\
&= \sum_{j=1}^{n} m(s_j)\pi(s_j),
\end{aligned}$$

from which (6.12) follows. Furthermore, we have equality in (6.12) precisely if $\widetilde{m}(I_j) = p(I_j)$ holds whenever $\pi(s_j) - \pi(s_{j+1}) > 0$. But this latter condition is equivalent to requiring that each strict π-top set is m-tight. ∎

Proposition 6.9. *Let p be an arbitrary set-function on S with $p(\emptyset) = 0$ and let m be an integral element of the polyhedron $B = B'(p)$. Then*

$$\sum_{s \in S} m(s)^2 \geq \hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil \tag{6.13}$$

whenever $\pi \in \mathbf{Z}^S$ is an integral vector. Furthermore, equality holds for m and π if and only if the following **optimality criteria** *hold:*

$$\text{(O1)} \qquad \text{(6.11) holds: } m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\} \quad \text{for every } s \in S, \tag{6.14}$$

$$\text{(O2)} \qquad \text{each strict } \pi\text{-top-set is } m\text{-tight with respect to } p. \tag{6.15}$$

Proof. Let $\pi \in \mathbf{Z}^S$. By the two preceding claims,

$$\sum_{s \in S} m(s)^2 = \sum_{s \in S} m(s)\pi(s) - \sum_{s \in S} m(s)[\pi(s) - m(s)] \geq \hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil, \tag{6.16}$$

from which (6.13) follows. The claims also immediately imply that we have equality in (6.13) precisely if the optimality criteria (O1) and (O2) hold. ∎

The min-max formula in the next theorem concerning min square-sum over the integral elements of an integral base-polyhedron can be derived from the more general Fenchel-type duality theorem in DCA (see [32] and also Theorem 8.21, page 222, in the book [33]), or from a recent framework [14] of separable discrete convex function minimization over the integer points in an integral box-TDI polyhedron. However, our proof relies only on the relatively simple characterization of dec-min elements described in Theorem 3.3. In particular, we need no results of Sections 4 and 5.

Theorem 6.10. *Let $B = B'(p)$ be a base-polyhedron defined by an integer-valued fully supermodular function p. Then*

$$\min\{\sum_{s \in S} m(s)^2 : m \in \ddot{B}\} = \max\{\hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil : \pi \in \mathbf{Z}^S\}. \tag{6.17}$$

Proof. By Proposition 6.9, min $\geq$ max holds in (6.17) and hence all what we have to prove is that there is an element $m \in \ddot{B}$ and an integral vector $\pi \in \mathbf{Z}^S$ meeting the two optimality criteria formulated in Proposition 6.9. Let m be an arbitrary dec-min element of $\ddot{B}$. By Property (B) of Theorem 3.3, there is a chain $(\emptyset \subset)\ C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ of m-tight and m-top sets for which the restrictions of m onto the difference sets $S_i := C_i - C_{i-1}$ $(i = 1, \dots, \ell)$ are near-uniform in S_i (where $C_0 := \emptyset$). Note that $\{S_1, \dots, S_\ell\}$ is a partition of S.

For $i = 1, \dots, \ell$, let $\beta_i(m) := \max\{m(s) : s \in S_i\}$. Define $\pi_m : S \to \mathbf{Z}$ by

$$\pi_m(s) := 2\beta_i(m) - 1 \ \text{ if } \ s \in S_i \ \ (i = 1, \dots, \ell).$$

We have

$$\lfloor \pi_m(s)/2 \rfloor = \beta_i(m) - 1 \leq m(s) \leq \beta_i(m) = \lceil \pi_m(s)/2 \rceil$$

for every $s \in S_i$, and hence Optimality criterion (O1) holds for m and π_m.

We claim that each strict π_m-top set Z is a member of chain C. Indeed, as π_m is uniform in each S_j, if Z contains an element of S_j, then Z includes the whole S_j. Furthermore, since each member of C is an m-top set, we have $\beta_1(m) \geq \beta_2(m) \geq \cdots \geq \beta_\ell(m)$, and hence if Z includes S_j, then it includes each S_i with $i < j$. Therefore every strict π_m-top set is indeed a member of the chain, implying Optimality criterion (O2). ■

It should be noted that the optimal dual solution π_m obtained in the proof of the theorem is actually an **odd** vector in the sense that each of its component is an odd integer.

Corollary 6.11. *There is an odd dual optimizer π in the min-max formula* (6.17)*, that is, the min-max formula in Theorem* 6.10 *can be re-written as follows:*

$$\min\{\sum_{s\in S} m(s)^2 : m \in \ddot{\ddot{B}}\} = \max\{\hat{p}(\pi) - \sum_{s\in S} \frac{\pi(s)^2 - 1}{4} \ : \ \pi \in \mathbf{Z}^S, \ \pi \text{ is odd } \}. \tag{6.18}$$

We emphasize that for the proof of Theorem 6.10 and Corollary 6.11 we relied only on Theorem 3.3 and did not need the characterization of the set of dec-min elements of $\ddot{\ddot{B}}$ given in Section 5.

In the proof of Theorem 6.10, we chose an arbitrary dec-min element m of $\ddot{\ddot{B}}$ and an arbitrary chain of m-tight and m-top sets such that m is near-uniform on each difference set. In Section 5, we proved that there is a single canonical chain C^* which meets these properties for every dec-min element of $\ddot{\ddot{B}}$. Therefore the dual optimal π^* assigned to C^* is also independent of m. Namely, consider the canonical S-partition $\{S_1, \ldots, S_q\}$ and the essential value-sequence $\beta_1 > \cdots > \beta_q$. Define π^* by

$$\pi^*(s) := 2\beta_i - 1 \ \text{ if } \ s \in S_i \ (i = 1, \ldots, q). \tag{6.19}$$

As we pointed out in the proof of Theorem 6.10, this π^* is also a dual optimum in (6.17). We shall prove in the next section that π^* is actually the unique smallest dual optimum in (6.17).

6.3 The set of optimal duals to integral square-sum minimization

We proved earlier that an element $m \in \ddot{\ddot{B}}$ is a square-sum minimizer precisely if it is a dec-min element. This and Theorem 5.1 imply that the square-sum minimizers of $\ddot{\ddot{B}}$ are the integral members of a base-polyhedron $B^\bullet$ obtained by intersecting a particular face of B with a special small box. This means that the integral square-sum minimizers form an M-convex set.

Our next goal is to reveal the structure of the set Π of the dual optima in Theorem 6.10 and we provide a description of Π as the integral solution set of feasible potentials in a box. This shows another connection to DCA, which is discussed after the proof of Theorem 6.10.

Recall that the optimality criteria for a dec-min element m of $\ddot{\ddot{B}}$ and for an integral vector π were given by (O1) and (O2) in (6.14)–(6.15). These immediately imply the following.

Proposition 6.12. *For an integral vector π, the following are equivalent.*

(A) *π is a dual optimum (that is, π belongs to Π).*

(B) *There is a dec-min element m of $\ddot{\ddot{B}}$ such that m and π meet the optimality criteria.*

(C) *For every dec-min m of $\ddot{\ddot{B}}$, m and π meet the optimality criteria.* ■

Consider the canonical S-partition $\{S_1, \dots, S_q\}$, the essential value-sequence $\beta_1 > \beta_2 > \cdots > \beta_q$, and the matroids M_i on S_i $(i = 1, \dots, q)$. We can use the notions and apply the results of Section 4.5 formulated for M_1 to each M_i $(i = 1, \dots, q)$. To follow the pattern of $\mathcal{F}_1$ introduced in (4.13), let

$$\mathcal{F}_i := \{X \subseteq S_i : \beta_i |X| = p_i(X)\}, \tag{6.20}$$

where p_i was defined by $p_i(X) = p(C_{i-1} \cup X) - p(C_{i-1})$ for $X \subseteq S_i$. Since $\beta_i|X| \geq p_i(X)$ for every $X \subseteq S_i$ and p_i is supermodular, $\mathcal{F}_i$ is closed under taking intersection and union. Let F_i denote the unique largest member of $\mathcal{F}_i$, that is, F_i is the union of the members of $\mathcal{F}_i$. Both $F_i = \emptyset$ and $F_i = S_i$ are possible.

Theorem 6.13. *For an element $s \in S_i$ $(i = 1, \dots, q)$, the following properties are pairwise equivalent.*

(A) *s is value-fixed.*

(B) *$m(s) = \beta_i$ holds for every dec-min element m of $\ddot{\ddot{B}}$.*

(C) *$s \in F_i$.*

(D) *s is a co-loop of M_i.* ■

Define a digraph $D_i = (F_i, A_i)$ on node-set F_i in which st is an arc if $s, t \in F_i$ and there is no $t\overline{s}$-set in $\mathcal{F}_i$. This implies that no arc of D_i enters any member of $\mathcal{F}_i$.

Theorem 6.14. *An integral vector $\pi \in \mathbf{Z}^S$ is an optimal dual solution to the integral minimum square-sum problem (that is, $\pi \in \Pi$) if and only if the following three conditions hold for each $i = 1, \dots, q$:*

$$\pi(s) = 2\beta_i - 1 \quad \text{for every } \ s \in S_i - F_i, \tag{6.21}$$

$$2\beta_i - 1 \leq \pi(s) \leq 2\beta_i + 1 \quad \text{for every } \ s \in F_i, \tag{6.22}$$

$$\pi(s) - \pi(t) \geq 0 \quad \text{whenever } \ s, t \in F_i \ \text{ and } \ st \in A_i. \tag{6.23}$$

Proof.

Claim 6.15. *Optimality criterion* (O1) *is equivalent to*

$$\text{(O1}'\text{)} \quad 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1 \ \text{ for } \ s \in S. \tag{6.24}$$

Proof. When $\pi(s)$ is even, we have the following equivalences:

$$\begin{aligned} m(s) \in \left\{\left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil\right\} \quad &\Leftrightarrow \quad \pi(s) = 2m(s) \\ &\Leftrightarrow \quad 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1. \end{aligned}$$

When $\pi(s)$ is odd, we have the following equivalences:

$$\begin{aligned} m(s) \in \left\{\left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil\right\} \quad &\Leftrightarrow \quad \pi(s) - 1 \leq 2m(s) \leq \pi(s) + 1 \\ &\Leftrightarrow \quad 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1. \end{aligned}$$ ■

Suppose first that $\pi \in \mathbf{Z}^S$ is an optimal dual solution. Then the optimality criteria (O1′) and (O2) formulated in (6.24) and (6.15) hold for every dec-min element m of $\ddot{\ddot{B}}$.

Let s be an element of $S_i - F_i$. Since s is not value-fixed, there are dec-min elements m and m' of $\ddot{\ddot{B}}$ for which $m(s) = \beta_i - 1$ and $m'(s) = \beta_i$. By applying (6.24) to m and to m', we obtain that

$$2\beta_i - 1 = 2m'(s) - 1 \leq \pi(s) \leq 2m(s) + 1 = 2(\beta_i - 1) + 1 = 2\beta_i - 1,$$

from which $\pi(s) = 2\beta_i - 1$ follows, and hence (6.21) holds indeed.

Let s be an element of F_i. As s is value-fixed, $m(s) = \beta_i$ holds for any dec-min element m of $\ddot{\ddot{B}}$. We obtain from (6.24) that

$$2\beta_i - 1 = 2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1 = 2\beta_i + 1$$

and hence (6.22) holds.

To derive (6.23), suppose indirectly that st is an arc in A_i for which $\pi(t) > \pi(s) \geq 2\beta_i - 1$. Let $Z := \{v \in S : \pi(v) \geq \pi(t)\}$. Then Z is a strict π-top set and hence $C_{i-1} \subseteq Z \subseteq C_{i-1} \cup F_i - s$. By Optimality criterion (O2), Z is m-tight with respect to p. Let $X := Z \cap S_i$. Then $X \subseteq F_i$ and hence

$$p(Z) = \widetilde{m}(Z) = \widetilde{m}(C_{i-1}) + \widetilde{m}(X) = p(C_{i-1}) + \beta_i|X|,$$

from which

$$\beta_i|X| = p(Z) - p(C_{i-1}) = p_i(X),$$

that is, X is in $\mathcal{F}_i$, in contradiction with the definition of A_i which requires that st enters no member of $\mathcal{F}_i$.

Suppose now that π meets the three properties formulated in Theorem 6.14. Let $m \in \ddot{\ddot{B}}$ be an arbitrary dec-min element. Consider an element s of S_i. If $s \in F_i$, that is, if s is value-fixed, then $m(s) = \beta_i$. By (6.22), we have $2m(s) - 1 \leq \pi(s) \leq 2m(s) + 1$, that is, Optimality criterion (O1′) holds. If $s \in S_i - F_i$, then $\pi(s) = 2\beta_i - 1$ by (6.21), from which

$$\left\lfloor \frac{\pi(s)}{2} \right\rfloor = \frac{\pi(s) - 1}{2} = \beta_i - 1 \leq m(s) \leq \beta_i = \frac{\pi(s) + 1}{2} = \left\lceil \frac{\pi(s)}{2} \right\rceil,$$

showing that Optimality criterion (O1′) holds.

To prove optimality criterion (O2), let Z be a strict π-top set and let $\mu := \min\{\pi(v) : v \in Z\}$. Let i denote the largest subscript for which $X := Z \cap S_i \neq \emptyset$. Then $\mu \leq 2\beta_i + 1 \leq 2\beta_{i-1} - 1 \leq \pi(u)$ holds for every $u \in C_{i-1}$, from which $C_{i-1} \subseteq Z$ as Z is a strict π-top set.

If $\mu = 2\beta_i - 1$, then $S_i \subseteq Z$ as Z is a strict π-top set, from which $Z = C_i$, implying that Z is an m-tight set in this case. Therefore we suppose $\mu \geq 2\beta_i$, from which $X \subseteq F_i$ follows. Now $X \in \mathcal{F}_i$, for otherwise there is an arc $st \in A_i$ $(s, t \in F_i)$ entering X, and then $\pi(t) \leq \pi(s)$ holds by Property (6.23); this contradicts the assumption that Z is a strict π-top set. By $X \in \mathcal{F}_i$ we have $\beta_i|X| = p_i(X)$ and hence

$$\begin{aligned} \widetilde{m}(Z) &= \widetilde{m}(X) + \widetilde{m}(C_{i-1}) = \beta_i|X| + p(C_{i-1}) \\ &= p_i(X) + p(C_{i-1}) = p(X \cup C_{i-1}) - p(C_{i-1}) + p(C_{i-1}) = p(Z), \end{aligned}$$

that is, Z is indeed m-tight. ■ ■

We now relate Theorem 6.10 to a concept from discrete convex analysis, where two kinds of discrete convexity play major roles as mutually ‘conjugate’ notions of discrete convexity [33]. One of them is M-convexity and the other is called L-convexity. One of the equivalent definitions says that a set L of integer vectors is an **L-convex set** if it is the set of integer-valued feasible potentials. Formally, $L = \{\pi \in \mathbf{Z}^S : \pi(v) - \pi(u) \leq g(uv)\ (u, v \in S)\}$, where g is an integer-valued function on the ordered pairs of elements of S. A set of integer vectors is called an **L$^\natural$-convex set** (pronounce L-natural convex set) if it is the intersection of an L-convex set with an integral box.

In (6.19), we defined a special dual optimal solution π^* by $\pi^*(s) = 2\beta_i - 1$ whenever $s \in S_i$ $(i = 1, \ldots, q)$. Theorem 6.14 and the definition we use for L$^\natural$-convex sets immediately implies the following.

Corollary 6.16. *The set* Π *of optimal dual integral vectors* π *in the min-max formula* (6.17) *of Theorem* 6.10 *is an* L$^\natural$*-convex set. The unique smallest element of* Π *(that is, the unique smallest dual optimum) is* π^*. ■

It will be worth mentioning that L$^\natural$-convexity of the set of optimal dual integral vectors is a general phenomenon that is true in separable convex function minimization on an M-convex set; see Section 5 of [11]. Indeed, this is a consequence of conjugacy between M-convexity and L-convexity. It is also known that every L$^\natural$-convex set has a unique smallest (and a unique largest) element.

7 Conclusion

The present work will be the first member of a series of papers concerning discrete decreasing minimization. In the companion paper [12] we give a strongly polynomial algorithm for finding a dec-min element of an M-convex set and discuss applications of discrete decreasing minimization to the ‘background problems’ mentioned in Section 1.1. The decreasing minimization on an M-convex set is a discrete counterpart of the lexico-graphical optimization on a base-polyhedron considered by Fujishige [15]. The relations between these discrete and continuous cases are clarified in [11], and in particular, the precise relation between the canonical partition (for the discrete case) and the principal partition (for the continuous case) is revealed and proximity theorems are shown. The DCA-based approach in [11] also enables us to consider discrete decreasing minimization with respect to a weight vector.

While the present framework of decreasing minimization on an M-convex set is effective for a fairly wide class of graph orientation problems [12], there are other important graph orientation problems that do not fit in this framework. For example, for strong orientations of mixed graphs, dec-min orientations and inc-max orientations do not coincide. The reason behind this phenomenon is that the set of in-degree vectors of strong orientations of a mixed graph is not an M-convex set anymore. It is, in fact, the intersection of two M-convex sets. By investigating the decreasing minimization problem over the intersection of two M-convex sets we can solve a broader class of graph orientation problems, which will be reported soon.

Decreasing minimization on an M-convex set contains the integer version of Megiddo's problem [31] of finding a maximum flow that is 'lexicographically optimal' on the set of edges leaving the source node. In [13] this problem is generalized to the problem of finding an integral feasible flow that is decreasing minimal on an arbitrarily specified subset of edges. The structure of decreasingly minimal integral feasible flows is clarified and a strongly polynomial algorithm for finding such a dec-min flow is developed. A further generalization to integral submodular flows will be reported elsewhere.

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