Complexity of the NTU International Matching Game

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Abstract

Motivated by the real-world problem of international kidney exchange, [Biró et al., Generalized Matching Games for International Kidney Exchange, 2019] introduced a generalized transferable utility matching game featuring a partition of the node set into countries, and analyzed its complexity. We explore the non-transferable utility (NTU) variant of the game and prove computational complexity results about the weak and strong cores under various assumptions on the countries.

1 Introduction

The NTU International Matching Game is defined by a graph $G = (V; E)$ and a partition $V = V_1 \cup V_2 \cup \cdots \cup V_m$ of V. There are m players, or countries, and the nodes in V_i belong to player i. Given a matching M in G, the utility of M for country i is given by

$$u_i(M) = |V(M) \cap V_i|,$$

where $V(M)$ denotes the node set of M.

A coalition $(i_1, i_2, \ldots, i_k)$ of countries is *strongly blocking* for matching M if there exists a matching M_0 in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $u_{i_j}(M_0) > u_{i_j}(M)$ for every $j \in [k]$. Similarly, a coalition $(i_1, i_2, \ldots, i_k)$ of countries is *weakly blocking* for a matching M if there exists a matching M_0 in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $u_{i_j}(M_0) \geq u_{i_j}(M)$ for every $j \in [k]$, and $u_{i_j}(M_0) > u_{i_j}(M)$ for at least one $j \in [k]$.

The International Matching Game, in a TU version, was introduced by Biró et al. [3] motivated by the kidney exchange problem. In that setting, the nodes correspond to patient-donor pairs, and edges represent the possible pairwise exchanges. The partition $V_1, \ldots, V_m$ can be thought of as a partitioning of the patients according to countries, but we can also think of hospitals instead of countries, with each hospital

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being interested in successful transplants for their own patients. The problem is studied from the point of view of cooperative games: are there matchings that are acceptable for all possible coalitions of countries?

In the NTU setting, a matching M is in the *weak core* if there is no strongly blocking coalition for it, and it is in the *strong core* if there is no weakly blocking coalition. It is easy to see that if M is in the strong core, then it is also in the weak core.

We study the computational complexity of membership and non-emptiness of the weak and strong cores, under various assumptions on the countries. In Section 2, we show that without further restrictions, it is NP-hard to decide whether a given matching is in the weak/strong core. We then study the problems with the restriction that the number of countries is constant (Section 3) or the size of countries is constant (Section 4). In the latter case, it is still hard to decide non-emptiness of both the weak and strong cores. In contrast, we show in Section 5 that the weak core is always non-empty if the size of each country is 2.

Related work

In [1, 2], Ashlagi and Roth considered individual rationality in the multi-hospital kidney exchange problem. In our terms, individual rationality means that there is no single blocking country. They also studied the problem of incentive-compatibility, i.e., whether hospitals are motivated to underreport their patient-donor pairs. Carvalho et al. [4] studied the properties of Nash-equilibria in the two-hospital case. Gourvès, Monnot and Pascual [5] considered a weighted bipartite graph model, where the profit of an edge in the matching is divided between the two endpoints according to a fixed ratio. They proved that it is NP-hard to decide if there is an individually rational maximum weight matching.

2 General hardness

In this section, we prove that it is hard to decide whether a given matching is in the weak or strong core. As we will see later, the problem can be decided in polynomial time if the number of countries is constant.

2.1 Membership in the weak core

Theorem 1. *It is coNP-complete to decide if a given matching M is in the weak core.*

The problem is in coNP because it can be decided in polynomial time if a given matching M_0 is a good witness for a given coalition being strongly blocking for M. We prove coNP-completeness of a hypergraph problem that is equivalent to a special case of our problem.

Let $G = \cup_{i=1}^{n} G_i$ be a union of complete graphs. The nodes of G are partitioned into m countries such that every country has one or two nodes in every complete graph. A max-size matching M in G leaves exactly one node unmatched in every

complete graph of odd size. We assume that every country has at least one node that is unmatched in M.

We define a hypergraph H_M with a weight function w_M on the hyperedges. The nodes of H_M are $[n] := \{1, 2, ..., n\}$ where i corresponds to the complete graph G_i. The hyperedges of H_M correspond to the countries. A node i belongs to a hyperedge A if the country corresponding to A has one node in G_i. The weight $w_M(A)$ is the number of nodes belonging to the country corresponding to A that is left unmatched in M minus 1. The motivation for this definition is the following: a coalition $(i_1, i_2, \ldots, i_k)$ of countries is blocking if and only if there is a matching in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that every country A of the coalition has at most $w_M(A)$ unmatched nodes. Note that the sum of weights are the number of odd sized complete graphs minus the number of countries.

Definition 2. A subhypergraph S of H is called a *blocking subhypergraph* if the number of odd-degree nodes in S is at most the sum of weights of the edges of S.

Claim 3. *A coalition of countries $B = (i_1, i_2, ...i_k)$ is blocking with respect to M if and only if the hyperedges corresponding to these countries form a blocking subhypergraph S.*

Proof. The subgraph $G_B = G[V_{i1} \cup V_{i2} \cup ... \cup V_{ik}]$ of G is also a union of complete graphs. The number of odd-degree nodes in S equals the number of odd size complete graphs in G_B which is the number of nodes left unmatched in any max-size matching of G_B. B is blocking if and only if there is a matching in G_B such that every country A in B has at most $w_M(A)$ unmatched nodes. From this it is clear that if B is blocking, then the inequality holds for S. If S is blocking, then B is blocking because every country has a node in every complete graph, therefore for every odd size complete graph we can choose which country should the unmatched node belong to. So we can distribute the nodes left unmatched in a max-size matching of M_B (their number is the number of odd-degree nodes in S) among the countries in B in a way that a country A has at most $w_M(A)$ unmatched nodes.   □

Claim 4. *Suppose we have a hypergraph H with n nodes and m hyperedges, and a weight function w on the hyperedges such that the sum of weights equals the number of odd-degree nodes in H minus m. Then there is a graph G and a max-size matching M of G with the following properties: G is a union of n complete graphs, and the nodes of G are partitioned into m countries, such that every complete graph contains one or two nodes from every country. Furthermore, there is at least one node from every country that is left unmatched by M, and $H = H_M$.*

Proof. If a node i belongs to the hyperedge A in H, then the corresponding complete graph G_i of G should contain one node that belongs to the country corresponding to A; otherwise it should contain two. The number of nodes left unmatched by any max-size matching is the number of odd size complete graphs in G which is equal to the number of odd-degree nodes in H which is equal to the sum of weights plus m. The number of nodes in a country corresponding to the hyperedge A left unmatched by M should be $w(A) + 1$ which can be achieved since for every odd size complete

graph we can choose which country should the unmatched node belong to (because every complete graph contains a node from every country). $\qquad\square$

Theorem 5. *For a hypergraph H with a weight function w on the hyperedges such that the sum of weights equals the number of odd-degree nodes in H minus the number of hyperedges, it is NP-complete to decide if there is a blocking subhypergraph S of H.*

We reduce from the 1-in-3 SAT problem. First we describe the construction. We are given a boolean formula with k clauses and n variables. For every clause we define a clause gadget which consists of

- three edges C_{i1}, C_{i2} and C_{i3}, such that the intersection of any two of them is $C_{i1} \cap C_{i2} \cap C_{i3}$, this is disjoint from any other edge, and $|C_{i1} \cap C_{i2} \cap C_{i3}| = c$. These edges represent the 3 literals l_{i1}, l_{i2} and l_{i3} of the ith clause.

- a fourth edge C_i such that $C_i \cap C_{ij}$ is disjoint from any other edge, its size is c' for $j = 1, 2, 3$, and $C_i \setminus (C_{i1} \cup C_{i2} \cup C_{i3}) = \{c_i, c_{i+1}\}$, where $c_{k+1} = c_1$. Note that $C_i \cap C_{i-1} = c_i$, $C_i \cap C_{i+1} = c_{i+1}$ and C_i is disjoint from C_j if $j \neq i-1, i, i+1$, where $C_0 = C_k$.

For every variable x_i, we define a variable gadget which consists of three edges, $X_i, \overline{X_i}$, and Y_i such that $|X_i \cap \overline{X_i}| = x$, $X_i \cap \overline{X_i} \subseteq Y_i$, $|Y_i \cap X_i| = |Y_i \cap \overline{X_i}| = x + x'$ and $Y_i \setminus (X_i \cup \overline{X_i}) = \{y_i, y_{i+1}\}$ where $y_{n+1} = y_1$. Note that $Y_i \cap Y_{i-1} = y_i$, $Y_i \cap Y_{i+1} = y_{i+1}$ and Y_i is disjoint from Y_j if $j \neq i-1, i, i+1$, where $Y_0 = Y_n$.

The clause gadgets intersect with the variable gadgets in the following way.

- The edge C_{ij} intersects X_i in 2 nodes if it represents the variable x_i in unnegated form, and it intersects $\overline{X_i}$ in 2 nodes if it represents the variable x_i in negated form. These intersections are disjoint from any other edges.

- The edges C_{ij}, X_i and $\overline{X_i}$ do not contain any nodes which do not belong to an intersection described previously.

Now we describe the weights. Let $w(C_i) = c'$ for $i \in [k]$, $w(Y_i) = x'$ for $i \in [n]$, and let the weight of every other edge be zero.

The sum of odd-degree nodes is $kc + nx$ (only the nodes of $C_{i1} \cap C_{i2} \cap C_{i3}$ and the nodes of $X_i \cap \overline{X_i}$ have odd degree, since their degree is 3, while all the other nodes have degree 2), and the number of edges is $4k + 3n$ (there are 4 edges in every clause gadget and 3 edges in every variable gadget). The sum of weights (which is $kc' + nx'$) has to be the number of odd-degree nodes minus the number of edges which is $k(c - 4) + n(x - 3)$, so we set $c' = c - 4$ and $x' = x - 3$. (We can set $c = 6$ and $x = 5$ for example).

Notation 6. Let $\overline{Z_i} = (\overline{X_i} \cap Y_i) \setminus (\overline{X_i} \cap X_i)$ and $Z_i = (X_i \cap Y_i) \setminus (\overline{X_i} \cap X_i)$.

Lemma 7. *For every hyperedge A with $w(A) > 0$ that belongs to a blocking subhypergraph S, there are at least $w(A)$ nodes in it (not contained in other hyperedges with positive weight) that have odd degree in S.*

Proof. The only edges with positive weight are Y_i for $i \in [n]$, $(w(Y_i) = x')$, and C_i for $i \in [k]$, $(w(C_i) = c')$.

If $A = Y_i$ for some $i \in [n]$, then Z_i and $\overline{Z_i}$ both have size x' and since they are subsets of Y_i which belongs to S, nodes in both of these sets only have even degree in S if X_i and $\overline{X_i}$ both belong to S. But then the nodes in $Y_i \cap X_i \cap \overline{X_i}$ have degree 3, and $|Y_i \cap X_i \cap \overline{X_i}| = x = x' + 3$.

If $A = C_i$ for some $i \in [k]$, then the set of nodes $C_{ij} \cap C_i$ for $j = 1, 2, 3$ have size c' and nodes in all three of these sets only have even degree in S if C_{ij} belongs to S for $j = 1, 2, 3$. But then the nodes in $C_{i1} \cap C_{i2} \cap C_{i3}$ have degree 3 and $|C_{i1} \cap C_{i2} \cap C_{i3}| = c = c' + 4$. □

Lemma 8. *Let us call the nodes that do not belong to Z_i or $\overline{Z_i}$ for any $i \in \{1, 2, ...n\}$ and do not belong to $C_i \cap C_{ij}$ for any $i \in [k]$, $j \in \{1, 2, 3\}$ ordinary.*

(i) If Y_i belongs to a blocking subhypergraph, then the nodes of Y_i that have odd degree in the subhypergraph are either Z_i or $\overline{Z_i}$, so the number of such nodes is $x' = w(Y_i)$.

(ii) If C_i belongs to a blocking subhypergraph, then the nodes of C_i that have odd degree in the subhypergraph are $C_i \cap C_{ij}$ for $j = 1, 2$ or 3, so the number of such nodes is $c' = w(C_i)$.

(iii) Ordinary nodes can not have odd degree in a blocking subhypergraph.

Proof. (iii) In a blocking subhypergraph S, the number of odd-degree nodes in S is at most the sum of weights in S, and Lemma 7 shows that in every edge A in S with positive weight, there are $w(A)$ nodes that have odd degree in S. This means that there cannot be more nodes in A with odd degree in S. Furthermore besides the $w(A)$ nodes for every edge A in S with positive weight that have odd degree in S, no other node can have odd degree in S so (iii) holds. (i) and (ii) follow from the proof of Lemma 7. □

Lemma 9. *If for two edges A and A' there is an ordinary node in $A \cap A'$ that does not belong to any other edge, then if A belongs to a blocking subhypergraph, A' has to belong to it too.*

Proof. This is a direct consequence of point (iii) of Lemma 8. □

Lemma 10. *If there is a blocking subhypergraph S, then it contains exactly one of X_i and $\overline{X_i}$ for $i = 1, \ldots, n$.*

Proof. If C_m is in S for some $m \in [k]$, then C_{mj} has to be in S too for some $j \in \{1, 2, 3\}$, otherwise C_m would have at least $3c'$ odd-degree nodes in S.

If C_{ij} is in S, then there exists a $t \in [n]$ such that $C_{ij} \cup X_t$ or $C_{ij} \cup \overline{X_t}$ is not empty, moreover it contains an ordinary node not contained in any other edge, so Lemma 9 implies that X_t or $\overline{X_t}$ has to belong to S too.

If X_t or $\overline{X_t}$ is in S, then Y_t has to be in S as well. Indeed, if Y_t does not belong to S, then the nodes of Z_t or $\overline{Z_t}$ have odd degree in S but then the sum of odd-degree nodes in S is greater than the sum of weights of the edges of S because of Lemma 7.

If Y_t belongs to S, then Y_i belongs to S for every $i \in [n]$. This is because $Y_t \cap Y_{t+1}$ contains an ordinary node that only belongs to these edges, so from Lemma 9 Y_{t+1} has to belong to S; then $Y_{t+1} \cap Y_{t+2}$ contains an ordinary node that only belongs to these edges, and so on.

If Y_i belongs to S, then exactly one of X_i and $\overline{X_i}$ belongs to S. This is because point (i) of Lemma 8.

$\square$

Lemma 11. *The 1-in-3 SAT instance is satisfiable if and only if the constructed hypergraph has a blocking subhypergraph.*

Proof. First suppose the 1-in-3 SAT instance is satisfiable. We prove that the subhypergraph S that consists of the following edges is a blocking subhypergraph:

- X_i and every C_{lm} intersecting X_i for x_i set to false,

- $\overline{X_j}$ and every C_{lm} intersecting $\overline{X_j}$ for x_j set to true,

- Y_i for $i \in [n]$,

- C_i for $i \in [k]$.

There are exactly two false literals in every clause, therefore in every clause gadget exactly two of C_{i1}, C_{i2} and C_{i3} are in the subhypergraph, so the nodes in $C_{i1} \cap C_{i2} \cap C_{i3}$ have even degree (they have degree 2). It is easy to see that all the other nodes in C_{ij} have degree 2 if C_{ij} is in the subgraph. If the jth literal of clause i was set to true, then the nodes of $C_{ij} \cap C_i$ have degree 1, and all the other nodes of C_i have degree 2. Therefore the number of nodes in a clause gadget with odd degree is c'.

For every $i \in [n]$, exactly one of X_i and $\overline{X_i}$ is chosen to be in the subhypergraph, say it is X_i. Then the nodes in $\overline{Z_i}$ have degree 1 (if $\overline{X_i}$ was chosen, then the nodes in Z_i have degree 1). Either way, the number of odd-degree nodes in a variable gadget is x' since it is easy to see that all the other nodes in the variable gadget have even degree.

We obtain that the total number of odd-degree nodes in the subhypergraph is $kc' + nx'$ which is equal to the sum of weights, which means that this subhypergraph is a blocking subhypergraph.

Now suppose that there is a blocking subhypergraph S in the constructed hypergraph. The following is a satisfying assignment of the 1-in-3 SAT instance: we set x_i to be false if and only if X_i belongs to S.

Because of Lemma 10, exactly one of x_i and $\overline{x_i}$ is set to false. For every C_{ij}, there is a t such that $X_t \cap C_{ij}$ or $\overline{X_t} \cap C_{ij}$ is not empty and contains an ordinary node that is only contained in these edges. Because of Lemma 9, C_{ij} belongs to S if and only if X_t belongs to S (if $X_t \cap C_{ij}$ was nonempty) or $\overline{X_i}$ belongs to S (if $\overline{X_t} \cap C_{ij}$ was nonempty). This means that C_{ij} belongs to S if and only if the literal l_{ij} it corresponds to is set to false.

If C_{ij} belongs to S, then C_i has to belong to S too, otherwise the nodes of $C_{ij} \cap C_i$ would have odd degree in S, but then the sum of odd-degree nodes in S would be greater than the sum of weights of the edges of S because of Lemma 7.

If C_i belongs to S, then C_m has to belong to S too for every $m \in [k]$. This is because $C_i \cap C_{i+1}$ contains an ordinary node that only belongs to these edges, so from Lemma 9 C_{i+1} has to belong to S, $C_{i+1} \cap C_{i+2}$ contains an ordinary node that only belongs to these edges, and so on.

If C_m belongs to S, then exactly two of C_{mj} for $j = 1, 2, 3$ belongs to S. This follows from point (ii) of Lemma 8.

The above statements prove that exactly two literals are set to false in every clause.

$\square$

2.2   Membership in the strong core

Theorem 12. *It is coNP-complete to decide if a given matching is in the strong core.*

We are given a graph $G = \bigcup\limits_{i=1}^{n} G_i$ and a matching M described in section 2.1 except that we do not assume that there is an unmatched node in every country. We define a hypergraph H in the same way as in section 2.1, but we define a different weight function w'_M on the hyperedges. Let $w'_M(A)$ be the number of nodes belonging to the country corresponding to A that is left unmatched in M.

Definition 13. We are given a hypergraph H with a weight function w on the hyperedges. A subhypergraph S of H is called a *weakly blocking subhypergraph* if the number of odd-degree nodes in S is less than the sum of weights of the edges of S.

Claim 14. *A set of countries $B = (i_1, i_2, \ldots, i_k)$ is weakly blocking with respect to M if an only if the hyperedges corresponding to these countries form a weakly blocking subhypergraph S.*

Proof. The subgraph $G_B = G[V_{i1} \cup V_{i2} \cup \cdots \cup V_{ik}]$ of G is also a union of complete graphs. The number of odd-degree nodes in S equals the number of odd-size complete graphs in G_B which is the number of nodes left unmatched in any max-size matching of G_B. B is weakly blocking if and only if there is a matching in G_B such that every country A in B has at most $w'_M(A)$ unmatched nodes, and at least one country has less than $w'_M(A)$. From this it is clear that if B is blocking, the inequality holds for S. If S is blocking, then B is blocking because every country has a node in every complete graph, therefore for every odd size complete graph we can choose which country should the unmatched node belong to. So we can distribute the nodes left unmatched in a max-size matching of M_B (their number is the number of odd-degree nodes in S) among the countries in B in a way that a country A has at most $w'_M(A)$ unmatched nodes and at least one country has less than $w'_M(A)$. $\square$

Claim 15. *Suppose we have a hypergraph H with n nodes and m hyperedges, and a weight function w' on the hyperedges such that the sum of weights equals the number of odd-degree nodes in H. Then there is a graph G and a max-size matching M in G*

with the following properties: G is a union of n complete graphs, and the nodes of G can be partitioned into m countries, such that every complete graph contains one or two nodes from every country, and $H = H_M$, $w' = w'_M$.

Proof. If a node i belongs to the hyperedge A in H, then the corresponding complete graph G_i of G should contain one node that belongs to the country corresponding to A, otherwise it should contain two. The number of nodes left unmatched by any max-size matching is the number of odd size complete graphs in G, which is equal to the number of odd-degree nodes in H, which is equal to the sum of weights. The number of nodes in a country corresponding to the hyperedge A left unmatched by M should be $w'(A)$ which can be achieved since for every odd size complete graph we can choose which country should the unmatched node belong to (because every complete graph contains a node from every country). □

Theorem 16. *For a hypergraph H with a weight function w' on the hyperedges such that the sum of weights equals the number of odd-degree nodes in H, it is NP-complete to decide if there is a weakly blocking subhypergraph S of H.*

Proof. We reduce from the 1-in-3 SAT problem. We will use the construction in the proof of Theorem 5 with slight modifications. The weights are the same except that we change the weight of Y_1 to $w'(Y_1) = x' + 1$ ($w'(A) = w(A)$ for all edges $A \neq Y_1$ of H.) Let $\sum w' = \sum w'(A)$: A is an edge of H. The new hypergraph H' is the same as H except we add an extra edge F of zero weight to the construction, which contains all the odd-degree nodes of H and besides these, it contains $\sum w'$ extra nodes that are not contained in any other edge. In this modified hypergraph H', the odd-degree nodes are these $\sum w'$ extra nodes. However, since F has zero weight, their number is equal the sum of weights in H', so H' satisfies the conditions of the theorem. F cannot belong to a weakly blocking subhypergraph, because it contains $\sum w'$ nodes that have odd degree in any subhypergraph that contains F. This means that a subhypergraph of H' is weakly blocking if and only if it is a weakly blocking subhypergraph of H. It is not hard to check that the lemmas of the proof of Theorem 5 still hold for weak blocking too (for the original weight w). □

3 Constant number of countries

If the number of countries, m, is constant, then all the questions that we study can be decided in polynomial time. This is a consequence of the following known result on matchings:

Lemma 17. *Let k be a constant. Given a graph $G = (V; E)$, a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ of V, and a vector $x \in \mathbb{Z}_+^k$, it can be decided in polynomial time if there is a matching N such that $|V(N) \cap V_i| \geq x_i$ for every $i \in [k]$.*

The lemma immediately implies that it can be decided in polynomial time if a given matching M is in the weak core. Indeed, the number of possible coalitions is polynomial, and for a given coalition $(i_1, i_2, \ldots, i_k)$, we can decide if there is a matching

N in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $|V(N) \cap V_{i_j}| \geq |V(M) \cap V_{i_j}| + 1$ for every $j \in [k]$.

Membership in the strong core can be decided similarly, but for every coalition $(i_1, i_2, \ldots, i_k)$ and every $\ell \in [k]$, we check if there is a matching N in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $|V(N) \cap V_{i_j}| \geq |V(M) \cap V_{i_j}|$ for every $j \in [k] \setminus \ell$ and $|V(N) \cap V_{i_\ell}| \geq |V(M) \cap V_{i_\ell}| + 1$.

Deciding non-emptiness of the weak and strong cores is somewhat more difficult, but still polynomial-time solvable. The crucial observation is that the membership of a matching M in these cores only depends on the values $|V(M) \cap V_i|$ $(i \in [m])$. Using Lemma 17, we can find in polynomial time all component-wise maximal vectors $x \in \mathbb{Z}_+^m$ for which a matching M with $|V(M) \cap V_i| = x_i$ for every $i \in [m]$ exists (we can check all possible vectors since m is constant, and $x_i \leq |V|$ $(i \in [m])$ can be assumed). For such a vector x, we can again check (by using Lemma 17 for every coalition) whether *any* matching M with $|V(M) \cap V_i| = x_i$ $(i \in [m])$ is in the strong or weak core. If the answer is negative for every maximal vector x, then the core is empty, otherwise it is non-empty.

4   Countries of constant size

In contrast to the polynomial-time solvability of problems with constant number of countries, restricting the size of the countries does not automatically make the problems tractable. In this section, we show that deciding emptiness of the core is hard even for small countries. In Section 5, we will separately discuss countries of size 2.

4.1   Emptiness of the weak core is NP-hard

The following example of an empty weak core was given by Zsuzsanna Jankó.

Example 18. *Let $A = \{a_1, a_2, \ldots, a_7\}$, $B = \{b_1, \ldots, b_7\}$, and $C = \{c_1, \ldots, c_7\}$ be three countries each having seven nodes. The graph consists of five disjoint complete graphs, three K_5's: $\{a_1, c_2, c_3, b_4, b_5\}$, $\{b_1, a_2, a_3, c_4, c_5\}$ and $\{c_1, b_2, b_3, a_4, a_5\}$, and two K_3's: $\{a_6, b_6, c_6\}$ and $\{a_7, b_7, c_7\}$.*

This instance does not admit a matching in the weak core. Indeed, if there was a matching in the weak core, then there would also be a max-size matching in it. Let M be any max-size matching, covering 16 nodes. We may assume that every country has at least 4 nodes covered, since otherwise that single country would block M. The two countries who have the least number of nodes covered have together at most $\frac{2}{3} \times 16 < 11$ nodes covered, therefore they have at most 10. If both of them have 5 covered in M, then they form a blocking coalition because there is a matching where both of them have 6 nodes covered. If one of them has 4 nodes covered while the other 6, then they form a blocking coalition because there is a matching where the first country has 5 nodes covered while the second has 7. Hence, there is always a strongly blocking coalition.

We will use this example in our gadgets to prove that deciding the emptiness of the weak core is hard.

Theorem 19. *It is NP-hard to decide whether a matching in the weak core exists.*

Before proving the theorem, we introduce the notion of *special edges*. We say that there is a *special edge* between two nodes u and v if there is a gadget described below between u and v. Let E be a copy of the instance in Example 18. Let S be a country with four nodes s_1, s_2, s_3 and s_4. There is an edge between s_2 and a_1. Let T be a country with two nodes, t_1 and t_2. The following edges belong to the gadget: s_1t_1, s_4t_2, us_1, s_4v and s_2s_3. See Figure 1.
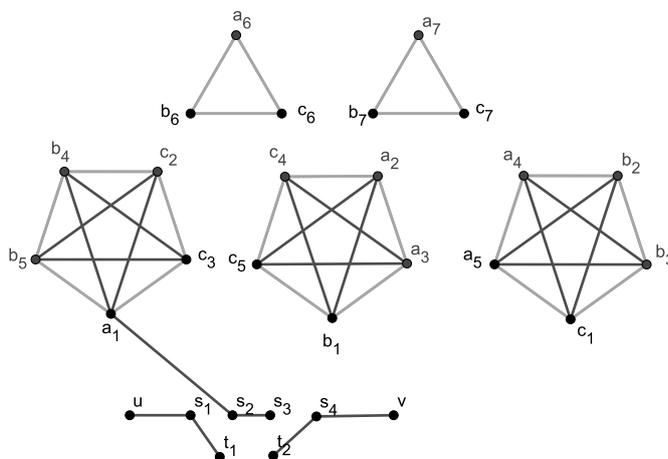
Figure 1

Lemma 20. *Suppose an instance of the NTU international matching game contains a special edge between u and v. This special edge cannot belong to a matching in the weak core, but it can belong to a strongly blocking coalition in the following sense:*

  i) *If there is a matching M in the weak core, s_2a_1, s_1t_1 and s_4t_2 has to belong to M. This means that us_1 and s_4v cannot belong to M.*

 ii) *Suppose the edges us_1 and s_4v do not belong to a matching M. If by replacing the special edge between u and v with an edge uv, u and v would belong to a strongly blocking coalition w.r.t. M restricted to the graph we get if we delete the special edge, then M does not belong to the weak core.*

Proof. i) We have seen that if there are no additional edges, then E cannot admit a matching in the weak core. Therefore if s_2a_1 does not belong to M, then the whole instance could not admit a matching in the weak core.

Suppose s_1t_1 does not belong to M, so t_1 is unmatched in M. We know that s_2a_1 belongs to M, therefore s_3 is unmatched in M. But this means that the countries S

and T with the matching $M' = \{s_1t_1,\ s_2s_3,\ s_4t_2\}$ form a blocking coalition, because M' covers every node of $S \cup T$.

ii) We have already seen that if the edges s_1t_1, s_2a_1 and s_4t_2 do not belong to M, then M is not in the weak core. Suppose these edges belong to M, so s_3 is unmatched in M. Let M' be the blocking matching. Let M'' be the matching we get if we leave out the edge uv from M' and add the edges us_1, s_4v and s_2s_3. The countries that belonged to the strongly blocking coalition would also be better off by M'', so together with S, they form a strongly blocking coalition: S prefers M'' because all nodes of S are covered by M''. □

Proof of Theorem 19. We reduce from 3-SAT. Given an instance I of 3-SAT, we construct an instance J of the NTU international matching game. For every variable of I, we construct a variable gadget, and for every clause we construct a clause gadget. After that, we describe the interconnecting edges between the clause gadgets and the variable gadgets.

Variable gadget. For a variable x_i, we define four sets of nodes, X_i, $\bar{X}_i$, Y_i and $\bar{Y}_i$. X_i contains a node for every occurrence of the variable x_i in unnegated form, and the nodes of X_i belong to one country. $\bar{X}_i$ contains a node for every occurrence of the variable x_i in negated form, and the nodes of $\bar{X}_i$ belong to one country. For every node in X_i, there is a separate node in Y_i so that there is an edge between them, and each node of Y_i belongs to a separate country of size one. Similarly for every node in $\bar{X}_i$, there is a separate node in $\bar{Y}_i$ so that there is an edge between them, and each node of $\bar{Y}_i$ belongs to a separate country of size one. Every node of Y_i is connected to every node of $\bar{Y}_i$ by a special edge. See Figure 2.
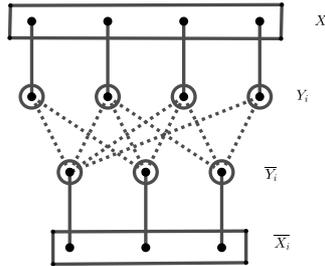
Figure 2: The variable gadget. The dotted lines represent special edges.

Clause gadget. To each clause c_j, we associate a copy of the instance in Example 18. We use the same notation for the nodes as in the example, but every node gets an upper index j. Let a_1^j, b_1^j and c_1^j correspond to the three literals in clause c_j.

Interconnecting edges Now we describe the edges between the clause gadgets and the variable gadgets. To every literal that appears in a clause, there is a corresponding node in the clause gadget and a corresponding node in the variable gadget. We connect these two nodes with an edge.

Claim 21. *For any matching M in the weak core of* J, *there is a truth assignment satisfying* I.

Proof. If M is in the weak core, Y_i or $\bar{Y}_i$ has all its nodes covered by M. Indeed, if there were a node $u \in Y_i$ and a node $v \in \bar{Y}_i$ so that neither is covered by M, then the countries $\{u\}$ and $\{v\}$ together with the special edge connecting them would form a strongly blocking coalition (see Lemma 20). Since the special edges cannot belong to M, every node of X_i (or $\bar{X}_i$) is matched to a node in Y_i (or $\bar{Y}_i$). In the first case we set the variable x_i to be false, and in the second case we set it to be true. If both hold simultaneously, we arbitrarily set x_i to be true or false, but it still holds that if x_i is set to false, every node in X_i is matched to a node in Y_i in M.

A clause gadget is a copy of the instance in Example 18, therefore without the additional (interconnecting) edges, it cannot admit a matching in the weak core, thus M has to contain an interconnecting edge leaving this clause gadget. The interconnecting edge corresponds to a literal which cannot be set to false, because then the node corresponding to it in X_i or $\bar{X}_i$ for some i would be matched to a node in Y_i or $\bar{Y}_i$ in M. This means that in this truth assignment, every clause contains a literal set to true, thus it satisfies I. □

Claim 22. *For any truth assignment satisfying* I, *it is possible to construct a matching M in the weak core of* J.

Proof. If a variable x_i is set to true (false), let M contain all the edges between $\bar{X}_i$ (X_i) and $\bar{Y}_i$ (Y_i), and all interconnecting edges incident to X_i ($\bar{X}_i$). See figure 3. In
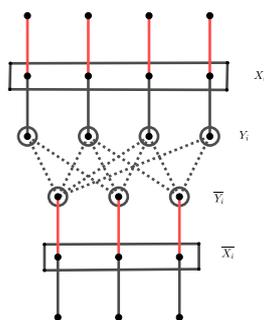
Figure 3: Red edges are in M. The dotted lines represent special edges.

a special edge uv, let s_1t_1, s_2a_1, s_4t_2 and c_2c_3, b_4b_5, a_2a_3, c_4c_5, b_2b_3, a_4a_5, b_6c_6, b_7c_7 belong to M. See figure 4.
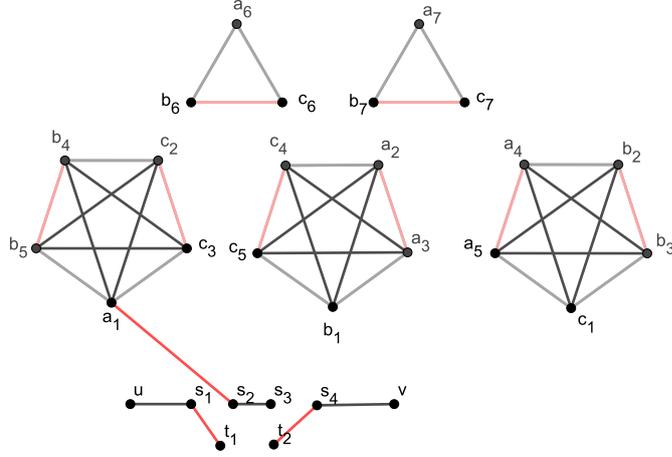
Figure 4: The red edges belong to M.

In every clause gadget, there is at least one node corresponding to a literal that is matched via an interconnecting edge in M since every clause contains a literal set to true. Suppose that a_1^j is such a node in the clause gadget corresponding to c_j. Let the edges $c_2^j c_3^j$, $b_4^j b_5^j$, $a_2^j a_3^j$, $c_4^j c_5^j$, $b_2^j b_3^j$, $a_4^j a_5^j$, $b_6^j c_6^j$ and $b_7^j c_7^j$ belong to M.

Now we show for every country that it cannot belong to a strongly blocking coalition. If the variable x_i is set to true, then all the nodes in X_i, $\bar{X}_i$ and $\bar{Y}_i$ are matched in M, therefore the countries X_i, $\bar{X}_i$ and the singleton countries of $\bar{Y}_i$ cannot belong to a strongly blocking coalition. Since a node (which is also a country) in Y_i is only connected to one of these countries, it also cannot belong to a blocking coalition. If x_i is set to false, a similar argument shows that these countries cannot belong to a strongly blocking coalition.

Next, we show that the countries of a special edge cannot belong to a strongly blocking coalition. T cannot, since it is fully matched in M. S cannot, since it has only one node unmatched in M, and s_1 is only connected to a node in Y_i or $\bar{Y}_i$ and a node in T, therefore it would be unmatched in any blocking coalition. It remains to check that some of the countries A, B and C do not form a blocking coalition. This is because the maximum matchings in A, B, C, $A \cup B \cup C$ and $B \cup C$ cover at most as many nodes as M, and a maximum matching in $A \cup B$ (or $A \cup C$) covers 12 nodes while M covers 11 nodes of $A \cup B$ (and $A \cup C$).

This last argument about the countries of the example shows that the countries of a clause gadget also cannot belong to a strongly blocking coalition, which completes the proof. $\qquad\square$

This concludes the proof of Theorem 19. $\qquad\square$

4.2   Emptiness of the strong core is NP-hard

Theorem 23. *It is NP-hard to decide whether a matching in the strong core exists.*

As in the previous section, we introduce the notion of *special edges*. The role of special edges is similar, but we need a different gadget for the definition. We say that there is a *special edge* between two nodes u and v if there is a gadget described below between u and v. Let $C_i = \{c_i, c_i'\}$ be countries of size two for $i = 1, 2, 3$, and let the following edges belong to the graph: uc_1, $c_1'c_2$, $c_2'v$, $c_2'c_3$, $c_3'c_1$. See Figure 5.
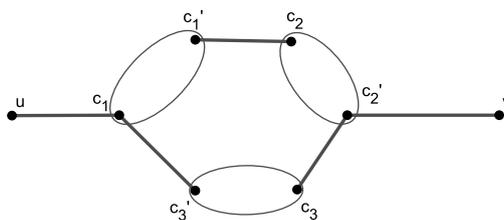
Figure 5: Gadget for special edge

Lemma 24. *Suppose an instance of the NTU international matching game contains a special edge between u and v. This special edge cannot belong to a matching in the strong core, but it can belong to a weakly blocking coalition in the following sense:*

  i) *If there is a matching M in the strong core, then $c_1'c_2$, $c_2'c_3$, and $c_3'c_1$ has to belong to M. This means that uc_1 and $c_2'v$ cannot belong to M.*

 ii) *Suppose the edges uc_1 and $c_2'v$ do not belong to a matching M. If by replacing the special edge between u and v with an edge uv, u and v would belong to a weakly blocking coalition w.r.t. M restricted to the graph we get if we delete the special edge, then M does not belong to the strong core.*

Proof. i) The countries C_i ($i = 1, 2, 3$) form a weakly blocking coalition unless all of their nodes are covered by M. The latter is possible only if the edges $c_1'c_2$, $c_2'c_3$, and $c_3'c_1$ belong to M.

ii) We add the countries C_1 and C_2 to the blocking coalition, and instead of the edge uv we add the edges uc_1, $c_1'c_2$ and $c_2'v$ to the blocking matching. □

Proof of Theorem 23. We reduce from 3-SAT. Given an instance I of the 3-SAT, we construct an instance J of the NTU international matching game. For every variable of I, we construct a variable gadget, and for every clause we construct a clause gadget. Then we describe the interconnecting edges between the clause gadgets and the variable gadgets.

Variable gadget. A variable x_i defines 3 countries $X_i = \{x_i\}$, $Y_i = \{y_i, y_i'\}$ and $Z_i = \{z_i, z_i'\}$. The edges x_iy_i', x_iz_i' and y_iz_i belong to the gadget.

Clause gadget. To a clause c_j, we associate 3 countries of size 5: $A_j = \{a_{j1}, a_{j2}, ..., a_{j5}\}$, $B_j = \{b_{j1}, b_{j2}, ..., b_{j5}\}$ and $D_j = \{d_{j1}, d_{j2}, ..., d_{j5}\}$. Each of these countries correspond to a literal of the clause c_j. The following edges belong to the gadget. $a_{j1}a_{j2}$, $a_{j2}b_{j3}$, $b_{j3}d_{j4}$, $b_{j1}b_{j2}$, $b_{j2}d_{j3}$, $d_{j3}a_{j4}$, $d_{j1}d_{j2}$, $d_{j2}a_{j3}$, $a_{j3}b_{j4}$, $a_{j5}b_{j5}$, $b_{j5}d_{j5}$, $d_{j5}a_{j5}$. See Figure 6.
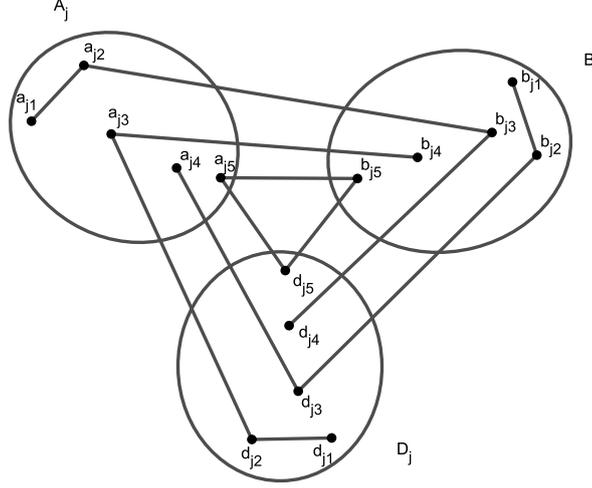
Figure 6: The clause gadget

Interconnecting edges Now we describe the edges between the clause gadgets and the variable gadgets. If a variable x_i is in a clause c_j in negated or unnegated form, and say the country A_j corresponds to this literal, then we connect a_{j1} with y_i by a special edge if x_i was in unnegated form, and we connect a_{j1} with z_i by a special edge if x_i was in negated form.

Claim 25. *For any truth assignment satisfying* I, *it is possible to construct a matching M in the strong core of* J.

Proof. For every variable gadget, the matching M contains the edge $y_i z_i$, and x_i is matched to y_i' in M if the variable x_i is set to true and it is matched to z_i' if x_i is set to false.

Every clause c_j contains a true literal, say D_j corresponds to a true literal. Let M contain the following edges of the clause gadget corresponding to c_j: $a_{j1}a_{j2}$, $b_{j3}d_{j4}$, $b_{j1}b_{j2}$, $d_{j3}a_{j4}$, $d_{j1}d_{j2}$, $a_{j3}b_{j4}$, and $a_{j5}bj5$. See Figure 7. This is the unique matching that covers all the nodes of $A_j \cup B_j \cup D_j$ except d_5. This completes the description of M; we now prove that M is in the strong core.

Suppose there is a weakly blocking coalition w.r.t. M. Then this contains a country who is better off by a matching M', therefore it was not fully matched by M, so either a) it is a country in a clause gadget who is not fully matched and therefore corresponds to a literal set to true, or b) it is Y_i (in this case x_i is set to false) or c) Z_i (in this case x_i is set to true) for some i.

Figure 7: The red edges belong to M

In case a) if in the clause c_j the country D_j is not fully matched by M, but D_j is in a weakly blocking coalition and it is better off by a matching M', then M' fully covers D_j (since M only left d_{j5} unmatched). The node d_{j4} can only be matched by the edge $b_{j3}d_{j4}$, therefore B_j has to belong to the blocking coalition, and has to be fully matched by M'. This means that $b_{j4}a_{j3}$ has to belong to M', implying that A_j has to belong to the blocking coalition as well, and has to be fully matched by M'. Thus A_j, B_j and D_j belong to the blocking coalition, and all of them are fully matched by M', but this is impossible, since one of a_{j5}, b_{j5} and d_{j5} is left unmatched by any matching.

In case b) the matching M' has to contain x_iy_i', which means that it leaves z_i' unmatched, therefore Z_i cannot belong to the blocking coalition. This means that y_i has to be matched by a special edge in M', say $y_ia_{j1} \in M'$. Since x_i was set to false, A_j corresponds to a false literal and therefore it is fully covered by M, so A_j has to be fully covered by M'. This means that $a_{j2}b_{j3} \in M'$ (thus d_{j4} is unmatched by M'), and $a_{j4}d_{j3} \in M'$. Therefore A_j, B_j and D_j all have to belong to the blocking coalition, and at least two nodes of $A_j \cup B_j \cup D_j$ are left unmatched by M' (d_{j4}, and one of a_{j5}, b_{j5} and d_{j5} is left unmatched by any matching). This is a contradiction, since M covers all but one node of $A_j \cup B_j \cup D_j$.

Case c) is similar to case b).   □

Claim 26. *For any matching M in the strong core of* J, *there is a truth assignment satisfying* I.

Proof. Suppose there is a matching M in the strong core. M has to cover all but one node of every clause gadget, since otherwise the 3 countries of the clause with the matching in Figure 7 (with the node left unmatched chosen appropriately) form a weakly blocking coalition. A matching that covers all but one node of $A_j \cup B_j \cup D_j$ is unique except that we can choose any edge of the triangle $a_{j5}b_{j5}d_{j5}$.

The matching M has to cover x_i for every i, otherwise X_i and Y_i would form a weakly blocking coalition. If x_i is matched to y_i', then we set x_i to be true, and if it is matched to z_i' we set x_i to be false.

We show that every clause contains a true literal. Suppose clause c_j does not. Let d_{j5} be the node left unmatched by M in $A_j \cup B_j \cup D_j$. If D_j corresponds to the variable x_i in unnegated form, then d_{j1} is connected to y_i by a special edge and since D_j corresponds to a false literal, $x_i z_i' \in M$. But then the countries X_i, Y_i, D_j and A_j form a weakly blocking coalition with the matching M' shown in figure 8. Similarly,
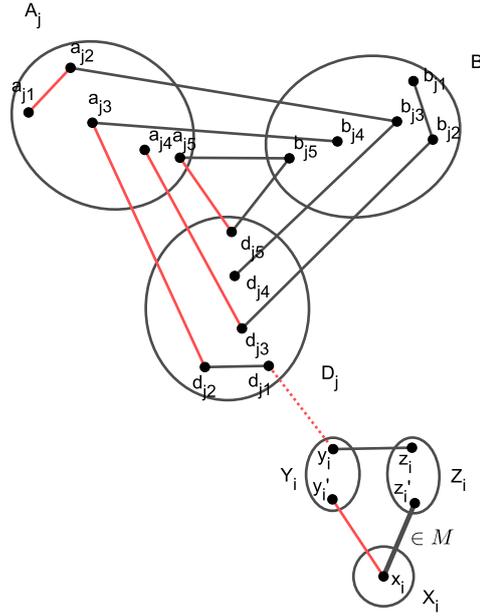
Figure 8: The blocking matching M'. The dotted line represents a special edge.

if D_j corresponds to the variable x_i in negated form, then d_{j1} is connected to z_i by a special edge, and the countries X_i, Z_i, D_j and A_j form a weakly blocking coalition. $\square$

This concludes the proof of Theorem 23. $\square$

5 Countries of size 2

In this section, we study the case when every country has size 2. We connect the two nodes of each country with a *country edge*. The prefect matching defined by the country edges is denoted by M_C.

Lemma 27. *A matching M is in the strong core if and only if*

a) *every country that is in an alternating cycle w.r.t. M_C has both its nodes covered by M,*

b) *there is no alternating path w.r.t. M_C from a country that has none of its nodes covered by M to a country that has at most one node covered by M,*

c) *there is no alternating path w.r.t. M_C that has 3 countries in it, such that each of them has one node covered by M.*

Proof. It is easy to see that if M is in the strong core, then these conditions hold, since otherwise the matching given by the alternating path or cycle would block M. Suppose the conditions hold, but the matching M is not in the strong core. Then there is a weakly blocking coalition of countries with a matching M' such that there is one country that is better off with M' than M, and the others are no worse off. Take the symmetric difference of M' and M_C. This is a disjoint union of alternating cycles and paths. If the country that is better off by M' has 0 nodes covered by M, then it has at least one node covered by M', so it is either in an alternating cycle, which contradicts condition a), or it is in an alternating path. In the latter case, the two red edges at the ends of the path are countries that have one node covered by M', so they have at most one node covered by M, which contradicts condition b). If the country that is better off by M' has 1 node covered by M, then it has two nodes covered by M', so it is either in an alternating cycle, which contradicts condition a), or it is in an alternating path (and it is not at the end of the path), but the two country edges at the ends of the path are countries that have one node covered by M', so they have at most one node covered by M, which contradicts condition b) or c). $\qquad\square$

Theorem 28. *If every country has size 2, then we can decide in polynomial time if a given matching M is in the strong core.*

Proof. We need to show that we can check the conditions given by Lemma 27 in polynomial time. We can check if a given red edge is in an alternating cycle, since it is equivalent to checking if, after the deletion of this edge, the remaining graph (the original graph G together with the red edges) has a perfect matching. We can do this for every red edge with at most one node covered by M. We can check whether there is an alternating path between two given country edges, knowing that these do not belong to an alternating cycle, since this is equivalent to checking if by deleting these two edges the remaining graph has a matching of size one less than a perfect matching. We can do this for any two country edges such that one of them has 0 nodes covered by M, and the other one has at most one node covered by M. For three given country edges, such that we know that none of them belong to an alternating cycle, we can check if there is an alternating path that contains all three of them, because this is equivalent to checking if by deleting all three of these edges, the remaining graph has a matching of size one less then a perfect matching. We can check this for any three country edges that have one node covered by M. $\qquad\square$

Lemma 29. *A matching M is in the weak core if and only if*

a) *there is no alternating cycle w.r.t. M_C such that every country edge in it has at most one node covered by M,*

b) there is no alternating path w.r.t. M_C such that the country edges at the two ends of the path have 0 nodes covered by M, and the country edges in the middle have one node covered by M.

Proof. If M is in the weak core, then the conditions a) and b) hold, since otherwise the matching defined by the alternating path or cycle would block M. Suppose a) and b) hold, but M is not in the weak core. There is a coalition of countries with a matching M' such that every country in the coalition is strictly better off with M', therefore those countries who have both of their nodes covered by M cannot belong to the coalition – we can delete these countries from the graph. Take the symmetric difference of the remaining country edges with M'. This consists of alternating cycles and paths. If there is an alternating cycle in the symmetric difference, then it contradicts a). If there is an alternating path, then the country edges at the two ends of the path have one node covered by M', therefore they have 0 nodes covered by M, which contradicts b). $\square$

Theorem 30. *If every country has size 2, then we can decide in polynomial time if a given matching M is in the weak core.*

Proof. We can check the conditions of Lemma 29 in polynomial time. $\square$

Theorem 31. *If every country has size 2, then the weak core is never empty.*

Proof. We can construct a matching M^* in the weak core the following way. Let M_C be as before. We check if there is an alternating cycle w.r.t. M_C in the current graph. If there is, then let the edges of the cycle belong to M, and delete the nodes of the cycle from the graph. Repeat this with the remaining graph, until there are no alternating cycles with respect to M_C. Let M^* be a maximum size matching that covers every node of M. We claim that M^* is in the weak core, since the conditions of Lemma 29 are met. Condition a) clearly holds, since the construction deleted countries that were covered twice by M (and hence by M^*), and the remaining graph did not contain an alternating cycle.

Suppose for contradiction that condition b) does not hold, i.e. there is an alternating path P w.r.t. M_C such that the country edges at the two ends of the path have 0 nodes covered by M^*, and the country edges in the middle have one node covered by M^*. Let N^* be the set of edges of M^* that contain a node from path P. We have $|N^*| \leq |P \cap M_C| - 2 = |P \cap E| - 1$. Therefore, $(M^* \setminus N^*) \cup (P \cap E)$ is a larger matching than M^*, contradicting the choice of M^* as a max-size matching. $\square$

It remains an intriguing open question whether the emptiness of the strong core can be decided in polynomial time if all countries have size 2.

Acknowledgement

References

[1] I. Ashlagi, A. Roth, *Individual rationality and participation in large scale, multihospital kidney exchange.* In: Proceedings of the 12th ACM Conference on Electronic Commerce, EC '11, pp. 321–322, ACM, New York, NY (2011)

[2] I. Ashlagi, A. Roth, *New challenges in multihospital kidney exchange*, American Economic Review 102(3) (2012), 354–59.

[3] P. Biró, W. Kern, D. Pálvölgyi, D. Paulusma, *Generalized Matching Games for International Kidney Exchange*, Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS'19), 2019, 413–421.

[4] M. Carvalho, A. Lodi, J.P. Pedroso, and A. Viana, *Nash equilibria in the twoplayer kidney exchange game*, Mathematical Programming, 161(1-2) (2017), 389–417

[5] L. Gourvès, J. Monnot, and F. Pascual, *Cooperation in multiorganization matching*, Algorithmic Operations Research, 7(2), 2012.