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A note on generic rigidity of graphs in higher dimension

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Abstract

The characterization of rigid graphs in $\mathbb{R}^d$ is known only in the low dimensional cases ($d = 1, 2$) and is a major open problem in higher dimensions. In this note we consider the other extreme case when d is close to n, the number of vertices of the graph. It turns out that there is a fairly simple characterization as long as $n - d$ is at most four. We also characterize globally rigid graphs in this range.

1 Introduction

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. It is also called a *realization* of G in $\mathbb{R}^d$. The *length* of an edge $uv \in E$ in (G, p) is the Euclidean distance between $p(u)$ and $p(v)$. We call the framework *rigid in* $\mathbb{R}^d$ if every continuous motion of the vertices in d-space which preserves all edge lengths preserves all pairwise distances. It is *globally rigid in* $\mathbb{R}^d$ if the edge lengths uniquely determine all pairwise distances in the framework.

A framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. We say that G is *rigid* (resp. *globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic d-dimensional realization of G is rigid (resp. globally rigid).

Rigid and globally rigid graphs in $\mathbb{R}^d$ are well-characterized for $d = 1, 2$. It remains an major open problem to extend these results to higher dimensions. There are some partial results in 3-space but the case when $d \geq 4$ is essentially unexplored. We refer the reader to [6, 8] for a general overview of rigid and globally rigid graphs and frameworks.

In this note we consider the case when d is high: so high that it differs from n, the number of vertices, by a constant. The motivation comes from the fact that in some problems (e.g. in the k-vertex-connectivity augmentation problem [2], or in the study of graphs with Colin de Verdière invariant μ [7]) the case when the parameter (k, or μ) is close to n gives rise to interesting and challenging problems.

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It turns out that the characterization of rigid and globally rigid graphs in $\mathbb{R}^d$ is fairly simple if $n - d$ is at most four. Nevertheless, it may be useful to summarize these results: for example, the new characterizations give rise to a new range of dimensions where old or new conjectures can be tested. They also lead to new families of (rigidity) matroids. Furthermore, an additional by-product is the fact that there is a polynomial time FPT algorithm for the problem of testing the rigidity or global rigidity of graphs in $\mathbb{R}^d$, when the parameter is $k := n - d$.

1.1 Preliminary results

We shall use the following results from rigidity theory. Let $G = (V, E)$ be a graph. The d-dimensional *0-extension* operation adds a new vertex to G and d new edges incident with v. The *1-extension* operation removes an edge uw, and adds a new vertex v and $d + 1$ new edges, including vu, vw. The 0-extension operation is known to preserve rigidity in $\mathbb{R}^d$. The 1-extension operation preserves rigidity as well as global rigidity in $\mathbb{R}^d$. We call the corresponding inverse operations *0-reduction* and *1-reduction*, respectively.

The *cone* of G is obtained from G by adding a new vertex v and new edges from v to every vertex of G. The graph G is rigid (resp. globally rigid) in $\mathbb{R}^d$ if and only if the cone of G is rigid (resp. globally rigid) in $\mathbb{R}^{d+1}$. We shall call these results, due to Whiteley [9] and Connelly and Whiteley [1], respectively, the Coning theorems.

2 Rigid graphs

Let $H = (V, E)$ be a graph. The complement graph and the maximum degree of H will be denoted by $\overline{H}$ and $\Delta(H)$, respectively. The degree of a vertex set X in H is denoted by $deg_H(X)$. If $X = \{v\}$ then we use $deg_H(v)$.

Since rigid graphs are dense, when d is close to n, it will be convenient to work with the complement graph. A graph on at most $d + 1$ vertices is rigid in $\mathbb{R}^d$ if and only if it is complete. Hence we shall consider the case $n \geq d + 2$.

Lemma 2.1. *Let $G = (V, E)$ be a rigid graph on n vertices in $\mathbb{R}^d$ and let $n = d + k$ for some $k \geq 2$. Then*

$$\Delta(\overline{G}) \leq k - 1, \tag{1}$$

$$|E(\overline{G})| \leq \binom{k}{2}, \tag{2}$$

and

$$deg_{\overline{G}}(\{u, v\}) \leq 2k - 3, \text{ for all } u, v \in V. \tag{3}$$

Proof. The minimum degree of a rigid graph on at least $d + 1$ vertices is at least d. Hence the maximum degree of its complement is at most $n - 1 - d = k - 1$. The number of edges of a rigid graph on at least d vertices is at least $dn - \binom{d+1}{2}$. Hence the complement graph has at most

$$\binom{n}{2} - dn + \binom{d+1}{2} = \binom{d+k}{2} - d(d+k) + \binom{d+1}{2} = \binom{k}{2}$$

edges. Finally, suppose, for a contradiction, that G is rigid but we have $deg_{\overline{G}}(\{u,v\}) \geq 2k-2$ for some pair $u, v \in V$. Let $X = \{u, v\}$. Then the total number of edges incident with X in G is at most $2(n-2)+1-deg_{\overline{G}}(X) = 2(d+k)-3-(2k-2) \leq 2d-1$. Since $G-X$ has at least d vertices, its rank (i.e. number of independent length constraints) is at most $d(n-2)-\binom{d+1}{2}$, which implies that the rank of G is at most $dn-\binom{d+1}{2}-1$. Thus G cannot be rigid (it has at least one degree of freedom). This completes the proof. □

Let J_{12} denote the graph obtained from the complete graph K_{12} by deleting a perfect matching (i.e. six disjoint edges). By computing the rank of the rigidity matrix of an appropriate 8-dimensional realization of J_{12} we can deduce the following fact (see the Appendix for a proof):

Lemma 2.2. *The graph J_{12} is rigid in $\mathbb{R}^8$.*

It turns out that J_{12} plays an important role in the proof of the next theorem, which is the main result of this section.

Theorem 2.3. *Let $G = (V, E)$ be a graph on n vertices and let $n = d + k$ for some $d \geq 1$, $2 \leq k \leq 4$. Then G is rigid in $\mathbb{R}^d$ if and only if it satisfies (1), (2), and (3).*

Proof. Necessity follows from Lemma 2.1. We prove sufficiency for each value $k \in \{2, 3, 4\}$ in increasing order. We shall assume that G satisfies (1), (2), and (3). Note that if $k \in \{2, 3\}$ then (3) follows from (1) and (2). If $k = 2$ then either G is complete (and hence rigid) or $G = K_{d+2} - e$. In the latter case G can be obtained from the rigid graph K_{d+1} by a 0-extension. Hence G is rigid. If $k = 3$ then G can be obtained from K_{d+3} be deleting at most three edges (by (2)) which are not allowed to form a star on three edges (by (1)). By deleting additional edges, if necessary, we may assume that $|E(\overline{G})| = 3$. First suppose that the edges of $\overline{G}$ are not pairwise disjoint. Then G contains a subgraph H isomorphic to K_{d+1}. By applying two 0-extensions to H we can obtain G. If the edges of $\overline{G}$ are pairwise disjoint then G can be obtained from K_{d+1} by two 1-extensions. In each of these cases G is rigid.

Finally, suppose that $k = 4$. We prove the theorem by induction on n. To verify the base case we can observe that if $n = 5$ (and hence $d = 1$) then (1) and (3) imply that G is connected, and hence rigid in $\mathbb{R}^1$. Consider the general case. If G has a vertex v with $deg_{\overline{G}}(v) = 0$ then we can apply induction to $G - v$ by using the Coning theorem. Note that in this situation we decrease n and d by one, while k is unchanged. Next suppose that G has a vertex v with $deg_{\overline{G}}(v) = 3$. Then v has degree d in G. Now we look at $G' = G - v$, which is obtained from G by a 0-reduction. To show that G' is rigid we use the fact that the theorem holds for $k = 3$. Note that the complement of G' has at most three edges and it has maximum degree two (for otherwise G violates (3)). So G', and hence also G is rigid.

It remains to consider the case when each vertex in the complement of G has degree one or two. Suppose there is a vertex v with $deg_{\overline{G}}(v) = 2$. Let vx, vy be the edges incident with v in $\overline{G}$. If $\overline{G}$ has an edge f which is disjoint from x, y then we can apply induction to $G' = G - v + f$. Observe that G' is obtained from G by a 1-reduction, and this operation decreases n and k by one, and the number of edges in the complement is decreased by three. Thus G' satisfies the conditions for $k = 3$, which gives that G' as well as G are rigid.

Now suppose that every edge of $\overline{G}$ is incident with v, x, or y. Then, since each vertex has degree one or two in $\overline{G}$ and $n \geq 6$, there exists a vertex z with $deg_{\overline{G}}(v) = 0$, a contradiction.

Finally we consider the case when $\overline{G}$ is a perfect matching. Then either $|E(\overline{G})| = 6$ and G is isomorphic to J_{12} (in which case we are done by Lemma 2.2) or $\overline{G}$ has less than six edges. In the latter case we can delete an edge from G and create a vertex v with $deg_{\overline{G}}(v) = 2$, preserving (1), (2), and (3). After the edge deletion we can apply the argument we used above to show that G is rigid. □

Remark 1 It may be possible to extend the theorem to a few more values $k \geq 5$. However, the next case $k = 5$ already includes the double banana graph (for $d = 3$), which shows that simple degree conditions are probably insufficient. On the other hand, at least in dimension 3, one can check by counting edges that up to $k = 8$ the graph must have at least one vertex of degree at most four. Thus the reduction methods used above should lead to some kind of characterization. This bound is tight, as shown by the graph of the icosahedron, which is minimally rigid and has minimum degree five ($d = 3$, $k = 9$).

Remark 2 Inequality (2) and the proof of Theorem 2.3 show that there is a very simple polynomial time algorithm for testing whether a graph G on n vertices is rigid in $\mathbb{R}^d$, if $k = n - d$ is fixed. Since the edge set of $\overline{G}$ spans at most $k(k-1)$ vertices, most of the vertices in G are of degree $n - 1$. We can delete these vertices and use the Coning theorem to reduce the graph (and the dimension) to a smaller instance, where the number of vertices and the dimension are bounded by a function of k. In this case even a symbolic rank computation of the rigidity matrix is efficient[1].

3 Globally rigid graphs

In this section we consider the globally rigid version of the problem. A globally rigid graph in $\mathbb{R}^d$ on at most $d + 2$ vertices is globally rigid if and only if it is complete. Hence we shall consider the case when $n \geq d + 3$.

Let J_{10} denote the graph obtained from K_{10} by deleting a perfect matching. By rank computations we can verify the following fact (see the Appendix for more details):

Lemma 3.1. *The graph J_{10} is globally rigid in $\mathbb{R}^6$.*

[1]This algorithm is called an FPT algorithm, since its running time can be bounded by a polynomial of n plus a(n exponential) function of k.

We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all $e \in E(G)$. A graph G is said to be $(d+1)$-connected if $G - S$ is connected for all $S \subseteq V(G)$ with $|S| \leq d$. Hendrickson [5] proved that if G is a globally rigid graph in $\mathbb{R}^d$ on at least $d+2$ vertices then G is redundantly rigid and $(d+1)$-connected.

The next result shows that these necessary conditions together are also sufficient to imply global rigidity in $\mathbb{R}^d$ when $n \leq d+4$, for all $d \geq 1$.

Theorem 3.2. *Let $G = (V, E)$ be a graph on n vertices and let $n = d + k$ for some $d \geq 1$, $3 \leq k \leq 4$. Then G is globally rigid in $\mathbb{R}^d$ if and only if G is redundantly rigid in $\mathbb{R}^d$ and $(d+1)$-connected.*

Proof. Necessity follows from Hendrickson's theorem. Suppose that G is redundantly rigid and $(d+1)$-connected. Let $k = 3$. Then, since $G - e$ is rigid for every edge $e \in E$, Lemma 2.1 implies that $\overline{G}$ has at most two edges, and these edges are disjoint. Thus G can be obtained from K_{d+3} by deleting (at most) two disjoint edges. Hence G can be obtained from K_{d+2} by a 1-extension operation. So G is globally rigid in $\mathbb{R}^d$.

Next consider the case $k = 4$. We can use Theorem 2.3 to deduce that the redundant rigidity and $(d+1)$-connectivity of G together are equivalent to the following conditions:

$$\Delta(\overline{G}) \leq 2, \tag{4}$$

$$|E(\overline{G})| \leq 5, \tag{5}$$

and

$$\overline{G} \text{ contains no } C_4. \tag{6}$$

We prove the theorem by induction on n. To verify the base case we can observe that if $n = 5$ then $d = 1$, in which case the fact that global rigidity in $\mathbb{R}^1$ is equivalent to 2-connectivity implies that G is globally rigid. Consider the general case. If G has a vertex v with $deg_{\overline{G}}(v) = 0$ then we can apply induction to $G - v$ by using the globally rigid Coning theorem. Note that in this situation we decrease n and d by one, while k is unchanged.

It remains to consider the case when each vertex in the complement of G has degree one or two. First suppose that there is a vertex v with $deg_{\overline{G}}(v) = 2$. Now $\overline{G}$ is a collection of paths and cycles with at most five edges in total and with no four-cycles by (6). It is easy to check that in this graph there exists a vertex v of degree two and an additional edge f, which is disjoint from the neighbours of v, for which $\overline{G} - v - f$ is a matching.

Let $G' = G - v + f$. Observe that G' satisfies the conditions of the theorem for $k = 3$, and hence it is globally rigid in $\mathbb{R}^d$. Since G can be obtained from G' by a 1-extension, it is also globally rigid in $\mathbb{R}^d$.

Finally we consider the case when $\overline{G}$ is a matching of size at most five. Then either G is isomorphic to J_{10} (in which case we are done by Lemma 3.1) or the matching has size at most four. In the latter case we can remove an edge from G without violating (4), (5), or (6), and then we we can apply the arguments used above to show that G is globally rigid. $\square$

Remark 3 We can also test global rigidity in $\mathbb{R}^d$ in polynomial time if k is fixed by an algorithm similar to that of rigidity testing. In this case the symbolic computation is more involved, as one has to compute the rank of the rigidity matrix as well as a corresponding stress matrix, see [3, Section 5.1].

4 Concluding remarks

Theorem 2.3 gives rise to a characterization of the d-dimensional rigidity matroid $\mathcal{R}_d(K_n)$ with $n \leq d+4$, since it describes the spanning sets (and bases). Thus it can be used to deduce various properties of this matroid. For example, we can obtain a complete list of circuits as follows.

Suppose that $n = d+4$ (the other cases are simpler). Theorem 2.3 says that a subgraph H of K_n is dependent if $K_n - E(H)$ does not contain a subgraph with six edges, maximum degree three, and which satisfies (3). A relatively simple case analysis shows that the minimal subgraphs with respect to these properties (i.e. the circuits of $\mathcal{R}_d(K_n)$) belong to one of the following categories: (i) they contain all but exactly five edges of K_n, or (ii) they are isomorphic to K_{d+3} minus two disjoint edges, or (iii) they are isomorphic to K_{d+2}.

Observe that each of these graphs is rigid in $\mathbb{R}^d$. It follows that the smallest non-rigid circuits of the d-dimensional rigidity matroid must have at least $d+5$ vertices. This bound is tight, as shown by the double banana graph (in $\mathbb{R}^3$) and its cones[2].

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Appendix

First we prove Lemma 2.2 by describing a realization of J_{12} in $\mathbb{R}^8$ that can be used to verify, by computing the rank of its rigidity matrix [8], that J_{12} is (infinitesimally rigid [8], and hence) rigid in $\mathbb{R}^8$. The rows of following matrix are indexed by the vertices of J_{12}. The entries in each row are the co-ordinates of the corresponding vertex.

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The rows of next matrix are indexed by the vertices of J_{10} and encode an infinitesimally rigid 6-dimensional realization (J_{10}, p) in a similar way.

$$\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 1 & 1 & 0\\
1 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 0\\
0 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1\\
1 & 1 & 1 & 0 & 1 & 1\\
0 & 1 & 0 & 0 & 1 & 1\\
1 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}$$

To verify that J_{10} is globally rigid in $\mathbb{R}^6$ it suffices to find an equilibrium stress on (J_{10}, p) whose stress matrix has rank 3, see [1, Theorem 5]. The following stress matrix will do. (The ordering of the rows of the stress matrix corresponds to that of the realization matrix above.) This proves Lemma 3.1.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1\\
0 & 0 & 2 & 2 & -2 & 0 & 0 & 0 & 0 & -2\\
0 & 2 & 2 & 0 & -2 & 0 & -1 & 1 & -1 & -1\\
0 & 2 & 0 & -2 & 0 & 0 & -1 & 1 & -1 & 1\\
0 & -2 & -2 & 0 & 2 & 0 & 1 & -1 & 1 & 1\\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1\\
0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -1\\
1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & 1 & 0\\
-1 & -2 & -1 & 1 & 1 & 1 & 1 & -1 & 0 & 1
\end{pmatrix}$$