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Rigidity of Random Subgraphs and Eigenvalues of Stiffness Matrices

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Abstract

In the random subgraph model we consider random subgraphs $G(t)$ of a graph G obtained as follows: for each edge in G we independently decide to retain the edge with probability t and discard the edge with probability $1-t$, for some $0 \leq t \leq 1$. A special case of this model is the Erdős-Rényi random graph model, where the host graph is the complete graph K_n.

In this paper we analyze the rigidity properties of random subgraphs and give new upper bounds on the threshold t_0 for which $G(t)$ is a.a.s. rigid or globally rigid when $t \geq t_0$. By specializing our results to complete host graphs we obtain, among others, that an Erdős-Rényi random graph is a.a.s. globally rigid in $\mathbb{R}^d$ if $t \geq \frac{C_d \log n}{n}$ for some constant C_d.

We also consider random subframeworks of (bar-and-joint) frameworks, which are geometric realizations of our graphs. Our bounds for the rigidity threshold of random subgraphs are in terms of the smallest non-zero eigenvalue of the stiffness matrix of the framework, which is the Gramian of its normalized rigidity matrix. Motivated by this connection, we introduce the concept of d-dimensional algebraic connectivity of graphs and provide upper and lower bounds for this value of several fundamental graph classes. The case $d = 1$ corresponds to the well-known algebraic connectivity, that is, the second smallest Laplacian eigenvalue of the graph.

We also consider the rigidity threshold in random molecular graphs, also called bond-bending networks, which are used in the study of rigidity properties of molecules. In this model we are concerned with the rigidity of the square graph of some graph G. We give an upper bound for the rigidity threshold of the square of random subgraphs in terms of the algebraic connectivity of the host graph. This enables us to derive an upper bound for the rigidity threshold for sparse host graphs.

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1 Introduction

A d-dimensional (bar-and-joint) *framework* (or geometric graph) is a pair $(G, \boldsymbol{p})$, where $G = (V, E)$ is a (finite, simple) graph on n vertices with vertex set $V = \{1, 2, \dots, n\}$ and edge set E, and $\boldsymbol{p} : V(G) \to \mathbb{R}^d$ is a map (or point configuration). We say that $(G, \boldsymbol{p})$ is a *realization* of G in $\mathbb{R}^d$. Each edge $uv \in E$ corresponds to a line segment between $\boldsymbol{p}(u)$ and $\boldsymbol{p}(v)$. The framework $(G, \boldsymbol{p})$ is *rigid*, if every continuous motion of the points of G in $\mathbb{R}^d$ which preserves all edge lengths takes the framework to a congruent realization of G. In other words, there is no continuous deformation of the framework. A d-dimensional framework $(G, \boldsymbol{p})$ is *globally rigid* if every other d-dimensional realization of G in which corresponding edges have the same length is congruent to $(G, \boldsymbol{p})$. Since continuous deformations give rise to infinitely many non-congruent realizations, it follows that global rigidity is stronger than rigidity.

Frameworks can be used to model various structures with fixed pairwise distances, including biomolecules and sensor networks. Thus results on the rigidity and flexibility properties of frameworks are vital in the analysis of the mechanical properties of biomolecules [38], in the localization problem of networks [20], and elsewhere. An active research area within the theory of rigid frameworks is concerned with the rigidity transition of random frameworks. These investigations are motivated by the analysis of rigidity percolation, which is the rigidity analogue of the conventional connectivity percolation. Rigidity percolation has been extensively studied by physicists in order to develop mathematical models for phase transitions of physical properties of different materials such as glass networks and proteins, see, e.g. [17, 23, 34]. Although several computational studies are available, the mathematical background is not fully explored.

In this paper we shall work with the *random subgraph model* (which is sometimes called the random dilution model or bond percolation model in physics). In this model we consider random subgraphs $G(t)$ of a graph G obtained as follows: for each edge in G we independently decide to retain the edge with probability t and discard the edge with probability $1 - t$, for some $0 \leq t \leq 1$. A special case of this model is the Erdős-Rényi random graph model, where the host graph is the complete graph K_n.

A fundamental problem is to determine (upper or lower bounds for) the critical value t_0 for which $G(t)$, with $t \geq t_0$, satisfies a given property with high probability: for example, we may require that it has a giant connected component, or that $G(t)$ itself is connected. We shall consider the critical value with respect to the rigidity and global rigidity of random subgraphs and random subframeworks in $\mathbb{R}^d$. It is well-known that the rigidity of a framework $(G, \boldsymbol{p})$ in $\mathbb{R}^1$ depends only on the underlying graph G: the framework is rigid if and only if G is connected. In this sense our problem is a d-dimensional extension of the connectivity problem of random graphs.

In the case of the connectivity and giant components problems Erdős and Rényi [10] determined the corresponding critical values for a complete host graph K_n. For general host graphs there are several partial results (on dense graphs, bounded degree graphs, regular graphs, expander graphs) concerning the giant component problem, see e.g. [5].

In higher dimensions, however, the rigidity of a framework $(G, \boldsymbol{p})$ depends on the realization, too. It gives rise to the version where we consider the rigidity (and a potential giant rigid component) of a random subframework. For generic (to be defined in the next section) frameworks rigidity is a graph property for all $d \geq 1$, and we can deal with random subgraphs in this context.

In the analysis of the rigidity properties of random graphs a frequently used approach is based on the so-called *Maxwell count*, which provides a simple combinatorial necessary condition for the rigidity of generic frameworks: if $(G, \boldsymbol{p})$ is rigid in $\mathbb{R}^d$ with $|V| \geq d+1$, then G has a spanning subgraph $H = (V, F)$ with $|F| = d|V| - \binom{d+1}{2}$ which satisfies $|F'| \leq d|V'| - \binom{d+1}{2}$ for every subgraph $H' = (V', F')$ of H with $|V'| \geq d+1$. Although this condition is, in general, not sufficient for $d \geq 3$, it gives rise to a purely combinatorial approach and can be used in heuristic results as well as in some special cases.

In $\mathbb{R}^2$ the above condition is also sufficient for generic frameworks: the existence of such a subgraph H implies rigidity [27, 29]. Based on this fact, it has been possible to obtain rigorous proofs for several results on the rigidity of random graphs. Consider for example the Erdős-Rényi random graph model and use $G_{n,t}$ to denote the random graph on n vertices, where each edge is present with probability t. Jackson, Servatius, and Servatius [21] proved that $G_{n,t}$ is asymptotically almost surely rigid (resp. globally rigid) in $\mathbb{R}^2$ if $t \geq \frac{\log n}{n} + \frac{(k+\delta)\log\log n}{n}$ for any $\delta > 0$, where $k = 2$ (resp. $k = 3$). Kasiviswanathan, Moore, and Theran [24] analysed the size of a giant rigid component of $G_{n,t}$ in $\mathbb{R}^2$. There are also results in different random framework models in $\mathbb{R}^2$, such as [9].

By using a completely different approach, Király and Theran [26] gave the first bound for the rigidity threshold in general dimension d in the Erdős-Rényi model. They proved that $G_{n,t}$ is a.a.s. rigid in $\mathbb{R}^d$ if $t \geq \frac{Cd\log n}{n}$ for some constant C. Instead of using the Maxwell count, they estimate the rank of the rigidity matrix (defined below) directly, applying methods from the matrix completion problem [4].

Although the random subgraph model is frequently used in experimental simulations in physics (see, e.g., [34]), little is known about the mathematical analysis, even in the 2-dimensional case. For infinite lattices the problem can be considered as a "d-dimensional version" of the classical connectivity percolation problem. There are a few results concerning the uniqueness of an infinite rigid component [15, 16] and a comparison of the rigidity threshold with the connectivity threshold for some lattices [19]. However, giving a quantitative analysis of the size of the giant rigid component or computing the exact threshold for general host graphs is a challenging open problem.

1.1 New results

In this paper we verify several algebraic properties of stiffness matrices of frameworks and, using these new results, we obtain new bounds for the critical values for the rigidity and global rigidity of random graphs in the random subgraph model.

In more detail, we give bounds for the smallest non-zero eigenvalue of the (normalized) stiffness matrix of the framework, which is the Gramian of its (normalized) rigidity matrix. We also extend the well-known concept of algebraic connectivity of graphs and introduce the d-dimensional algebraic connectivity of a graph G, which is the supremum of the smallest non-zero eigenvalues over all d-dimensional realizations of G. This parameter plays a key role in our approach to random graphs.

Our bounds for the critical values are in terms of the d-dimensional algebraic connectivity of the graph. In the special case of the Erdős-Rényi random graph model, we show that $G_{n,t}$ is a.a.s. *globally rigid* in $\mathbb{R}^d$ if $t \geq \frac{C_d \log n}{n}$ for some constant C_d. When $d \leq 100$, we have $C_d = \frac{8(d+1)^3}{2d^2+d}$.

We shall also consider the rigidity of random *molecular graphs*, also called bond-bending networks, which are frequently used in the study of rigidity percolation (see, e.g., [34]). In this model the goal is to analyze the rigidity of the *square* G^2 of G. We give a better upper bound for the rigidity threshold of the square G^2 in terms of the second smallest Laplacian eigenvalue of the underlying graph G. This enables us to derive an upper bound for the rigidity threshold even for some sparse host graphs.

The structure of the paper is as follows. In Section 2 we introduce a few more basic notions of rigidity theory, including infinitesimal rigidity and the rigidity matrix. In Section 3 we define the stiffness matrix of a framework and introduce the concept of d-dimensional algebraic connectivity of a graph. Sections 4, 5, and 6 contain our results on the eigenvalues of stiffness matrices. In Section 6 we focus on regular graphs. In the rest of the paper we consider random subframeworks and random graphs. We start with a preliminary result on random submatrices, given in Section 7. Sections 8 and 9 contain the main results on rigidity and global rigidity properties of random graphs and random molecular graphs, respectively.

2 Rigid Frameworks and Graphs

Testing or analysing the rigidity of a framework $(G, \boldsymbol{p})$ is a hard problem in $\mathbb{R}^2$ and in higher dimensions. A standard approach is to work with the stronger and more tractable property of *infinitesimal rigidity*, defined as follows. Let $(G, \boldsymbol{p})$ be a d-dimensional framework. For simplicity we shall always suppose that there is no proper affine subspace of $\mathbb{R}^d$ which contains the whole point configuration $\boldsymbol{p}(V)$. In particular, this implies $|V| \geq d+1$.

The *normalized rigidity matrix* $R(G, \boldsymbol{p})$ of $(G, \boldsymbol{p})$ is a matrix of size $d|V| \times |E|$, where each vertex has an associated d-tuple of rows and each edge $ij \in E(G)$ is associated with a *column* vector $\boldsymbol{r}_{ij}$ which has the following form:

$$\boldsymbol{r}_{ij}^{\top} := \begin{array}{ccccc} & i & & j & \\ \big[\; 0\dots0 & \boldsymbol{d}_{ij}^{\top} & 0\dots0 & -\boldsymbol{d}_{ij}^{\top} & 0\dots0 \;\big] \end{array}, \qquad (1)$$

where $\boldsymbol{d}_{ij} \in \mathbb{R}^d$ is the *edge-direction vector* defined by $\boldsymbol{d}_{ij} = \frac{\boldsymbol{p}(i)-\boldsymbol{p}(j)}{\|\boldsymbol{p}(i)-\boldsymbol{p}(j)\|}$, if $\boldsymbol{p}(i) \neq \boldsymbol{p}(j)$, and $\boldsymbol{d}_{ij} = 0$ otherwise. (Note that our definition is different from the usual definition of the rigidity matrix in the sense that we consider the transpose and normalize it.) It is well-known that the rank of $R(G, \boldsymbol{p})$ is at most $d|V| - \binom{d+1}{2}$. The framework $(G, \boldsymbol{p})$ is said to be *infinitesimally rigid* if the rank is equal to $d|V| - \binom{d+1}{2}$. Furthermore, infinitesimal rigidity implies rigidity. For *generic* frameworks (where the set of $d|V|$

coordinates of the points is algebraically independent over the rationals) $(G, \boldsymbol{p})$ is rigid if and only if $(G, \boldsymbol{p})$ is infinitesimally rigid. Thus, in this case, the rigidity and infinitesimal rigidity of the framework depend only on the graph G. Hence we call a graph G *rigid* in $\mathbb{R}^d$ if the rank of $R(G, \boldsymbol{p})$ is equal to $d|V| - \binom{d+1}{2}$ for some (or equivalently, for all generic) d-dimensional realization of G. Gortler, Healy, and Thurston [14] showed that global rigidity is also a generic property, i.e., $(G, \boldsymbol{p})$ is globally rigid in $\mathbb{R}^d$ for some generic $\boldsymbol{p}$ if and only if $(G, \boldsymbol{p})$ is globally rigid in $\mathbb{R}^d$ for all generic $\boldsymbol{p}$. Hence, we say that a graph G is *globally rigid* in $\mathbb{R}^d$ if $(G, \boldsymbol{p})$ is globally rigid in $\mathbb{R}^d$ for some generic $\boldsymbol{p}$. Note that the characterization of rigid graphs (resp. globally rigid graphs) as well as the complexity of testing whether a graph is rigid (resp. globally rigid) in $\mathbb{R}^d$, for $d \geq 3$, are still major open problems in this area.

The rigidity matrix $R(G, \boldsymbol{p})$ represents the following system of linear equations with variables $\boldsymbol{x} : V \to \mathbb{R}^d$:

$$\langle \boldsymbol{x}(i) - \boldsymbol{x}(j), \boldsymbol{d}_{ij} \rangle = 0 \qquad (ij \in E).$$

A solution $\boldsymbol{x}$ of the system or equivalently, a vector in the left kernel of $R(G, \boldsymbol{p})$, is called an *infinitesimal motion* of $(G, \boldsymbol{p})$. (Here we regard $\boldsymbol{x}$ as a vector in $\mathbb{R}^{dn}$, for $n = |V|$, in which $\boldsymbol{x}(i)$ occupies d consecutive entries for each $i \in V$.)

Every framework admits infinitesimal motions that do not depend on the graph: infinitesimal translations and infinitesimal rotations. An *infinitesimal translation* $\boldsymbol{x}_t : V \to \mathbb{R}^d$ satisfies that $\boldsymbol{x}_t(i) = \boldsymbol{x}_t(j)$ for every $i, j \in V$. An *infinitesimal rotation* (about the origin) $\boldsymbol{x}_r : V \to \mathbb{R}^d$ has the form $\boldsymbol{x}_r(i) = S\boldsymbol{p}(i)$, for every $i \in V$, where S is a skew-symmetric matrix of size d. A linear combination of infinitesimal translations and infinitesimal rotations is called a *trivial infinitesimal motion* (or *trivial motion*, for short). The space of all trivial motions of $(G, \boldsymbol{p})$ has dimension $\binom{d+1}{2}$ (provided $\boldsymbol{p}(V)$ affinely spans $\mathbb{R}^d$). Thus $\operatorname{rank} R(G, \boldsymbol{p}) \leq d|V| - \binom{d+1}{2}$. In what follows we shall use $D = \binom{d+1}{2}$, when the positive integer d is clear from the context.

We refer the reader to [22, 30] for more details of the theory of rigid and globally rigid graphs and frameworks.

3 Stiffness Matrices and the Generalized Algebraic Connectivity

Given a d-dimensional framework $(G, \boldsymbol{p})$, its *stiffness matrix* $L(G, \boldsymbol{p})$ is defined to be

$$L(G, \boldsymbol{p}) := R(G, \boldsymbol{p}) R(G, \boldsymbol{p})^\top.$$

This square matrix of size $dn \times dn$ (where n is the number of vertices) can be considered as a natural geometric-graph version of the Laplacian. Indeed, for a graph $G = (V, E)$, consider a 1-dimensional realization $(G, \boldsymbol{p})$ of G on the line, and orient each edge $uv \in E$ from left to right (assuming $\boldsymbol{p}(u) \neq \boldsymbol{p}(v)$ for each edge $uv \in E$). Then the vertex-edge incidence matrix $I(\vec{G})$ of the resulting oriented graph $\vec{G}$ is equal to $R(G, \boldsymbol{p})$, and the *Laplacian* $L(G)$ of G satisfies

$$L(G) = I(\vec{G}) I(\vec{G})^\top = R(G, \boldsymbol{p}) R(G, \boldsymbol{p})^\top = L(G, \boldsymbol{p}).$$

In this sense the Laplacian of a graph is exactly the stiffness matrix of its 1-dimensional realizations. Hence the results of spectral graph theory can be used to analyze the 1-dimensional case. However, for $d \geq 2$ one obtains several new and interesting questions.

This connection to the Laplacian motivates us to investigate the following graph parameter. For a symmetric matrix A, let $\lambda_k(A)$ be the k-th smallest eigenvalue of A (counted with multiplicities). Let G be a graph and $d \geq 1$. Recall that $D = \binom{d+1}{2}$. The *d-dimensional algebraic connectivity* of G, denoted by $a_d(G)$, is defined by

$$a_d(G) := \sup \left\{ \lambda_{D+1}(L(G, \boldsymbol{p})) \mid \boldsymbol{p} : V(G) \to \mathbb{R}^d \right\}.$$

The 1-dimensional algebraic connectivity $a_1(G)$ of a graph G is equal to the second smallest eigenvalue of the Laplacian of G, which is often called the *algebraic connectivity* of G, see [11, 12].

Let $(G, \boldsymbol{p})$ be a d-dimensional framework, and suppose that $\boldsymbol{p}(V)$ spans $\mathbb{R}^d$ affinely. The first observation is that, since $L(G, \boldsymbol{p}) = R(G, \boldsymbol{p}) R(G, \boldsymbol{p})^\top$, we have $\lambda_{D+1}(L(G, \boldsymbol{p})) \neq 0$ if and only if $(G, \boldsymbol{p})$ is infinitesimally rigid.

Let $\boldsymbol{e}_i \in \mathbb{R}^n$ be the n-dimensional unit vector whose i-th coordinate is one and zero elsewhere. Then the column vector $\boldsymbol{r}_{ij}$ of $R(G, \boldsymbol{p})$ associated with an edge ij is written by $\boldsymbol{r}_{ij} = (\boldsymbol{e}_i - \boldsymbol{e}_j) \otimes \boldsymbol{d}_{ij}$, where $\otimes$ denotes the Kronecker product of matrices. Hence the stiffness matrix is written by

$$L(G, \boldsymbol{p}) = \sum_{e=ij \in E} \boldsymbol{r}_{ij} \boldsymbol{r}_{ij}^\top = \sum_{e=ij \in E} (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \otimes \boldsymbol{d}_{ij} \boldsymbol{d}_{ij}^\top. \tag{2}$$

In the one-dimensional case, $\boldsymbol{d}_{ij}$ is either 1 or -1, and hence we obtain, as above, that $L(G, \boldsymbol{p}) = \sum_{e=ij \in E} (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top = L(G)$.

The quadratic form becomes

$$\boldsymbol{x}^\top L(G, \boldsymbol{p}) \boldsymbol{x} = \sum_{e=ij \in E} \langle \boldsymbol{x}(i) - \boldsymbol{x}(j), \boldsymbol{d}_{ij} \rangle^2. \tag{3}$$

This value has the following physical interpretation. When we consider $\boldsymbol{x}(i)$ as an infinitesimal displacement of $\boldsymbol{p}(i)$, then $\langle \boldsymbol{x}(i) - \boldsymbol{x}(j), \boldsymbol{d}_{ij} \rangle$ is the strain along an edge ij and $\boldsymbol{x}^\top L(G, \boldsymbol{p}) \boldsymbol{x}$ is the potential energy caused by the displacement. In this sense, $\lambda_{D+1}(L(G, \boldsymbol{p}))$ can be considered as a quantitive measure for the stiffness of $(G, \boldsymbol{p})$.

Stiffness matrices (and their submatrices) are also used in the stability analysis of truss structures in structural engineering [3, 7]. They occurred in rigidity theory, too, in certain sufficient conditions for the rigidity of frameworks [8, 18]. Recently, stiffness matrices have been studied by the control theory community [35, 39, 40, 41].

4 Eigenvalues of Stiffness Matrices

In this section we give various bounds for the eigenvalue λ_{D+1} of the stiffness matrix of a d-dimensional framework.

Throughout the paper we use the following notation. Let I_d denote the identity matrix of size d, and let J_d denote the all-one matrix of size d. For symmetric matrices A and B, we use $A \succeq B$ to denote that $A-B$ is positive semidefinite. The largest and the smallest eigenvalues of a square matrix A are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively.

4.1 Upper bounds based on graph Laplacians

We shall frequently use the following key result, called the Courant-Fisher min-max theorem.

Theorem 4.1 (Courant-Fisher)**.** *For a symmetric matrix A of size n,*

$$\lambda_k(A) = \max_U \min_{x \in U^\perp \setminus \{0\}} \frac{x^\top A x}{x^\top x}$$

where the maximum is taken over all $(k-1)$-dimensional subspaces U of $\mathbb{R}^n$. Similarly,

$$\lambda_{n-k+1}(A) = \min_U \max_{x \in U^\perp \setminus \{0\}} \frac{x^\top A x}{x^\top x}$$

where the minimum is taken over all $(k-1)$-dimensional subspaces U of $\mathbb{R}^n$.

Our first upper bound is based on the Laplacian of the graph.

Theorem 4.2. *Let $(G, \boldsymbol{p})$ be a d-dimensional realization of a graph G on n vertices. Then*

$$\lambda_k(L(G, \boldsymbol{p})) \leq \lambda_{\lceil \frac{k}{d} \rceil}(L(G)).$$

Proof.

$$\begin{aligned}
\lambda_k(L(G, \boldsymbol{p})) &= \max_U \min_{x \in U^\perp \setminus \{0\}} \frac{x^\top L(G, \boldsymbol{p}) x}{x^\top x} \qquad \text{(by Theorem 4.1)} \\
&= \max_U \min_{x \in U^\perp \setminus \{0\}} \frac{\sum_{ij \in E} \langle x_i - x_j, \frac{p_i - p_j}{\|p_i - p_j\|} \rangle^2}{x^\top x} \\
&\leq \max_U \min_{x \in U^\perp \setminus \{0\}} \frac{\sum_{ij \in E} \|x_i - x_j\|^2}{x^\top x} \qquad \text{(by the Cauchy-Schwarz inequality)} \\
&= \max_U \min_{x \in U^\perp \setminus \{0\}} \frac{x^\top (L(G) \otimes I_d) x}{x^\top x} \\
&= \lambda_k(L(G) \otimes I_d) \qquad \text{(by Theorem 4.1)} \\
&= \lambda_{\lceil \frac{k}{d} \rceil}(L(G)),
\end{aligned}$$

where each maximum is taken over all $(k-1)$-dimensional subspaces of $\mathbb{R}^{dn}$. □

By Theorem 4.2 we can use several well-known results on the spectrum of the Laplacian of a graph in order to estimate the spectrum of the stiffness matrices of its realizations.

4.2 Complete graphs in the plane

Let K_n be the complete graph on n vertices. As an application of Theorem 4.2 we shall give the explicit value of the 2-dimensional algebraic connectivity $a_2(K_n)$ of K_n.

Lemma 4.3. *For any $\boldsymbol{p} : V(K_n) \to \mathbb{R}^d$, the largest eigenvalue of $L(K_n, \boldsymbol{p})$ is n.*

Proof. We may suppose that the center of gravity of $(K_n, \boldsymbol{p})$ is at the origin, i.e., $\sum_{i\in V} \boldsymbol{p}(i) = 0$. Then we have

$$\sum_{ij\in E(K_n)} \|\boldsymbol{p}(i)-\boldsymbol{p}(j)\|^2 = (n-1)\sum_{i\in V}\|\boldsymbol{p}(i)\|^2 - \sum_{i\in V}\sum_{j\neq i}\langle \boldsymbol{p}(i), \boldsymbol{p}(j)\rangle = n\sum_{i\in V}\|\boldsymbol{p}(i)\|^2 = n\|\boldsymbol{p}\|^2.$$

Furthermore, we can use (3) to deduce

$$\frac{\boldsymbol{p}^\top L(K_n, \boldsymbol{p})\boldsymbol{p}}{\|\boldsymbol{p}\|^2} = \frac{\sum_{ij\in E(K_n)} \|\boldsymbol{p}(i) - \boldsymbol{p}(j)\|^2}{\|\boldsymbol{p}\|^2} = n,$$

which implies $\lambda_{\max}(L(K_n, \boldsymbol{p})) \geq n$ by Theorem 4.1. On the other hand, it is well-known that $\lambda_{\max}(L(K_n)) = n$. Hence Theorem 4.2 gives $\lambda_{\max}(L(K_n, \boldsymbol{p})) = n$. □

A 2-dimensional framework $(K_n, \boldsymbol{p})$ is said to be *planar regular* if the set of points forms the vertices of a regular n-gon on the circle with radius $\frac{1}{\sqrt{n}}$, whose center is the origin. The following result is essentially due to G. Zhu [40, Remark 3.4.1], who determined the eigenvalues of $L(K_n, \boldsymbol{p})$, when $(K_n, \boldsymbol{p})$ is planar regular.

Theorem 4.4. *Let $(K_n, \boldsymbol{p})$ be the planar regular realization of K_n with $n \geq 3$. Then*

$$a_2(K_n) = \lambda_4(L(K_n, \boldsymbol{p})) = \frac{n}{2}.$$

Proof. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote the infinitesimal translations to the x-direction and to the y-direction, respectively, with $\|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1$, and let $\boldsymbol{x}_{12}$ denote the infinitesimal rotation about the origin with $\|\boldsymbol{x}_{12}\| = 1$. Zhu [40] observed that

$$L(K_n, \boldsymbol{p}) = \frac{n}{2}(I_{2n} + \boldsymbol{p}\boldsymbol{p}^\top - \boldsymbol{x}_1\boldsymbol{x}_1^\top - \boldsymbol{x}_2\boldsymbol{x}_2^\top - \boldsymbol{x}_{12}\boldsymbol{x}_{12}^\top)$$

by regarding each of $\boldsymbol{x}_1, \boldsymbol{x}_2$ and $\boldsymbol{x}_{12}$ as a $2n$-dimensional vector, see [40, Remark 3.4.1] for more details. Since $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_{12}$ are trivial motions, they belong to the kernel of $L(K_n, \boldsymbol{p})$. Hence, for any unit vector $\boldsymbol{y} \in (\ker L(K_n, \boldsymbol{p}))^\perp$ we have $\boldsymbol{y}^\top L(K_n, \boldsymbol{p})\boldsymbol{y} \geq \frac{n}{2}$, where equality holds if $\boldsymbol{y}$ is orthogonal to $\boldsymbol{p}$. By using this fact and Theorem 4.1 we obtain that the fourth smallest eigenvalue is $\frac{n}{2}$.

On the other hand, for any 2-dimensional framework $(K_n, \boldsymbol{q})$ of K_n, the largest eigenvalue is n by Lemma 4.3. Since the trace of $L(K_n, \boldsymbol{q})$ is $2|E(K_n)| = n(n-1)$, the fourth smallest eigenvalue is at most $\frac{n(n-1)-n}{2n-4} = \frac{n}{2}$. Hence λ_4 is maximized by the planar regular realization, as claimed. □

4.3 Vertex removal

The following vertex-removal lemma provides a lower bound on λ_{D+1} of the induced subgraphs of a graph. It extends the one-dimensional result due to Fiedler [12]. For some vertex i of graph G the set of vertices adjacent to i (the *neighbours* of i) is denoted by $N_G(i)$.

Lemma 4.5. *Let $(G, \boldsymbol{p})$ be a d-dimensional framework and $(G-i, \boldsymbol{p}')$ be the subframework of $(G, \boldsymbol{p})$ obtained by removing $i \in V$. Then $\lambda_{D+1}(L(G-i, \boldsymbol{p}')) \geq \lambda_{D+1}(L(G, \boldsymbol{p})) - 1$.*

Proof. Let $\boldsymbol{x}' : V \setminus \{i\} \to \mathbb{R}^d$ be the unit eigenvector of $\lambda_{D+1}(L(G-i, \boldsymbol{p}'))$, and extend it to $\boldsymbol{x} : V \to \mathbb{R}^d$ by setting $\boldsymbol{x}(i) = 0$. Then $\boldsymbol{x}$ is in the orthogonal complement of the space of trivial motions, and hence $\boldsymbol{x}$ is in the orthogonal complement of the kernel of $L(G, \boldsymbol{p})$ by the construction. Therefore,

$$\begin{aligned}
\lambda_{D+1}(L(G, \boldsymbol{p})) &\leq \boldsymbol{x}^\top L(G, \boldsymbol{p}) \boldsymbol{x} \qquad \text{(by Theorem 4.1)} \\
&= \boldsymbol{x}'^\top \left(L(G-i, \boldsymbol{p}') + \sum_{j \in N_G(i)} \boldsymbol{e}_j \boldsymbol{e}_j^\top \otimes \boldsymbol{d}_{ij} \boldsymbol{d}_{ij}^\top \right) \boldsymbol{x}' \\
&= \lambda_{D+1}(L(G-i, \boldsymbol{p}')) + \boldsymbol{x}'^\top \left(\sum_{j \in N_G(i)} (\boldsymbol{e}_j \otimes \boldsymbol{d}_{ij})(\boldsymbol{e}_j \otimes \boldsymbol{d}_{ij})^\top \right) \boldsymbol{x}' \\
&\leq \lambda_{D+1}(L(G-i, \boldsymbol{p}')) + 1,
\end{aligned}$$

where the last inequality follows since $\{\boldsymbol{d}_{ij} \otimes \boldsymbol{e}_j : j \in N_G(i)\}$ is an orthonormal set of vectors. $\square$

4.4 The adjacency matrix for frameworks

The Laplacian matrix $L(G)$ of a graph G can be written as $D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose i-th diagonal entry is equal to $deg_G(i)$ (the *degree* of vertex i in G), and $A(G)$ is the adjacency matrix of G. In a similar manner, we define

$$D(G, \boldsymbol{p}) = \begin{pmatrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & B_{nn} \end{pmatrix} \quad \text{and} \quad A(G, \boldsymbol{p}) = \begin{pmatrix} 0 & B_{12} & \dots & B_{1n} \\ B_{12} & 0 & & \vdots \\ \vdots & & \ddots & \\ B_{1n} & \dots & & 0 \end{pmatrix}$$

where $B_{ij} = \boldsymbol{d}_{ij} \boldsymbol{d}_{ij}^\top$, if $i \neq j$, and $B_{ii} = \sum_{j \in N_G(i)} \boldsymbol{d}_{ij} \boldsymbol{d}_{ij}^\top$. Recall that $\boldsymbol{d}_{ij} \in \mathbb{R}^d$ is the edge direction vector defined in Section 2. Note that we have $L(G, \boldsymbol{p}) = D(G, \boldsymbol{p}) - A(G, \boldsymbol{p})$. We have the following inequality for the eigenvalues.

Lemma 4.6. *Let G be a graph on n vertices and let $(G, \boldsymbol{p})$ be a d-dimensional realization of G. Then for every $1 \leq i \leq dn$ we have*

$$\lambda_i(L(G, \boldsymbol{p})) \leq \lambda_{\max}(D(G, \boldsymbol{p})) - \lambda_{dn-i+1}(A(G, \boldsymbol{p})).$$

Proof. By Theorem 4.1, $\lambda_i(L(G,\boldsymbol{p})) = \min_{x\in\hat{U}^\perp\setminus\{0\}} \frac{x^\top L(G,\boldsymbol{p})x}{x^\top x}$ for some $(i-1)$-dimensional subspace $\hat{U}$ of $\mathbb{R}^{dn}$. Hence

$$\begin{aligned}
\lambda_i(L(G,\boldsymbol{p})) &= \min_{x\in\hat{U}^\perp\setminus\{0\}} \frac{x^\top L(G,\boldsymbol{p})x}{x^\top x} \\
&= \min_{x\in\hat{U}^\perp\setminus\{0\}} \frac{x^\top (D(G,\boldsymbol{p})-A(G,\boldsymbol{p}))x}{x^\top x} \\
&\leq \lambda_{\max}(D(G,\boldsymbol{p})) - \max_{x\in\hat{U}^\perp\setminus\{0\}} \frac{x^\top A(G,\boldsymbol{p})x}{x^\top x} \\
&\leq \lambda_{\max}(D(G,\boldsymbol{p})) - \min_{U:\dim U=i-1}\ \max_{x\in U^\perp\setminus\{0\}} \frac{x^\top A(G,\boldsymbol{p})x}{x^\top x} \\
&= \lambda_{\max}(D(G,\boldsymbol{p})) - \lambda_{dn-i+1}(A(G,\boldsymbol{p})) \qquad \text{(by Theorem 4.1)}
\end{aligned}$$

□

5 Balanced Complete Multipartite Graphs

As we showed in Theorem 4.2, we can obtain an upper bound on $a_d(G)$, for some graph G, in terms of certain eigenvalues of $L(G)$. It turns out that proving lower bounds on $a_d(G)$ is substantially more difficult. In particular, deciding whether $a_d(G) \neq 0$ is equivalent to deciding whether G is rigid in $\mathbb{R}^d$. Since the complexity of testing whether G is rigid in $\mathbb{R}^d$ is still open for $d \geq 3$, this indicates the potential difficulties. In fact it seems that finding reasonable lower bounds on $a_d(G)$ is not easy even if G is dense and rigid. Computing $a_d(K_n)$ for $d \geq 3$ is already a challenging open problem.

In this section we give a lower bound for $a_d(G)$ when G is a balanced complete multipartite graph. As a corollary, we obtain a lower bound for $a_d(K_n)$. The proof of this bound is not short, but having such a lower bound is essential: we shall need it later in the proofs of our results on the rigidity of random subgraphs.

For $n \geq q \geq 2$, the *balanced complete multipartite graph*, denoted by $K_{n:q}$ is the graph on n vertices whose vertex set V can be partitioned into q parts $\{V_1,\dots,V_q\}$ in such a way that $||V_i|-|V_j|| \leq 1$ holds for every $1 \leq i,j \leq q$, and two vertices $u,v \in V$ are adjacent if and only if they belong to different members of this partition.

A *regular d-simplex* in $\mathbb{R}^d$ is a simplex (with $d+1$ vertices) in which the edge lengths are the same. It will be convenient to consider a specific regular simplex, or equivalently, a specific realization of K_{d+1}. Let $(K_{d+1},\boldsymbol{p}^*)$ be the framework in which all points lie on the unit sphere with center at the origin, and induce a regular d-simplex in $\mathbb{R}^d$. Let $s_d = \lambda_{D+1}(L(K_{d+1},\boldsymbol{p}^*))$.

The main result of this section is as follows. The proof will be given in Subsection 5.2.

Theorem 5.1. *Let $d \geq 1$ and $n \geq d+1$ be integers. Then $a_d(K_{n:d+1}) \geq \frac{2d^2+d}{2(d+1)^3} s_d n - d$.*

Theorem 5.1 implies the following by using the monotonicity of a_d.

Theorem 5.2. *Let $d \geq 1$ and $n \geq d+1$ be integers. Then $a_d(K_n) \geq \frac{2d^2+d}{2(d+1)^3} s_d n - d$.*

In the special case $d = 2$, Theorem 4.4 implies that $s_2 = \frac{3}{2}$. By using a computer we have checked that $s_d = 1$ holds for $3 \leq d \leq 100$, and we strongly believe that $s_d = 1$ for all $d \geq 3$. In fact our computational results suggest the following more general conjecture.

Conjecture 1. *Let $d \geq 2$. Then the spectrum of $L(K_{d+1}, \boldsymbol{p}^*)$ is given by*

$$\begin{bmatrix} 0 & 1 & \frac{d+1}{2} & d+1 \\ \frac{(d+1)d}{2} & \frac{(d+1)(d-2)}{2} & d & 1 \end{bmatrix}$$

where the first row is the list of the eigenvalues and the second row contains their multiplicities.

When $d = 1$, $a_1(G)$ is equal to the algebraic connectivity of G. In this case better bounds can be found in the literature: $a_1(K_n) = n$ and $a_1(K_{n:2}) = a_1(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) = \lfloor \frac{n}{2} \rfloor$.

5.1 Trivial motions of the regular simplex

In the proof of Theorem 5.1 we shall use properties of the space of trivial motions of the regular d-simplex $(K_{d+1}, \boldsymbol{p}^*)$ defined above.

Lemma 5.3. *The framework $(K_{d+1}, \boldsymbol{p}^*)$ satisfies*

$$\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top = \left(1 + \frac{1}{d}\right) I_d.$$

Proof. For $i, j \in V(K_{d+1})$ with $i \neq j$, let $\theta = \langle \boldsymbol{p}^*(i), \boldsymbol{p}^*(j) \rangle$. The regularity implies that θ is independent of the choice of i and j. Since the center of gravity of $(K_{d+1}, \boldsymbol{p}^*)$ is the origin, we have $\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i) = 0$. By taking the inner product with $\boldsymbol{p}^*(1)$, we get $0 = \langle \boldsymbol{p}^*(1), \sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i) \rangle = 1 + d\theta$, implying $\theta = -\frac{1}{d}$.

For every $j \in V(K_{d+1})$, we have

$$\left(\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top \right) \boldsymbol{p}^*(j) = \boldsymbol{p}^*(j) + \sum_{i : i \neq j} \theta \boldsymbol{p}^*(i) = \boldsymbol{p}^*(j) - \theta \boldsymbol{p}^*(j) = \left(1 + \frac{1}{d}\right) \boldsymbol{p}^*(j).$$

Hence every $\boldsymbol{p}^*(j)$ is an eigenvector of $\left(\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top \right)$ whose corresponding eigenvalue is $\left(1 + \frac{1}{d}\right)$. Since the set $\{\boldsymbol{p}^*(j) : j \in V(K_{d+1})\}$ of vectors spans $\mathbb{R}^d$, this in turn implies that all eigenvalues of $\left(\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top \right)$ are equal to $\left(1 + \frac{1}{d}\right)$. Hence the lemma follows. □

Denote the standard basis of $\mathbb{R}^d$ by $\{\boldsymbol{e}_i : 1 \leq i \leq d\}$. Given a d-dimensional framework $(G, \boldsymbol{p})$, we define the canonical infinitesimal translations $\boldsymbol{x}_a^* : V(G) \to \mathbb{R}^d$ and the canonical infinitesimal rotations $\boldsymbol{x}_{ab}^* : V(G) \to \mathbb{R}^d$ as follows. Let

$$\boldsymbol{x}_a^*(i) := \frac{1}{\sqrt{d+1}} \boldsymbol{e}_a \qquad (i \in V(G)) \tag{4}$$

for each a with $1 \le a \le d$, and

$$\boldsymbol{x}^*_{ab}(i) = \sqrt{\frac{d}{2(d+1)^2}}\left(\boldsymbol{e}_a\boldsymbol{e}_b^\top - \boldsymbol{e}_b\boldsymbol{e}_a^\top\right)\boldsymbol{p}(i) \qquad (i \in V(G)) \tag{5}$$

for each a, b with $1 \le a < b \le d$. (Note that $\boldsymbol{e}_a\boldsymbol{e}_b^\top - \boldsymbol{e}_b\boldsymbol{e}_a^\top$ is skew-symmetric.) Since $\boldsymbol{x}^*_a$ and $\boldsymbol{x}^*_{ab}$ are trivial motions of $(G, \boldsymbol{p})$, they belong to the kernel of $L(G, \boldsymbol{p})$.

We can say more when the framework comes from the regular simplex.

Lemma 5.4. *The vectors* $\{\boldsymbol{x}^*_a : 1 \le a \le d\} \cup \{\boldsymbol{x}^*_{ab} : 1 \le a < b \le d\}$ *form an orthonormal basis of the kernel of* $L(K_{d+1}, \boldsymbol{p}^*)$.

Proof. We have already seen that each vector in the statement belongs to the kernel of $L(K_{d+1}, \boldsymbol{p}^*)$. We also know that the dimension of the kernel is D, which is equal to the cardinality of this set of vectors.

Let us prove that they are orthogonal. One can easily check that $\boldsymbol{x}^*_a$ and $\boldsymbol{x}^*_b$ are orthogonal if $a \neq b$, and $\boldsymbol{x}^*_a$ and $\boldsymbol{x}^*_{bc}$ are orthogonal (by using the fact that the center of gravity is the origin). To show the orthogonality of $\boldsymbol{x}^*_{ab}$ and $\boldsymbol{x}^*_{a'b'}$, let $\boldsymbol{y}_{ab} := \sqrt{\frac{2(d+1)^2}{d}}\boldsymbol{x}^*_{ab}$. We show that

$$\langle \boldsymbol{y}_{ab}, \boldsymbol{y}_{a'b'}\rangle := \sum_{i \in V(K_{d+1})} \langle \boldsymbol{y}_{ab}(i), \boldsymbol{y}_{a'b'}(i)\rangle = 0 \tag{6}$$

for any (a, b) and (a', b') with $(a, b) \neq (a', b')$. Denote the a-th coordinate of $\boldsymbol{p}^*(i)$ by $(\boldsymbol{p}^*(i))_a$. Then, by (5), $\boldsymbol{y}_{ab}(i) = (\boldsymbol{p}^*(i))_b\boldsymbol{e}_a - (\boldsymbol{p}^*(i))_a\boldsymbol{e}_b$. Hence, (6) obviously holds if $\{a, b\} \cap \{a', b'\} = \emptyset$. If $a = a'$ and $b \neq b'$, then $\langle \boldsymbol{y}_{ab}, \boldsymbol{y}_{a'b'}\rangle = \sum_{i \in V(K_{d+1})}(\boldsymbol{p}^*(i))_b(\boldsymbol{p}^*(i))_{b'}$, which is equal to zero since the right term is equal to the (b, b')-th entry of $\sum_{i \in V(K_{d+1})} \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top$, which is zero by Lemma 5.3. Thus (6) holds.

It remains to check that each vector has unit length. Indeed, $\langle \boldsymbol{x}^*_a, \boldsymbol{x}^*_a\rangle = \frac{1}{d+1}(d+1) = 1$ for every a by the definition (4). Also, by (5), we have

$$\langle \boldsymbol{x}^*_{ab}, \boldsymbol{x}^*_{ab}\rangle = \frac{d}{2(d+1)^2}\sum_{i \in V(K_{d+1})}\left((\boldsymbol{p}^*(i))_a^2 + (\boldsymbol{p}^*(i))_b^2\right) = \frac{d}{2(d+1)^2}\cdot(d+1)\cdot 2\cdot\left(1 + \frac{1}{d}\right) = 1,$$

where the second equation follows from Lemma 5.3. □

5.2 Proof of Theorem 5.1

Proof of Theorem 5.1. For simplicity, we shall use the notation $L_\Delta = L(K_{d+1}, \boldsymbol{p}^*)$ and $V(K_{d+1}) = \{1, \ldots, d+1\} = [d+1]$. We first verify some properties of L_Δ. Recall that s_d denotes the smallest nonzero eigenvalue of L_Δ. By Lemma 5.4 the vectors $\{\boldsymbol{x}^*_a : 1 \le a \le d\} \cup \{\boldsymbol{x}^*_{ab} : 1 \le a < b \le d\}$ form an orthonormal basis of $\ker L_\Delta$. Hence we have the following.

Claim 5.5. *Define a $d(d+1) \times d(d+1)$-matrix by*

$$T_\Delta = \sum_{a:1\le a\le d} \boldsymbol{x}_a^*(\boldsymbol{x}_a^*)^\top + \sum_{a,b:1\le a<b\le d} \boldsymbol{x}_{ab}^*(\boldsymbol{x}_{ab}^*)^\top,$$

where each of the trivial motions $\boldsymbol{x}_a^* : [d+1] \to \mathbb{R}^d$ and $\boldsymbol{x}_{ab}^* : [d+1] \to \mathbb{R}^d$ is regarded as a $d(d+1)$-dimensional vector. Then $L_\Delta \succeq s_d(I_{d+1} - T_\Delta)$ holds.

We shall use the following decomposition of T_Δ into $d \times d$-blocks T_{ij}:

$$T_\Delta = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ d \\ d+1 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & \dots & d & d+1 \end{array} \\ \begin{bmatrix} T_{11} & T_{12} & \dots & \dots & T_{1,d+1} \\ T_{21} & T_{22} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & T_{d,d} & T_{d,d+1} \\ T_{d+1,1} & \dots & \dots & T_{d+1,d} & T_{d+1,d+1} \end{bmatrix} \end{array}, \tag{7}$$

where each T_{ij} is associated with a pair (i,j) of vertices of K_{d+1}. Then

$$T_{ij} = \sum_{a:1\le a\le d+1} \boldsymbol{x}_a^*(i)\boldsymbol{x}_a^*(j)^\top + \sum_{a,b:1\le a<b\le d+1} \boldsymbol{x}_{ab}^*(i)\boldsymbol{x}_{ab}^*(j)^\top. \tag{8}$$

Claim 5.6. *For each i,*

$$T_{ii} \preceq \frac{3d+2}{2(d+1)^2} I_d.$$

Proof. Plugging (4) and (5) into (8), we have $T_{ii} = \frac{1}{d+1}I_d + \frac{d}{2(d+1)^2}(I_d - \boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top) = \frac{3d+2}{2(d+1)^2}I_d - \frac{d}{2(d+1)^2}\boldsymbol{p}^*(i)\boldsymbol{p}^*(i)^\top \preceq \frac{3d+2}{2(d+1)^2}I_d$. □

Now we start the main part of the proof of Theorem 5.1. In light of Lemma 4.5, we may focus on proving

$$a_d(K_{n:d+1}) \ge \frac{2d^2+d}{2(d+1)^3} s_d n \tag{9}$$

in the case when $\frac{n}{d+1} = k$ for some integer $k \ge 1$. Let $V_1, \dots, V_{d+1}$ denote the balanced partition of the vertex set of $K_{n:d+1}$ into $d+1$ parts of size k each.

Now define a realization $\boldsymbol{p} : V(K_{n:d+1}) \to \mathbb{R}^d$ of $K_{n:d+1}$ by putting

$$\boldsymbol{p}(v) = \boldsymbol{p}^*(i) \qquad (v \in V_i, i \in [d+1]).$$

In other words, all vertices of V_i are mapped to a vertex of the regular d-simplex. Then (9) will follow if we can show that

$$\lambda_{D+1}(L(K_{n:d+1}, \boldsymbol{p})) \ge \frac{2d^2+d}{2(d+1)^3} s_d n. \tag{10}$$

In order to prove (10), we consider a vertex-disjoint packing of d-simplices in $K_{n:d+1}$. Formally, a family $\{\Delta_1, \Delta_2, \dots, \Delta_k\}$ of subgraphs of $K_{n:d+1}$ is said to be a *simplex* *packing* if each Δ_i is isomorphic to K_{d+1} and $V(\Delta_i) \cap V(\Delta_j) = \emptyset$ for any i, j with $i \neq j$. Note that $\bigcup_{i=1}^{k} V(\Delta_i) = V(K_{n:d+1})$.

Let $\mathcal{P} = \{\Delta_1, \Delta_2, \ldots, \Delta_k\}$ be a simplex packing. Let $L_{\mathcal{P}}$ be the stiffness matrix of the union of the subframeworks induced by the members of $\mathcal{P}$. Since the simplices are vertex-disjoint, $L_{\mathcal{P}}$ can be written as

$$L_{\mathcal{P}} = \sum_{i=1}^{k} \sum_{e \in E(\Delta_i)} \boldsymbol{r}_e \boldsymbol{r}_e^{\top} = \bigoplus_{i=1}^{k} L(\Delta_i, p_{|V(\Delta_i)}), \tag{11}$$

where $\bigoplus$ denotes a block-diagonalized matrix consisting of the matrices in the summation. (Recall also that $\boldsymbol{r}_e$ denotes the column vector of the rigidity matrix associated with e.) Since each $(\Delta_i, p_{|V(\Delta_i)})$ is identical to $(K_{d+1}, \boldsymbol{p}^*)$, (11) and Claim 5.5 imply

$$L_{\mathcal{P}} = \bigoplus_{i=1}^{k} L_{\Delta} \succeq \bigoplus_{i=1}^{k} s_d(I_{d(d+1)} - T_{\Delta}) = s_d(I_{dn} - \bigoplus_{i=1}^{k} T_{\Delta}). \tag{12}$$

Let α be the number of all possible simplex packings in $K_{n:d+1}$ and β be the number of all possible simplex packings containing a specified edge. A simple calculation shows $\alpha = \frac{(k!)^{d+1}}{k!}$ and $\beta = \frac{((k-1)!)^2 (k!)^{d-1}}{(k-1)!}$, and hence

$$\frac{\alpha}{\beta} = k = \frac{n}{d+1}. \tag{13}$$

If we consider all possible simplex packings in $K_{n:d+1}$, then each edge is used in β packings, which means

$$\beta L(K_{n:d+1}, \boldsymbol{p}) = \sum_{\mathcal{P}:\text{a simplex packing}} L_{\mathcal{P}}. \tag{14}$$

Combining (12) and (14), we have

$$\beta L(K_{n:d+1}, \boldsymbol{p}) \succeq s_d \left(\alpha I_{dn} - \sum_{\mathcal{P}:\text{a simplex packing}} \bigoplus T_{\Delta} \right),$$

but this relation is not precise because the block-structures (when taking $\bigoplus$) depend on $\mathcal{P}$ and they are not consistent over all packings. To get a precise relation, recall that T_{Δ} is decomposed into T_{ij}'s as described in (7), where each T_{ij} is associated with a pair (i, j) of vertices of K_{d+1}. In $K_{n:d+1}$, each vertex is covered by α simplices over all $\mathcal{P}$, while each edge is covered by β simplices over all $\mathcal{P}$. Hence, the precise relation obtained from (12) and (14) is

$$\beta L(K_{n:d+1}, \boldsymbol{p}) \succeq s_d(\alpha I_{dn} - T), \tag{15}$$

where

$$T = \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_d \\ V_{d+1} \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & \cdots & V_d & V_{d+1} \end{array} \\ \begin{bmatrix} \alpha T_{11} \otimes I_k & \beta T_{12} \otimes J_k & \ldots & \ldots & \beta T_{1,d+1} \otimes J_k \\ \beta T_{21} \otimes J_k & \alpha T_{22} \otimes I_k & & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & \alpha T_{d,d} \otimes I_k & \beta T_{d,d+1} \otimes J_k \\ \beta T_{d+1,1} \otimes J_k & \ldots & \ldots & \beta T_{d+1,d} \otimes J_k & \alpha T_{d+1,d+1} \otimes I_k \end{bmatrix} \end{array}.$$

In order to analyze T, we define infinitesimal motions $\boldsymbol{x}_a$ and $\boldsymbol{x}_{ab}$ of $(K_{n:d+1}, \boldsymbol{p})$ by

$$\boldsymbol{x}_a(v) = \boldsymbol{x}_a^*(i) \qquad (v \in V_i, i \in [d+1] = V(K_{d+1}))$$

for $1 \leq a \leq d$, and

$$\boldsymbol{x}_{ab}(v) = \boldsymbol{x}_{ab}^*(i) \qquad (v \in V_i, i \in [d+1] = V(K_{d+1}))$$

for $1 \leq a < b \leq d$. Then, the fact that $\boldsymbol{x}_a^*$ and $\boldsymbol{x}_{ab}^*$ are trivial motions of $(K_{d+1}, \boldsymbol{p}^*)$ implies that $\boldsymbol{x}_a$ and $\boldsymbol{x}_{ab}$ are trivial motions of $(K_{n:d+1}, \boldsymbol{p})$. Let

$$S := \sum_{a:1\leq a\leq d} \boldsymbol{x}_a \boldsymbol{x}_a^\top + \sum_{a,b:1\leq a<b\leq d} \boldsymbol{x}_{ab}\boldsymbol{x}_{ab}^\top. \tag{16}$$

Then we have

$$S = \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_d \\ V_{d+1} \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & \cdots & V_d & V_{d+1} \end{array} \\ \begin{bmatrix} T_{11}\otimes J_k & T_{12}\otimes J_k & \dots & \dots & T_{1,d+1}\otimes J_k \\ T_{21}\otimes J_k & T_{22}\otimes J_k & & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & T_{d,d}\otimes J_k & T_{d,d+1}\otimes J_k \\ T_{d+1,1}\otimes J_k & \dots & \dots & T_{d+1,d}\otimes J_k & T_{d+1,d+1}\otimes J_k \end{bmatrix} \end{array}.$$

By comparing S and T we can deduce that

$$T = \beta S + \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_{d+1} \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & \cdots & V_{d+1} \end{array} \\ \begin{bmatrix} T_{11}\otimes(\alpha I_k - \beta J_k) & 0 & \dots & 0 \\ 0 & T_{22}\otimes(\alpha I_k - \beta J_k) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & T_{d+1,d+1}\otimes(\alpha I_k - \beta J_k) \end{bmatrix} \end{array} \tag{17}$$

Since each T_{ii} is positive semidefinite by (8), so is $T_{ii} \otimes J_k$. Hence (17) implies

$$T \preceq \beta S + \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_{d+1} \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & \cdots & V_{d+1} \end{array} \\ \begin{bmatrix} T_{11}\otimes(\alpha I_k) & 0 & \dots & 0 \\ 0 & T_{22}\otimes(\alpha I_k) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & T_{d+1,d+1}\otimes(\alpha I_k) \end{bmatrix} \end{array}$$

Furthermore, by Claim 5.6, we also have

$$T \preceq \beta S + \frac{3d+2}{2(d+1)^2}\alpha I_{dn}. \tag{18}$$

By (15) and (18), we finally obtain

$$L(K_{n:d+1}, \boldsymbol{p}) \succeq \frac{s_d}{\beta}(\alpha I_{dn} - T) \succeq s_d \left(\frac{\alpha}{\beta} \frac{2d^2+d}{2(d+1)^2} I_{dn} - S \right).$$

By the definition of S from (16), and the fact that $\boldsymbol{x}_a$ and $\boldsymbol{x}_{ab}$ are trivial motions, this relation implies that the smallest nonzero eigenvalue of $L(K_{n:d+1}, \boldsymbol{p})$ is at least $s_d\left(\frac{\alpha}{\beta}\frac{2d^2+d}{2(d+1)^2}\right)$. Since $\frac{\alpha}{\beta} = \frac{n}{d+1}$ by (13), we conclude that $\lambda_{D+1}(L(K_{n:d+1}, \boldsymbol{p})) \geq \frac{2d^2+d}{2(d+1)^3} s_d n$ as we stated in (10). This completes the proof of Theorem 5.1. □

6 Regular Graphs

In this section we consider the d-dimensional algebraic connectivity of regular graphs. The one-dimensional version (the algebraic connectivity of regular graphs) is one of the central topics in spectral graph theory. The Laplacian version of (a strengthening of) a key result of Alon and Boppana is as follows, see [1].

Theorem 6.1. *Let G be a k-regular graph. Then*

$$\lambda_2(L(G)) \leq k - 2\sqrt{k-1} + O\left(\frac{\sqrt{k-1}}{\operatorname{diam}(G) - 4}\right).$$

Since the diameter grows when $n \to +\infty$ (assuming k is fixed), this gives an asymptotic upper bound $k - 2\sqrt{k-1}$ for the algebraic connectivity. In particular it follows that for $k = 2$ the algebraic connectivity converges to zero as the size of the graph increases. (This fact can also be deduced from the explicit formula $2(1 - \cos\frac{2\pi}{n})$ for the algebraic connectivity of the cycle on n vertices.)

Given our d-dimensional extension of the algebraic connectivity of a graph, one natural question is whether Theorem 6.1 can be extended to stiffness matrices of d-dimensional realizations of graphs. For simplicity let us consider the case $d = 2$ and a two-dimensional realization $(G, \boldsymbol{p})$ of a k-regular graph G. Theorem 4.2 implies that $\lambda_4(L(G, \boldsymbol{p}))$ is less than or equal to the upper bound given in Theorem 6.1. We believe that the tight bound is (asymptotically) much smaller. In fact, based on our computational results, we conjecture that the asymptotic bound is actually 0 if $k = 4$.

Conjecture 2. *Let*

$$a_{2,4,n}^{\max} = \max\{a_2(G) : G \text{ is a 4-regular graph on } n \text{ vertices}\}.$$

Then $\lim_{n\to+\infty} a_{2,4,n}^{\max} = 0$.

Friedman [13] proved that the algebraic connectivity of a random k-regular graph is essentially equal to the upper bound of Theorem 6.1. Again, based on a computational experiment, we expect that the tight asymptotic upper bound for the d-dimensional algebraic connectivity of k-regular graphs is attained by the random k-regular graphs. The general case seems to be hard to attack. Our next result at least suggests that the tight bound is better than the bound of Theorem 6.1 for large k.

Theorem 6.2. *Let G be a k-regular graph on n vertices, for some $k \geq 2$ and $n \geq 3k$, and let $(G, \boldsymbol{p})$ be a two-dimensional realization of G in which the points $\boldsymbol{p}(v)$, $v \in V(G)$, form a regular n-gon on the unit circle centered around the origin. Then*

$$\lambda_4(L(G, \boldsymbol{p})) \leq \frac{n}{2n-2}\left(k - \sqrt{\frac{kn - 3k^2}{2n-3}}\right)$$

Proof. Recall the definitions of $D(G,\boldsymbol{p})$, $A(G,\boldsymbol{p})$, and Lemma 4.6 from Section 4. The lemma shows that in order to upper bound $\lambda_4(L(G,\boldsymbol{p}))$ it suffices to bound $\lambda_{\max}(D(G,\boldsymbol{p}))$ from above and bound $\lambda_{2n-3}(A(G,\boldsymbol{p}))$ from below. Note that $D(G,\boldsymbol{p})$ is block diagonal, and hence

$$\lambda_{\max}(D(G,\boldsymbol{p})) = \max_{i\in V} \lambda_{\max}(B_{ii}),$$

where B_{ii} is as defined in Subsection 4.4. Here $\lambda_{\max}(D(G,\boldsymbol{p}))$ can be as large as k. To obtain an improved bound we show the following.

Lemma 6.3. *Suppose that $\lambda_{\max}(B_{ii}) \geq a$ for some positive number a and $i \in V$. Then*

$$\lambda_4(L(G,\boldsymbol{p})) \leq \frac{n(k-a)}{n-2}.$$

Proof. Informally, the idea of the proof is based on the observation that if $\lambda_{\max}(B_{ii})$ is large, then $N_G(i)\cup\{i\}$ induces a thin subframework in $(G,\boldsymbol{p})$ and hence it is easy to deform the framework by applying a force to i in the direction orthogonal to this thin part.

Formally, suppose that $\lambda_{\max}(B_{ii}) \geq a$. Let $\boldsymbol{v} \in \mathbb{R}^2$ be the unit eigenvector of the smallest eigenvalue of B_{ii}, and define $\boldsymbol{z} : V(G) \to \mathbb{R}^2$ by

$$\boldsymbol{z}(j) = \begin{cases} (n-1)\boldsymbol{v} & (j = i) \\ -\boldsymbol{v} & (j \neq i). \end{cases}$$

Just like in (4) and (5), we define the canonical trivial motions $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{12}$ of $(G,\boldsymbol{p})$. Namely, let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ denote the translations of $(G,\boldsymbol{p})$ to the x- and y-directions, respectively, with $\|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1$, and let $\boldsymbol{x}_{12}$ denote the infinitesimal rotation about the origin with $\|\boldsymbol{x}_{12}\| = 1$. Since the center of gravity is the origin, it is easy check that $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{12}$ are pairwise orthogonal, and hence $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_{12}\}$ is an orthonormal basis of the space of trivial infinitesimal motions of $(G,\boldsymbol{p})$. One can easily check that $\langle \boldsymbol{z}, \boldsymbol{x}_1\rangle = \langle \boldsymbol{z}, \boldsymbol{x}_2\rangle = 0$. Also, since $\boldsymbol{x}_{12}$ is the infinitesimal rotation about the origin with $\|\boldsymbol{x}_{12}\| = 1$, $\boldsymbol{x}_{12}(j) = \sqrt{\frac{1}{n}}\boldsymbol{p}(j)^{\perp}$ for $j \in V(G)$, where $\boldsymbol{p}(j)^{\perp}$ denotes the 90 degree rotation of $\boldsymbol{p}(j)$. Thus

$$\langle \boldsymbol{z}, \boldsymbol{x}_{12}\rangle = \sqrt{\frac{1}{n}}\left(\langle (n-1)\boldsymbol{v}, \boldsymbol{p}(i)^{\perp}\rangle + \sum_{j:j\neq i}\langle -\boldsymbol{v}, \boldsymbol{p}(j)^{\perp}\rangle\right) = \sqrt{n}\langle \boldsymbol{v}, \boldsymbol{p}(i)^{\perp}\rangle,$$

where the second equation follows from $\sum_{j=1}^{n} \boldsymbol{p}(j)^{\perp} = 0$. Hence by setting $\bar{\boldsymbol{z}} = \boldsymbol{z} - \langle \boldsymbol{z}, \boldsymbol{x}_{12}\rangle \boldsymbol{x}_{12}$, $\bar{\boldsymbol{z}}$ is orthogonal to the space of trivial infinitesimal motions of $(G,\boldsymbol{p})$.

We have

$$\begin{aligned}
\bar{\boldsymbol{z}}^\top L(G,\boldsymbol{p})\bar{\boldsymbol{z}} &= \sum_{ij\in E}\langle \boldsymbol{d}_{ij}, \boldsymbol{z}(i)-\boldsymbol{z}(j)\rangle^2 \qquad \text{(by (3))}\\
&= \sum_{j\in N_G(i)}\langle \boldsymbol{d}_{ij}, (n-1)\boldsymbol{v}-(-\boldsymbol{v})\rangle^2 \qquad \text{(by definition of } \boldsymbol{z})\\
&= n^2\boldsymbol{v}^\top\left(\sum_{j\in N_G(i)} \boldsymbol{d}_{ij}\boldsymbol{d}_{ij}^\top\right)\boldsymbol{v}\\
&= n^2\boldsymbol{v}^\top B_{ii}\boldsymbol{v} \qquad \text{(by definition of } B_{ii})\\
&= n^2\lambda_{\min}(B_{ii}) \qquad \text{(by definition of } \boldsymbol{v}),
\end{aligned}$$

and

$$\bar{\boldsymbol{z}}^\top\bar{\boldsymbol{z}} = \|\boldsymbol{z}\|^2-\langle \boldsymbol{z},\boldsymbol{x}_{12}\rangle^2 = (n-1)^2+n-1-n\langle \boldsymbol{v},\boldsymbol{p}(i)^\perp\rangle^2 \geq n^2-n-n\|\boldsymbol{p}(i)^\perp\|^2 = n^2-2n,$$

where the third inequality follows from the Cauchy-Schwarz inequality. Moreover

$$k = \mathrm{Tr}(B_{ii}) \geq a + \lambda_{\min}(B_{ii}).$$

Since $\bar{z}$ is orthogonal to the space of trivial motions of $(G,\boldsymbol{p})$, $\bar{z}$ is in the orthogonal complement of $\ker L(G,\boldsymbol{p})$. Hence, the Courant-Fisher theorem implies that

$$\lambda_4(L(G,\boldsymbol{p})) \leq \frac{\bar{\boldsymbol{z}}^\top L(G,\boldsymbol{p})\bar{\boldsymbol{z}}}{\bar{\boldsymbol{z}}^\top\bar{\boldsymbol{z}}} \leq \frac{n^2\lambda_{\min}(B_{ii})}{n^2-2n} \leq \frac{n(k-a)}{n-2}.$$

□

We next derive another bound based on a standard trace argument.

Lemma 6.4. $\mathrm{Tr}(A(G,\boldsymbol{p})^2) = kn$.

Proof. Observe that the (i,i)-th block of $A(G,\boldsymbol{p})^2$ is $\sum_{j\in N_G(i)} B_{ij}^2$. Moreover, $B_{ij}^2 = \boldsymbol{d}_{ij}\boldsymbol{d}_{ij}^\top\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}^\top = \boldsymbol{d}_{ij}\boldsymbol{d}_{ij}^\top = B_{ij}$. As $\mathrm{Tr}(B_{ij}) = \|\boldsymbol{d}_{ij}\|^2 = 1$, $\mathrm{Tr}(A(G,\boldsymbol{p})^2)$ is equal to twice the number of edges, which is kn as G is k-regular. □

Lemma 6.5. $\lambda_{\max}(A(G,\boldsymbol{p})) \leq k$.

Proof. Note that $\lambda_{\max}(B_{ii}) \leq k$ for each i as $\mathrm{Tr}(B_{ii}) = k$. Hence $\lambda_{\max}(D(G,\boldsymbol{p})) \leq k$. This and Lemma 4.6 imply $0 = \lambda_{\min}(L(G,\boldsymbol{p})) \leq \lambda_{\max}(D(G,\boldsymbol{p})) - \lambda_{\max}(A(G,\boldsymbol{p})) \leq k - \lambda_{\max}(A(G,\boldsymbol{p}))$, implying the claim. □

Lemma 6.6. *Suppose $n \geq 3k$. Then $\lambda_{2n-3}(A(G,\boldsymbol{p})) \geq \sqrt{\frac{kn-3k^2}{2n-3}}$.*

Proof. By Lemmas 6.4 and 6.5,

$$kn = \mathrm{Tr}(A(G,\boldsymbol{p})^2) = \sum_{j=1}^{2n}\lambda_j(A(G,\boldsymbol{p})^2) \leq 3k^2 + (2n-3)\lambda_{2n-3}(A(G,\boldsymbol{p}))^2,$$

implying $\lambda_{2n-3}(A(G,\boldsymbol{p})) \geq \sqrt{\frac{kn-3k^2}{2n-3}}$ by $n \geq 3k$. □

By Lemma 4.6 and Lemma 6.6, we obtain

$$\lambda_4(L(G,\boldsymbol{p})) \le \max_{i\in V} \lambda_{\max}(B_{ii}) - \sqrt{\frac{kn-3k^2}{2n-3}}.$$

By combining this with Lemma 6.3, we have

$$\lambda_4(L(G,\boldsymbol{p})) \le \min\left\{\frac{n(k-a)}{n-2}, a - \sqrt{\frac{kn-3k^2}{2n-3}}\right\}.$$

for some number a with $0 \le a \le k$. Thus $\lambda_4(L(G,\boldsymbol{p})) \le \max_{a:0\le a\le k} \min\{\frac{n(k-a)}{n-2}, a - \sqrt{\frac{kn-3k^2}{2n-3}}\} \le \max_{a\in\mathbb{R}} \min\{\frac{n(k-a)}{n-2}, a - \sqrt{\frac{kn-3k^2}{2n-3}}\}$. By setting a such that $\frac{n(k-a)}{n-2} = a - \sqrt{\frac{kn-3k^2}{2n-3}}$, we get $\lambda_4(L(G,\boldsymbol{p})) \le \frac{n}{2n-2}\left(k - \sqrt{\frac{kn-3k^2}{2n-3}}\right)$. This completes the proof of Theorem 6.2. □

7 Random Submatrices

In the rest of the paper we apply our algebraic results on stiffness matrices to obtain new results on the rigidity of random graphs and frameworks. In this section we prove a theorem on random submatrices that we shall employ later. We need the following Matrix Chernoff bound[1].

Theorem 7.1. *Let $\{X_i\}$ be a finite sequence of independent, random, positive semidefinite matrices of size $m \times m$, and suppose that $\lambda_{\max}(X_i) \le L$ for every i. Let $Y = \sum_i X_i$ and $k = \dim\ker\mathbb{E}(Y)$. Then, for any $\varepsilon \in [0,1)$,*

$$\mathbb{P}\left[\lambda_{k+1}(Y) \le (1-\varepsilon)\lambda_{k+1}(\mathbb{E}(Y))\right] \le (m-k)\left(\frac{\mathrm{e}^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\frac{\lambda_{k+1}(\mathbb{E}(Y))}{L}} \le (m-k)\mathrm{e}^{-\frac{\varepsilon^2\lambda_{k+1}(\mathbb{E}(Y))}{2L}}.$$

Oliveira [28] showed how to use a matrix concentration inequality for analyzing the connectivity in the random subgraph model. A simpler argument based on the Matrix Chernoff bound was used to construct spectral sparsifiers in spectral graph theory [31], and now the technique is widely used for designing fast matrix approximation algorithms, see, e.g. [6]. The following theorem (Theorem 7.2) is obtained by adapting some arguments from [31, 6]. We give a formal proof since the technique is new in the rigidity context.

Let $A \in \mathbb{R}^{n\times m}$ be a matrix and let E be a finite set such that each element $e \in E$ is associated with a submatrix A_e of A of size $n \times k_e$ in such a way that the columns of these submatrices form a partition of the columns of A. Thus $m = \sum_e k_e$ holds. In this case we simply write $A = [A_e : e \in E]$. For $t \in [0,1]$, let $A(t)$ be the matrix obtained by deleting (the columns of) A_e from A with probability $1-t$, independently for each $e \in E$.

[1] A standard form of the Matrix Chernoff bound gives a deviation bound for the minimum eigenvalue of the sum of random positive semidefinite matrices, see e.g. [36, Corollary 5.2]. The present form is obtained by restricting the underlying vector space to the orthogonal complement of $\ker\mathbb{E}(Y)$ and then applying the standard form.

Theorem 7.2. *Let E be a finite set, let $c \geq 1$, and let $A \in \mathbb{R}^{n \times m}$ with $A = [A_e : e \in E]$ and $r = \operatorname{rank} A$. Suppose that $\lambda_{\max}(A_e A_e^\top) \leq h$ for every $e \in E$, and*

$$1 \geq t \geq \frac{h \log(rc)}{\lambda_{n-r+1}(AA^\top)}.$$

Then $\operatorname{rank} A(t)A(t)^\top = \operatorname{rank} AA^\top$ *holds with probability at least* $1 - \frac{1}{c}$.

Before presenting the proof we recall a key notion that we shall need. For a matrix $A \in \mathbb{R}^{n \times m}$, the *Moore-Penrose inverse* of A is denoted by $A^\dagger$. It is defined as the unique matrix satisfying $AA^\dagger A = A, (AA^\dagger)^\top AA^\dagger, A^\dagger AA^\dagger = A^\dagger$, and $(A^\dagger A)^\top = A^\dagger A$. Several fundamental properties of $A^\dagger$ can be deduced from the singular value decomposition of A. Suppose that $A = U\Sigma V^\top$ is the singular value decomposition of A, where U and V are orthogonal matrices of size n and m, and Σ is a diagonal matrix whose diagonal entries are the singular values of A. It is known that

$$A^\dagger = V\Sigma^\dagger U^\top, \tag{19}$$

where $\Sigma^\dagger$ is obtained by replacing each non-zero diagonal entry of Σ with its reciprocal, and then taking the transpose, see e.g. [2]. By using (19) it can be seen that $A^\dagger$ represents a non-singular map from image A to $(\ker A)^\perp$.

Since $A = U\Sigma V^\top$, we have $A^\top A = V\tilde{\Sigma}^2 V^\top$, where $\tilde{\Sigma}^2 := \Sigma^\top\Sigma$. Note that $\tilde{\Sigma}^2$ is a diagonal matrix whose diagonal contains the eigenvalues of $A^\top A$ by the definition of the singular values of A. Hence, by applying (19) to $A^\top A$, we get $(A^\top A)^\dagger = V(\tilde{\Sigma}^2)^\dagger V^\top = V\tilde{\Sigma}^{-2}V^\top$. This implies that the non-zero eigenvalues of $(A^\top A)^\dagger$ are the reciprocals of the non-zero eigenvalues of $A^\top A$. Also, a further calculation gives $(A^\top A)^\dagger = A^\dagger (A^\dagger)^\top$ by using $(A^\top A)^\dagger = V\tilde{\Sigma}^{-2}V^\top = V\Sigma^\dagger U^\top U(\Sigma^\dagger)^\top V^\top = A^\dagger(A^\dagger)^\top$. Now we are ready to prove Theorem 7.2.

Proof. Let $L = AA^\top$. Then $L = \sum_{e \in E} A_e A_e^\top$. For each $e \in E$, define

$$V_e = \frac{A^\dagger A_e}{\sqrt{t}}.$$

We first show that, for any subset E' of E,

$$\operatorname{rank} \sum_{e \in E'} V_e V_e^\top = \operatorname{rank} \sum_{e \in E'} A_e A_e^\top. \tag{20}$$

To see this, let A' be a matrix obtained by aligning A_e for all $e \in E'$ as column submatrices. Then $A'(A')^\top = \sum_{e \in E'} A_e A_e^\top$. Let $V' = \frac{A^\dagger A'}{\sqrt{t}}$. Then,

$$V'(V')^\top = A^\dagger \left(\sum_{e \in E'} \frac{A_e A_e^\top}{t} \right) (A^\dagger)^\top = \sum_{e \in E'} V_e V_e^\top. \tag{21}$$

By $V' = \frac{A^\dagger A'}{\sqrt{t}}$, $\operatorname{rank} V' \leq \operatorname{rank} A'$ holds. Moreover, since image $A' \subseteq$ image A holds by definition and $A^\dagger$ represents a non-singular linear map from image A to $(\ker A)^\perp$, the dimension of the image of $A^\dagger A'$ is equal to that of A', that is, $\operatorname{rank} V' = \operatorname{rank} A'$. This and (21) imply (20).

Recall that $A(t)A(t)^\top$ is obtained by adding $A_e A_e^\top$ with probability t. Let X_e be a random matrix which is equal to $V_e V_e^\top$ with probability t and is equal to the zero matrix with probability $1-t$, and let $Y = \sum_{e\in E} X_e$. Then, it follows from (20) that

$$\mathbb{P}\left[\operatorname{rank} A(t)A(t)^\top < r \ (= \operatorname{rank} L)\right] \leq \frac{1}{c} \tag{22}$$

if and only if

$$\mathbb{P}\left[\operatorname{rank} Y < r\right] \leq \frac{1}{c}. \tag{23}$$

We shall prove (23) by applying the Matrix Chernoff bound, Theorem 7.1 to the matrices $\{X_e\}$, $e \in E$. First we show the following.

Claim 7.3. $\lambda_{\max}(X_e) \leq \frac{1}{\log(rc)}$.

Proof. It suffices to show that $\lambda_{\max}(V_e V_e^\top) \leq \frac{1}{\log(rc)}$. By Theorem 4.1 we have

$$\lambda_{\max}(V_e V_e^\top) = \frac{1}{t}\lambda_{\max}(A^\dagger A_e A_e^\top (A^\dagger)^\top) = \frac{1}{t}\left(\max_{y:\|y\|=1} y^\top A^\dagger A_e A_e^\top (A^\dagger)^\top y\right). \tag{24}$$

Let $x = (A^\dagger)^\top y$. Then $\|x\|^2 = y^\top (A^\dagger (A^\dagger)^\top) y = y^\top (A^\top A)^\dagger y$. Note that the maximum eigenvalue of $(A^\top A)^\dagger$ is equal to the reciprocal of the minimum nonzero eigenvalue of $A^\top A$. Since the minimum nonzero eigenvalue of $A^\top A$ is equal to that of $AA^\top (= L)$, we have $\lambda_{\max}((A^\top A)^\dagger) = \frac{1}{\lambda_{n-r+1}(L)}$. Hence, for y with $\|y\| = 1$,

$$\|x\|^2 = y^\top (A^\top A)^\dagger y \leq \lambda_{\max}((A^\top A)^\dagger)\|y\|^2 = \frac{1}{\lambda_{n-r+1}(L)}.$$

Hence, by relaxing the domain of the maximization in (24),

$$t\lambda_{\max}(V_e V_e^\top) \leq \max_{x:\|x\|\leq \frac{1}{\lambda_{n-r+1}(L)}} x^\top A_e A_e^\top x \leq \lambda_{\max}(A_e A_e^\top)\frac{1}{\lambda_{n-r+1}(L)} \leq \frac{h}{\lambda_{n-r+1}(L)}.$$

By $t \geq \frac{h\log(rc)}{\lambda_{n-r+1}(L)}$, we get $\lambda_{\max}(V_e V_e^\top) \leq \frac{h}{t\lambda_{n-r+1}(L)} \leq \frac{1}{\log(rc)}$. $\square$

Claim 7.3 gives the upper bound on $\lambda_{\max}(X_e)$.

Since $A = AA^\dagger A$ and $A^\dagger A = (A^\dagger A)^\top$ by the definition of $A^\dagger$, we have $A = A(A^\dagger A)^\top = AA^\top (A^\dagger)^\top$. Hence,

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{e\in E} tV_e V_e^\top = \sum_{e\in E} A^\dagger A_e A_e^\top (A^\dagger)^\top = A^\dagger\left(\sum_{e\in E} A_e A_e^\top\right)(A^\dagger)^\top \\ &= A^\dagger AA^\top (A^\dagger)^\top = A^\dagger A.\end{aligned}$$

Moreover, by $A = AA^\dagger A$,

$$(\mathbb{E}[Y])^2 = A^\dagger AA^\dagger A = A^\dagger A = \mathbb{E}[Y].$$

This implies that $\mathbb{E}[Y]$ is a symmetric projection matrix, and each eigenvalue of $\mathbb{E}[Y]$ is either 1 or 0. Moreover, since $A^\dagger$ represents a non-singular linear map from image A to $(\ker A)^\top$, $\operatorname{rank}\mathbb{E}[Y] = \operatorname{rank} A^\dagger A = \operatorname{rank} A = r$.

We now apply Theorem 7.1 to $\{X_e\}$. Then, for any $\varepsilon \in [0,1)$,

$$\mathbb{P}\left[\lambda_{n-r+1}(Y) \le (1-\varepsilon)\lambda_{n-r+1}(\mathbb{E}(Y))\right] \le r\left(\frac{\mathrm{e}^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\frac{\lambda_{n-r+1}(\mathbb{E}(Y))}{L}} = r\left(\frac{\mathrm{e}^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\log(rc)},$$

where, in the last equation, we used $\lambda_{n-r+1}(\mathbb{E}[Y]) = 1$ and $\lambda_{\max}(X_e) \le \frac{1}{\log(rc)} =: L$ from Claim 7.3. Since $\lambda_{n-r+1}(\mathbb{E}(Y))$ is the smallest nonzero eigenvalue of $\mathbb{E}(Y)$, $\operatorname{rank} Y < \operatorname{rank}\mathbb{E}[Y]$ holds if and only if $\lambda_{n-r+1}(Y) = 0$. Thus,

$$\mathbb{P}\left[\operatorname{rank} Y < \operatorname{rank}\mathbb{E}[Y] = r\right] \le r\left(\frac{\mathrm{e}^{-\varepsilon}}{(1-\varepsilon)^{1-\varepsilon}}\right)^{\log(rc)} \tag{25}$$

for any $\varepsilon \in [0,1)$. By letting $\varepsilon \to 1$, we obtain $\mathbb{P}\left[\operatorname{rank} Y < r\right] \le \frac{1}{c}$, which yields (23). This completes the proof. □

8 Rigidity of Random Subgraphs

Given a graph G, the *random subgraph* $G(t)$ with parameter $t \in [0,1]$ is the probability distribution over the spanning subgraphs of G obtained by picking each edge with probability t. Given a framework $(G,\boldsymbol{p})$ in $\mathbb{R}^d$, the *random subframework* $(G(t),\boldsymbol{p})$ is defined similarly.

Theorem 7.2 gives the following bound.

Theorem 8.1. *Let G be a rigid graph in $\mathbb{R}^d$ on $n \ge d+1$ vertices and let $(G,\boldsymbol{p})$ be an infinitesimally rigid realization of G in $\mathbb{R}^d$. Let $c \ge 1$. If*

$$1 \ge t \ge \frac{2\log((dn-D)c)}{\lambda_{D+1}(L(G,\boldsymbol{p}))},$$

then $(G(t),\boldsymbol{p})$ is infinitesimally rigid with probability at least $1-\frac{1}{c}$.

Proof. We apply Theorem 7.2 to the rigidity matrix $R(G,\boldsymbol{p})$ with the partition $R(G,\boldsymbol{p}) = [\boldsymbol{r}_e : e \in E(G)]$ corresponding to the edges of G. Note that $\lambda_{\max}(\boldsymbol{r}_e\boldsymbol{r}_e^\top) = 2$ for each $e \in E$, and $\operatorname{rank} R(G,\boldsymbol{p}) = dn - D$. Hence the theorem follows from Theorem 7.2 by choosing $h = 2$ and $r = dn - D$. □

We next consider the rigidity of random subgraphs $G(t)$ (rather than subframeworks $(G(t),\boldsymbol{p})$). The main point is that when we use Theorem 8.1 to analyze the rigidity of $G(t)$, we are free to choose the realization. Recall that the existence of a single infinitesimally rigid realization of G implies that G is rigid. So by choosing $\boldsymbol{p}$ so that $\lambda_{D+1}(L(G,\boldsymbol{p}))$ is larger, we obtain better upper bounds on the rigidity threshold t. To make this idea work we can use our bounds on the d-dimensional algebraic connectivity.

Theorem 8.1 implies the following.

Corollary 8.2. *Let $G = (V, E)$ be a rigid graph in $\mathbb{R}^d$ on $n \geq d + 1$ vertices. Let $c \geq 1$. If*

$$1 \geq t > \frac{2\log((dn - D)c)}{a_d(G)},$$

then $G(t)$ is rigid in $\mathbb{R}^d$ with probability at least $1 - \frac{1}{c}$.

With the help of the following sufficient condition for global rigidity, we shall obtain a globally rigid version of this corollary. We call a graph G *vertex-redundantly rigid* in $\mathbb{R}^d$ if $G - v$ is rigid in $\mathbb{R}^d$ for all $v \in V(G)$.

Theorem 8.3. *[32] Suppose that G is vertex-redundantly rigid in $\mathbb{R}^d$. Then G is globally rigid in $\mathbb{R}^d$.*

Thus we have the following bound for the global rigidity of random subgraphs.

Corollary 8.4. *Let $G = (V, E)$ be a vertex-redundantly rigid graph in $\mathbb{R}^d$ on $n \geq d+2$ vertices. Let $c \geq 1$. If*

$$1 \geq t > \frac{2\log(n(dn - D)c)}{a_d(G) - 1},$$

then $G(t)$ is globally rigid with probability at least $1 - \frac{1}{c}$.

Proof. By Theorem 8.3 it suffices to show that $(G - v)(t)$ is rigid, for all $v \in V$, with probability at least $1 - \frac{1}{c}$.

Consider a vertex $v \in V$. By Lemma 4.5, $a_d(G - v) \geq a_d(G) - 1$. Hence, by the choice of t, Corollary 8.2 implies that $(G - v)(t)$ is rigid with probability at least $1 - \frac{1}{cn}$. Hence, by the union bound, $(G - v)(t)$ is rigid for all $v \in V$ with probability at least $1 - \frac{1}{c}$. □

Recall that $K_{n:d+1}$ denotes the balanced complete $(d+1)$-partite graph on n vertices. This graph is vertex-redundantly rigid in $\mathbb{R}^d$, provided $n \geq (d+2)(d+1)$. To see this observe that in this case each partition class has size at least $d+2$, and hence for each vertex v the graph $K_{n:d+1} - v$ contains a spanning subgraph isomorphic to a complete bipartite graph $K_{d+1,n-(d+1)}$ on more than $\binom{d+2}{2}$ vertices, which is rigid in $\mathbb{R}^d$, see [30, Theorem 61.1.5]. As we defined earlier, s_d denotes the smallest nonzero eigenvalue of the regular d-simplex $(K_{d+1}, \boldsymbol{p}^*)$. By combining Corollaries 8.2, 8.4, and Theorem 5.1 we obtain the following inequalities.

Theorem 8.5. *Suppose that*

$$1 \geq t \geq \left(\frac{4(d+1)^3}{(2d^2+d)s_d} + \varepsilon\right) \cdot \frac{\log n}{n}$$

for some $\varepsilon > 0$. Then $K_{n:d+1}(t)$ is a.a.s. rigid in $\mathbb{R}^d$.

Theorem 8.6. *Suppose that*

$$1 \geq t \geq \left(\frac{8(d+1)^3}{(2d^2+d)s_d} + \varepsilon\right) \cdot \frac{\log n}{n}$$

for some $\varepsilon > 0$. Then $K_{n:d+1}(t)$ is a.a.s. globally rigid in $\mathbb{R}^d$

As we remarked earlier, we have $s_2 = \frac{3}{2}$ and we conjecture that $s_d = 1$ for all $d \geq 3$. By using a computer we have verified this up to $d = 100$.

Theorem 8.5 and Theorem 8.6 can be used to obtain the same upper bounds for the rigidity and global rigidity thresholds of the Erdős-Rényi random graph $G_{n,t}$ in $\mathbb{R}^d$. In the case of rigidity, we get a bound which is essentially the same as that of Király and Theran [26] (assuming that s_d can be bounded by an absolute constant). For global rigidity we have the first upper bound in general dimension as follows.

Theorem 8.7. *Suppose that*

$$1 \geq t \geq \left(\frac{8(d+1)^3}{(2d^2+d)s_d} + \varepsilon \right) \cdot \frac{\log n}{n}$$

for some $\varepsilon > 0$*. Then* $G_{n,t}$ *is a.a.s. globally rigid in* $\mathbb{R}^d$*.*

As we mentioned in the Introduction, the 2-dimensional special case (with a better – essentially best possible – constant) was settled in [21]. It is known that a graph with at least three vertices is globally rigid in $\mathbb{R}^1$ if and only if it is 2-connected. Hence in the 1-dimensional special case the bounds on the 2-connectivity threshold apply.

9 Rigidity of Frames and the Bond-bending Model

In this section we consider the rigidity properties of frames and introduce their stiffness matrices. The concept of k-frame was introduced by W. Whiteley (and also implicitly and independently by Tay in [33]) in his analysis of the so-called body-bar frameworks [37]. These objects, which are closely related to bar-and-joint frameworks, can be used to analyze the 3-dimensional rigidity of (random subgraphs of) squares of graphs, also called molecular graphs, or bond-bending networks.

9.1 (k, ℓ)-frames and tree packings

Let k and ℓ be two positive integers with $k \geq \ell$. A (k,ℓ)-*frame* is a pair $(G, \boldsymbol{b})$, where $G = (V, E)$ is a simple graph and $\boldsymbol{b} : E \to \mathbb{R}^{k \times \ell}$ is a map, which assigns a $k \times \ell$ semi-orthogonal matrix $\boldsymbol{b}_e$ to each edge $e \in E$. A matrix is *semi-orthogonal* if the column vectors form an orthonormal set. The *frame matrix* $F(G, \boldsymbol{b})$ of a (k,ℓ)-frame $(G, \boldsymbol{b})$ is a matrix of size $k|V| \times \ell|E|$ in which the ℓ-tuple of columns associated with edge $e = ij \in E$ has the following entries:

$$\begin{array}{cccccccc} & & & i & & j & & \\ \boldsymbol{f}_e^\top := & [& 0 \dots 0 & \boldsymbol{b}_e^\top & 0 \dots 0 & -\boldsymbol{b}_e^\top & 0 \dots 0 &] \end{array} \qquad (26)$$

In the special case $\ell = 1$ we obtain the familiar concept of a $(k, 1)$-frame, or simply k-*frame*, which is used in the analysis of body-bar frameworks [33, 37]. Note that, for a d-dimensional framework $(G, \boldsymbol{p})$ we have $F(G, \boldsymbol{b}) = R(G, \boldsymbol{p})$ if we take $\boldsymbol{b}_{ij} = \boldsymbol{d}_{ij}$ for each $e = ij \in E$. In this sense, every d-dimensional framework is a special d-frame.

It is easy to see that if we assign the same k-dimensional vector to each vertex of G then (by concatenating these vectors) we obtain a vector in the left kernel of $F(G, \boldsymbol{b})$

for every (k, ℓ)-frame $(G, \boldsymbol{b})$. Hence $\dim\ker F(G, \boldsymbol{b}) \geq k$ and $\operatorname{rank} F(G, \boldsymbol{b}) \leq k|V| - k$. We say that a (k, ℓ)-frame $(G, \boldsymbol{b})$ is *rigid* if $\operatorname{rank} F(G, \boldsymbol{b}) = k|V| - k$.

A (k, ℓ)-frame $(G, \boldsymbol{b})$ is said to be *generic* if each column-induced submatrix of $F(G, \boldsymbol{b})$ has maximum rank over all (k, ℓ)-frame realizations of G. The rigidity of generic (k, ℓ)-frames can be characterized by the following combinatorial property. For a graph G and integer s we use sG to denote the graph obtained from G by replacing each edge with s parallel copies.

Theorem 9.1 ([37]). *Let $(G, \boldsymbol{b})$ be a generic (k, ℓ)-frame. Then $(G, \boldsymbol{b})$ is rigid if and only if ℓG contains k edge-disjoint spanning trees.*

Notice that the special case $\ell = 1$ implies that for a generic (k, ℓ)-frame $(G, \boldsymbol{b})$ the frame matrix $F(G, \boldsymbol{b})$ is a linear representation of the union of k copies of the graphic matroid of G.

As we shall see below, the case $k = 6$, $\ell = 5$ is important in the rigidity analysis of molecular graphs in $\mathbb{R}^3$.

9.2 Stiffness matrices of (k, ℓ)-frames and random subgraphs

In order to analyze the rigidity of (random sub)frames, we shall follow the approach we used for frameworks. First we define the *stiffness matrix* $T(G, \boldsymbol{b})$ of a (k, l)-frame $(G, \boldsymbol{b})$ by letting $T(G, \boldsymbol{b}) = F(G, \boldsymbol{b})F(G, \boldsymbol{b})^\top$, c.f. (2). By expanding this formula, we get

$$T(G, \boldsymbol{b}) = \sum_{e=ij\in E(G)} (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \otimes \boldsymbol{b}_e \boldsymbol{b}_e^\top.$$

Since the kernel of $F(G, \boldsymbol{b})$ is at least k-dimensional, the k-th smallest eigenvalues of $T(G, \boldsymbol{b})$ are zeros. Furthermore, for generic frames, we have that $\lambda_{k+1}(T(G, \boldsymbol{b})) > 0$ if and only if ℓG contains k edge-disjoint spanning trees by Theorem 9.1.

The definition of a random subframe $(G(t), \boldsymbol{b})$ of a (k, ℓ)-frame $(G, \boldsymbol{b})$ is similar to that of random subframeworks. The previous observations and Theorem 7.2 give the following bound for random subframes.

Theorem 9.2. *Let $(G, \boldsymbol{b})$ be a rigid (k, ℓ)-frame with n vertices and let $c \geq 1$. Suppose that*

$$1 \geq t \geq \frac{2\log((kn - k)c)}{\lambda_{k+1}(T(G, \boldsymbol{b}))}.$$

Then $(G(t), \boldsymbol{b})$ is rigid with probability at least $1 - \frac{1}{c}$.

Proof. We apply Theorem 7.2 to $F(G, \boldsymbol{b})$ with $F(G, \boldsymbol{b}) = [\boldsymbol{f}_e : e \in E(G)]$, where $\boldsymbol{f}_e$ is as defined in (26). Note that $\lambda_{\max}(\boldsymbol{f}_e \boldsymbol{f}_e^\top) = 2$ for every $e \in E(G)$ since $\boldsymbol{b}_e$ is semi-orthogonal (and hence the column vectors $\boldsymbol{f}_e$ are pairwise orthogonal with length equal to two). Also $\operatorname{rank} F(G, \boldsymbol{b}) = kn - k$. Hence the statement follows from Theorem 7.2 by putting $h = 2$ and $r = kn - k$. □

Given a graph G and a pair (k, ℓ), we define

$$t_{k,l}(G) := \sup\{\lambda_{k+1}(T(G, \boldsymbol{b})) : (G, \boldsymbol{b}) \text{ is a } (k, \ell)\text{-frame}\}.$$

By using the proof method of Theorem 4.2, one can easily prove that $t_{k,\ell}(G) \leq \lambda_2(L(G))$. Recall that $\lambda_2(L(G))$ denotes the second smallest eigenvalue of the ordinary Laplacian of G. Interestingly, in the case of $t_{k,\ell}(G)$, we can also derive a lower bound by using the same Laplacian eigenvalue.

Lemma 9.3. *Let G be a graph, let k, ℓ be positive integers, and let $\varepsilon \in (0,1)$. Let $q = k/\ell$. Suppose that $\lambda_2(L(G)) > \frac{4q\log(kn-k)}{\varepsilon^2}$. Then*

$$\frac{1-\varepsilon}{q}\lambda_2(L(G)) \leq t_{k,\ell}(G).$$

Proof. We shall consider the case when $\ell = 1$ since the proof can be easily adapted to the general case. Let $\boldsymbol{u}_{i'}$ be the i'-th column vector of the $k \times k$ identity matrix I_k. We shall consider a random k-frame $(G, \boldsymbol{b})$ of G defined by setting $\boldsymbol{b}_e = \boldsymbol{u}_{i'}$ with probability $\frac{1}{k}$ for $1 \leq i' \leq k$ and $e \in E(G)$. Then the corresponding stiffness matrix $T(G, \boldsymbol{b})$ is a probability distribution over matrices of the form $\sum_{e \in E} X_e$, where for each $e = ij \in E$

$$\mathbb{P}\left[X_e = (\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \otimes \boldsymbol{u}_{i'}\boldsymbol{u}_{i'}^\top\right] = \frac{1}{k} \qquad (1 \leq i' \leq k).$$

Since $\|(\boldsymbol{e}_i - \boldsymbol{e}_j) \otimes \boldsymbol{u}_{i'}\| = \sqrt{2}$, we have $\lambda_{\max}(X_e) = 2$. Moreover,

$$\mathbb{E}[T(G,\boldsymbol{b})] = \sum_{e \in E}\sum_{i'=1}^{k} \frac{1}{k}(\boldsymbol{e}_i - \boldsymbol{e}_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \otimes \boldsymbol{u}_{i'}\boldsymbol{u}_{i'}^\top = \frac{1}{k}L(G) \otimes I_k,$$

implying that the smallest nonzero eigenvalue of $\mathbb{E}[T(G,\boldsymbol{b})]$ is $\frac{\lambda_2(L(G))}{k}$. Thus, by the Matrix Chernoff Bound (Theorem 7.1), we have

$$\mathbb{P}\left[\lambda_{k+1}(T(G,\boldsymbol{b})) \leq (1-\varepsilon)\frac{\lambda_2(L(G))}{k}\right] \leq kn \cdot \exp\left(-\frac{\varepsilon^2\lambda_2(L(G))}{4k}\right) < 1,$$

where the last inequality follows from our assumption $\lambda_2(L(G)) > \frac{4q\log(kn-k)}{\varepsilon^2}$. This in particular implies that there is a k-frame $(G, \boldsymbol{b}^*)$ such that $\lambda_{k+1}(T(G,\boldsymbol{b}^*)) \geq \frac{(1-\varepsilon)\lambda_2(L(G))}{k}$. □

Combining Theorems 9.1, 9.2, and Lemma 9.3, we obtain the following.

Corollary 9.4. *Let $q = k/\ell$, $\varepsilon \in (0,1)$, $c \geq 1$, and G a graph on n vertices such that ℓG contains k edge-disjoint spanning trees. Suppose that*

$$\lambda_2(L(G)) > \frac{4q\log(kn-k)}{\varepsilon^2} \text{ and } 1 \geq t \geq \frac{2q\log((kn-k)c)}{(1-\varepsilon)\lambda_2(L(G))}.$$

Then $\ell G(t)$ contains k edge-disjoint spanning trees with probability at least $1 - \frac{1}{c}$.

9.3 Squares of graphs and the rigidity of molecules

The *square* G^2 of a graph G is obtained from G by adding the edges uv for all pairs u, v of non-adjacent vertices of G which have a common neighbour in G (i.e. by connecting second neighbours). Squares of graphs, which are sometimes called *molecular graphs*, can be used in the analysis of the rigidity or flexibility of a molecule in $\mathbb{R}^3$. In this model the molecule is represented by a graph G in which each vertex represents an atom and each edge represents a covalent bond. The edges represent fixed inter-atomic distances. Furthermore, since the angles between incident bonds uw, wv are also fixed, or equivalently, the distance between the atoms u and v is fixed, we add an edge uv for each incident edge pair uw, wv in order to represent these additional constraints. By adding these edges to G, we obtain the square G^2. This model is also known as the *bond-bending model* in the literature, and frequently appears in the context of rigidity transition or percolation [34, 38].

It turns out that for the three-dimensional rigidity of squares there is a purely combinatorial characterization. This result is due to N. Katoh and the second author [25]. Let $\delta(G)$ denote the smallest degree over all vertices of G.

Theorem 9.5. *Let $(G, \boldsymbol{p})$ be a generic 3-dimensional framework with $\delta(G) \geq 2$. Then $(G^2, \boldsymbol{p})$ is rigid if and only if $5G$ contains 6 edge-disjoint spanning trees.*

We can combine the previous theorem with Corollary 9.4 to deduce the next result. To the best of our knowledge, this is the first theoretical result on "bond-bending rigidity" in the random subgraph model.

Corollary 9.6. *Let G be a graph on n vertices with $\delta(G) \geq 2$. Suppose that $5G$ contains 6 edge-disjoint spanning trees. Let $\epsilon \in (0, 1)$ and $c \geq 1$. If*

$$\lambda_2(L(G)) > \frac{4.8 \log(6n)}{\varepsilon^2} \text{ and } 1 \geq t \geq \frac{2.4 \log(6nc)}{(1-\varepsilon)\lambda_2(L(G))},$$

then $(G(t))^2$ is rigid in $\mathbb{R}^3$ with probability at least $1 - \frac{1}{c}$.

We may obtain explicit lower bounds by using results on the algebraic connectivity of some families of graphs. For example, by using the well-known equality $\lambda_2(K_{n,n}) = n$, we can deduce that $(K_{n,n}(t))^2$ is a.a.s. rigid in $\mathbb{R}^3$ if $t \geq \frac{2.4 \log n}{n}$.

Remarkably the bound is still applicable for sparse graphs G with $\Theta(n \log n)$ edges as long as $\lambda_2(L(G)) \geq 9 \log(6nc)$. The reader is referred to [31] for sparse graphs with large spectral gap.

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