Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2019-12. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

# Complexity of the NTU International Matching Game

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September 2019
Revised: November 2020

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**Abstract**

Motivated by the real-world problem of international kidney exchange, [Biró et al., Generalized Matching Games for International Kidney Exchange, 2019] introduced a generalized transferable utility matching game featuring a partition of the node set into countries, and analyzed its complexity. We explore the non-transferable utility (NTU) variant of the game and prove computational complexity results about the weak and strong cores under various assumptions on the countries.

## 1 Introduction

The NTU International Matching Game is defined by a graph $G = (V; E)$ and a partition $V = V_1 \cup V_2 \cup \cdots \cup V_m$ of $V$. There are $m$ players, or countries, and the nodes in $V_i$ belong to player $i$. Given a matching $M$ in $G$, the utility of $M$ for country $i$ is given by

$$u_i(M) = |V(M) \cap V_i|,$$

where $V(M)$ denotes the node set of $M$.

A coalition $(i_1, i_2, \ldots, i_k)$ of countries is *strongly blocking* for matching $M$ if there exists a matching $M_0$ in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $u_{i_j}(M_0) > u_{i_j}(M)$ for every $j \in [k]$. Similarly, a coalition $(i_1, i_2, \ldots, i_k)$ of countries is *weakly blocking* for a matching $M$ if there exists a matching $M_0$ in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $u_{i_j}(M_0) \geq u_{i_j}(M)$ for every $j \in [k]$, and $u_{i_j}(M_0) > u_{i_j}(M)$ for at least one $j \in [k]$.

The International Matching Game, in a TU version, was introduced by Biró et al. [3] motivated by the kidney exchange problem. In that setting, the nodes correspond to patient-donor pairs, and edges represent the possible pairwise exchanges. The partition $V_1, \ldots, V_m$ can be thought of as a partitioning of the patients according to countries, but we can also think of hospitals instead of countries, with each hospital

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*September 2019. Revised: November 2020*

being interested in successful transplants for their own patients. The problem is studied from the point of view of cooperative games: are there matchings that are acceptable for all possible coalitions of countries?

In the NTU setting, a matching $M$ is in the *weak core* if there is no strongly blocking coalition for it, and it is in the *strong core* if there is no weakly blocking coalition. It is easy to see that if $M$ is in the strong core, then it is also in the weak core.

We study the computational complexity of membership and non-emptiness of the weak and strong cores, under various assumptions on the countries. In Section 2, we show that without further restrictions, it is NP-hard to decide whether a given matching is in the weak/strong core. We then study the problems with the restriction that the number of countries is constant (Section 3) or the size of countries is constant (Section 4). In the latter case, it is still hard to decide non-emptiness of both the weak and strong cores. In contrast, we show in Section 5 that the weak core is always non-empty if the size of each country is 2.

### Related work

In [1, 2], Ashlagi and Roth considered individual rationality in the multi-hospital kidney exchange problem. In our terms, individual rationality means that there is no single blocking country. They also studied the problem of incentive-compatibility, i.e., whether hospitals are motivated to underreport their patient-donor pairs. Carvalho et al. [4] studied the properties of Nash-equilibria in the two-hospital case. Gourvès, Monnot and Pascual [5] considered a weighted bipartite graph model, where the profit of an edge in the matching is divided between the two endpoints according to a fixed ratio. They proved that it is NP-hard to decide if there is an individually rational maximum weight matching.

# 2 General hardness

In this section, we prove that it is hard to decide whether a given matching is in the weak or strong core. As we will see later, the problem can be decided in polynomial time if the number of countries is constant.

## 2.1 Membership in the weak core

**Theorem 1.** *It is coNP-complete to decide if a given matching $M$ is in the weak core.*

The problem is in coNP because it can be decided in polynomial time if a given matching $M_0$ is a good witness for a given coalition being strongly blocking for $M$. We prove coNP-completeness of a hypergraph problem that is equivalent to a special case of our problem.

Let $G = \cup_{i=1}^{n} G_i$ be a union of complete graphs. The nodes of $G$ are partitioned into $m$ countries such that every country has one or two nodes in every complete graph. A max-size matching $M$ in $G$ leaves exactly one node unmatched in every complete graph of odd size. We assume that every country has at least one node that is unmatched in $M$.

We define a hypergraph $H_M$ with a weight function $w_M$ on the hyperedges. The nodes of $H_M$ are $[n] := \{1, 2, ..., n\}$ where $i$ corresponds to the complete graph $G_i$. The hyperedges of $H_M$ correspond to the countries. A node $i$ belongs to a hyperedge $A$ if the country corresponding to $A$ has one node in $G_i$. The weight $w_M(A)$ is the number of nodes belonging to the country corresponding to $A$ that is left unmatched in $M$ minus 1. The motivation for this definition is the following: a coalition $(i_1, i_2, \ldots, i_k)$ of countries is blocking if and only if there is a matching in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that every country $A$ of the coalition has at most $w_M(A)$ unmatched nodes. Note that the sum of weights are the number of odd sized complete graphs minus the number of countries.

**Definition 2.** A subhypergraph $S$ of $H$ is called a *blocking subhypergraph* if the number of odd-degree nodes in $S$ is at most the sum of weights of the edges of $S$.

**Claim 3.** *A coalition of countries $B = (i_1, i_2, ...i_k)$ is blocking with respect to $M$ if and only if the hyperedges corresponding to these countries form a blocking subhypergraph $S$.*

*Proof.* The subgraph $G_B = G[V_{i1} \cup V_{i2} \cup ... \cup V_{ik}]$ of $G$ is also a union of complete graphs. The number of odd-degree nodes in $S$ equals the number of odd size complete graphs in $G_B$ which is the number of nodes left unmatched in any max-size matching of $G_B$. $B$ is blocking if and only if there is a matching in $G_B$ such that every country $A$ in $B$ has at most $w_M(A)$ unmatched nodes. From this it is clear that if $B$ is blocking, then the inequality holds for $S$. If $S$ is blocking, then $B$ is blocking because every country has a node in every complete graph, therefore for every odd size complete graph we can choose which country should the unmatched node belong to. So we can distribute the nodes left unmatched in a max-size matching of $M_B$ (their number is the number of odd-degree nodes in $S$) among the countries in $B$ in a way that a country $A$ has at most $w_M(A)$ unmatched nodes. $\square$

**Claim 4.** *Suppose we have a hypergraph $H$ with $n$ nodes and $m$ hyperedges, and a weight function $w$ on the hyperedges such that the sum of weights equals the number of odd-degree nodes in $H$ minus $m$. Then there is a graph $G$ and a max-size matching $M$ of $G$ with the following properties: $G$ is a union of $n$ complete graphs, and the nodes of $G$ are partitioned into $m$ countries, such that every complete graph contains one or two nodes from every country. Furthermore, there is at least one node from every country that is left unmatched by $M$, and $H = H_M$.*

*Proof.* If a node $i$ belongs to the hyperedge $A$ in $H$, then the corresponding complete graph $G_i$ of $G$ should contain one node that belongs to the country corresponding to $A$; otherwise it should contain two. The number of nodes left unmatched by any max-size matching is the number of odd size complete graphs in $G$ which is equal to the number of odd-degree nodes in $H$ which is equal to the sum of weights plus $m$. The number of nodes in a country corresponding to the hyperedge $A$ left unmatched by $M$ should be $w(A) + 1$ which can be achieved since for every odd size complete graph we can choose which country should the unmatched node belong to (because every complete graph contains a node from every country). $\square$

**Theorem 5.** *For a hypergraph $H$ with a weight function $w$ on the hyperedges such that the sum of weights equals the number of odd-degree nodes in $H$ minus the number of hyperedges, it is NP-complete to decide if there is a blocking subhypergraph $S$ of $H$.*

We reduce from the 1-in-3 SAT problem. First we describe the construction. We are given a boolean formula with $k$ clauses and $n$ variables. For every clause we define a clause gadget which consists of

- three edges $C_{i1}, C_{i2}$ and $C_{i3}$, such that the intersection of any two of them is $C_{i1} \cap C_{i2} \cap C_{i3}$, this is disjoint from any other edge, and $|C_{i1} \cap C_{i2} \cap C_{i3}| = c$. These edges represent the 3 literals $l_{i1}, l_{i2}$ and $l_{i3}$ of the $i$th clause.

- a fourth edge $C_i$ such that $C_i \cap C_{ij}$ is disjoint from any other edge, its size is $c'$ for $j = 1, 2, 3$, and $C_i \setminus (C_{i1} \cup C_{i2} \cup C_{i3}) = \{c_i, c_{i+1}\}$, where $c_{k+1} = c_1$. Note that $C_i \cap C_{i-1} = c_i$, $C_i \cap C_{i+1} = c_{i+1}$ and $C_i$ is disjoint from $C_j$ if $j \neq i-1, i, i+1$, where $C_0 = C_k$.

For every variable $x_i$, we define a variable gadget which consists of three edges, $X_i, \overline{X_i}$, and $Y_i$ such that $|X_i \cap \overline{X_i}| = x$, $X_i \cap \overline{X_i} \subseteq Y_i$, $|Y_i \cap X_i| = |Y_i \cap \overline{X_i}| = x + x'$ and $Y_i \setminus (X_i \cup \overline{X_i}) = \{y_i, y_{i+1}\}$ where $y_{n+1} = y_1$. Note that $Y_i \cap Y_{i-1} = y_i$, $Y_i \cap Y_{i+1} = y_{i+1}$ and $Y_i$ is disjoint from $Y_j$ if $j \neq i-1, i, i+1$, where $Y_0 = Y_n$.

The clause gadgets intersect with the variable gadgets in the following way.

- The edge $C_{ij}$ intersects $X_i$ in 2 nodes if it represents the variable $x_i$ in unnegated form, and it intersects $\overline{X_i}$ in 2 nodes if it represents the variable $x_i$ in negated form. These intersections are disjoint from any other edges.

- The edges $C_{ij}, X_i$ and $\overline{X_i}$ do not contain any nodes which do not belong to an intersection described previously.

Now we describe the weights. Let $w(C_i) = c'$ for $i \in [k]$, $w(Y_i) = x'$ for $i \in [n]$, and let the weight of every other edge be zero.

The sum of odd-degree nodes is $kc + nx$ (only the nodes of $C_{i1} \cap C_{i2} \cap C_{i3}$ and the nodes of $X_i \cap \overline{X_i}$ have odd degree, since their degree is 3, while all the other nodes have degree 2), and the number of edges is $4k + 3n$ (there are 4 edges in every clause gadget and 3 edges in every variable gadget). The sum of weights (which is $kc' + nx'$) has to be the number of odd-degree nodes minus the number of edges which is $k(c-4) + n(x-3)$, so we set $c' = c - 4$ and $x' = x - 3$. (We can set $c = 6$ and $x = 5$ for example).

**Notation 6.** Let $\overline{Z_i} = (\overline{X_i} \cap Y_i) \setminus (\overline{X_i} \cap X_i)$ and $Z_i = (X_i \cap Y_i) \setminus (\overline{X_i} \cap X_i)$.

**Lemma 7.** *For every hyperedge $A$ with $w(A) > 0$ that belongs to a blocking subhypergraph $S$, there are at least $w(A)$ nodes in it (not contained in other hyperedges with positive weight) that have odd degree in $S$.*

*Proof.* The only edges with positive weight are $Y_i$ for $i \in [n]$, $(w(Y_i) = x')$, and $C_i$ for $i \in [k]$, $(w(C_i) = c')$.

If $A = Y_i$ for some $i \in [n]$, then $Z_i$ and $\overline{Z_i}$ both have size $x'$ and since they are subsets of $Y_i$ which belongs to $S$, nodes in both of these sets only have even degree in $S$ if $X_i$ and $\overline{X_i}$ both belong to $S$. But then the nodes in $Y_i \cap X_i \cap \overline{X_i}$ have degree 3, and $|Y_i \cap X_i \cap \overline{X_i}| = x = x' + 3$.

If $A = C_i$ for some $i \in [k]$, then the set of nodes $C_{ij} \cap C_i$ for $j = 1, 2, 3$ have size $c'$ and nodes in all three of these sets only have even degree in $S$ if $C_{ij}$ belongs to $S$ for $j = 1, 2, 3$. But then the nodes in $C_{i1} \cap C_{i2} \cap C_{i3}$ have degree 3 and $|C_{i1} \cap C_{i2} \cap C_{i3}| = c = c' + 4$. □

**Lemma 8.** *Let us call the nodes that do not belong to $Z_i$ or $\overline{Z_i}$ for any $i \in \{1, 2, ...n\}$ and do not belong to $C_i \cap C_{ij}$ for any $i \in [k]$, $j \in \{1, 2, 3\}$* ordinary*.*

- *(i) If $Y_i$ belongs to a blocking subhypergraph, then the nodes of $Y_i$ that have odd degree in the subhypergraph are either $Z_i$ or $\overline{Z_i}$, so the number of such nodes is $x' = w(Y_i)$.*
- *(ii) If $C_i$ belongs to a blocking subhypergraph, then the nodes of $C_i$ that have odd degree in the subhypergraph are $C_i \cap C_{ij}$ for $j = 1, 2$ or $3$, so the number of such nodes is $c' = w(C_i)$.*
- *(iii) Ordinary nodes can not have odd degree in a blocking subhypergraph.*

*Proof.* (iii) In a blocking subhypergraph $S$, the number of odd-degree nodes in $S$ is at most the sum of weights in $S$, and Lemma 7 shows that in every edge $A$ in $S$ with positive weight, there are $w(A)$ nodes that have odd degree in $S$. This means that there cannot be more nodes in $A$ with odd degree in $S$. Furthermore besides the $w(A)$ nodes for every edge $A$ in $S$ with positive weight that have odd degree in $S$, no other node can have odd degree in $S$ so (iii) holds. (i) and (ii) follow from the proof of Lemma 7. □

**Lemma 9.** *If for two edges $A$ and $A'$ there is an ordinary node in $A \cap A'$ that does not belong to any other edge, then if $A$ belongs to a blocking subhypergraph, $A'$ has to belong to it too.*

*Proof.* This is a direct consequence of point (iii) of Lemma 8. □

**Lemma 10.** *If there is a blocking subhypergraph $S$, then it contains exactly one of $X_i$ and $\overline{X_i}$ for $i = 1, \ldots, n$.*

*Proof.* If $C_m$ is in $S$ for some $m \in [k]$, then $C_{mj}$ has to be in $S$ too for some $j \in \{1, 2, 3\}$, otherwise $C_m$ would have at least $3c'$ odd-degree nodes in $S$.

If $C_{ij}$ is in $S$, then there exists a $t \in [n]$ such that $C_{ij} \cup X_t$ or $C_{ij} \cup \overline{X_t}$ is not empty, moreover it contains an ordinary node not contained in any other edge, so Lemma 9 implies that $X_t$ or $\overline{X_t}$ has to belong to $S$ too.

If $X_t$ or $\overline{X_t}$ is in $S$, then $Y_t$ has to be in $S$ as well. Indeed, if $Y_t$ does not belong to $S$, then the nodes of $Z_t$ or $\overline{Z_t}$ have odd degree in $S$ but then the sum of odd-degree nodes in $S$ is greater than the sum of weights of the edges of $S$ because of Lemma 7.

If $Y_t$ belongs to $S$, then $Y_i$ belongs to $S$ for every $i \in [n]$. This is because $Y_t \cap Y_{t+1}$ contains an ordinary node that only belongs to these edges, so from Lemma 9 $Y_{t+1}$ has to belong to $S$; then $Y_{t+1} \cap Y_{t+2}$ contains an ordinary node that only belongs to these edges, and so on.

If $Y_i$ belongs to $S$, then exactly one of $X_i$ and $\overline{X_i}$ belongs to $S$. This is because point (i) of Lemma 8.

□

**Lemma 11.** *The 1-in-3 SAT instance is satisfiable if and only if the constructed hypergraph has a blocking subhypergraph.*

*Proof.* First suppose the 1-in-3 SAT instance is satisfiable. We prove that the subhypergraph $S$ that consists of the following edges is a blocking subhypergraph:

- $X_i$ and every $C_{lm}$ intersecting $X_i$ for $x_i$ set to false,
- $\overline{X_j}$ and every $C_{lm}$ intersecting $\overline{X_j}$ for $x_j$ set to true,
- $Y_i$ for $i \in [n]$,
- $C_i$ for $i \in [k]$.

There are exactly two false literals in every clause, therefore in every clause gadget exactly two of $C_{i1}$, $C_{i2}$ and $C_{i3}$ are in the subhypergraph, so the nodes in $C_{i1} \cap C_{i2} \cap C_{i3}$ have even degree (they have degree 2). It is easy to see that all the other nodes in $C_{ij}$ have degree 2 if $C_{ij}$ is in the subgraph. If the $j$th literal of clause $i$ was set to true, then the nodes of $C_{ij} \cap C_i$ have degree 1, and all the other nodes of $C_i$ have degree 2. Therefore the number of nodes in a clause gadget with odd degree is $c'$.

For every $i \in [n]$, exactly one of $X_i$ and $\overline{X_i}$ is chosen to be in the subhypergraph, say it is $X_i$. Then the nodes in $\overline{Z_i}$ have degree 1 (if $\overline{X_i}$ was chosen, then the nodes in $Z_i$ have degree 1). Either way, the number of odd-degree nodes in a variable gadget is $x'$ since it is easy to see that all the other nodes in the variable gadget have even degree.

We obtain that the total number of odd-degree nodes in the subhypergraph is $kc' + nx'$ which is equal to the sum of weights, which means that this subhypergraph is a blocking subhypergraph.

Now suppose that there is a blocking subhypergraph $S$ in the constructed hypergraph. The following is a satisfying assignment of the 1-in-3 SAT instance: we set $x_i$ to be false if and only if $X_i$ belongs to $S$.

Because of Lemma 10, exactly one of $x_i$ and $\overline{x_i}$ is set to false. For every $C_{ij}$, there is a $t$ such that $X_t \cap C_{ij}$ or $\overline{X_t} \cap C_{ij}$ is not empty and contains an ordinary node that is only contained in these edges. Because of Lemma 9, $C_{ij}$ belongs to $S$ if and only if $X_t$ belongs to $S$ (if $X_t \cap C_{ij}$ was nonempty) or $\overline{X_i}$ belongs to $S$ (if $\overline{X_t} \cap C_{ij}$ was nonempty). This means that $C_{ij}$ belongs to $S$ if and only if the literal $l_{ij}$ it corresponds to is set to false.

If $C_{ij}$ belongs to $S$, then $C_i$ has to belong to $S$ too, otherwise the nodes of $C_{ij} \cap C_i$ would have odd degree in $S$, but then the sum of odd-degree nodes in $S$ would be greater than the sum of weights of the edges of $S$ because of Lemma 7.

If $C_i$ belongs to $S$, then $C_m$ has to belong to $S$ too for every $m \in [k]$. This is because $C_i \cap C_{i+1}$ contains an ordinary node that only belongs to these edges, so from Lemma 9 $C_{i+1}$ has to belong to $S$, $C_{i+1} \cap C_{i+2}$ contains an ordinary node that only belongs to these edges, and so on.

If $C_m$ belongs to $S$, then exactly two of $C_{mj}$ for $j = 1, 2, 3$ belongs to $S$. This follows from point (ii) of Lemma 8.

The above statements prove that exactly two literals are set to false in every clause.

□

## 2.2 Membership in the strong core

**Theorem 12.** *It is coNP-complete to decide if a given matching is in the strong core.*

We are given a graph $G = \bigcup_{i=1}^{n} G_i$ and a matching $M$ described in section 2.1 except that we do not assume that there is an unmatched node in every country. We define a hypergraph $H$ in the same way as in section 2.1, but we define a different weight function $w'_M$ on the hyperedges. Let $w'_M(A)$ be the number of nodes belonging to the country corresponding to $A$ that is left unmatched in $M$.

**Definition 13.** We are given a hypergraph $H$ with a weight function $w$ on the hyperedges. A subhypergraph $S$ of $H$ is called a *weakly blocking subhypergraph* if the number of odd-degree nodes in $S$ is less than the sum of weights of the edges of $S$.

**Claim 14.** *A set of countries $B = (i_1, i_2, \dots, i_k)$ is weakly blocking with respect to $M$ if an only if the hyperedges corresponding to these countries form a weakly blocking subhypergraph $S$.*

*Proof.* The subgraph $G_B = G[V_{i1} \cup V_{i2} \cup \cdots \cup V_{ik}]$ of $G$ is also a union of complete graphs. The number of odd-degree nodes in $S$ equals the number of odd-size complete graphs in $G_B$ which is the number of nodes left unmatched in any max-size matching of $G_B$. $B$ is weakly blocking if and only if there is a matching in $G_B$ such that every country $A$ in $B$ has at most $w'_M(A)$ unmatched nodes, and at least one country has less than $w'_M(A)$. From this it is clear that if $B$ is blocking, the inequality holds for $S$. If $S$ is blocking, then $B$ is blocking because every country has a node in every complete graph, therefore for every odd size complete graph we can choose which country should the unmatched node belong to. So we can distribute the nodes left unmatched in a max-size matching of $M_B$ (their number is the number of odd-degree nodes in $S$) among the countries in $B$ in a way that a country $A$ has at most $w'_M(A)$ unmatched nodes and at least one country has less than $w'_M(A)$. □

**Claim 15.** *Suppose we have a hypergraph $H$ with $n$ nodes and $m$ hyperedges, and a weight function $w'$ on the hyperedges such that the sum of weights equals the number of odd-degree nodes in $H$. Then there is a graph $G$ and a max-size matching $M$ in $G$*

*with the following properties: $G$ is a union of $n$ complete graphs, and the nodes of $G$ can be partitioned into $m$ countries, such that every complete graph contains one or two nodes from every country, and $H = H_M$, $w' = w'_M$.*

*Proof.* If a node $i$ belongs to the hyperedge $A$ in $H$, then the corresponding complete graph $G_i$ of $G$ should contain one node that belongs to the country corresponding to $A$, otherwise it should contain two. The number of nodes left unmatched by any max-size matching is the number of odd size complete graphs in $G$, which is equal to the number of odd-degree nodes in $H$, which is equal to the sum of weights. The number of nodes in a country corresponding to the hyperedge $A$ left unmatched by $M$ should be $w'(A)$ which can be achieved since for every odd size complete graph we can choose which country should the unmatched node belong to (because every complete graph contains a node from every country). □

**Theorem 16.** *For a hypergraph $H$ with a weight function $w'$ on the hyperedges such that the sum of weights equals the number of odd-degree nodes in $H$, it is NP-complete to decide if there is a weakly blocking subhypergraph $S$ of $H$.*

*Proof.* We reduce from the 1-in-3 SAT problem. We will use the construction in the proof of Theorem 5 with slight modifications. The weights are the same except that we change the weight of $Y_1$ to $w'(Y_1) = x' + 1$ ($w'(A) = w(A)$ for all edges $A \neq Y_1$ of $H$.) Let $\sum w' = \sum w'(A)$: $A$ is an edge of $H$. The new hypergraph $H'$ is the same as $H$ except we add an extra edge $F$ of zero weight to the construction, which contains all the odd-degree nodes of $H$ and besides these, it contains $\sum w'$ extra nodes that are not contained in any other edge. In this modified hypergraph $H'$, the odd-degree nodes are these $\sum w'$ extra nodes. However, since $F$ has zero weight, their number is equal the sum of weights in $H'$, so $H'$ satisfies the conditions of the theorem. $F$ cannot belong to a weakly blocking subhypergraph, because it contains $\sum w'$ nodes that have odd degree in any subhypergraph that contains $F$. This means that a subhypergraph of $H'$ is weakly blocking if and only if it is a weakly blocking subhypergraph of $H$. It is not hard to check that the lemmas of the proof of Theorem 5 still hold for weak blocking too (for the original weight $w$). □

# 3 Constant number of countries

If the number of countries, $m$, is constant, then all the questions that we study can be decided in polynomial time. This is a consequence of the following known result on matchings:

**Lemma 17.** *Let $k$ be a constant. Given a graph $G = (V; E)$, a partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ of $V$, and a vector $x \in \mathbb{Z}_+^k$, it can be decided in polynomial time if there is a matching $N$ such that $|V(N) \cap V_i| \geq x_i$ for every $i \in [k]$.*

The lemma immediately implies that it can be decided in polynomial time if a given matching $M$ is in the weak core. Indeed, the number of possible coalitions is polynomial, and for a given coalition $(i_1, i_2, \ldots, i_k)$, we can decide if there is a matching

$N$ in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $|V(N) \cap V_{i_j}| \geq |V(M) \cap V_{i_j}| + 1$ for every $j \in [k]$.

Membership in the strong core can be decided similarly, but for every coalition $(i_1, i_2, \ldots, i_k)$ and every $\ell \in [k]$, we check if there is a matching $N$ in the induced subgraph $G[V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_k}]$ such that $|V(N) \cap V_{i_j}| \geq |V(M) \cap V_{i_j}|$ for every $j \in [k] \setminus \ell$ and $|V(N) \cap V_{i_\ell}| \geq |V(M) \cap V_{i_\ell}| + 1$.

Deciding non-emptiness of the weak and strong cores is somewhat more difficult, but still polynomial-time solvable. The crucial observation is that the membership of a matching $M$ in these cores only depends on the values $|V(M) \cap V_i|$ $(i \in [m])$. Using Lemma 17, we can find in polynomial time all component-wise maximal vectors $x \in \mathbb{Z}_+^m$ for which a matching $M$ with $|V(M) \cap V_i| = x_i$ for every $i \in [m]$ exists (we can check all possible vectors since $m$ is constant, and $x_i \leq |V|$ $(i \in [m])$ can be assumed). For such a vector $x$, we can again check (by using Lemma 17 for every coalition) whether *any* matching $M$ with $|V(M) \cap V_i| = x_i$ $(i \in [m])$ is in the strong or weak core. If the answer is negative for every maximal vector $x$, then the core is empty, otherwise it is non-empty.

# 4 Countries of constant size

In contrast to the polynomial-time solvability of problems with constant number of countries, restricting the size of the countries does not automatically make the problems tractable. In this section, we show that deciding emptiness of the core is hard even for small countries. In Section 5, we will separately discuss countries of size 2.

## 4.1 Emptiness of the weak core is NP-hard

The following example of an empty weak core was given by Zsuzsanna Jankó.

**Example 18.** *Let $A = \{a_1, a_2, \ldots, a_7\}$, $B = \{b_1, \ldots, b_7\}$, and $C = \{c_1, \ldots, c_7\}$ be three countries each having seven nodes. The graph consists of five disjoint complete graphs, three $K_5$'s: $\{a_1, c_2, c_3, b_4, b_5\}$, $\{b_1, a_2, a_3, c_4, c_5\}$ and $\{c_1, b_2, b_3, a_4, a_5\}$, and two $K_3$'s: $\{a_6, b_6, c_6\}$ and $\{a_7, b_7, c_7\}$.*

*This instance does not admit a matching in the weak core. Indeed, if there was a matching in the weak core, then there would also be a max-size matching in it. Let $M$ be any max-size matching, covering 16 nodes. We may assume that every country has at least 4 nodes covered, since otherwise that single country would block $M$. The two countries who have the least number of nodes covered have together at most $\frac{2}{3} \times 16 < 11$ nodes covered, therefore they have at most 10. If both of them have 5 covered in $M$, then they form a blocking coalition because there is a matching where both of them have 6 nodes covered. If one of them has 4 nodes covered while the other 6, then they form a blocking coalition because there is a matching where the first country has 5 nodes covered while the second has 7. Hence, there is always a strongly blocking coalition.*

We will use this example in our gadgets to prove that deciding the emptiness of the weak core is hard.

**Theorem 19.** *It is NP-hard to decide whether a matching in the weak core exists, even if every country has size at most 7.*

Before proving the theorem, we introduce the notion of *special edges*. We say that there is a *special edge* between two nodes $u$ and $v$ if there is a gadget described below between $u$ and $v$. Let $E$ be a copy of the instance in Example 18. Let $S$ be a country with four nodes $s_1, s_2, s_3$ and $s_4$. There is an edge between $s_2$ and $a_1$. Let $T$ be a country with two nodes, $t_1$ and $t_2$. The following edges belong to the gadget: $s_1t_1$, $s_4t_2$, $us_1$, $s_4v$ and $s_2s_3$. See Figure 1.

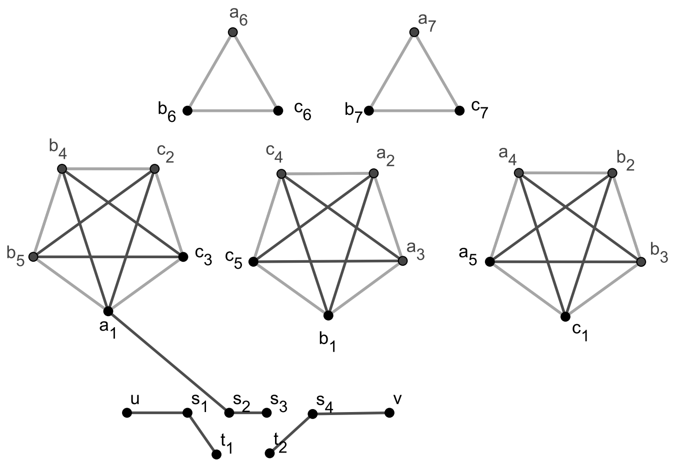


Figure 1

**Lemma 20.** *Suppose an instance of the NTU international matching game contains a special edge between u and v. This special edge cannot belong to a matching in the weak core, but it can belong to a strongly blocking coalition in the following sense:*

*i) If there is a matching $M$ in the weak core, $s_2a_1$, $s_1t_1$ and $s_4t_2$ has to belong to $M$. This means that $us_1$ and $s_4v$ cannot belong to $M$.*

*ii) Suppose the edges $us_1$ and $s_4v$ do not belong to a matching $M$. If by replacing the special edge between $u$ and $v$ with an edge $uv$, $u$ and $v$ would belong to a strongly blocking coalition w.r.t. $M$ restricted to the graph we get if we delete the special edge, then $M$ does not belong to the weak core.*

*Proof.* i) We have seen that if there are no additional edges, then $E$ cannot admit a matching in the weak core. Therefore if $s_2a_1$ does not belong to $M$, then the whole instance could not admit a matching in the weak core.

Suppose $s_1t_1$ does not belong to $M$, so $t_1$ is unmatched in $M$. We know that $s_2a_1$ belongs to $M$, therefore $s_3$ is unmatched in $M$. But this means that the countries $S$ and $T$ with the matching $M' = \{s_1t_1,\ s_2s_3,\ s_4t_2\}$ form a blocking coalition, because $M'$ covers every node of $S \cup T$.

ii) We have already seen that if the edges $s_1t_1$, $s_2a_1$ and $s_4t_2$ do not belong to $M$, then $M$ is not in the weak core. Suppose these edges belong to $M$, so $s_3$ is unmatched in $M$. Let $M'$ be the blocking matching. Let $M''$ be the matching we get if we leave out the edge $uv$ from $M'$ and add the edges $us_1$, $s_4v$ and $s_2s_3$. The countries that belonged to the strongly blocking coalition would also be better off by $M''$, so together with $S$, they form a strongly blocking coalition: $S$ prefers $M''$ because all nodes of $S$ are covered by $M''$. □

*Proof of Theorem 19.* We reduce from 3-SAT. Given an instance $I$ of 3-SAT, we construct an instance $J$ of the NTU international matching game. For every variable of $I$, we construct a variable gadget, and for every clause we construct a clause gadget. After that, we describe the interconnecting edges between the clause gadgets and the variable gadgets.

**Variable gadget.** For a variable $x_i$, we define four sets of nodes, $X_i$, $\bar{X}_i$, $Y_i$ and $\bar{Y}_i$. $X_i$ contains a node for every occurrence of the variable $x_i$ in unnegated form, and the nodes of $X_i$ belong to one country. $\bar{X}_i$ contains a node for every occurrence of the variable $x_i$ in negated form, and the nodes of $\bar{X}_i$ belong to one country. For every node in $X_i$, there is a separate node in $Y_i$ so that there is an edge between them, and each node of $Y_i$ belongs to a separate country of size one. Similarly for every node in $\bar{X}_i$, there is a separate node in $\bar{Y}_i$ so that there is an edge between them, and each node of $\bar{Y}_i$ belongs to a separate country of size one. Every node of $Y_i$ is connected to every node of $\bar{Y}_i$ by a special edge. See Figure 2.

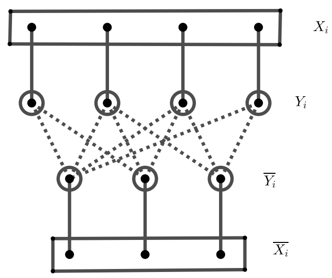


Figure 2: The variable gadget. The dotted lines represent special edges.

**Clause gadget.** To each clause $c_j$, we associate a copy of the instance in Example

18. We use the same notation for the nodes as in the example, but every node gets an upper index $j$. Let $a_1^j$, $b_1^j$ and $c_1^j$ correspond to the three literals in clause $c_j$.

**Interconnecting edges** Now we describe the edges between the clause gadgets and the variable gadgets. To every literal that appears in a clause, there is a corresponding node in the clause gadget and a corresponding node in the variable gadget. We connect these two nodes with an edge.

**Claim 21.** *For any matching $M$ in the weak core of* J*, there is a truth assignment satisfying* I.

*Proof.* If $M$ is in the weak core, $Y_i$ or $\bar{Y}_i$ has all its nodes covered by $M$. Indeed, if there were a node $u \in Y_i$ and a node $v \in \bar{Y}_i$ so that neither is covered by $M$, then the countries $\{u\}$ and $\{v\}$ together with the special edge connecting them would form a strongly blocking coalition (see Lemma 20). Since the special edges cannot belong to $M$, every node of $X_i$ (or $\bar{X}_i$) is matched to a node in $Y_i$ (or $\bar{Y}_i$). In the first case we set the variable $x_i$ to be false, and in the second case we set it to be true. If both hold simultaneously, we arbitrarily set $x_i$ to be true or false, but it still holds that if $x_i$ is set to false, every node in $X_i$ is matched to a node in $Y_i$ in $M$.

A clause gadget is a copy of the instance in Example 18, therefore without the additional (interconnecting) edges, it cannot admit a matching in the weak core, thus $M$ has to contain an interconnecting edge leaving this clause gadget. The interconnecting edge corresponds to a literal which cannot be set to false, because then the node corresponding to it in $X_i$ or $\bar{X}_i$ for some $i$ would be matched to a node in $Y_i$ or $\bar{Y}_i$ in $M$. This means that in this truth assignment, every clause contains a literal set to true, thus it satisfies $I$. □

**Claim 22.** *For any truth assignment satisfying* I*, it is possible to construct a matching $M$ in the weak core of* J.

*Proof.* If a variable $x_i$ is set to true (false), let $M$ contain all the edges between $\bar{X}_i$ ($X_i$) and $\bar{Y}_i$ ($Y_i$), and all interconnecting edges incident to $X_i$ ($\bar{X}_i$). See figure 3. In

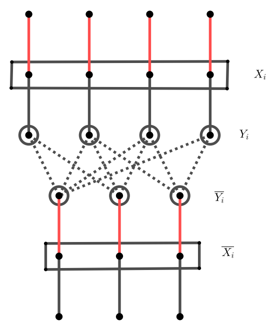

Figure 3: Red edges are in $M$. The dotted lines represent special edges.

a special edge $uv$, let $s_1t_1$, $s_2a_1$, $s_4t_2$ and $c_2c_3$, $b_4b_5$, $a_2a_3$, $c_4c_5$, $b_2b_3$, $a_4a_5$, $b_6c_6$, $b_7c_7$ belong to $M$. See figure 4.

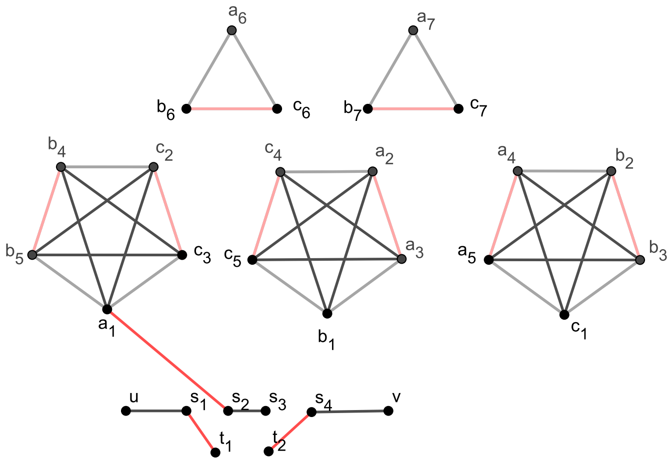


Figure 4: The red edges belong to $M$.

In every clause gadget, there is at least one node corresponding to a literal that is matched via an interconnecting edge in $M$ since every clause contains a literal set to true. Suppose that $a_1^j$ is such a node in the clause gadget corresponding to $c_j$. Let the edges $c_2^jc_3^j$, $b_4^jb_5^j$, $a_2^ja_3^j$, $c_4^jc_5^j$, $b_2^jb_3^j$, $a_4^ja_5^j$, $b_6^jc_6^j$ and $b_7^jc_7^j$ belong to $M$.

Now we show for every country that it cannot belong to a strongly blocking coalition. If the variable $x_i$ is set to true, then all the nodes in $X_i$, $\bar{X}_i$ and $\bar{Y}_i$ are matched in $M$, therefore the countries $X_i$, $\bar{X}_i$ and the singleton countries of $\bar{Y}_i$ cannot belong to a strongly blocking coalition. Since a node (which is also a country) in $Y_i$ is only connected to one of these countries, it also cannot belong to a blocking coalition. If $x_i$ is set to false, a similar argument shows that these countries cannot belong to a strongly blocking coalition.

Next, we show that the countries of a special edge cannot belong to a strongly blocking coalition. $T$ cannot, since it is fully matched in $M$. $S$ cannot, since it has only one node unmatched in $M$, and $s_1$ is only connected to a node in $Y_i$ or $\bar{Y}_i$ and a node in $T$, therefore it would be unmatched in any blocking coalition. It remains to check that some of the countries $A$, $B$ and $C$ do not form a blocking coalition. This is because the maximum matchings in $A$, $B$, $C$, $A \cup B \cup C$ and $B \cup C$ cover at most as many nodes as $M$, and a maximum matching in $A \cup B$ (or $A \cup C$) covers 12 nodes while $M$ covers 11 nodes of $A \cup B$ (and $A \cup C$).

This last argument about the countries of the example shows that the countries of a clause gadget also cannot belong to a strongly blocking coalition, which completes the proof. □

This concludes the proof of Theorem 19. □

## 4.2 Emptiness of the strong core is NP-hard

**Theorem 23.** *It is NP-hard to decide whether a matching in the strong core exists, even if every country has size at most 7.*

As in the previous section, we introduce the notion of *special edges.* The role of special edges is similar, but we need a different gadget for the definition. We say that there is a *special edge* between two nodes $u$ and $v$ if there is a gadget described below between $u$ and $v$. Let $C_i = \{c_i, c'_i\}$ be countries of size two for $i = 1, 2, 3$, and let the following edges belong to the graph: $uc_1$, $c'_1c_2$, $c'_2v$, $c'_2c_3$, $c'_3c_1$. See Figure 5.

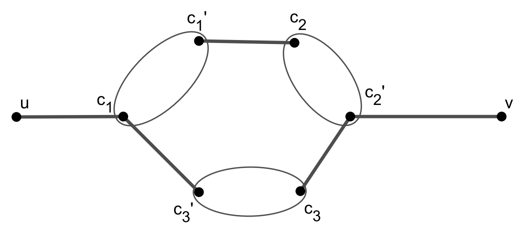


Figure 5: Gadget for special edge

**Lemma 24.** *Suppose an instance of the NTU international matching game contains a special edge between u and v. This special edge cannot belong to a matching in the strong core, but it can belong to a weakly blocking coalition in the following sense:*

*i) If there is a matching $M$ in the strong core, then $c'_1c_2$, $c'_2c_3$, and $c'_3c_1$ has to belong to $M$. This means that $uc_1$ and $c'_2v$ cannot belong to $M$.*

*ii) Suppose the edges $uc_1$ and $c'_2v$ do not belong to a matching $M$. If by replacing the special edge between $u$ and $v$ with an edge $uv$, $u$ and $v$ would belong to a weakly blocking coalition w.r.t. $M$ restricted to the graph we get if we delete the special edge, then $M$ does not belong to the strong core.*

*Proof.* i) The countries $C_i$ $(i = 1, 2, 3)$ form a weakly blocking coalition unless all of their nodes are covered by $M$. The latter is possible only if the edges $c'_1c_2$, $c'_2c_3$, and $c'_3c_1$ belong to $M$.

ii) We add the countries $C_1$ and $C_2$ to the blocking coalition, and instead of the edge $uv$ we add the edges $uc_1$, $c'_1c_2$ and $c'_2v$ to the blocking matching. □

*Proof of Theorem 23.* We reduce from 3-SAT. Given an instance $I$ of the 3-SAT, we construct an instance $J$ of the NTU international matching game. For every variable of $I$, we construct a variable gadget, and for every clause we construct a clause gadget. Then we describe the interconnecting edges between the clause gadgets and the variable gadgets.

**Variable gadget.** A variable $x_i$ defines 3 countries $X_i = \{x_i\}$, $Y_i = \{y_i, y_i'\}$ and $Z_i = \{z_i, z_i'\}$. The edges $x_i y_i'$, $x_i z_i'$ and $y_i z_i$ belong to the gadget.

**Clause gadget.** To a clause $c_j$, we associate 3 countries of size 5: $A_j = \{a_{j1}, a_{j2}, ..., a_{j5}\}$, $B_j = \{b_{j1}, b_{j2}, ..., b_{j5}\}$ and $D_j = \{d_{j1}, d_{j2}, ..., d_{j5}\}$. Each of these countries correspond to a literal of the clause $c_j$. The following edges belong to the gadget. $a_{j1}a_{j2}$, $a_{j2}b_{j3}$, $b_{j3}d_{j4}$, $b_{j1}b_{j2}$, $b_{j2}d_{j3}$, $d_{j3}a_{j4}$, $d_{j1}d_{j2}$, $d_{j2}a_{j3}$, $a_{j3}b_{j4}$, $a_{j5}b_{j5}$ , $b_{j5}d_{j5}$, $d_{j5}a_{j5}$. See Figure 6.

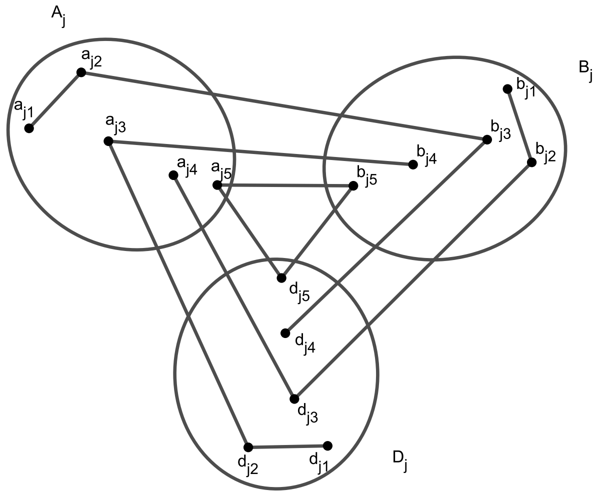


Figure 6: The clause gadget

**Interconnecting edges** Now we describe the edges between the clause gadgets and the variable gadgets. If a variable $x_i$ is in a clause $c_j$ in negated or unnegated form, and say the country $A_j$ corresponds to this literal, then we connect $a_{j1}$ with $y_i$ by a special edge if $x_i$ was in unnegated form, and we connect $a_{j1}$ with $z_i$ by a special edge if $x_i$ was in negated form.

**Claim 25.** *For any truth assignment satisfying* I, *it is possible to construct a matching* $M$ *in the strong core of* J.

*Proof.* For every variable gadget, the matching $M$ contains the edge $y_i z_i$, and $x_i$ is matched to $y_i'$ in $M$ if the variable $x_i$ is set to true and it is matched to $z_i'$ if $x_i$ is set to false.

Every clause $c_j$ contains a true literal, say $D_j$ corresponds to a true literal. Let $M$ contain the following edges of the clause gadget corresponding to $c_j$: $a_{j1}a_{j2}$, $b_{j3}d_{j4}$, $b_{j1}b_{j2}$, $d_{j3}a_{j4}$, $d_{j1}d_{j2}$, $a_{j3}b_{j4}$, and $a_{j5}bj5$. See Figure 7. This is the unique matching that covers all the nodes of $A_j \cup B_j \cup D_j$ except $d_5$. This completes the description of $M$; we now prove that $M$ is in the strong core.

Suppose there is a weakly blocking coalition w.r.t. $M$. Then this contains a country who is better off by a matching $M'$, therefore it was not fully matched by $M$, so either

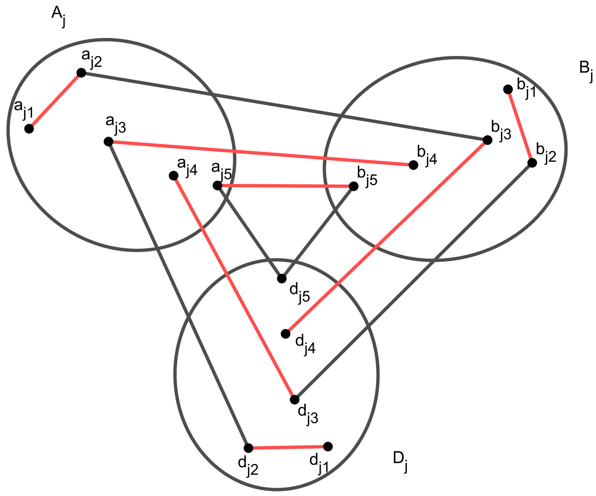


Figure 7: The red edges belong to $M$

a) it is a country in a clause gadget who is not fully matched and therefore corresponds to a literal set to true, or b) it is $Y_i$ (in this case $x_i$ is set to false) or c) $Z_i$ (in this case $x_i$ is set to true) for some $i$.

In case a) if in the clause $c_j$ the country $D_j$ is not fully matched by $M$, but $D_j$ is in a weakly blocking coalition and it is better off by a matching $M'$, then $M'$ fully covers $D_j$ (since $M$ only left $d_{j5}$ unmatched). The node $d_{j4}$ can only be matched by the edge $b_{j3}d_{j4}$, therefore $B_j$ has to belong to the blocking coalition, and has to be fully matched by $M'$. This means that $b_{j4}a_{j3}$ has to belong to $M'$, implying that $A_j$ has to belong to the blocking coalition as well, and has to be fully matched by $M'$. Thus $A_j$, $B_j$ and $D_j$ belong to the blocking coalition, and all of them are fully matched by $M'$, but this is impossible, since one of $a_{j5}$, $b_{j5}$ and $d_{j5}$ is left unmatched by any matching.

In case b) the matching $M'$ has to contain $x_iy_i'$, which means that it leaves $z_i'$ unmatched, therefore $Z_i$ cannot belong to the blocking coalition. This means that $y_i$ has to be matched by a special edge in $M'$, say $y_ia_{j1} \in M'$. Since $x_i$ was set to false, $A_j$ corresponds to a false literal and therefore it is fully covered by $M$, so $A_j$ has to be fully covered by $M'$. This means that $a_{j2}b_{j3} \in M'$ (thus $d_{j4}$ is unmatched by $M'$), and $a_{j4}d_{j3} \in M'$. Therefore $A_j$, $B_j$ and $D_j$ all have to belong to the blocking coalition, and at least two nodes of $A_j \cup B_j \cup D_j$ are left unmatched by $M'$ ($d_{j4}$, and one of $a_{j5}$, $b_{j5}$ and $d_{j5}$ is left unmatched by any matching). This is a contradiction, since $M$ covers all but one node of $A_j \cup B_j \cup D_j$.

Case c) is similar to case b). □

**Claim 26.** *For any matching $M$ in the strong core of* J*, there is a truth assignment satisfying* I.

*Proof.* Suppose there is a matching $M$ in the strong core. $M$ has to cover all but one node of every clause gadget, since otherwise the 3 countries of the clause with the matching in Figure 7 (with the node left unmatched chosen appropriately) form a weakly blocking coalition. A matching that covers all but one node of $A_j \cup B_j \cup D_j$ is unique except that we can choose any edge of the triangle $a_{j5}b_{j5}d_{j5}$.

The matching $M$ has to cover $x_i$ for every $i$, otherwise $X_i$ and $Y_i$ would form a weakly blocking coalition. If $x_i$ is matched to $y_i'$, then we set $x_i$ to be true, and if it is matched to $z_i'$ we set $x_i$ to be false.

We show that every clause contains a true literal. Suppose clause $c_j$ does not. Let $d_{j5}$ be the node left unmatched by $M$ in $A_j \cup B_j \cup D_j$. If $D_j$ corresponds to the variable $x_i$ in unnegated form, then $d_{j1}$ is connected to $y_i$ by a special edge and since $D_j$ corresponds to a false literal, $x_i z_i' \in M$. But then the countries $X_i$, $Y_i$, $D_j$ and $A_j$ form a weakly blocking coalition with the matching $M'$ shown in figure 8. Similarly,

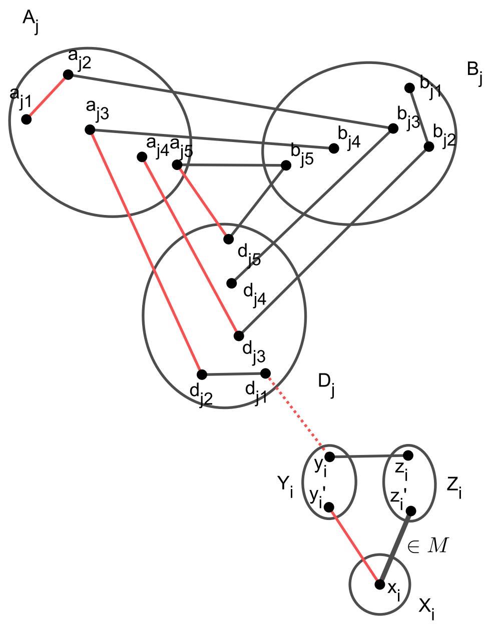


Figure 8: The blocking matching $M'$. The dotted line represents a special edge.

if $D_j$ corresponds to the variable $x_i$ in negated form, then $d_{j1}$ is connected to $z_i$ by a special edge, and the countries $X_i$, $Z_i$, $D_j$ and $A_j$ form a weakly blocking coalition. □

This concludes the proof of Theorem 23. □

# 5 Countries of size 2

In this section, we study the case when every country has size 2. We connect the two nodes of each country with a *country edge*. The prefect matching defined by the country edges is denoted by $M_C$.

**Lemma 27.** *A matching $M$ is in the strong core if and only if* *a) every country that is in an alternating cycle w.r.t. $M_C$ has both its nodes covered by $M$,*

*b) there is no alternating path w.r.t. $M_C$ from a country that has none of its nodes covered by $M$ to a country that has at most one node covered by $M$,*

*c) there is no alternating path w.r.t. $M_C$ that has 3 countries in it, such that each of them has one node covered by $M$.*

*Proof.* It is easy to see that if $M$ is in the strong core, then these conditions hold, since otherwise the matching given by the alternating path or cycle would block $M$. Suppose the conditions hold, but the matching $M$ is not in the strong core. Then there is a weakly blocking coalition of countries with a matching $M'$ such that there is one country that is better off with $M'$ than $M$, and the others are no worse off. Take the symmetric difference of $M'$ and $M_C$. This is a disjoint union of alternating cycles and paths. If the country that is better off by $M'$ has 0 nodes covered by $M$, then it has at least one node covered by $M'$, so it is either in an alternating cycle, which contradicts condition a), or it is in an alternating path. In the latter case, the two red edges at the ends of the path are countries that have one node covered by $M'$, so they have at most one node covered by $M$, which contradicts condition b). If the country that is better off by $M'$ has 1 node covered by $M$, then it has two nodes covered by $M'$, so it is either in an alternating cycle, which contradicts condition a), or it is in an alternating path (and it is not at the end of the path), but the two country edges at the ends of the path are countries that have one node covered by $M'$, so they have at most one node covered by $M$, which contradicts condition b) or c). □

**Theorem 28.** *If every country has size 2, then we can decide in polynomial time if a given matching $M$ is in the strong core.*

*Proof.* We need to show that we can check the conditions given by Lemma 27 in polynomial time. We can check if a given red edge is in an alternating cycle, since it is equivalent to checking if, after the deletion of this edge, the remaining graph (the original graph $G$ together with the red edges) has a perfect matching. We can do this for every red edge with at most one node covered by $M$. We can check whether there is an alternating path between two given country edges, knowing that these do not belong to an alternating cycle, since this is equivalent to checking if by deleting these two edges the remaining graph has a matching of size one less than a perfect matching. We can do this for any two country edges such that one of them has 0 nodes covered by $M$, and the other one has at most one node covered by $M$. For three given country edges, such that we know that none of them belong to an alternating cycle, we can check if there is an alternating path that contains all three of them, because this is equivalent to checking if by deleting all three of these edges, the remaining graph has a matching of size one less then a perfect matching. We can check this for any three country edges that have one node covered by $M$. □

**Lemma 29.** *A matching $M$ is in the weak core if and only if* *a) there is no alternating cycle w.r.t. $M_C$ such that every country edge in it has at most one node covered by $M$,*

*b) there is no alternating path w.r.t. $M_C$ such that the country edges at the two ends of the path have 0 nodes covered by $M$, and the country edges in the middle have one node covered by $M$.*

*Proof.* If $M$ is in the weak core, then the conditions a) and b) hold, since otherwise the matching defined by the alternating path or cycle would block $M$. Suppose a) and b) hold, but $M$ is not in the weak core. There is a coalition of countries with a matching $M'$ such that every country in the coalition is strictly better off with $M'$, therefore those countries who have both of their nodes covered by $M$ cannot belong to the coalition – we can delete these countries from the graph. Take the symmetric difference of the remaining country edges with $M'$. This consists of alternating cycles and paths. If there is an alternating cycle in the symmetric difference, then it contradicts a). If there is an alternating path, then the country edges at the two ends of the path have one node covered by $M'$, therefore they have 0 nodes covered by $M$, which contradicts b). □

**Theorem 30.** *If every country has size 2, then we can decide in polynomial time if a given matching $M$ is in the weak core.*

*Proof.* We can check the conditions of Lemma 29 in polynomial time. □

**Theorem 31.** *If every country has size 2, then the weak core is never empty.*

*Proof.* We can construct a matching $M^*$ in the weak core the following way. Let $M_C$ be as before. We check if there is an alternating cycle w.r.t. $M_C$ in the current graph. If there is, then let the edges of the cycle belong to $M$, and delete the nodes of the cycle from the graph. Repeat this with the remaining graph, until there are no alternating cycles with respect to $M_C$. Let $M^*$ be a maximum size matching that covers every node of $M$. We claim that $M^*$ is in the weak core, since the conditions of Lemma 29 are met. Condition a) clearly holds, since the construction deleted countries that were covered twice by $M$ (and hence by $M^*$), and the remaining graph did not contain an alternating cycle.

Suppose for contradiction that condition b) does not hold, i.e. there is an alternating path $P$ w.r.t. $M_C$ such that the country edges at the two ends of the path have 0 nodes covered by $M^*$, and the country edges in the middle have one node covered by $M^*$. Let $N^*$ be the set of edges of $M^*$ that contain a node from path $P$. We have $|N^*| \leq |P \cap M_C| - 2 = |P \cap E| - 1$. Therefore, $(M^* \setminus N^*) \cup (P \cap E)$ is a larger matching than $M^*$, contradicting the choice of $M^*$ as a max-size matching. □

It remains an intriguing open question whether the emptiness of the strong core can be decided in polynomial time if all countries have size 2.

## Acknowledgement

The authors would like to thank Péter Biró and Zsuzsanna Jankó for the fruitful discussions on the topic. The research was supported by the Hungarian National Research, Development and Innovation Office – NKFIH, grant number K120254, by the Hungarian Academy of Sciences under the research grant no. KEP-6/2017, and by Thematic Excellence Programme, Industry and Digitization Subprogramme, NRDI Office, 2019.


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