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A Discrete Convex Min-Max Formula for Box-TDI Polyhedra

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Abstract

A min-max formula is proved for the minimum of an integer-valued separable discrete convex function where the minimum is taken over the set of integral elements of a box total dual integral (box-TDI) polyhedron. One variant of the theorem uses the notion of conjugate function (a fundamental concept in non-linear optimization) but we also provide another version that avoids conjugates, and its spirit is conceptually closer to the standard form of classic min-max theorems in combinatorial optimization. The presented framework provides a unified background for separable convex minimization over the set of integral elements of the intersection of two integral base-polyhedra, submodular flows, L-convex sets, and polyhedra defined by totally unimodular (TU) matrices. As an application, we show how inverse combinatorial optimization problems can be covered by this new framework.

Keywords: Min-max formula, Discrete convex function, Combinatorial inverse problem, Integral base-polyhedron, M-convex set, Total dual integrality.
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1 Introduction

A central aspect of convex optimization is minimizing a convex function over a convex set. Discrete convex analysis [22, 23] considers discrete convex functions. It turned out that there are two strongly interrelated general classes, M-convex and L-convex functions, for which fundamental min-max theorems can be formulated. It is important to distinguish between the cases when we minimize over real or over integer vectors. For example, one may be interested in finding a minimum ℓ_2-norm element of an integral base-polyhedron B (say) or a minimum ℓ_2-norm integral element of B. These are pretty different problems as the continuous version has a unique solution [16], while the structure of integral optima is rich [12, 13, 14, 15]. In the present work, we discuss the second type of minimization when the function to be minimized is an integer-valued separable discrete convex function. It was proved in [23] that these functions are exactly those which are both $\mathrm{M}^\natural$-convex and $\mathrm{L}^\natural$-convex. In this sense separable discrete convex functions are rather special but this speciality makes it possible that we can develop min-max theorems when we minimize over a discrete box-TDI set. Box-TDI linear systems and polyhedra (defined formally below) were introduced by Edmonds an Giles [10], studied in detail by Cook [4, 5], and recently by Chervet, Grappe, and Robert [3]. We shall call the set of integral elements of an integral box-TDI polyhedron a **discrete box-TDI set**, or just a **box-TDI set**.

Our main goal is to develop a general min-max formula for the minimum of an integer-valued separable discrete convex function Φ over a discrete box-TDI set. Actually, we exhibit two equivalent forms. One of them makes use of the discrete version of Fenchel conjugate, a fundamental concept from non-linear (continuous) optimization (see [2, 19, 24]). But we also develop another form which does not rely on the concept of conjugate, and therefore this version is conceptually closer to classic min-max theorems of combinatorial optimization like the ones of Menger, Kőnig, Egerváry, Dilworth, Ford + Fulkerson, Tutte, Edmonds, Lucchesi+Younger, etc.

Our general framework includes as a special case the corresponding optimization problems for totally unimodular (TU) matrices, in particular, circulations and potentials. The results can also be applied to submodular flows, in particular to the intersection of two base-polyhedra. As a special case, we derive a min-max theorem for the minimum square-sum of an integer-valued (!) feasible circulation or maximum flow.

It is our important goal to bring those readers closer to discrete convex optimization who are not particularly familiar with the notion of conjugate. The present work, apart from one exception, does not deal with algorithmic issues, but we hope that our min-max formulas pave the way to forthcoming researches for constructing strongly polynomial algorithms to compute the optima in question.

As an unexpected application, we shall show in Section 6 how a significant part of inverse combinatorial optimization problems can be modelled in this new framework. We provide a min-max theorem for the minimum total change (measured in ℓ_1-norm) of a given cost function w_0 for which a specified element of a discrete box-TDI set (for example, a spanning tree of a graph) becomes a cheapest one with respect to the modified cost function w. Even the more general inverse problem fits into our framework when each element from a specified list is expected to be a cheapest one with respect to the wanted w.

In the present work, for the sake of technical simplicity, we concentrate on integer-valued functions. It should, however, be emphasized that all the results can be extended in a natural way to real-valued separable discrete convex functions, as well.

1.1 Notions and notation

Let $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{Z}$ denote the set of reals, rationals, and integers, respectively. When it does not make any confusion, we do not distinguish between row- and column-vectors. For example, if u and v are vectors from $\mathbf{R}^n$, then $uv = vu$ denotes their scalar product. For a vector w, we use the notation w^2 for the scalar product ww, and will refer to w^2 as the **square-sum** of w. If Q is an m-by-n matrix while $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are vectors, then x is considered a column-vector in the product Qx, while y is considered a row-vector in yQ.

Throughout we work with a ground-set S with n elements. The incidence or characteristic vector χ_S of S will be briefly denoted by $\underline{\mathbf{1}}$. For elements $s, t \in S$, we call a subset $X \subset S$ an $s\bar{t}$**-set** if $s \in X \subset S - t$. For a function f on S, the set-function $\widetilde{f}$ is defined by $\widetilde{f}(X) := \sum[f(s) : s \in X]$ $(X \subseteq S)$.

For a polyhedron $R := \{x : Qx \geq p\} \subseteq \mathbf{R}^S$, $\ddddot{R}$ denotes the set of integral elements of R, that is,

$$\ddddot{R} := R \cap \mathbf{Z}^S. \tag{1.1}$$

For a cost function w on S, let $\mu_R(w)$ denote the minimum of $\{wx : x \in R\}$, while $\mu_{\ddddot{R}}(w) = \{wx : x \in \ddddot{R}\}$. We say that an element z^* of R is a w**-minimizer** if $wz^* \leq wx$ holds for every $x \in R$, that is, $wz^* = \mu_R(w)$.

The **effective domain** [23, 24] (or sometimes just **domain** [2, 19]) $\mathrm{dom}(\varphi)$ of an integer-valued function $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{-\infty, +\infty\}$ is the set of integers where φ is finite. When we say that a function φ is integer-valued, we allow that some values of φ may be $-\infty$ or $+\infty$. A function $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ is called **discrete convex** if $\varphi(k-1) + \varphi(k+1) \geq 2\varphi(k)$ for each $k \in \mathrm{dom}(\varphi)$. Let φ' denote the function defined on $\mathbf{Z}$ by

$$\varphi'(k) := \varphi(k+1) - \varphi(k) \quad (k \in \mathbf{Z}). \tag{1.2}$$

The function φ' may intuitively be considered the discrete right derivative of φ. Clearly, φ is discrete convex precisely if φ' is monotone non-decreasing. The effective domain of a discrete convex function is the set of integers in a (possibly unbounded) interval.

When we are given a function φ_s for every $s \in S$, the functions $\Phi : \mathbf{Z}^S \to \mathbf{Z} \cup \{+\infty\}$ and $\Phi' : \mathbf{Z}^S \to \mathbf{Z} \cup \{-\infty, +\infty\}$ are defined by:

$$\Phi(z) := \sum_{s \in S} \varphi_s(z(s)), \qquad \Phi'(z) := \sum_{s \in S} \varphi'_s(z(s)). \tag{1.3}$$

When each φ_s is discrete convex, Φ is called a **separable discrete convex** function. The **discrete conjugate** function $\varphi^\bullet$ of a function $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ is defined for any integer ℓ by

$$\varphi^\bullet(\ell) := \max\{k\ell - \varphi(k) : k \in \mathbf{Z}\}, \tag{1.4}$$

while the discrete conjugate $\Phi^\bullet$ of Φ is defined for $w \in \mathbf{Z}^S$ by

$$\Phi^\bullet(w) := \sum_{s \in S} \varphi^\bullet_s(w(s)).$$

Note that $\Phi^\bullet(w) = \max\{wz - \Phi(z) : z \in \mathbf{Z}^S\}$, and this latter expression is actually the definition of the discrete conjugate of an arbitrary integer-valued function Φ on $\mathbf{Z}^S$.

Note that $\varphi^\bullet(\ell)$ may be $+\infty$ (when $\{k\ell - \varphi(k) : k \in \mathbf{Z}\}$ is not bounded from above) and hence using supremum would be formally a bit more precise but we keep the term maximum. It should be emphasized that in the original definition of Fenchel conjugate in continuous optimization [2], the maximum is taken over all real values k and not only on integer k's.

Let $p : 2^S \to \mathbf{Z} \cup \{-\infty\}$ be an integer-valued (fully) supermodular function on a ground-set S for which the value $p(S)$ is finite. When we say that a function p is supermodular, we always mean that the supermodular inequality $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ holds for every pair $\{X, Y\}$ of subsets of S. Since weaker supermodular functions (e.g., intersecting, crossing) are also important in applications, sometimes we (over-) emphasize by saying that p is 'fully' supermodular.

Let

$$B := B'(p) := \{x : \widetilde{x}(Z) \geq p(Z) \text{ for every } Z \subset S, \text{ and } \widetilde{x}(S) = p(S)\}$$

be the **base-polyhedron** defined by p. Note that the complementary function $\overline{p}$, defined by $\overline{p}(X) := p(S) - p(S - X)$, is submodular and $B'(p) = B(\overline{p}) := \{x : \widetilde{x}(Z) \leq \overline{p}(Z)$ for every $Z \subset S$, and $\widetilde{x}(S) = \overline{p}(S)\}$. That is, a base polyhedron can be defined by a submodular function as well.

In discrete convex analysis [23], the set $\overset{....}{B}$ of integral elements of B is called an **M-convex set** and the intersection of two M-convex sets an $\mathbf{M_2}$**-convex set**. A fundamental theorem of Edmonds [7] states that a set is M_2-convex precisely if it is the set of integral elements of the intersection of two integral base-polyhedra.

1.2 Starting points

A starting point of the present work is the problem of finding/characterizing an element of an M-convex set $\overset{....}{B}$ for which an integer-valued separable discrete convex function $\Phi(z)$ in (1.3) is minimum. It is a basic property of base-polyhedra (see, e.g., [11]) that the intersection of an (integral) box with an (integral) base-polyhedron is itself an (integral) base-polyhedron. Since the effective domain of Φ is a box, it follows that we can replace B with this intersection, or in other words, we may assume that Φ is finite-valued on the whole M-convex set $\overset{....}{B}$.

A min-max theorem for separable discrete convex functions on an M-convex set can be obtained as a special case of the Fenchel-type discrete duality theorem [23] (Theorem 8.21) concerning discrete convex functions which are not necessarily separable. The formulation needs the well-known concept of linear (or Lovász) extension $\hat{p}$ of p which is recalled in Section 4.3. We also hasten to recall a basic theorem of Edmonds [7, 8] asserting that $\hat{p}(w) = \min\{wz : z \in \overset{....}{B}\}$ $(= \min\{wz : z \in B\})$. For an element $z \in \overset{....}{B}$, we call a subset $X \subseteq S$ z**-tight** if $\widetilde{z}(X) = p(X)$. For a vector $w \in \mathbf{Z}^S$, we call a non-empty set $X \subseteq S$ a **strict** w**-top set** if $w(s) > w(t)$ holds whenever $s \in X$ and $t \in S - X$. Note that the strict w-top sets form a chain.

Recall that an M-convex set $\overset{....}{B}$ was defined as the set of integral elements of an integral base-polyhedron B, that is,

$$\overset{....}{B} := B \cap \mathbf{Z}^S . \tag{1.5}$$

Theorem 1.1 ([15])**.** *Suppose that an integer-valued separable discrete convex function Φ is finite-valued and bounded from below on an M-convex set $\overset{....}{B}$ defined by an integer-valued (fully) supermodular function p (allowing $-\infty$ values). Then*

$$\min\{\Phi(z) : z \in \overset{....}{B}\} = \max\{\hat{p}(w) - \Phi^{\bullet}(w) : \ w \in \mathbf{Z}^S\}, \tag{1.6}$$

where $\Phi^{\bullet}$ denotes the discrete conjugate of Φ and $\hat{p}$ denotes the linear extension of p (and hence $\hat{p}(w) = \mu_B(w)$). Moreover, an element $z^* \in \overset{....}{B}$ is a Φ-minimizer if and only if there is an integer-valued function w^* on S meeting the following optimality criteria:

$$\text{each strict } w^*\text{-top set is } z^*\text{-tight}, \tag{1.7}$$

$$\varphi'_s(z^*(s) - 1) \leq w^*(s) \leq \varphi'_s(z^*(s)) \ \text{ for each } s \in S, \tag{1.8}$$

or writing (1.8) *concisely:*

$$\Phi'(z^* - \underline{\mathbf{1}}) \ \leq \ w^* \ \leq \ \Phi'(z^*). \tag{1.9}$$

Actually, the general Fenchel-type min-max theorem in [23] also implies the following extension of Theorem 1.1 to M_2-convex sets.

Theorem 1.2 ([15])**.** *Let $B_1 := B'(p_1)$ and $B_2 := B'(p_2)$ be base-polyhedra defined by integer-valued supermodular functions p_1 and p_2 for which $B := B_1 \cap B_2$ is non-empty. Let Φ be a finite integer-valued separable discrete convex function on B which is bounded from below on B. Then one has:*

$$\min\{\Phi(z) : z \in \overset{....}{B}\} = \max\{\hat{p}_1(w_1) + \hat{p}_2(w_2) - \Phi^{\bullet}(w_1 + w_2) : \ w_1, w_2 \in \mathbf{Z}^S\}. \tag{1.10}$$

In Section 4, we shall derive these theorems from the new min-max formula concerning discrete box-TDI sets. It is worth mentioning already at this point that in important special cases the discrete conjugate of Φ can be explicitly given. For example, let $\Phi(z) := z^2$ $(= \sum[z(s)^2 \ : \ s \in S])$. For any real number $\alpha \in \mathbf{R}$, let $\lfloor \alpha \rfloor$ denote the largest integer not larger than α, and $\lceil \alpha \rceil$ the smallest integer not smaller than α. Then Theorems 1.1 and 1.2 can be specialized, as follows.

Theorem 1.3 ([12, 13, 14, 15])**.** *Let $B = B'(p)$ be an integral base-polyhedron. Then*

$$\min\{z^2 : z \in \overset{....}{B}\} \ = \ \max\{\hat{p}(w) - \sum_{s \in S} \left\lfloor \frac{w(s)}{2} \right\rfloor \left\lceil \frac{w(s)}{2} \right\rceil : \ w \in \mathbf{Z}^S\}. \tag{1.11}$$

Let $B_1 := B'(p_1)$ and $B_2 := B'(p_2)$ be integral base-polyhedra for which $B := B_1 \cap B_2$ is non-empty. Then

$$\begin{aligned} &\min\{z^2 : z \in \overset{....}{B}\} \\ &= \max\{\hat{p}_1(w_1) + \hat{p}_2(w_2) - \sum_{s \in S} \left\lfloor \frac{w_1(s) + w_2(s)}{2} \right\rfloor \left\lceil \frac{w_1(s) + w_2(s)}{2} \right\rceil : \ w_1, w_2 \in \mathbf{Z}^S\}. \end{aligned} \tag{1.12}$$

These results were formulated first in [15] with a proof relying on the general discrete Fenchel-type duality theorem [23]. We shall directly derive not only Theorems 1.1 and 1.2 but a variant, as well, which does not use the concept of conjugate. Furthermore, we shall show that the role of the M-convex or M_2-convex set in these theorems is only that they are discrete box-TDI sets. Note that it is a basic property of base-polyhedra that they are box-TDI and a theorem of Edmonds and Giles [9] implies that the intersection of two base-polyhedra is also a box-TDI polyhedron. Therefore our main min-max theorem concerning discrete box-TDI sets will imply these special cases.

As mentioned above, the present work does not consider algorithmic aspects, apart from one exception. In Section 4.3, we shall provide an algorithmic approach to compute the dual optimum in Theorem 1.1, but even that algorithm can work only if a primal optimal solution is already available. But constructing a strongly polynomial algorithm for computing the primal optimum (that is, a Φ-minimizer element of an M-convex set) already in the special case of weighted square-some (when $\Phi(z) := \sum[c(s)w(s)^2 : s \in S]$, each $c(s)$ is positive) remains a major research problem. In the more general Theorem 1.2, the even more special case when $\Phi(w) = w^2$ is wide open from an algorithmic point of view.

2 Box-TDI systems and polyhedra

In what follows, Q is an integral matrix and p is an integral vector. Throughout we assume that there is a one-to-one correspondence between the columns of Q and the elements of ground-set S.

Edmonds and Giles [9, 10] called a (rational) linear system $Qx \geq p$ **totally dual integral** (**TDI**) if the maximum in the linear programming duality equation

$$\min\{cx : Qx \geq p\} \ = \ \max\{yp : y \geq 0, yQ = c\} \tag{2.1}$$

has an integral optimal solution y for every integral vector c for which the maximum is finite. They called the system $Qx \geq p$ **box total dual integral** (**box-TDI**) if the system $\{Qx \geq p, f \leq x \leq g\}$ is TDI for every choice of rational (finite-valued) bounding functions $f \leq g$. This definition can be extended to linear systems including equations, as follows. A linear system $\{Q_1x \geq p_1, Q_2x = p_2\}$ is **box-TDI** if the system $\{Q_1x \geq p_1, Q_2x = p_2, f \leq x \leq g\}$ is TDI for every choice of rational (finite-valued) bounding functions $f \leq g$.

A polyhedron is called a **box-TDI polyhedron** if it can be described by a box-TDI system. Edmonds and Giles proved basic properties of box-TDI systems, while the paper of Cook [5] includes further important results on box-TDI polyhedra. For a rich overview of the topic, see the book of Schrijver [25] and the recent paper of Chervet, Grappe, and Robert [3].

Our goal is to show that a result analogous to Theorem 1.1 holds for the set $\ddot{R}$ of integral elements of a box-TDI polyhedron R. An important special case is when Q is a TU (totally unimodular) matrix. This includes the special case of L-convex or $L^\natural$-convex sets. It can be proved that $L_2^\natural$-convex (in particular, L_2-convex) sets are also discrete box-TDI sets. Another special case is the one of integral submodular flows, in particular, M_2-convex and $M_2^\natural$-convex sets.

2.1 Properties and operations

In this section, we collect some basic properties of box-TDI systems and polyhedra, which shall serve as useful tools for our later investigations.

Proposition 2.1. *The linear system $\{Q_1x \geq p_1,\ Q_2x = p_2\}$ is box-TDI if and only if the linear system $\{Q_1x \geq p_1,\ Q_2x \geq p_2,\ -Q_2x \geq -p_2\}$ is box-TDI.*

Proof. Let c be an integral cost function (for the primal programs) and f, g rational bounding functions with $f \leq g$. Consider the following two linear programs which are the duals of the problem to minimize cx under the respective constraints:

$$\max\{y_1p_1 + y_2p_2 + sf - tg : y_1Q_1 + y_2Q_2 + s - t = c,\ (y_1, s, t) \geq 0\}, \tag{2.2}$$

$$\max\{y_1p_1 + y_2'p_2 - y_2''p_2 + sf - tg : y_1Q_1 + y_2'Q_2 - y_2''Q_2 + s - t = c,\ (y_1, y_2', y_2'', s, t) \geq 0\}. \tag{2.3}$$

Note that program (2.3) has an optimal solution in which

$$\text{at least one of } y_2'(i) \text{ and } y_2''(i) \text{ is zero for each index } i, \tag{2.4}$$

since if i were a bad index in the sense that both values are positive, then reducing both values by the smaller of them would result in another optimal solution in which the number of bad indices is smaller. Therefore we may restrict ourselves to solutions of (2.3) meeting (2.4). We call such solution 'restricted.'

Claim 2.2. *There is a one-to one correspondence between the integral optimal solutions to* (2.2) *and the restricted integral optimal solutions to* (2.3).

Proof. If (y_1, y_2, s, t) is an integral feasible solution to (2.2), then (y_1, y_2', y_2'', s, t) is a restricted integral feasible solution to (2.3), where

$$y_2'(i) := \begin{cases} y_2(i) & \text{if } \ y_2(i) \geq 0, \\ 0 & \text{if } \ y_2(i) < 0, \end{cases}$$

$$y_2''(i) := \begin{cases} -y_2(i) & \text{if } \ y_2(i) \leq 0, \\ 0 & \text{if } \ y_2(i) > 0. \end{cases}$$

Clearly,

$$y_1p_1 + y_2p_2 + sf - tg = y_1p_1 + y_2'p_2 - y_2''p_2 + sf - tg, \tag{2.5}$$

implying that if (y_1, y_2, s, t) is an integral feasible solution to (2.2), then (y_1, y_2', y_2'', s, t) is a restricted integral feasible solution to (2.3) admitting the same objective value

If (y_1, y_2', y_2'', s, t) is a restricted integral feasible solution to (2.3), then (y_1, y_2, s, t) is an integral feasible solution to (2.2), where $y_2 := y_2' - y_2''$, for which (2.5) holds.

These correspondences imply that (y_1, y_2, s, t) is optimal if and only if (y_1, y_2', y_2'', s, t) is optimal, and the claim follows. ■

The one-to-one correspondence ensured by the claim implies the proposition. ■■

Proposition 2.3 ([25] Theorem 22.7)**.** *A box-TDI system is TDI.* ■

Proposition 2.4 ([5])**.** *Any TDI linear system defining a box-TDI polyhedron is box-TDI.* ■

Let $Qx \geq p$ be a box-TDI system and let $R := \{x : Qx \geq p\}$. For technical simplicity, we formulate the next propositions only for this form but emphasize that, due to Proposition 2.1, each proposition below extends to the case when the system is given in the more general form $\{Q_1x \geq p_1, Q_2x = p_2\}$.

Proposition 2.5. *For an element $z^* \in \overset{....}{R}$, let $p_0 := p - Qz^*$. Then the system $Qx \geq p_0$ is box-TDI.* ■

Proposition 2.6. *If Q' is a matrix obtained from Q by negating some columns of Q, then the system $Q'x' \geq p$ is also box-TDI.*

Proof. It suffices to prove the special case when we negate the first column of Q. Let Q' denote the matrix arising in this way. Let $f' \leq g'$ be rational bounding vectors and c' an integer cost function. We have to show that the dual program

$$\max\{yp + f'u - g'v : yQ' + u - v = c', (y, u, v) \geq 0\} \tag{2.6}$$

has an integral optimal solution $(y, u, v) \geq 0$. Let c denote the vector obtained from c' by negating its first component. Let f denote the vector obtained from f' by replacing its first component $f'(1)$ to $-g'(1)$, and let g denote the vector obtained from g' by replacing its first component $g'(1)$ to $-f'(1)$. Then $yQ' + u' - v' = c'$ if and only if $yQ + u - v = c$, where (u', v') arises from (u, v) by interchanging their first components. Furthermore, $yp + f'u' - g'v' = yp + fu - gv$. By the box-TDI-ness of the system $Qx \geq p$, there is an integer-valued optimal solution (y, u, v) to

$$\max\{yp + fu - gv : yQ + u - v = c, (y, u, v) \geq 0\} \tag{2.7}$$

and hence (y, u', v') is an integer-valued optimal solution to (2.6). ■

Proposition 2.7 ([25] p. 323)**.** *The system obtained from a box-TDI system $Qx \geq p$ by deleting some columns of Q is box-TDI.* ■

Proposition 2.8 ([10])**.** *If Q' is a matrix obtained from Q by duplicating some columns of Q, then the system $Q'x' \geq p$ is also box-TDI.* ■

Proposition 2.9 ([25] p. 323)**.** *The projection of a box-TDI polyhedron along a coordinate axis is box-TDI.* ■

Proposition 2.10. *Let $Qx \geq p$ be a (box-) TDI system defining polyhedron $R := \{x : Qx \geq p\}$. Let $qx \geq \beta$ be an inequality which is superfluous in the sense that every member x of R satisfies $qx \geq \beta$. Then the system $\{Qx \geq p, qx \geq \beta\}$ is also (box-) TDI.*

Proof. Let c be an integral cost function and let y_0 be an integral dual optimum ensured by the TDI-ness of $Qx \geq p$. Since by adding a superfluous inequality to a linear system does not change the primal optimum value, the dual optimum value does not change either. Therefore, by extending y_0 by a new zero-valued dual component corresponding to the primal inequality $qx \geq \beta$, we obtain an integral dual solution $(y_0, 0)$ to the dual of the primal problem $\min\{cx : Qx \geq p, qx \geq \beta\}$.

The statement for box-TDI-ness follows from the first part since if $qx \leq \beta$ is superfluous with respect to the system $Qx \geq p$, then it is superfluous, as well, for the system $\{Qx \geq p, f \leq x \leq g\}$ for any pair of bounding functions $f \leq g$. ■

Proposition 2.11. *Let $Qx \geq p$ be a box-TDI system. Let $f' : S \to \mathbf{Q} \cup \{-\infty\}$ and $g' : S \to \mathbf{Q} \cup \{+\infty\}$ be rational bounding vectors with $f' \leq g'$. Then $\{Qx \geq p, f' \leq x \leq g'\}$ is also box-TDI, and (hence) TDI.*

Proof. We have to show for any choice $f : S \to \mathbf{Q}$ and $g : S \to \mathbf{Q}$ of finite-valued rational bounds that the system

$$\{Qx \geq p, f' \leq x \leq g', f \leq x \leq g\} \tag{2.8}$$

is TDI. Let f_0 be the componentwise maximum of f and f', and let g_0 be the componentwise minimum of g and g'. Then f_0 and g_0 are finite-valued and hence the system

$$\{Qx \geq p, f_0 \leq x \leq g_0\} \tag{2.9}$$

is box-TDI. By Proposition 2.3, the system in (2.9) is TDI. Since the system in (2.8) arises from the system in (2.9) by adding superfluous inequalities, Proposition 2.10 implies that (2.8) is indeed TDI, as required. ■.

Proposition 2.12. *Let $Qx \geq p$ be a box-TDI system defining the box-TDI polyhedron $R := \{x : Qx \geq p\}$, let $z^* \in \ddddot{R}$, and $p_0 := p - Qz^*$. Then the system*

$$\{(x_1, x_2) \geq 0,\ Qx_2 - Qx_1 \geq p_0\} \tag{2.10}$$

is box-TDI.

Proof. By Proposition 2.5, the system $Qx \geq p_0$ is box-TDI. By Proposition 2.8, $Qx_2 + Qx_1 \geq p_0$ is box-TDI. By applying Proposition 2.6 to the matrix (Q, Q), we get that $Qx_2 - Qx_1 \geq p_0$ is box-TDI. And finally, by Proposition 2.11, the system $\{(x_1, x_2) \geq 0,\ Qx_2 - Qx_1 \geq p_0\}$ is box-TDI. ■

2.2 The main tool

The following result is the main tool in proving the min-max theorem in Section 3.

Theorem 2.13. *Let Q be an integral matrix, p an integral vector, and suppose that the linear system $Qx \geq p$ is box-TDI. Let z^* be an integral element of polyhedron $R := \{x : Qx \geq p\} \subseteq \mathbf{R}^S$, and let $\ell : S \to \mathbf{Z} \cup \{-\infty\}$ and $u : S \to \mathbf{Z} \cup \{+\infty\}$ be integer-valued bounding vectors on S for which $\ell \leq u$. There exists an integer-valued non-negative vector y^* such that $\ell \leq y^*Q \leq u$ and $y^*(Qz^* - p) = 0$ if and only if*

$$\widetilde{\ell}(S^-) \leq \widetilde{u}(S^+) \tag{2.11}$$

holds for every pair $\{S^-, S^+\}$ of disjoint subsets of S for which

$$z' := z^* + \chi_{S^+} - \chi_{S^-} \in R. \tag{2.12}$$

Proof. Necessity of (2.11). Let y^* be a function meeting the requirements, $w^* := y^*Q$, and let $\{S^-, S^+\}$ be a pair meeting (2.12). Then $y^*(Qz^* - p) = 0$ implies that z^* is w^*-minimizer of R, and hence

$$w^*z^* \leq w^*z' = w^*z^* + \widetilde{w}^*(S^+) - \widetilde{w}^*(S^-) \leq w^*z^* + \widetilde{u}(S^+) - \widetilde{\ell}(S^-),$$

from which (2.11) follows. (Note that $\widetilde{u}(S^+) = +\infty$ and $\widetilde{\ell}(S^-) = -\infty$ may occur.)

Sufficiency of (2.11). Let $p_0 := p - Qz^*$. By the linear programming duality theorem, we have

$$\min\{ux_2 - \ell x_1 : (x_1, x_2) \geq 0,\ Qx_2 - Qx_1 \geq p_0\} \tag{2.13}$$

$$= \max\{yp_0 : y \geq 0,\ yQ \leq u,\ y(-Q) \leq -\ell\}. \tag{2.14}$$

Formally, this is correct only if both u and ℓ are finite-valued. To get the right pair of dual programs for the general case, one must remove the columns of Q corresponding to elements s with $u(s) = +\infty$ and remove the columns of $-Q$ corresponding to elements s with $\ell(s) = -\infty$. But in order to avoid notational difficulties, with this remark in mind, we work with the dual linear programs (2.13) and (2.14).

By Proposition 2.12, the linear system in (2.13) is box-TDI. Let M denote the common optimum value of the primal and the dual programs. Since $y \geq 0$ and $p_0 \leq 0$, we have $M \leq 0$.

Claim 2.14. $M = 0$.

Proof. Suppose indirectly that $M < 0$. Then there is a solution (x'_1, x'_2) to (2.13) for which $ux'_2 - \ell x'_1 = M < 0$. By the definition of p_0, the primal constraint $Qx'_2 - Qx'_1 \geq p_0$ is equivalent to $z' := z^* + x'_2 - x'_1 \in R$. Since both z^* and z' are in R, the line segment connecting z^* and z' also lies in R, that is, for any ε with $0 \leq \varepsilon \leq 1$, the vector $z^* + \varepsilon(x'_2 - x'_1)$ belongs to R, or equivalently $\varepsilon(Qx'_2 - Qx'_1) \geq p_0$. We can choose ε in such a way that $0 < \varepsilon \leq 1$,

$$x''_1(s) := \varepsilon x'_1(s) \leq 1 \ \text{ and } \ x''_2(s) := \varepsilon x'_2(s) \leq 1 \ \text{ for every } \ s \in S. \tag{2.15}$$

Clearly, $Qx''_2 - Qx''_1 \geq p_0$ and

$$ux''_2 - \ell x''_1 = \varepsilon(ux'_2 - \ell x'_1) = \varepsilon M < 0. \tag{2.16}$$

These imply that the linear system $\{(x_1, x_2) \geq 0, Qx_2 - Qx_1 \geq p_0\}$ in (2.13) has a solution meeting (2.15) and (2.16). The box total dual integrality of the linear system in (2.13) implies that there is $\{0, 1\}$-valued solution (x^*_1, x^*_2) for which $M^* := ux^*_2 - \ell x^*_1 < 0$.

Furthermore, we can also assume that no element $s \in S$ exists with $x^*_1(s) = 1 = x^*_2(s)$ since in this case we could reduce both values by 1, and then $\ell(s) \leq u(s)$ would imply for the revised (x^*_1, x^*_2) that $ux^*_2 - \ell x^*_1 = M^* - u(s) + \ell(s) \leq M^* < 0$.

Let $S^+ := \{s \in S : x^*_2(s) > 0\}$ and $S^- := \{s \in S : x^*_1(s) > 0\}$. Then S^+ and S^- are disjoint for which $\widetilde{u}(S^+) = ux^*_2 < \ell x^*_1 = \widetilde{\ell}(S^-)$, contradicting (2.11). ■

As $M = 0$, the box-TDI-ness of the linear system in (2.13) implies that the dual problem in (2.14) has an integer-valued solution y^* for which $y^* p_0 = M = 0$, that is, $y^*(Qz^* - p) = 0$. Furthermore $\ell \leq w^* \leq u$ holds for $w^* := y^* Q$, as required. ■■

Corollary 2.15. *Let Q, p, R, ℓ, u, and z^* be the same as in Theorem* 2.13. *There exists an integer-valued cost function w^* on S for which $\ell \leq w^* \leq u$ and z^* is a w^*-minimizer of R if and only if* (2.11) *holds for every pair $\{S^-, S^+\}$ of disjoint subsets of S meeting* (2.12).

Proof. The corollary follows immediately from Theorem 2.13 once we make the standard observation from linear programming that a primal solution z^* is a w^*-minimizer of R if and only if there is a dual solution y^* meeting the optimality criteria, that is, $y^* \geq 0$, $y^* Q = w^*$, and $y^*(Qz^* - p) = 0$. ■

3 Min-max theorem for Φ

3.1 Preparation

Let $\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ be an arbitrary integer-valued function on $\mathbf{Z}$ allowing the $+\infty$ value. We say that an ordered pair $\{k^*, \ell^*\}$ of integers is **φ-fitting** if

$$\varphi(k^*) - \varphi(k^* - 1) \leq \ell^* \leq \varphi(k^* + 1) - \varphi(k^*) \tag{3.1}$$

or more concisely

$$\varphi'(k^* - 1) \leq \ell^* \leq \varphi'(k^*). \tag{3.2}$$

Let Φ be a separable function on $\mathbf{Z}^S$ defined by univariate integer-valued functions φ_s ($s \in S$). We say that an ordered pair $\{z^*, w^*\}$ of vectors from $\mathbf{Z}^S$ is **Φ-fitting** if $\{z^*(s), w^*(s)\}$ is φ_s-fitting for each $s \in S$, that is,

$$\varphi_s(z^*(s)) - \varphi_s(z^*(s) - 1) \leq w^*(s) \leq \varphi_s(z^*(s) + 1) - \varphi_s(z^*(s)) \ \text{ for every } \ s \in S, \tag{3.3}$$

which can concisely be written as follows:

$$\Phi'(z^* - \underline{\mathbf{1}}) \ \leq \ w^* \ \leq \ \Phi'(z^*). \tag{3.4}$$

As a preparation, we need the following proposition.

Proposition 3.1. *Let φ be an integer-valued discrete convex function and let $\{k^*, \ell^*\}$ be a φ-fitting pair of integers. Then*

$$\ell^* k^* - \varphi(k^*) \geq \ell^* k - \varphi(k) \quad \text{for every integer } \ k \tag{3.5}$$

(or equivalently $\varphi^\bullet(\ell^*) = \ell^* k^* - \varphi(k^*)$ where $\varphi^\bullet$ denotes the discrete conjugate of φ).

Proof. Suppose indirectly that there is an integer k_0 for which

$$\ell^* k^* - \varphi(k^*) < \ell^* k_0 - \varphi(k_0). \tag{3.6}$$

If $k_0 > k^*$, we may assume that k_0 is minimal, and hence

$$\ell^* k^* - \varphi(k^*) \geq \ell^*(k_0 - 1) - \varphi(k_0 - 1). \tag{3.7}$$

By subtracting (3.7) from (3.6), we get $0 < \ell^* - (\varphi(k_0) - \varphi(k_0 - 1))$. This and the convexity of φ imply that $\ell^* > \varphi(k_0) - \varphi(k_0 - 1) \geq \varphi(k^* + 1) - \varphi(k^*)$, in contradiction to the second inequality in (3.1).

Analogously, if $k_0 < k^*$, we may assume that k_0 is maximal, and hence

$$\ell^* k^* - \varphi(k^*) \geq \ell^*(k_0 + 1) - \varphi(k_0 + 1). \tag{3.8}$$

By subtracting (3.6) from (3.8), we get $0 > \ell^* - (\varphi(k_0 + 1) - \varphi(k_0))$. This and the convexity of φ imply that $\ell^* < \varphi(k_0 + 1) - \varphi(k_0) \leq \varphi(k^*) - \varphi(k^* - 1)$, in contradiction to the first inequality in (3.1). ■

Remark 3.1. There is a standard concept and terminology in (discrete) convex analysis that is equivalent in the present case to φ-fitting. Namely, Condition (3.2) is equivalent to saying that ℓ^* is a subgradient of φ at k^*, which is usually denoted as $\ell^* \in \partial\varphi(k^*)$. Therefore, $\{k^*, \ell^*\}$ is φ-fitting if and only if $\ell^* \in \partial\varphi(k^*)$. Proposition 3.1 is a restatement of the well-known fact that $\varphi(k^*) + \varphi^\bullet(\ell^*) = k^* \ell^*$ holds if and only if $\ell^* \in \partial\varphi(k^*)$. •

3.2 Main results

Let $R \subseteq \mathbf{R}^S$ be an arbitrary integral polyhedron and z^* an element of $\ddddot{R}$. Let φ_s be an integer-valued discrete convex function on $\mathbf{Z}$ for each $s \in S$ and let Φ denote the separable discrete convex function defined in (1.3) by the univariate functions φ_s $(s \in S)$.

Let y^* be a vector whose components correspond to the rows of Q. We say that the ordered pair $\{z^*, y^*\}$ of integral vectors is **Φ-compatible** with respect to Q (or, shortly Φ-compatible) if $\{z^*, w^*\}$ is Φ-fitting where $w^* := y^*Q$, that is,

$$\Phi'(z^* - \underline{\mathbf{1}}) \;\leq\; y^*Q \;\leq\; \Phi'(z^*). \tag{3.9}$$

Remark 3.2. In the special case when Φ is a linear function, that is, $\Phi(z) = cz$ for a given vector $c \in \mathbf{Z}^S$, one has $\Phi'(z) = c$ for every $z \in \mathbf{Z}^S$. Therefore, in this case, Φ-compatibility given in (3.9) is equivalent to $c \leq y^*Q \leq c$, that is, $c = y^*Q$. •

Lemma 3.2. *Let Φ be an integer-valued separable discrete convex function on $\mathbf{Z}^S$. Suppose for $z^* \in \ddddot{R}$ and $y^* \geq 0$ that the pair $\{z^*, y^*\}$ is Φ-compatible. Then*

$$\Phi(z) \geq \Phi(z^*) - y^*(Qz^* - p) \tag{3.10}$$

holds for every $z \in \ddddot{R}$.

Proof. Let $w^* := y^*Q$. Since $\{z^*, y^*\}$ is Φ-compatible, $\{z^*, w^*\}$ is a Φ-fitting pair, and we can apply Proposition 3.1 to $\varphi := \varphi_s$, $k^* := z^*(s)$, $\ell^* := w^*(s)$, and $k := z(s)$:

$$w^*(s)z^*(s) - \varphi_s(z^*(s)) \geq w^*(s)z(s) - \varphi_s(z(s)), \tag{3.11}$$

that is,

$$\varphi_s(z(s)) \geq w^*(s)z(s) - [w^*(s)z^*(s) - \varphi_s(z^*(s))]. \tag{3.12}$$

Since

$$\sum_{s \in S} w^*(s)z(s) = w^*z = (y^*Q)z = y^*(Qz) \geq y^*p$$

and $\{z^*, w^*\}$ is a Φ-fitting pair, we have

$$\begin{aligned}
\Phi(z) &= \sum_{s \in S} \varphi_s(z(s)) \\
&\geq \sum_{s \in S} w^*(s)z(s) - \Big[\sum_{s \in S} w^*(s)z^*(s) - \sum_{s \in S} \varphi_s(z^*(s))\Big] \\
&\geq y^*p - \Big[\sum_{s \in S} w^*(s)z^*(s) - \sum_{s \in S} \varphi_s(z^*(s))\Big] \\
&= y^*p - [(y^*Q)z^* - \Phi(z^*)] = \Phi(z^*) - y^*(Qz^* - p),
\end{aligned}$$

as required. ∎

The new min-max theorem for the case when R is an integral box-TDI polyhedron is as follows.

Theorem 3.3. *Let $\varphi_s : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ be an integer-valued discrete convex function on $\mathbf{Z}$ for each $s \in S$ and let Φ denote the separable discrete convex function defined by the univariate functions φ_s $(s \in S)$. Suppose for an integral matrix Q and integral vector p that $Qx \geq p$ is a box-TDI system defining a non-empty integral (box-TDI) polyhedron $R := \{x : Qx \geq p\} \subseteq \mathbf{R}^S$ such that Φ is finite-valued on $\overset{....}{R}$. Then Φ is bounded from below on $\overset{....}{R}$ if and only if there exists an element $z \in \overset{....}{R}$ and an integral vector $y \geq 0$ for which $\{z, y\}$ is Φ-compatible with respect to Q. Moreover, if Φ is bounded from below on $\overset{....}{R}$, then the following min-max formula holds:*

$$\min\{\Phi(z) : z \in \overset{....}{R}\} \tag{3.13}$$

$$= \max\{\Phi(z) - y(Qz - p) : \ z \in \overset{....}{R}, \ y \geq 0 \ \text{ integer-valued}, \ \{z, y\} \ \Phi\text{-compatible}\}. \tag{3.14}$$

In addition, an optimal vector y^* in (3.14) can be chosen in such a way that the number of its positive components is at most $2|S| - 1$.

Proof. Suppose first that there is a requested Φ-compatible pair $\{z^*, y^*\}$. Then Lemma 3.2 implies that Φ is bounded from below and that min $\geq$ max.

Suppose now that Φ is bounded from below on $\overset{....}{R}$. Since Φ is integer-valued, $\overset{....}{R}$ has a Φ-minimizer element z^*. We are going to show that there is an integer-valued vector $y^* \geq 0$ for which the following optimality criteria hold:

$$y^*(Qz^* - p) = 0, \tag{3.15}$$

$$\Phi'(z^* - \underline{\mathbf{1}}) \leq y^* Q \leq \Phi'(z^*). \tag{3.16}$$

This will imply that a Φ-compatible pair in question indeed exists for which equality holds in (3.13) and (3.14). Define bounding vectors ℓ and u on S, as follows. For $s \in S$, let

$$\ell(s) := \varphi'_s(z^*(s) - 1) \quad \text{ and } \quad u(s) := \varphi'_s(z^*(s)),$$

where $\ell(s)$ may be $-\infty$ and $u(s)$ may be $+\infty$. The discrete convexity of φ_s implies that $\ell(s) \leq u(s)$. Note that $\{z^*, y^*\}$ is Φ-compatible precisely if $\ell \leq y^* Q \leq u$.

Claim 3.4. *The inequality $\widetilde{\ell}(S^-) \leq \widetilde{u}(S^+)$ in (2.11) holds for every pair of disjoint subsets S^+, S^- of S for which $z' := z^* + \chi_{S^+} - \chi_{S^-} \in R$.*

Proof. As z^* is a Φ-minimizer, we have $\Phi(z^*) \leq \Phi(z')$. Furthermore

$$\begin{aligned}
\Phi(z') &= \sum_{s \in S} \varphi_s(z'(s)) \\
&= \sum_{s \in S - (S^+ \cup S^-)} \varphi_s(z^*(s)) + \sum_{s \in S^+} \varphi_s(z^*(s) + 1) + \sum_{s \in S^-} \varphi_s(z^*(s) - 1) \\
&= \sum_{s \in S} \varphi_s(z^*(s)) + \sum_{s \in S^+} [\varphi_s(z^*(s) + 1) - \varphi_s(z^*(s))] - \sum_{s \in S^-} [\varphi_s(z^*(s)) - \varphi_s(z^*(s) - 1)] \\
&= \Phi(z^*) + \sum_{s \in S^+} \varphi'_s(z^*(s)) - \sum_{s \in S^-} \varphi'_s(z^*(s) - 1) \\
&= \Phi(z^*) + \widetilde{u}(S^+) - \widetilde{\ell}(S^-) \\
&\leq \Phi(z') + \widetilde{u}(S^+) - \widetilde{\ell}(S^-),
\end{aligned}$$

from which $\widetilde{\ell}(S^-) \le \widetilde{u}(S^+)$, as required. ■

Theorem 2.13 implies the existence of the requested y^*. The last statement about the number of positive components is a consequence of a theorem of Cook, Fonlupt, and Schrijver [6] (see also Theorem 5.30 in the book of Schrijver [26]). ■■

Remark 3.3. Note that the dual objective function in (3.14) can be rewritten, as follows:

$$\Phi(z) - y(Qz - p) = yp - [(yQ)z) - \Phi(z)]. \tag{3.17}$$

Furthermore, the characterization of boundedness in Theorem 3.3 can be interpreted as a special case of the min-max formula when the minimum in (3.13) is $-\infty$ and the maximum in (3.14), when taken over the empty set, is defined to be $-\infty$. Therefore, in the variations and applications of Theorem 3.3 below, we shall not explicitly formulate the condition for the lower boundedness of Φ. •

Remark 3.4. At first sight, this min-max theorem looks a bit strange in the sense that in the maximization part, not only the usual dual variable y appears but integral members z of the primal polyhedron R also show up. Still, this form may be viewed as a proper min-max theorem since the right-hand side is a straightforward lower bound for the minimum, and for given z^* and y^*, the validity of optimality criteria (3.15) and (3.16) is easily checkable. •

Remark 3.5. In the special case when $\Phi(z) = cz$, the compatibility of z and y, as observed in Remark 3.2, is equivalent to $yQ = c$. Furthermore, the dual objective function in (3.14) is as follows:

$$\Phi(z) - y(Qz - p) = yp - [(yQ)z) - \Phi(z) = yp - [cz - cz] = yp,$$

showing that in this case we are back at the integral version of the linear programming duality theorem formulated for box-TDI polyhedra. •

It is useful to formulate separately the optimality criteria appearing in (3.15) and (3.16).

Corollary 3.5 (Optimality criteria)**.** *An element $z^* \in \ddot{R}$ is a Φ-minimizer if and only if there exists a non-negative integer-valued vector y^* meeting the optimality criteria in* (3.15) *and* (3.16). ■

3.3 Using discrete conjugate

The min-max formula for the minimum of Φ can be described in a more concise way in term of discrete conjugates. To this end, we need some easy observations. In Proposition 3.1, we proved for a univariate discrete convex function φ that if $\{k^*, \ell^*\}$ is a φ-fitting pair of integers, then $\varphi^\bullet(\ell^*) = \ell^* k^* - \varphi(k^*)$. The reverse implication holds for an arbitrary integer-valued function φ on $\mathbf{Z}$.

Proposition 3.6. *Let φ be an arbitrary integer-valued function on* $\mathbf{Z}$, *and k^*, ℓ^* integers for which $\varphi^\bullet(\ell^*) = \ell^* k^* - \varphi(k^*)$. Then the pair $\{k^*, \ell^*\}$ is φ-fitting,*

Proof. The definition of $\varphi^\bullet$ implies that

$$\ell^* k^* - \varphi(k^*) = \varphi^\bullet(\ell^*) \geq \ell^*(k^* + 1) - \varphi(k^* + 1),$$

from which $\varphi(k^* + 1) - \varphi(k^*) \geq \ell^*$. Analogously, we have

$$\ell^* k^* - \varphi(k^*) = \varphi^\bullet(\ell^*) \geq \ell^*(k^* - 1) - \varphi(k^* - 1),$$

from which $\ell^* \geq \varphi(k^*) - \varphi(k^* - 1)$. ■

Proposition 3.6 results in the following estimation (that may be viewed as a discrete counterpart of a standard lower bound in continuous optimization.)

Proposition 3.7. *Let $R = \{x : Qx \geq p\} \subseteq \mathbf{R}^S$ be an integral polyhedron and φ_s an arbitrary integer-valued function on $\mathbf{Z}$ for each $s \in S$. Let $\varphi_s^\bullet$ denote the discrete conjugate of φ_s. For any element z of $\overset{....}{R}$ and for any integer-valued vector $y \geq 0$ (whose components correspond to the rows of Q) one has:*

$$\Phi(z) \geq yp - \Phi^\bullet(yQ). \tag{3.18}$$

If equality holds for z^* and y^*, then z^* is a Φ-minimizer of $\overset{....}{R}$ and the pair $\{z^*, y^*\}$ is Φ-compatible.

Proof. Let $w := yQ$. By the definition of discrete conjugate, we have $\varphi_s^\bullet(w(s)) + \varphi_s(z(s)) \geq w(s)z(s)$ from which

$$\Phi(z) = \sum_{s \in S} \varphi_s(z(s)) = wz - \Big[\sum_{s \in S} w(s)z(s) - \sum_{s \in S} \varphi_s(z(s))\Big] \geq yp - \Phi^\bullet(yQ). \tag{3.19}$$

To see the second part, observe that (3.18) implies that $\Phi(z) \geq y^* p - \Phi^\bullet(y^* Q) = \Phi(z^*)$, showing that z^* is indeed a Φ-minimizer element of $\overset{....}{R}$. Since we have equality in (3.19) for z^* and y^*, it follows for each $s \in S$ that $w^*(s)z^*(s) - \varphi_s(z^*(s)) = \varphi_s^\bullet(w^*(s))$ where $w^* := y^* Q$. By applying Proposition 3.6 to $\varphi := \varphi_s$, $\ell^* := w^*(s)$, and $k^* := z^*(s)$, we obtain that

$$\varphi(k^*) - \varphi(k^* - 1) \leq \ell^* \leq \varphi(k^* + 1) - \varphi(k^*), \tag{3.20}$$

that is,

$$\varphi_s(z^*(s)) - \varphi_s(z^*(s) - 1) \leq w^*(s) \leq \varphi_s(z^*(s) + 1) - \varphi_s(z^*(s)), \tag{3.21}$$

and hence the pair $\{z^*, w^*\}$ is Φ-fitting, showing that $\{z^*, y^*\}$ is Φ-compatible. ■

Theorem 3.8. *Under the same assumptions as in Theorem 3.3, one has the following min-max formula:*

$$\min\{\Phi(z) : z \in \overset{....}{R}\} = \max\{yp - \Phi^\bullet(yQ) : y \geq 0 \ \text{ integer-valued}\}. \tag{3.22}$$

The optimal dual vector y can be can be chosen so as to have at most $2|S| - 1$ positive components.

Proof. Let $z^* \in \overset{....}{R}$ be a minimizer element to (3.13) and $y^* \geq 0$ a maximizer element to (3.14). Then $\{z^*, w^*\}$ is a Φ-fitting pair where $w^* := y^* Q$ and by Theorem 3.3 we have

$$\Phi(z^*) = y^* p - [(y^* Q)z^*) - \Phi(z^*)]. \tag{3.23}$$

For $s \in S$, consider φ_s and its discrete conjugate $\varphi_s^\bullet$. For $k^* := z^*(s)$ and $\ell^* := w^*(s)$, $\{k^*, \ell^*\}$ is a φ_s-fitting pair. Proposition 3.1, when applied to φ_s in place of φ, implies that $\ell^* k^* - \varphi_s(k^*) \geq \ell^* k - \varphi_s(k)$ holds for every integer k. Hence

$$\ell^* k^* - \varphi_s(k^*) = \max\{\ell^* k - \varphi_s(k) : k \in \mathbf{Z}\} = \varphi_s^\bullet(\ell^*), \tag{3.24}$$

from which $(y^* Q)z^* - \Phi(z^*) = y^* p - \Phi^\bullet(w^*) = \Phi^\bullet(y^* Q)$ follows. This, (3.23), and (3.18) (with (z^*, y^*) in place of (z, y)) imply:

$$\Phi(z^*) = y^* p - [(y^* Q)z^*) - \Phi(z^*)] = y^* p - \Phi^\bullet(y^* Q) \leq \Phi(z^*),$$

from which (3.22) follows. ■

Corollary 3.9. *Let z^* be Φ-minimizer element of $\overset{....}{R}$. If y^* is an optimal solution to* (3.14)*, then y^* is an optimal solution to* (3.22)*. If y^* is an optimal solution to* (3.22)*, then the pair $\{z^*, y^*\}$ is Φ-compatible and it is an optimal solution to* (3.14)*.*

Proof. The first part is an immediate consequence of the proof of Theorem 3.8. The second part follows from Theorem 3.8 and the second half of Proposition 3.7. ■

Remark 3.6. In the special case when Φ is linear and defined by $\Phi(w) = cw$, one can easily observe that $\Phi^\bullet(w) = 0$ when $w = c$ and $\Phi^\bullet(w)$ has a $+\infty$ summand when $w \neq c$. Therefore $\Phi^\bullet(yQ)$ in (3.22) is finite only if $yQ = c$ and in this case $\Phi^\bullet(yQ) = 0$. This means that the maximum in (3.22) is equal to $\max\{yp : yQ = c, y \geq 0\}$, showing that Theorem 3.8 also specializes to the integral version of the linear programming duality theorem formulated for box-TDI polyhedra. •

The results above can be extended to the case when R is defined by a box-TDI system $\{Q'x \geq p',\ Q^= x = p^=\}$ because Proposition 2.1 implies that the system $\{Q'x \geq p', Q^= x \geq p^=, -Q^= x \geq -p^=\}$ is also box-TDI and defines the same polyhedron R. We call a dual vector $y = (y', y^=)$ **sign-feasible** if $y' \geq 0$. That is, we require non-negativity of those components that correspond to the rows of Q'.

Theorem 3.10. *Suppose that in Theorem* 3.3 *the box-TDI polyhedron is given in form $R = \{x : Q'x \geq p', Q^= x = p^=\}$, where each of Q', $Q^=$, p', $p^=$ is integer-valued. Then*

$$\begin{aligned}
&\min\{\Phi(z) : z \in \overset{....}{R}\}\\
&= \max\{\Phi(z) - y(Qz - p) :\ z \in \overset{....}{R},\ y \text{ sign-feasible and integer-valued, } \{z, y\}\ \Phi\text{-compatible}\}\\
&= \max\{yp - \Phi^\bullet(yQ) : y \text{ sign-feasible and integer-valued}\},
\end{aligned} \tag{3.25}$$

where $Q = \begin{pmatrix} Q' \\ Q^= \end{pmatrix}$ and $p = \begin{pmatrix} p' \\ p^= \end{pmatrix}$. An element $z^* \in \overset{....}{R}$ is a Φ-minimizer if and only if there exists a sign-feasible integer-valued vector y^* meeting the optimality criteria in (3.15) *and* (3.16)*. Moreover, y^* can be chosen in such a way that the number of its non-zero components is at most* $2|S| - 1$. ■

We formulate yet another variant for the maximum in the min-max theorem. This version is useful in cases when there is a simple formula for $\mu_R(w) := \{wx : x \in R\}$, see the next section on special box-TDI polyhedra.

Theorem 3.11. *Let $R = \{x : Q'x \geq p', Q^{=}x = p^{=}\}$ be a box-TDI polyhedron, where each of Q', $Q^{=}$, p', $p^{=}$ is integer-valued. Let Φ be an integer-valued separable discrete convex function which is bounded from below on R. Then*

$$\min\{\Phi(z) : z \in \overset{....}{R}\} = \max\{\mu_R(w) - \Phi^{\bullet}(w) : w \in \mathbf{Z}^S\}. \tag{3.26}$$

Proof. For $z \in \overset{....}{R}$ and $w \in \mathbf{Z}^S$, we have

$$\Phi(z) = wz - [wz - \Phi(z)] \geq \mu_R(w) - \Phi^{\bullet}(w), \tag{3.27}$$

from which min $\geq$ max follows.

To see the reverse direction, we show that there is an element z^* of $\overset{....}{R}$ and an integral vector w^* meeting (3.27) with equality. Let z^* be a Φ-minimizer of $\overset{....}{R}$, y^* a maximizer in (3.25), and let $w^* := y^*Q$. Then $\Phi(z^*) = py^* - \Phi^{\bullet}(y^*Q)$ holds by Theorem 3.10, and a straightforward estimation (the weak duality theorem of linear programming) shows that $\mu_R(w^*) \geq y^*p$. This and (3.27) (when applied to z^* and w^*) imply

$$\Phi(z^*) \geq \mu_R(w^*) - \Phi^{\bullet}(w^*) \geq y^*p - \Phi^{\bullet}(y^*Q) = \Phi(z^*), \tag{3.28}$$

from which equality follows throughout, and hence (3.26) holds indeed. ■

Remark 3.7. The maximization form in Theorem 3.11 has the advantage that it does not need explicitly the polyhedral description of R. For example, if $Qx \geq p$ is an explicitly given box-TDI system but we are interested in the polyhedron

$$R' := \{w : w = yQ,\ y \geq 0,\ f \leq w \leq g\},$$

then it is known from [3] that R' is box-TDI, and hence Theorem 3.11 can be applied, without the explicit knowledge of the polyhedral description of R'. Note that it may be the case that the size of Q is 'small' (for example, if Q is the signed incidence matrix of a digraph, when Q is actually totally unimodular) but R' can be described only with an exponential number of inequalities. •

Remark 3.8. Theorem 3.10 can be further extended to the formally more general framework where primal non-negativity constraints are written separately. In this case, the primal polyhedron R is defined by a box-TDI system as follows:

$$R := \{(x_1, x_2) : Q'x_1 + A'x_2 \geq p',\ Q^{=}x_1 + A^{=}x_2 = p^{=},\ x_2 \geq 0\}.$$

The min-max theorem for the minimum of Φ and the optimality criteria, though technically more complex, can also be described by applying Theorem 3.10. •

4 Special box-TDI polyhedra

In this section, we consider special box-TDI polyhedra.

4.1 Polyhedra defined by TU-matrices

It is known that if $Q = \begin{pmatrix} Q' \\ Q^= \end{pmatrix}$ is a totally unimodular (TU) matrix and $p = \begin{pmatrix} p' \\ p^= \end{pmatrix}$ is an integral vector, then the linear system $\{Q'x \geq p',\ Q^= x = p^=\}$ (and the polyhedron $\{x : Q'x \geq p',\ Q^= x = p^=\}$) is box-TDI. But a TU-matrix Q may define box-TDI polyhedra in other ways, as well.

Proposition 4.1. *Let Q be a totally unimodular matrix, and let $f \leq g$ be integer-valued bounding vectors (of appropriate dimension) where f may have $-\infty$ while g may have $+\infty$ components. Then the polyhedron*

$$R' := \{z : z = Qx \ \text{ for some } x \text{ meeting } \ f \leq x \leq g\}$$

is box-TDI. Analogously, if $\ell \leq u$ are integer-valued bounding vectors (of appropriate dimension), then the polyhedron

$$R'' := \{w : w = yQ \ \text{ for some } y \text{ meeting } \ \ell \leq y \leq u\}$$

is box-TDI.

Proof. Since the operation of adding a unit vector $(1, 0, \dots, 0)$ to Q as a new row or a new column preserves total unimodularity, the system $\{Qx - z = 0, f \leq x \leq g\}$ is box-TDI. But then R' is the projection of the polyhedron $\{(x, z) : Qx - z = 0,\ f \leq x \leq g\}$ along the coordinate axes of x (to the components of z), and since projection, by Proposition 2.9, preserves box-TDI-ness, R' is indeed box-TDI. The second part follows from the first one since the transpose of a TU matrix is also totally unimodular. ∎

It follows that Theorem 3.11 (for example) can be applied to the box-TDI polyhedra occurring in Proposition 4.1. A special case is the polyhedron of feasible flows defined on the edge-set of a digraph $D = (V, A)$ by $\{x \in \mathbf{R}^A : \varrho_x(v) - \delta_x(v) = m(v) \text{ for each } v \in V,\ f \leq x \leq g\}$, where $m : V \to \mathbf{Z}$ is a function on V with $\widetilde{m}(V) = 0$ while $f : A \to \mathbf{Z} \cup \{-\infty\}$ and $g : A \to \mathbf{Z} \cup \{+\infty\}$ are bounding functions on A with $f \leq g$. Here $\varrho_x(v) := \sum[x(uv) : uv \in A]$ and $\delta_x(v) := \sum[x(vu) : vu \in A]$. The classic notions of feasible st-flows with given flow-amount as well as feasible circulations fit into this framework. Another special case of TU-polyhedra is the one of feasible potentials.

A network matrix Q is a more general TU-matrix which is defined by a digraph whose underlying undirected graph is connected. For a spanning tree T of D, the rows of Q correspond to the elements of T, the columns correspond to the edges in $A - T$, and the column corresponding to e is the signed characteristic vector of the fundamental circuit belonging to e.

4.2 M-convex and M$_2$-convex sets

Let $B := B'(p) = \{\widetilde{x}(Z) \geq p(Z) \text{ for } Z \subset S \text{ and } \widetilde{x}(S) = p(S)\}$ be a base-polyhedron defined by an integral supermodular function p for which $p(S)$ is finite. Recall that the set $\overset{....}{B}$ of integral elements of B was called an M-convex set. A basic property of base-polyhedra is that they are box-TDI.

Recall that the linear extension (Lovász extension) $\hat{p}$ of p is defined by

$$\hat{p}(w) = p(S_n)w(s_n) + \sum_{j=1}^{n-1} p(S_j)[w(s_j) - w(s_{j+1})],$$

where $n = |S|$, the elements of S are indexed in such a way that $w(s_1) \geq \cdots \geq w(s_n)$, and $S_j = \{s_1, \dots, s_j\}$ for $j = 1, \dots, n$. (Here $p(S_j)[w(s_j) - w(s_{j+1})]$ is defined 0 when $w(s_j) - w(s_{j+1}) = 0$ even if $p(S_j)$ is not finite.)

Recall the definitions of z-tight sets and strict w-top sets. For supermodular function p, a theorem of Edmonds [7] is as follows.

Claim 4.2. *Let $p : 2^S \to \mathbf{Z} \cup \{-\infty\}$ be a supermodular function for which $p(S)$ is finite. For an integral cost-function w on S, one has*

$$\hat{p}(w) = \mu_B(w) = \mu_{\ddot{B}}(w),$$

where $\mu_B(w) := \min\{wx : x \in B\}$ and $\mu_{\ddot{B}}(w) := \min\{wx : x \in \ddot{B}\}$. In particular, $\hat{p}(w) = -\infty$ if and only if wz is unbounded from below over $\ddot{B}$. When $\hat{p}(w) > -\infty$, an element $z \in \ddot{B}$ is a w-minimizer if and only if each strict w-top set is z-tight. ■

By combining Claim 4.2 with Theorem 3.11, we arrive at the starting min-max formula described in Theorem 1.1.

Theorem 1.2 can also be derived in an analogous way. Let $B_1 := B'(p_1)$ and $B_2 := B'(p_2)$ be base-polyhedra defined by integer-valued supermodular functions p_1 and p_2 for which $B := B_1 \cap B_2$ is non-empty. A fundamental theorem of Edmonds states that B is box-TDI. A version of the well-known weight-splitting theorem ([11], Theorem 16.1.8) states for an integral vector w that

$$\mu_B(w) = \mu_{\ddot{B}}(w) = \max\{\hat{p}_1(w_1) + \hat{p}_2(w_2) : w_1 + w_2 = w,\ w_1, w_2 \text{ integral}\}.$$

Combining this formula with Theorem 3.11, we arrive at Theorem 1.2.

It should be noted that the intersection of two integral g-polymatroids is also box-TDI and so is a submodular flow polyhedron (by a theorem of Edmonds and Giles [9]). Therefore the general min-max formulas described in Theorem 3.8 can be specialized to these cases as well.

4.3 Direct proof for M-convex sets

The goal of this section is to provide a direct proof of the non-trivial part of Theorem 1.1. The proof is independent of the results in Section 3 and gives rise to a strongly polynomial algorithm to compute the optimal dual, provided that an optimal solution to the primal problem is available. Namely, we prove the following.

Theorem 4.3. *Let Φ be an integer-valued separable discrete convex function. Let z^* be a Φ-minimizer element of an M-convex set $\ddot{B}$ defined by a finite-valued supermodular function p. There exists an integer-valued vector $w^* \in \mathbf{Z}^S$ for which z^* and w^* meet the optimality criteria* (1.7) *and* (1.8) *(or* (1.9)*).*

Proof. For $s \in S$, let $T(s)$ denote the unique smallest z^*-tight set containing s.

Claim 4.4. *For a* Φ*-minimizer element* z^**,*

$$\varphi'_s(z^*(s) - 1) \leq \varphi'_t(z^*(t)) \tag{4.1}$$

holds whenever $t \in T(s)$*.*

Proof. If $t \in T(s) - s$, then $z' := z^* - \chi_s + \chi_t$ belongs to $\ddot{B}$. This implies that $\Phi(z^*) \leq \Phi(z')$ from which

$$\varphi_s(z^*(s)) + \varphi_t(z^*(t)) \leq \varphi_s(z^*(s) - 1) + \varphi_t(z^*(t) + 1),$$

that is,

$$\varphi_s(z^*(s)) - \varphi_s(z^*(s) - 1) \leq \varphi_t(z^*(t) + 1) - \varphi_t(z^*(t)), \tag{4.2}$$

which is exactly (4.1).

By the discrete convexity of φ_s, we have $\varphi_s(z^*(s)) - \varphi_s(z^*(s) - 1) \leq \varphi_s(z^*(s) + 1) - \varphi_s(z^*(s))$, implying (4.2) (and hence (4.1)) for the case $s = t$. ■

Our goal is to find an integer-valued w^* meeting the optimality criteria in the theorem. Define w^* as follows:

$$w^*(s) := \min\{\varphi'_t(z^*(t)) : t \in T(s)\}. \tag{4.3}$$

Claim 4.5. w^* *and* z^* *meet Optimality criterion* (1.8).

Proof. The definition of $w^*(s)$ in (4.3) and $s \in T(s)$ imply that $w^*(s) = \min\{\varphi'_t(z^*(t)) : t \in T(s)\} \leq \varphi'_s(z^*(s))$, from which $w^*(s) \leq \varphi'_s(z^*(s))$ follows. Furthermore, (4.3) and (4.1) imply that $w^*(s) = \min\{\varphi'_t(z^*(t)) : t \in T(s)\} \geq \varphi'_s(z^*(s) - 1)$. Hence (1.8) holds. ■

Claim 4.6. w^* *and* z^* *meet Optimality criterion* (1.7).

Proof. Let $\beta_1 > \beta_1 > \cdots > \beta_\ell$ denote the distinct values of the components of w^*, and let $C_i := \{s : w^*(s) \geq \beta_i\}$ for $i = 1, \cdots, \ell$. Let $S'_1 := C_1$ and $S'_i := C_i - C_{i-1}$ for $i = 2, \ldots, \ell$. Then $(\emptyset \neq)\ C_1 \subset C_2 \subset \cdots \subset C_\ell\ (= S)$ is a chain whose members are the strict w^*-top sets, while $\{S'_1, \ldots, S'_\ell\}$ is a partition of S for which $w^*(s) = \beta_i$ holds for every $s \in S'_i$.

For every $t \in T(s)$, we have $T(t) \subseteq T(s)$ and hence $w^*(t) \geq w^*(s)$, implying that $T(s) \subseteq C_i$ whenever $s \in S'_i$. Therefore $C_i = \cup\{T(s) : w^*(s) \geq \beta_i\}$ and hence each C_i is z^*-tight, showing that Optimality criterion (1.7) holds. ■

As z^* and w^* meet the optimality criteria, the proof of the theorem is complete. ■ ■

In order to compute w^*, we have to be able to determine the unique smallest z^*-tight set $T(s)$ containing an element $s \in S$. This is easy once we are able to decide for a given pair $\{s, t\}$ of elements of S whether there is a z^*-tight $s\bar{t}$-set. But this can be done by minimizing the submodular function $\widetilde{z^*} - p$ over the $s\bar{t}$-sets, which is doable in strongly polynomial time with the help of a general subroutine to minimize a submodular function.

Remark 4.1. We formulated and proved Theorem 4.3 for the special case when the defining supermodular function p is finite-valued. But the arguments above can easily be extended to the general case when p may have $-\infty$ values (but preserving the finiteness of $p(S)$), that is, $B'(p)$ may be unbounded. •

5 Special discrete convex functions

5.1 Minimizing the square-sum

Consider the special case when $\varphi_s(k) := \varphi(k) := k^2$ for each $s \in S$ $(= \{1, 2, \ldots, n\})$ and hence the separable discrete convex function Φ to be minimized is given by $\Phi(z) := z^2$, where $z^2 = \sum[z(i)^2 : i = 1, 2, \ldots, n]$. That is, we want to minimize the square-sum of the components of z. In this case, the discrete conjugate is explicitly available, namely, for integer ℓ:

$$\varphi^\bullet(\ell) = \left\lfloor \frac{\ell}{2} \right\rfloor \left\lceil \frac{\ell}{2} \right\rceil,$$

and hence Theorem 3.10 can be specialized. To this end, observe that $\varphi'(k) = (k+1)^2 - k^2 = 2k + 1$ and $\varphi'(k-1) = k^2 - (k-1)^2 = 2k - 1$. For an integral vector $z \in \mathbf{Z}^S$, we have $\Phi'(z) = 2z + \underline{1}$ and $\Phi'(z - \underline{1}) = 2z - \underline{1}$ (where $\underline{1} = \chi_S$.) For a vector $x = (x(1), \ldots, x(n))$, let $\lfloor x \rfloor := (\lfloor x(1) \rfloor, \ldots, \lfloor x(n) \rfloor)$ and $\lceil x \rceil := (\lceil x(1) \rceil, \ldots, \lceil x(n) \rceil)$. Then, for an integral vector $w \in \mathbf{Z}^S$, $\Phi^\bullet(w) = \lfloor w/2 \rfloor \lceil w/2 \rceil$. Theorem 3.10 specializes as follows.

Theorem 5.1. *Let* $Q = \begin{pmatrix} Q' \\ Q^= \end{pmatrix}$ *be an integral matrix and* $p = \begin{pmatrix} p' \\ p^= \end{pmatrix}$ *an integral vector, and suppose that the linear system* $\{Q'x \geq p', Q^=x = p^=\}$ *is box-TDI. Let* $R := \{x : Q'x \geq p', Q^=x = p^=\} \subseteq \mathbf{R}^S$ *be the (box-TDI) polyhedron defined by this system. Then*

$$\min\{z^2 : z \in \ddot{R}\} \tag{5.1}$$
$$= \max\{yp - \left\lfloor \frac{yQ}{2} \right\rfloor \left\lceil \frac{yQ}{2} \right\rceil : \ y = (y', y^=) \ \text{sign-feasible and integer-valued}\}, \tag{5.2}$$

where the sign-feasibility of y means that $y' \geq 0$. Moreover, an integral element $z^* \in \ddot{R}$ is a square-sum minimizer if and only if there exists a sign-feasible integral vector y^* for which the following optimality criteria hold:

$$y^*(Qz^* - p) = 0, \tag{5.3}$$
$$2z^* - \underline{1} \ \leq \ y^*Q \leq \ 2z^* + \underline{1}, \tag{5.4}$$

where (5.4) *is (trivially) equivalent to*

$$\left\lfloor \frac{y^*Q}{2} \right\rfloor \ \leq \ z^* \ \leq \ \left\lceil \frac{y^*Q}{2} \right\rceil. \tag{5.5}$$

The optimal (integral) dual solution y^* can be chosen in such a way that the number of its non-zero components is at most $2|S| - 1$. ■

Remark 5.1. For a simple understanding, it is worth providing a direct proof of the trivial inequality min $\geq$ max that relies neither on Φ-compatibility nor on conjugacy. For real vectors w and z in $\mathbf{R}^n$, one has the obvious estimation $z(w - z) \leq (w/2)(w/2)$. For integral vectors w and z, the stronger inequality $z(w-z) \leq \lfloor w/2 \rfloor \lceil w/2 \rceil$ holds, with equality precisely if $\lfloor w/2 \rfloor \leq z \leq \lceil w/2 \rceil$, that is, $z(s) \in \{\lfloor w(s)/2 \rfloor, \lceil w(s)/2 \rceil\}$ for each $s \in S$. This implies for any $z \in \ddot{R}$ and for any integral vector $y = (y', y^=)$ with $y' \geq 0$ that

$$z^2 \ = \ (yQ)z - ((yQ)z - z^2) \ = \ y(Qz) - (yQ - z)z \ \geq \ yp - \left\lfloor \frac{yQ}{2} \right\rfloor \left\lceil \frac{yQ}{2} \right\rceil, \tag{5.6}$$

from which min $\geq$ max follows. Moreover, equality holds in (5.6) for z^* and y^* in place of z and y precisely if the optimality criteria (5.3) and (5.5) hold. •

Remark 5.2. In linear programming, the problem of minimizing a linear cost-function cx in the special case when the polyhedron $R = \{x : Qx = p\}$ is an affine subspace is uninteresting since the minimum is either $-\infty$ or every element of R is a minimizer. But minimizing the square-sum over the elements of R is a standard problem of linear algebra. In this case, there is a unique minimizer. If R is an integral box-TDI affine subspace (that is, Q' is empty in Theorem 5.1), then we get the following min-max formula:

$$\min\{z^2 : z \in \dddot{R}\} = \max\{yp - \left\lfloor \frac{yQ}{2} \right\rfloor \left\lceil \frac{yQ}{2} \right\rceil : \ y \text{ integer-valued}\}. \tag{5.7}$$

For the case when Q is totally unimodular, McCormick et al. [21] described a polynomial algorithm for computing the minimum. •

Remark 5.3. Theorem 5.1 can easily be extended to the slightly more general case when the goal is to minimize the sum of squares over a given subset S' of coordinates. In this case (5.1) turns to

$$\min\{\sum_{s \in S'} z(s)^2 : z \in \dddot{R}\}.$$

Let Q'' denote a matrix consisting of the columns of $Q = \begin{pmatrix} Q' \\ Q^= \end{pmatrix}$ corresponding to the elements of $S - S'$. Then (5.2) transforms to the following:

$$\max\{yp - \left\lfloor \frac{yQ}{2} \right\rfloor \left\lceil \frac{yQ}{2} \right\rceil : \ y \ \text{ sign-feasible and integer-valued}, \ yQ'' = 0\}. \tag{5.8}$$

•

5.2 Flows and circulations

In this section, we specialize Theorem 5.1 to network flows. Let $D = (V, A)$ be a digraph and let m be an integral function on V for which $\widetilde{m}(V) = 0$. A function x on A is called an m-**flow** if

$$\varrho_x(v) - \delta_x(v) = m(v) \ \text{ for every } \ v \in V. \tag{5.9}$$

Note that this is equivalent to $Q_D x = m$ where Q_D denotes the signed incidence matrix of D. The columns of Q_D correspond to the edges of D while the rows correspond to the nodes, an entry of Q_D corresponding to edge a, and node v is $+1$ or -1 according as a enters or leaves v, and 0 otherwise. By the assumption $\widetilde{m}(V) = 0$, (5.9) is equivalent to

$$\varrho_x(v) - \delta_x(v) \geq m(v) \ \text{ for every } \ v \in V, \tag{5.10}$$

or concisely $Q_D x \geq m$.

By Hoffman's circulation theorem, there is a non-negative integral m-flow if and only if $\widetilde{m}(X) \geq 0$ holds for every subset $X \subseteq V$ for which $\delta_D(X) = 0$. We assume that there is a non-negative integral m-flow z and we want to characterize those minimizing the square-sum $z^2 = \sum[z(a)^2 : a \in A]$. We are going to specialize Theorem 5.1. In this case, y is a $(|V| + |A|)$-dimensional vector but in order to have a better fit to the standard notation in network flow theory, we replace y by a vector (π, h) where π (a 'potential') is defined on V while h is defined on A.

Theorem 5.2. *The minimum square-sum of a non-negative integral m-flow is equal to*

$$\max\{m\pi - \left\lfloor \frac{\Delta_\pi + h}{2} \right\rfloor \left\lceil \frac{\Delta_\pi + h}{2} \right\rceil : \pi : V \to \mathbf{Z}_+,\ h : A \to \mathbf{Z}_+\} \tag{5.11}$$

$$= \max\{m\pi - \left\lfloor \frac{\max(\Delta_\pi, 0)}{2} \right\rfloor \left\lceil \frac{\max(\Delta_\pi, 0)}{2} \right\rceil : \pi : V \to \mathbf{Z}_+\}, \tag{5.12}$$

where Δ_π denotes the potential difference defined by π, that is, $\Delta_\pi(uv) = \pi(v) - \pi(u)$ for every edge $uv \in A$, or concisely, $\Delta_\pi = \pi Q_D$. The minimum square-sum of an integral m-flow is equal to

$$\max\{m\pi - \left\lfloor \frac{\Delta_\pi}{2} \right\rfloor \left\lceil \frac{\Delta_\pi}{2} \right\rceil : \pi : V \to \mathbf{Z}_+\}. \tag{5.13}$$

Proof. Apply Theorem 5.1 to the special case when the system is $Q'x \geq p'$ (and $Q^=$ is empty) where $Q' = \begin{pmatrix} Q_D \\ I \end{pmatrix}$ and p' is defined by $p'(v) := m(v)$ for $v \in V$ and $p'(a) := 0$ when $a \in A$. (Here I denotes the $|A|$ by $|A|$ unit-matrix). The optimal dual vector $y = y'$ in Theorem 5.1 can be written in the form $y = (\pi, h)$ where π corresponds to the sub-vector of y whose components are assigned to the rows of Q_D (that is, to the nodes of D) while the components of h are assigned to the rows of I (that is to the edges of D).

To see that the maxima in (5.11) and (5.12) are equal, it suffices to observe that in an optimal solution (π, h) to (5.11), if $\Delta_\pi(a)$ is negative for an edge a of D, then $h(a)$ may be chosen to be $|\Delta_\pi(a)|$, while if $\Delta_\pi(a)$ is non-negative, then $h(a)$ may be chosen to be zero, and hence $\Delta_\pi + h = \max\{\Delta_\pi, 0\}$.

The last part of the theorem follows analogously from (5.7) in Remark 5.2. ■

Remark 5.4. Theorem 5.2 can also be derived from the network dualilty in discrete convex analysis (Section 9.6 of [23]), see Proposition 7.14 in [15]. Analogously to Remark 5.3 on a slight extension of Theorem 5.1, Theorem 5.2 can also be easily extended to the case when A' is a specified subset of edges of D, and we are interested in a non-negative integer-valued m-flow z for which $\sum[z(a)^2 : a \in A']$ is minimum. •

Remark 5.5. We worked out the details of min-max formulas concerning the minimum square-sum of a non-negative m-flow. It is only a technical matter to derive analogous min-max theorems for the minimum square-sum of feasible (= (f, g)-bounded) integral m-flow, in particular, a circulation or a maximum st-flow. Our general framework also permits the derivation of a min-max formula for the minimum square-sum of feasible integral tension (= potential difference), even in the case when not only the potential-difference but the potential itself is required to meet upper and lower bounds. •

5.3 Minimizing the weighted square-sum

Technically slightly more complicated, but the same approach works for the weighted square-sum problem. Let a be a positive integer and consider the discrete convex function

$$\varphi(k) := ak^2 \quad (k \in \mathbf{Z}). \tag{5.14}$$

Proposition 5.3 ([15]). *The discrete conjugate function $\varphi^\bullet$ of φ defined in* (5.14) *is given for integers ℓ by the following:*

$$\varphi^\bullet(\ell) = \left\lfloor \frac{\ell - a}{2a} \right\rfloor \left(\ell - a \left\lfloor \frac{\ell - a}{2a} \right\rfloor \right). \tag{5.15}$$

■

The right derivative φ' of φ (as introduced in (1.2)) for the function in (5.14) is given by $\varphi'(k) := \varphi(k+1) - \varphi(k) = a(k+1)^2 - ak^2 = a(2k+1)$. Clearly, $\varphi'(k-1) := a(2k-1)$. In this case, Theorem 3.8 can be written in the following more specific form.

Theorem 5.4. *Let $R := \{x : Qx \geq p\} \subseteq \mathbf{R}^S$ be a box-TDI polyhedron where Q is an integral matrix and p is an integral vector. Let c be a positive integral vector in $\mathbf{Z}^S$. Then*

$$\min\{\sum_{s \in S} c(s) z(s)^2 : z \in \ddot{R}\} \tag{5.16}$$

$$= \max\{yp - \sum_{s \in S} \left\lfloor \frac{w(s) - c(s)}{2c(s)} \right\rfloor \left(w(s) - c(s) \left\lfloor \frac{w(s) - c(s)}{2c(s)} \right\rfloor \right),\ w = yQ :\ y \geq 0 \ \text{integral}\}. \tag{5.17}$$

Moreover, an integral element $z^* \in \ddot{R}$ is a minimizer of (5.16) *if and only if there exists a non-negative integral vector y^* (whose components correspond to the rows of Q) for which the following optimality criteria hold:*

$$y^*(Qz^* - p) = 0, \tag{5.18}$$

$$2c(s)z^*(s) - 1 \leq w^*(s) \leq 2c(s)z^*(s) + 1 \quad \text{for each } s \in S, \tag{5.19}$$

where $w^* := y^*Q$. The optimal (integral) dual solution y^* can be chosen in such a way that the number of its positive components is at most $2|S| - 1$. ■

Remark 5.6. It is not difficult to work out a direct formula for other concrete separable discrete convex functions. For example, one may consider the functions $\varphi(k) := ak^2 + bk$, where $a > 0$ and b are integers. In another example, for a specified integer k_0, let

$$\varphi(k) := \begin{cases} k_0 - k & \text{if} \ \ k \leq k_0, \\ k - k_0 & \text{if} \ \ k \geq k_0. \end{cases} \tag{5.20}$$

Or more generally,

$$\varphi(k) := \begin{cases} c_-(k - k_0) & \text{if} \ \ k \leq k_0, \\ c_+(k - k_0) & \text{if} \ \ k \geq k_0, \end{cases} \tag{5.21}$$

with integers $c_- \leq c_+$. In these cases, we can easily derive explicit expressions of the conjugate $\varphi^\bullet$ but we omit the technical details. •

6 Inverse combinatorial optimization

Given a linear weight or cost function w_0, find a cheapest st-path, a spanning tree, spanning arborescence, perfect matching, common basis of two matroids, etc. These are standard and well-solved combinatorial optimization problems. In an inverse combinatorial optimization problem, beside w_0, we are given an input object z_0 (path, tree, matching) and the objective is to modify w_0 as little as possible so that the input object z_0 becomes a cheapest one with respect to the new cost function w. If w_0 is integer-valued, one may require that the modified w should also be integer-valued, and in this section we concentrate exclusively on this case. There may be various ways to measure the deviation of w from w_0. For example, in ℓ_1-norm the deviation is defined by $\sum[|w(s) - w_0(s)| : s \in S]$. The ℓ_2-norm or the ℓ_∞-norm are also natural choices for measuring the deviation, but one may consider weighted versions as well, when, for example, the deviation is defined by $\sum[c_1(s)(w_0(s) - w(s)) : w_0(s) > w(s)] + \sum[c_2(s)(w(s) - w_0(s)) : w(s) > w_0(s)]$, where $c_1(s)$ and $c_2(s)$ are non-negative integers. Even more, imposing lower and upper bounds for the wanted w is also a natural requirement, or, instead of a single input z_0, we may have an input set $\{z_1, \dots, z_k\}$ of solutions and want to find w in such a way that each z_i is a w-minimizer and the deviation of w from w_0 is minimum. Several further versions of inverse combinatorial optimization problems have been investigated. A relatively early survey paper [18] is due to Heuberger, while the work of Ahmadian et al. [1] includes recent developments, for example the one where the members of $\{z_1, \dots, z_k\}$ are required to be exactly the w-minimizers in the considered set. Note that Corollary 2.15 may be viewed as a solution to a feasibility-type inverse optimization problem.

In this section, we show that the framework in previous sections for minimizing separable discrete convex functions over a discrete box-TDI set covers and even extends an essential part of inverse combinatorial optimization problems (though not the framework of [1]). Here we concentrate exclusively on the theoretical background and establish a min-max theorem for the minimum deviation, where the deviation is measured by an arbitrary separable discrete convex function. Our hope is that this theoretical background will provide a good service in developing efficient algorithms to compute the wanted optimal modification of the input cost-function w_0.

6.1 A general framework for inverse problems

Let $Qx \geq p$ be a box-TDI system and $R = \{x : Qx \geq p\}$ an integral polyhedron. As before, the columns of Q are associated with the elements of ground-set S. Let $z_0 \in \ddddot{R}$ be a specified element. Let φ_s be an integer-valued discrete convex functions ($s \in S$) defining the separable discrete convex function Φ as given in (1.3). Let $\ell : S \to \mathbf{Z} \cup \{-\infty\}$ and $u : S \to \mathbf{Z} \cup \{+\infty\}$ be integral bounding vectors on S for which $\ell \leq u$.

The **inverse separable discrete convex** problem seeks for an integer-valued cost vector (objective function) w on S for which z_0 is a w-minimizer of R (that is, $wz_0 \leq wx$ for every $x \in R$), $\ell \leq w \leq u$ and $\Phi(w)$ is minimum. In Corollary 2.15, we provided a necessary and sufficient condition for the existence of a cost-function w on S for which $\ell \leq w \leq u$ and z_0 is a w-minimizer of $\ddddot{R}$. Observe that the bounding vectors ℓ and u can easily be built into Φ by changing $\varphi_s(k)$ to $+\infty$ whenever $k > u(s)$ or $k < \ell(s)$ $(s \in S)$, and hence we do not have to work explicitly with the bounding vectors ℓ and u.

Our main goal is to characterize those (linear) cost-functions w for which the input z_0 is a w-minimizer over R and $\Phi(w)$ is minimum. We emphasize that Φ is integer-valued (along with the bounds ℓ and u that can be built into Φ) and expect that the wanted optimal cost-function w is also integer-valued.

In the standard inverse combinatorial optimization problem, as indicated above, the goal is to modify a starting cost function w_0 as little as possible in l_1-norm so that the input $z_0 \in R$ is a w-minimizer, where w is the new cost-function. For $s \in S$, let $\varphi_s(k) := |w_0(s) - k|$. Then a solution to the general inverse problem (which minimizes Φ) will provide the wanted solution w for the standard problem. With an analogous approach, the general inverse problems can also be built into our framework of minimizing Φ over a discrete box-TDI set. As a result, the deviation of w from the starting w_0 may be measured in other norms. Moreover, instead of a single initial cost function w_0, we may specify an interval $[\ell_0(s), u_0(s)]$ for each $s \in S$ and strive to minimize the total deviation of the wanted w from the box defined by these intervals.

6.2 Preparation

In order to embed the general inverse problem into the framework of discrete box-TDI sets and apply then the min-max results of Section 3, we overview some further properties of box-TDI systems and polyhedra. Let $Kx \geq 0$ be a box-TDI system defining the box-TDI cone $C := \{x : Kx \geq 0\}$. Let C^* denote the **dual cone** of C, that is, $C^* := \{w : w = yK, y \geq 0\}$. The **polar cone** of C is $-C^*$.

Proposition 6.1 (Chervet, Grappe, Robert [3], Lemma 6)**.** *A cone is box-TDI if and only if its dual cone is box-TDI.* ∎

By specializing Theorem 3.10 to the case of box-TDI cones, we obtain the following.

Theorem 6.2. *Let C be a box-TDI integral cone and let C^* denote its dual cone. Let Φ be an integer-valued separable discrete convex function on $\mathbf{Z}^S$. Then*

$$\min\{\Phi(z) : z \in \ddddot{C}\} \tag{6.1}$$

$$= \max\{\Phi(z) - wz : \ z \in \ddddot{C},\ w \in \ddddot{C}^*,\ \ \{z, w\}\ \ \Phi\text{-fitting}\} \tag{6.2}$$

$$= \max\{-\Phi^\bullet(w) : w \in \ddddot{C}^*\}. \tag{6.3}$$

An element $z^* \in \ddddot{C}$ is a Φ-minimizer if and only if there exists a $w^* \in \ddddot{C}^*$ for which $w^*z^* = 0$ and

$$\Phi'(z^* - \underline{1}) \leq w^* \leq \Phi'(z^*). \tag{6.4}$$

∎

Note that we defined cone C as a polyhedral cone but in the present formulation we did not make use of this description of C. Therefore, by relying on Proposition 6.1, Theorem 6.2 can be applied to the dual cone C^* of C.

Theorem 6.3. *Let C be a box-TDI integral cone and let C^* denote its dual cone. Let Φ be an integer-valued separable discrete convex function on $\mathbf{Z}^S$. Then*

$$
\begin{aligned}
&\min\{\Phi(w) : w \in \ddddot{C}^*\}\\
&= \max\{\Phi(w) - zw : \ w \in \ddddot{C}^*, \ z \in \ddddot{C}, \ \{w, z\} \ \Phi\text{-fitting}\}\\
&= \max\{-\Phi^\bullet(z) : z \in \ddddot{C}\}.
\end{aligned}
$$

An element $w^* \in \ddddot{C}^*$ is a Φ-minimizer if and only if there exists a $z^* \in \ddddot{C}$ for which $w^* z^* = 0$ and

$$\Phi'(w^* - \underline{1}) \leq z^* \leq \Phi'(w^*). \tag{6.5}$$

■

A polyhedron is called **box-integer** [3, 26] if its intersection with any integral box is integral. For a positive integer k the k**-dilation** kR of a polyhedron $R = \{x : Qx \geq p\}$ is defined by $\{x : Qx \geq kp\}$. Any k-dilation is called an (integer) **dilation** of R.

Proposition 6.4 ([3])**.** *An integer polyhedron R is box-TDI if and only if each of its integer dilation is box-integer.* ■

This immediately implies the following.

Proposition 6.5 ([3])**.** *An integer cone is box-TDI if and only if it is box-integer.* ■

Proposition 6.6 ([3])**.** *Let x_1 be a solution to a box-TDI system $Qx \geq p$. Let $Q_1 x \geq p_1$ denote the subsystem of $Qx \geq p$ consisting of those inequalities which are met by x_1 with equality. Then the system $Q_1 x \geq p_1$ is box-TDI.* ■

Remark 6.1. The polyhedron $C_1 := \{x : Q_1 x \geq p_1\}$ (called the tangent cone of R at x_1 in [3]) is the translation of the cone $C := \{x : Q_1 x \geq 0\}$ by vector x_1. Proposition 6.6 is equivalent to stating that a tangent cone of a box-TDI polyhedron R is box-TDI. This result was formulated and proved in Lemma 5 of [3] for minimal tangent cones of R. But the proof of Lemma 5 works word for word for arbitrary tangent cones of R. •

Proposition 6.7. *Let Q, p, p_1 ,x_1 be the same as in Proposition* 6.6. *Then the cone $C = \{x : Q_1 x \geq 0\}$ is box-TDI.*

Proof. As mentioned in Remark 6.1, C is a translation of $C_1 = \{x : Q_1 x \geq p_1\}$. By Proposition 6.6, C_1 is box-TDI and hence Proposition 6.5 implies that C is also box-TDI. ■.

6.3 Min-max theorem for the general inverse problem

Recall that an ordered pair $\{w, z\}$ of vectors from $\mathbf{Z}^S$ was called Φ-fitting if $\Phi'(w - \underline{1}) \leq z \leq \Phi'(w)$. Note that we introduced this notion in Section 3 for $\{z, w\}$ but we use it here for $\{w, z\}$. The following result provides a min-max formula for the minimum in the inverse separable discrete convex optimization problem in which we want to determine the minimum of $\Phi(w)$ over those integer-valued linear objective functions w for which the input vector $z_0 \in \ddddot{R}$

minimizes wx over R, that is, $wz_0 \leq wx$ for each $x \in R$. Note that the total dual integrality of the system $Qx \geq p$ implies that $z_0 \in \ddddot{R}$ minimizes wx over R if and only if z_0 minimizes wx over $\ddddot{R}$. We also remark that the duality theorem of linear programming implies that z_0 minimizes wx over R if and only if w belongs to the cone C^* generated by those rows ${}_iq$ of Q for which ${}_iqz_0 = p(i)$.

Theorem 6.8. *Let $Qx \geq p$ be a box-TDI system defining the integral box-TDI polyhedron $R = \{x : Qx \geq p\}$, and let Φ be an integer-valued separable discrete convex function on $\mathbf{Z}^S$. Let $z_0 \in \ddddot{R}$ and let $Q_0x \geq p_0$ be the subsystem of $Qx \geq p$ consisting of those inequalities which are met by z_0 with equalities. Let $C := \{x : Q_0x \geq 0\}$ and let $C^* := \{w : w = yQ_0, y \geq 0\}$ be the dual cone of C. Then*

$$\begin{aligned} &\min\{\Phi(w) : z_0 \text{ is a } w\text{-minimizer of } \ddddot{R},\ w \text{ integer-valued}\} \\ &= \max\{\Phi(w) - zw :\ w \in \ddddot{C}^*,\ z \in \ddddot{C},\ \ \{w, z\}\ \ \Phi\text{-fitting}\} \\ &= \max\{-\Phi^\bullet(z) : z \in \ddddot{C}\}. \end{aligned}$$

An integral cost-function w^* for which z_0 is a w^*-minimizer over R is a Φ-minimizer if and only if there exists a $z^* \in \ddddot{C}$ for which $w^*z^* = 0$ and the ordered pair $\{w^*, z^*\}$ is Φ-fitting.

Proof. We mentioned before the theorem that z_0 is a w^*-minimizer element of R precisely if $w^* \in C^*$. By Proposition 6.7, C is box-TDI and hence Theorem 6.3 implies the theorem. ∎

Note that, by Proposition 2.1, Theorem 6.8 can easily be extended to the case when the box-TDI system is given in the more general form $\{Q'x \geq p', Q^= x = p^=\}$.

A natural extension of the problem is when, instead of a single element z_0, we have a subset $Z_0 := \{z_1, z_2, \dots, z_k\}$ of elements of $\ddddot{R}$, and the goal is to characterize those integer-valued weight-functions w for which each $z_i \in Z_0$ is a w-minimizer element of $\ddddot{R}$ and $\Phi(w)$ is minimum. (It is allowed that $\ddddot{R}$ may have other w-minimizer elements.) To treat this case let $R_k := kR$ denote the k-dilation of R. By Proposition 6.4, R_k is also a box-TDI polyhedron containing $z_0 := z_1 + \cdots + z_k$. It is a straightforward observation for a cost function w that z_0 is a w-minimizer element of R_k precisely if each z_i is a w-minimizer of R. Therefore we can apply Theorem 6.8 to k-dilation R_k of R and to $z_0 := z_1 + \cdots + z_k$.

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