Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2019-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Compressed frameworks and compressible graphs

Tibor Jordán and Jialin Zhang

August 2019

Compressed frameworks and compressible graphs

Tibor Jordán[⋆] and Jialin Zhang[⋆⋆]

Abstract

A d-dimensional framework is a pair (G, p), where $G = (V, E)$ is a graph and p maps the vertices of G to points in $\mathbb{R}^d$. The edges correspond to the line segments that connect the points of their end-vertices. We say that (G, p) is compressed if in every other framework (G, q), with the same graph G and with the same edge-lengths, we have $||p(u) - p(v)|| \leq ||q(u) - q(v)||$ for all pairs $u, v \in V$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. A graph G is said to be compressible in $\mathbb{R}^d$ if there exists a compressed d-dimensional framework (G, p).

We characterize the compressible graphs in $\mathbb{R}^1$ and give some partial results in the two-dimensional case. We also consider the case when the coordinates of the points are generic.

Keywords: framework, graph realization, rigid, globally rigid

1 Introduction

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$ satisfying that $p(u) \neq p(v)$ for all pairs u, v with $uv \in E$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$.

We say that (G, p) is *compressed* if for all equivalent realizations (G, q) we have $||p(u) - p(v)|| \leq ||q(u) - q(v)||$ for all pairs u, v with $u, v \in V$ [1]. A graph G is called *compressible* in $\mathbb{R}^d$ if it has a compressed d-dimensional realization (G, p).

A framework (G, p) is *generic* if the set of the $d|V(G)|$ coordinates of the points $p(v)$, $v \in V(G)$, is algebraically independent over the rationals[2]. We say that a graph

[⋆]Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. Email: `jordan@cs.elte.hu`

[⋆⋆]Boston University, 111 Cummington Mall, Boston MA 02215, USA. Email: `zjleric@bu.edu`

[1]We may reverse the inequality in the definition and call a framework (G, p) *expanded* if for all equivalent realizations (G, q) we have $||p(u) - p(v)|| \geq ||q(u) - q(v)||$ for all pairs u, v with $u, v \in V$. In this paper we focus on compressed frameworks.

[2]Recall that a set of real numbers is said to be algebraically independent over the rationals if they do not satisfy any non-zero polynomial with rational coefficients.

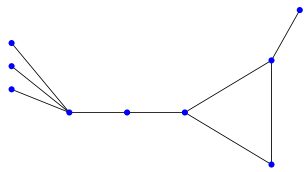

Figure 1: A compressible graph in $\mathbb{R}^1$.

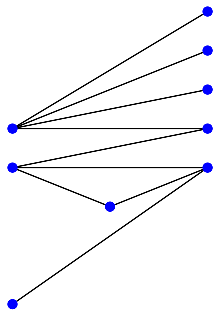

Figure 2: A compressed realization of the graph above.

G is *generically compressible* in $\mathbb{R}^d$ if there exists a compressed generic realization (G,p) of G in $\mathbb{R}^d$.

Our goal is to establish necessary conditions for d-dimensional compressibility and generic compressibility and to give a complete characterization of these properties in $\mathbb{R}^1$ as well as partial results in $\mathbb{R}^2$.

It turns out that the new notion of compressibility is sandwiched between rigidity and global rigidity, which are central concepts in rigidity theory. In the next section we give the related definitions and study this connection.

2 Rigid and globally rigid frameworks

Frameworks (G,p), (G,q) are *congruent* if $||p(u)-p(v)|| = ||q(u)-q(v)||$ holds for all pairs u, v with $u, v \in V$. We say that (G,p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional realization of G which is equivalent to (G,p) is congruent to (G,p).

The framework (G,p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G,q) is equivalent to (G,p) and $||p(v)-q(v)|| < \epsilon$ for all $v \in V$, then (G,q) is congruent to (G,p). Intuitively, this means that if we think of a d-dimensional framework (G,p) as a collection of bars and joints where points correspond to joints and each edge to a rigid (i.e. fixed length) bar joining its end-points, then the framework is globally rigid if its bar lengths determine the realization up to congruence. It is rigid if every continuous motion of the joints that preserves all bar lengths must preserve all pairwise distances between the joints.

It is a hard problem to decide if a given framework is rigid or globally rigid. It is NP-hard to decide if even a 1-dimensional framework is globally rigid [10], and the rigidity problem is NP-hard for 2-dimensional frameworks [1]. Our first result shows that testing whether a given framework (G, p) in $\mathbb{R}^d$ is compressed is also hard, even in $\mathbb{R}^1$.

Theorem 1. *It is NP-hard to decide whether a given 1-dimensional framework is compressed (resp. expanded).*

Proof. We show that the PARTITION problem[3] can be reduced to the problem of testing whether a given framework on the line is compressed (resp. expanded).

Let $\{a_1, a_2, ..., a_n\}$ be an instance of PARTITION. Let C be the cycle of length $n+2$ on vertex set $\{v_0, v_1, ..., v_n, w\}$ and edge set $\{v_i v_{i+1}, 0 \leq i \leq n-1\} \bigcup \{v_n w, w v_0\}$. Let $b = \frac{\sum_{1 \leq i \leq n} a_i}{2}$. Construct a 1-dimensional framework (C, p) in which the length of edge $v_i v_{i+1}$ is equal to a_{i+1} for $0 \leq i \leq n-1$, the length of each of the edges incident with vertex w is equal to b, and the map p satisfies $p(v_0) = 0, p(w) = b$, and $p(v_n) = 2b$. Note that these positions uniquely determine the coordinates of the other vertices and lead to a "stretched" realization of the path $C - w$.

The first observation is that if there is an equivalent, but not congruent realization (C, q), then in this realization the edges incident with w must overlap, $q(v_0) = q(v_n)$ must hold, and the edge lengths and vertex positions of the path $C - w$ give rise to a solution of the given instance of the PARTITION problem. A similar argument shows that if there is solution of the PARTITION problem then there exists an equivalent, but not congruent realization (C, q). (This observation shows that global rigidity testing is NP-hard.)

The next observation is that if there exists such a realization (C, q) then $||p(v_0) - p(v_n)|| > ||q(v_0) - q(v_n)||$ and for at least one vertex v_i we have $||p(v_i) - p(w)|| < ||q(v_i) - q(w)||$ (unless vertex w is coincident with some other vertex in (C, p), but we may assume that this situation - in which the PARTITION problem has a solution - does not hold). Thus (C, p) is globally rigid if and only if it is compressed if and only if it is expanded if and only if the PARTITION problem has a solution. This completes the proof. □

These problems become more tractable, however, if we consider *generic frameworks*. It is known that the rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic, see [12]. We say that the graph G is *rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid. The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$, and is a major open problem for $d \geq 3$.

A similar situation holds for global rigidity. Gortler, Healy and Thurston [3] proved that the global rigidity of d-dimensional frameworks is a generic property for all $d \geq 1$. We say that a graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid.

[3]The input of the PARTITION problem is a set $\{a_1, a_2, ..., a_n\}$ of n positive integers and the goal is to decide whether there is a subset $I \subseteq \{1, 2, ..., n\}$ with $\sum_{1 \leq i \leq n, i \in I} a_i = \sum_{1 \leq i \leq n, i \notin I}$. It is known to be NP-complete.

It is clear from the definition that every globally rigid d-dimensional framework is compressed in $\mathbb{R}^d$. In the case of generic frameworks it is not hard to see that compressed frameworks are rigid.

Proposition 2. *Let (G, p) be a d-dimensional compressed generic framework. Then (G, p) is rigid.*

Proof. For a contradiction suppose that (G, p) is not rigid. By adding new edges, if necessary, we may assume that $(G + uv, p)$ is rigid for some non-adjacent pair u, v of vertices of G. Since (G, p) is not rigid, it has a continuous motion which results in an equivalent but non-congruent realization (G, q). By the choice of u, v it is not hard to see that we must have $||p(u) - p(v)|| \neq ||q(u) - q(v)||$.

Furthermore, it follows from [9, Theorems 5.2, 5.8] that such a motion exists even if we add a "cable" connecting u and v, that is, we do not allow the distance from u to v to increase. This way we can achieve $||p(u) - p(v)|| > ||q(u) - q(v)||$, which shows that (G, p) is not compressed, a contradiction. This implies the theorem. □

In $\mathbb{R}^1$ Proposition 2 remains valid without assuming that the framework is generic[4]. For $d \geq 2$ there exist non-rigid compressed frameworks. Consider for example a d-dimensional realization (C_4, p) of the four-cycle with vertices a, b, c, d and $p(a) = p(c)$ and $p(b) = p(d)$. This realization is not rigid for $d \geq 2$, but it is "universally" compressed, i.e. it is compressed in every dimension.

Compressibility is not a generic property in $\mathbb{R}^d$ for all $d \geq 1$. Hence we may also define a stronger property by calling a graph G *generically compressed* in $\mathbb{R}^d$ if every generic realization (G, p) is compressed. The fact that every globally rigid graph is generically compressed, together with Proposition 2, gives the following chain of containment relations in every fixed dimension:

$$\text{Globally rigid} \subseteq \text{Generically compressed} \subseteq \text{Generically compressible} \subseteq \text{Rigid}$$

The first containment relation is in fact an equality, showing that this stronger property is less interesting for generic frameworks.

Proposition 3. *Let G be a generically compressed graph in $\mathbb{R}^d$. Then G is globally rigid in $\mathbb{R}^d$.*

Proof. Let G be generically compressed in $\mathbb{R}^d$ and suppose, for a contradiction, that G is not globally rigid in $\mathbb{R}^d$. Consider a generic d-dimensional realization (G, p). Then there is an equivalent realization (G, q), in which $||q(u) - q(v)|| \neq ||p(u) - p(v)||$ for some pair $u, v \in V(G)$. Since (G, p) is rigid by Proposition 2, we can use [6, Corollary 3.7] to deduce that (G, q) is quasi-generic, which means there is an isometry T of $\mathbb{R}^d$ for which the framework $T(G, q)$ is generic. Since the two generic realizations (G, p) and $T(G, q)$ are equivalent and satisfy $||T(q)(u) - T(q)(v)|| \neq ||p(u) - p(v)||$, at least one of them is not compressed. Thus G is not generically compressed in $\mathbb{R}^d$, a contradiction. □

[4]This observation follows from the simple fact that a framework (G, p) is rigid in $\mathbb{R}^1$ if and only if G is connected, see e.g. [4].

3 Necessary conditions

A graph $G = (V, E)$ is said to be *k-connected* if $|V| \geq k+1$ and $G-S$ is connected for all $S \subset V$ with $|S| \leq k-1$. A vertex set S of size k for which $G-S$ is disconnected is a *k-separator*. For $k = 1, 2$ a k-separator is sometimes called a *cut-vertex* or a *cut-pair*, respectively.

Consider a d-dimensional realization (G, p) of graph $G = (V, E)$. Let $conv(G, p)$ denote the convex hull of the points $\{p(v) : v \in V\}$. We say that $u \in V$ is an *extreme* vertex in (G, p) is $p(u)$ is on the boundary of polytope $conv(G, p)$.

Lemma 4. *Let (G, p) be a compressed d-dimensional framework and suppose that G is d-connected and the points $\{p(v) : v \in V\}$ are in general position. Let S be a d-separator in G. Then each vertex of S is extreme.*

Proof. The proof is based on the following simple claim.

Claim 5. *Let $x, y \in \mathbb{R}^d$ be two points and let $\mathcal{H} \subset \mathbb{R}^d$ be a hyperplane. Suppose that $x, y \notin \mathcal{H}$. Let $\bar{y}$ be the point obtained by reflecting y to the other side of $\mathcal{H}$. Then we have $||x-y|| < ||x, \bar{y}||$ if and only if x, y are in the same half-space determined by $\mathcal{H}$.*

Proof. Suppose that x, y are in the same half-space and let L denote the line through $x, \bar{y}$. Let $z = \mathcal{H} \cap L$ and consider the triangle $\triangle xyz$. Then we have $||x-y|| < ||x-z|| + ||y-z|| = ||x-\bar{y}||$. A similar argument shows that if x, y are on different sides of $\mathcal{H}$ then $||x, \bar{y}|| < ||x-y||$. □

Let $\mathcal{H}$ be the hyperplane containing $conv(S, p|_S)$. Note that the general position assumption implies that $\mathcal{H}$ is unique and it contains no other points of the framework. Suppose that some point of S is not extreme. Then $\mathcal{H}$ is not a supporting hyperplane of the polytope $conv(G, p)$ and hence we can use Claim 5 to show that by reflecting a component of $G-S$ about $\mathcal{H}$ we obtain an equivalent realization (G, q) in which the distance between some pair of vertices is strictly smaller than that in (G, p). □

It is easy to see that in $\mathbb{R}^1$ Lemma 4 remains valid without assuming that the points are in general position. Thus we have the following corollary. For a graph $G = (V, E)$ and vertex set $X \subseteq V$ we use $G[X]$ to denote the subgraph of G induced by X.

Lemma 6. *Let (G, p) be a compressed one-dimensional framework and let X denote the set of cut-vertices of G. Then*
(i) u is extreme for every $u \in X$,
(ii) $G[X]$ is bipartite,
(iii) if (G, p) is generic then $|X| \leq 2$.

Proof. (i) is clear from Lemma 4. Since (G, p) is one-dimensional, all the extreme vertices are mapped to two specific points of $\mathbb{R}^1$. Since coincident extreme points must be non-adjacent, (ii) follows. In a generic framework there are no coincident pairs of points. This gives (iii). □

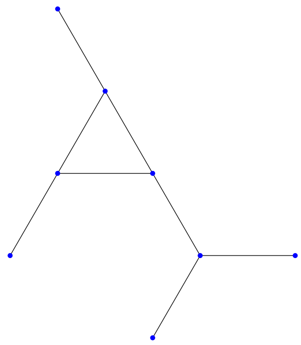

Figure 3: A graph which is not compressible in $\mathbb{R}^1$.

Let G be a 2-connected graph on vertex set V. Let V' denote the vertices of G which belong to at least one cut-pair of G and define a graph $\bar{G}$ on V' by letting $uv \in E(\bar{G})$ if and only if $\{u, v\}$ is a cut-pair in G.

Since the convex hull of a two-dimensional framework in general position is a strictly convex polygon, Lemma 4 implies that $\bar{G}$ is a subgraph of a cycle:

Lemma 7. *Let (G, p) be a compressed two-dimensional framework in general position. Then $\bar{G}$ is either*
(i) a cycle, or
(ii) a collection of pairwise vertex-disjoint paths.

We shall characterize compressible graphs in $\mathbb{R}^1$ (resp. generically compressible graphs in $\mathbb{R}^1$) by proving that the necessary conditions in Lemma 6(ii) (resp. Lemma 6(iii)) are also sufficient.

In the two-dimensional case we shall obtain a partial result by proving that if the necessary condition of Lemma 7(i) holds and G is an M-connected graph (we define M-connected later) then G is compressible in $\mathbb{R}^2$.

3.1 Ear-decompositions

The one-dimensional compressed realizations we shall construct will be obtained inductively by using the well-known decomposition resp. construction method that builds up a graph by adding ears. We recall the definitions.

Let $G = (V, E)$ be a graph. An *ear-decomposition* of G is a sequence $P_1, P_2, ..., P_t$ of subgraphs (called *ears*) of G satisfying the following properties:
(i) every ear is a (closed or open) path,
(ii) the first ear is closed (that is, P_1 is a cycle),
(iii) every edge of G belongs to exactly one ear,
(iv) the intersection of the vertex set of P_i and the union of the vertex sets of $P_1, P_2, ..., P_{i-1}$ coincides with the end-vertices of P_i, for $2 \leq i \leq t$.
The ear-decomposition is *open* if every ear, except for the first one, is open. The next theorem of H. Whitney is well-known in graph theory.

Theorem 8. *[13] A graph G has an open ear-decomposition if and only if G is 2-connected.*

4 Semi-generic realizations of graphs

It follows from Lemma 6 that if G is compressible then $G[X]$ is bipartite, where X denotes the set of cut-vertices of G. Furthermore, in any compressed realization (G, p) each vertex of X is mapped to one of the two boundary points of the realization. Hence (G, p) is typically highly non-generic with several coincident points.

Non-generic realizations, in general, are difficult to deal with, even in $\mathbb{R}^1$. In order to obtain a managable set of frameworks we shall introduce the concept of stretched semi-generic realizations of graphs and verify some properties of their equivalent realizations. Based on the lemmas proved in this section we shall be able to prove that the necessary condition of compressibility, mentioned above, is also sufficient. In the proof of this result we shall construct special stretched semi-generic realizations for graphs in which the cut-vertices induce a bipartite subgraph, and show that they are compressed.

In this section we consider one-dimensional realizations (G, p) of 2-connected graphs G *in standard position*, which means that for a designated vertex x_0 of G we have $p(x_0) = 0$. Since every framework can be moved to standard position by a translation, we preserve (a congruent copy of) every equivalent realization of every framework with underlying graph G by pinning x to the origin.

Let $G = (V, E)$ be a graph and let $X \subseteq V$ be a vertex set. A one-dimensional realization (G, p) of G is said to be *semi-generic (with respect to X)* if
(i) $p(x)$ is an integer for all $x \in X$, and
(ii) the set $\{p(v) : v \in V - X\}$ is algebraically independent over the rationals.

We shall assume that X is non-empty and the pinned vertex x_0 belongs to X.

Consider a one-dimensional semi-generic realization (H, p) of some graph H with respect to a set $X \subseteq V(H)$. Suppose that graph G is obtained from H by adding a new open ear, that is, by attaching a path P to H along the end-vertices s, t of P. Let X_P be a (possibly empty) subset of the internal vertices of P. Now we define a specific method for extending (H, p) to a semi-generic realization of G with respect to $X \cup X_P$. We assume that if $p(s) = p(t)$ holds then P has at least two edges (for otherwise no extended framework exists). We need the following notions.

Consider a path $T = (a_1, a_2, ..., a_k)$ on k vertices, in this order, and a one-dimensional realization (T, p) of T. We say that (T, p) is *stretched* if the sequence $p(a_1), p(a_2), \ldots, p(a_k)$ is strictly monotone (increasing or decreasing). Note that in this case $||p(a_k)-p(a_1)|| = \sum_{i=2}^{k} ||p(a_i) - p(a_{i-1})||$. We call a realization (T, p) *loop-stretched* if $p(a_1) = p(a_k)$ and the realization restricted to the subpath $(a_1, a_2, ..., a_{k-1})$ is stretched. In this case $||p(a_k) - p(a_{k-1})|| = \sum_{i=2}^{k-1} ||p(a_i) - p(a_{i-1})||$. See Figure 4.

The vertex set X_P decomposes the (edge-set of) P into subpaths $T_1, T_2, ..., T_m$ that connect pairs of vertices of the set $X_P \cup \{s, t\}$. We say that an extension (G, p') of (H, p) is *stretched* if the realization restricted to T_i is stretched or loop-stretched, for all $1 \leq i \leq m$. It is a *semi-generic stretched extension* if it is a stretched extension in which the coordinates of the vertices of X_P are integers and the set $\{p(v) : v \in (V(H) - X) \cup (V(P) - \{s, t\} - X_P)\}$ is generic, i.e. algebraically independent over the rationals. See Figure 5.

It is useful to observe that in the above construction of a semi-generic stretched

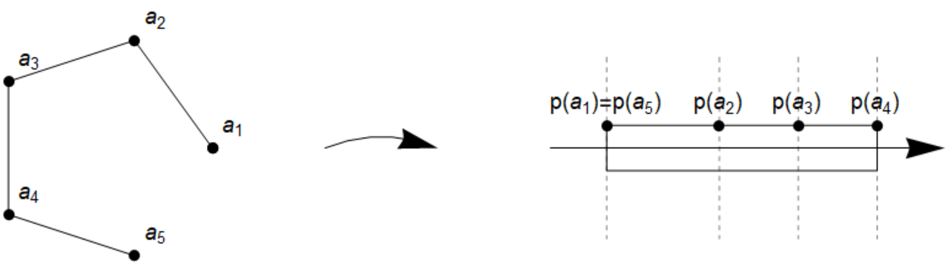

Figure 4: A looped-stretched realization of a path.

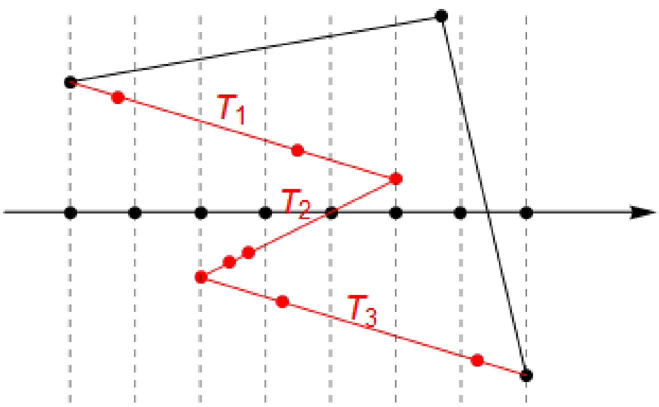

Figure 5: A semi-generic stretched extension.

extension of (H, p) we may guarantee that $\{p(v) : v \in (V(H) - X) \cup (V(P) - \{s, t\} - X_P)\}$ is generic by choosing generic edge lengths for all but one of the edges of T_i, for each $1 \leq i \leq m$. Let us declare some edge of each path T_i *inessential* and let us call all the other edges *essential*, for $1 \leq i \leq m$. Let S_i denote the set of essential edges of T_i, $1 \leq i \leq m$.

Lemma 9. *Let (H, p) be semi-generic with respect to $X \subseteq V(H)$ and let (G, p) be a stretched extension of (H, p). Then (G, p) is semi-generic if and only if*
(i) $p(x)$ is an integer for all $x \in X_P$, and
(ii) the set $\{p(v) : v \in V(H) - X\}$, together with the edge lengths of all the essential edges on P, is algebraically independent over the rationals.

Finally, we define a property of frameworks that will turn out to be crucial when we show, in the next section, that certain semi-generic frameworks are compressed. We say that a framework (G, p) is *well-behaved* if for every equivalent realization (G, q), and for all $v \in V$, we have that

$$||q(v) - p(v)|| \text{ or } ||q(v) + p(v)|| \text{ is an even integer.} \tag{1}$$

Thus either the position $q(v)$ can be obtained from $p(v)$ by an even valued translation or by a reflection about an integer point. In particular, if $p(v)$ is an integer then $q(v)$ is also an integer.

The main result of this section is as follows.

Theorem 10. *Let (H, p) be a well-behaved semi-generic framework with respect to X and let (G, p') be a stretched semi-generic extension of (H, p). Then (G, p') is well-behaved.*

Proof. As above, let P, s, t, X_P, and T_i, $1 \leq i \leq m$ be the extending path, its end-vertices, its designated set of internal vertices, and the corresponding subpaths, respectively. Let us define some edge of T_i to be inessential for all $1 \leq i \leq m$.

Consider a realization (G, q) of G which is equivalent to (G, p'). For an edge $e = uv$ of G let $l(e) = ||p(u) - p(v)|| = ||q(u) - q(v)||$ denote its length in these realizations. We first show that (G, q) is also stretched.

Claim 11. *If (T_i, p) is stretched (resp. looped-stretched) then (T_i, q) is also stretched (resp. looped-stretched), for all $1 \leq i \leq m$.*

Proof. Since the restriction of (G, q) to H is an equivalent realization of (H, p), and (H, p) is well-behaved, it follows that $||q(s) - p(s)||$ or $||q(s) + p(s)||$ is equal to an even integer $Z(s)$ and that $||q(t) - p(t)||$ or $||q(t) + p(t)||$ is equal to an even integer $Z(t)$.

Observe that the distance between the end-vertices of P is equal to the signed sum of the distances between the end-vertices s_i, t_i of the T_i's, in each of the two realizations. Thus

$$||p(s) - p(t)|| = \sum_{i=1}^{m} \pm ||p(s_i) - p(t_i)||,$$

and a similar equality holds for the q's. The next key observation is that for each T_i either we have $||p(s_i) - p(t_i)|| = ||q(s_i) - q(t_i)||$ (in which case the claim holds for T_i) or $||p(s_i) - p(t_i)|| - ||q(s_i) - q(t_i)|| = \pm 2 \sum_{e \in R_i} l(e)$, where $R_i \subseteq S_i$. Thus, in the latter case (which occurs, roughly speaking, when some edges of T_i are traversed in the opposite direction in (G, q) when we move from s_i to t_i) the difference is equal to twice the sum of the edge lengths of some essential edges of T_i. Note that we may assume that the single inessential edge of T_i is traversed the same way.

Suppose, for a contradiction, that the claim is false and hence the two distances in question are different for some subpath T_i. Then we can use the previous observations and the fact that the coordinates of the end-vertices s_i, t_i (except, possibly, for $s = s_1$ and $t = t_m$) are integers, to deduce that

$$||q(s) - q(t)|| - ||p(s) - p(t)|| = Z \pm p(s) \pm p(t) = \sum_{e \in R} \pm 2l(e),$$

where Z is an integer and R is a non-empty subset of the essential edges of P. This gives rise to a non-trivial algebraic dependence between some essential edge lengths and $p(s), p(t)$. By Lemma 9 this contradicts the fact that (G, p) is semi-generic. This completes the proof of the claim. □

We next show that (G, q) is also semi-generic. Since (H, p) is well-behaved, and by Lemma 9, it suffices to show the following claim.

Claim 12. *$q(v)$ is an integer for all $v \in X_P$.*

Proof. We may suppose that X_P is not empty. Let v_1 (resp. v_r) be the first (resp. last) internal vertex of P that belongs to X_P. We may have $v_1 = v_r$. By Claim 11 the distance between the end-vertices of each subpath T_i is the same in (G, p) and in (G, q). Hence it suffices to show that $q(v_1)$ (or $q(v_r)$) is an integer. It is clear if s or t belongs to X, which includes the case when $p(s) = p(t)$ (since only X-vertices can be coincident by the semi-generic property of (H, p)).

Thus we may assume that $s, t \notin X$. We can use Claim 11 and the fact that (H, p) is well-behaved to deduce that $q(v_1)$ is either an integer or it is equal to an integer plus or minus $2p(s)$. Similarly, we obtain that $q(v_r)$ is either an integer or it is equal to an integer plus or minus $2p(t)$. Since $v_1, v_r \in X_P$, Claim 11 implies that $||q(v_1) - q(v_r)||$ is an integer. Thus $q(v_1)$ (as well as $q(v_r)$) must be an integer, for otherwise we obtain an algebraic dependence between $p(s)$ and $p(t)$, contradicting the fact that (H, p) is semi-generic and $s, t \notin X$. □

Finally we show that (G, p') is well-behaved. Since (H, p) is well-behaved, (1) is satisfied for all vertices of H. Thus if P has no internal vertices then (G, p') is clearly well-behaved. So we may assume that P has at least one internal vertex.

First suppose that no internal vertex of P belongs to X_P. Then there is only one subpath $T_1 = P$. By Claim 11 P is stretched or loop stretched in $(G.p')$ as well as in (G, q). If P is loop stretched, then its end-vertices are coincident, with integer coordinates. This shows that (1) holds for all internal vertices of P. If P is stretched, we can also deduce that (1) holds for all internal vertices of P by using that we have $||q(t) - q(s)|| = ||p(t) - p(s)||$ and that (H, p) is well-behaved.

Next suppose that at least one internal vertex of P is part of X_P. By Claim 12 $q(v)$ is an integer for all $v \in X_P$. Let v_1 be the first internal vertex of P that belongs to X_P. By Claim 11 the distance from v_1 to s is the same in (G, p') and in (G, q). Since (H, p) is well-behaved, $q(s)$ and $p(s)$ satisfy one of the two alternatives in (1). Furthermore, $q(v_1)$ is either on the left side or on the right side of $q(s)$. This gives us four cases to consider. It is easy to check that (1) holds for v_1 in the two cases when s and the first subpath T_1 "move together". In the remaining two cases we have that $q(v_1) = p(s) - Z_e + (p(s) - p(v_1)) = 2p(s) - Z_e + p(v_1)$ and $q(v_1) = p(s) - 2(p(s) - Z) - (p(s) - p(v_1) = p(v_1) - 2(p(s) - Z)$, respectively, where Z_e is an even integer and Z is an integer. By Claim 12 $q(v_1)$ is an integer. Moreover, by the semi-generic property of (H, p), if $p(s)$ is rational then it is an integer. Thus in each of these cases we obtain that (1) holds for v_1.

This implies, by considering the subpaths T_i along P one by one, that (1) holds for all $v \in X_P$ and also for all internal vertices of P. Thus (G, p') is well-behaved, as claimed. □

We can deduce the following corollary on realizations of cycles.

Lemma 13. *Let C be a cycle and let X be a set of vertices of C. Then there exists a well-behaved realization of C which is semi-generic with respect to X.*

Proof. Let $x_0 \in X$ be the vertex fixed to the origin. Apply Theorem 10 so that H is equal to the one-vertex graph consisting of x_0 and $G = C$. □

Theorem 14. *Let G be a 2-connected graph and let X be a set of vertices of G. Then G has a well-behaved semi-generic realization with respect to X.*

Proof. By induction on the number of ears. We can use Theorem 8 and Lemma 13, together with Theorem 10 to construct the required realization as long as we make sure that whenever a one-edge ear is added, the end-vertices of the ear are not coincident. This can be achieved, for example, by using different integers for the coordinates of the vertices in X. □

5 Compressible graphs in $\mathbb{R}^1$

Note that the proofs of Theorems 10 and 14 show that when we create a well-behaved semi-generic framework by iteratively adding ears, then in every iteration we can choose arbitrary integers for the coordinates of the vertices in X_P and arbitrary generic coordinates for the internal vertices of the subpaths T_i, provided we respect the (loop)-stretched property of T_i. We shall use this freedom in the proof of the next theorem.

Let G be a 2-connected graph and let X be a designated vertex set for which $G[X]$ is bipartite with bipartition $X = A \cup B$. We assume that $G[X]$ has at least one edge (hence A and B are both non-empty).

We say that a one-dimensional well-behaved semi-generic realization (G, p^*), with respect to X, is *proper* if it is obtained by iteratively adding ears and respecting the following properties:
(i) $p(a) = 0$, for all $a \in A$,
(ii) $p(b) = 1$, for all $b \in B$,
(iii) $p(v) \in (0, 1)$, for all $v \in V - X$.

By the remarks above, and by Theorem 8, (G, p^*) exists. Note that, since $G[X]$ is bipartite, (i) and (ii) implies that if a one-edge ear is added, the end-vertices of the ear cannot be coincident. Also note that in this construction every subpath T_i (of every ear P) that connects two vertices from A (or two vertices from B) is loop stretched, while all the other subpaths are stretched.

Lemma 15. *Let (G, p^*) be a proper one-dimensional well-behaved semi-generic realization of a 2-connected graph G. Then (G, p^*) is compressed.*

Proof. Let (G, q) be a realization of G equivalent to (G, p^*) and consider two vertices $x, y \in V$. Suppose, without loss of generality, that $0 \leq p(x) \leq p(y) \leq 1$. We use the fact that (G, p^*) is well-behaved to deduce that the distance from $q(x)$ to some even integer is at most $p(x)$, and, similarly, the distance from $q(y)$ to some odd integer is at most $1 - p(y)$. These bounds imply that $||q(x) - q(y)|| \geq ||p(x) - p(y)||$. □

If G is not 2-connected, we can construct a compressed realization (G, p) by taking a proper well-behaved semi-generic realization for every maximal 2-connected subgraph H, with respect to the set X_H of cut-vertices of G included by H. Since the subgraph of H spanned by X_H is bipartite, and due to the tree structure of the maximal 2-connected subgraphs, this can be done. Note that if H has only two vertices then one vertex of H is mapped to 0, the other vertex is mapped to 1. Furthermore, observe that the construction of a proper realization is symmetric in the sense that the pinned vertex may be chosen from B as well. When we merge two pinned frameworks, only one vertex remains pinned. A proof similar to that of Lemma 15 shows that this realization is compressed.

Thus we obtain the main result of this section. Necessity follows from (the remark after) Proposition 2 and Lemma 6.

Theorem 16. *Let $G = (V, E)$ be a graph. Then G is compressible in $\mathbb{R}^1$ if and only if G is connected and the subgraph spanned by its cut-vertices is bipartite.*

5.1 Generic realizations

The case of generic compressibility is much simpler.

Theorem 17. *Let G be a graph. Then G has a compressed generic realization in $\mathbb{R}^1$ if and only if G is connected and has at most two cut-vertices.*

Proof. Necessity follows from Proposition 2 and Lemma 6(iii). To prove sufficiency, first recall that every 2-connected graph has a globally rigid (and hence compressed)

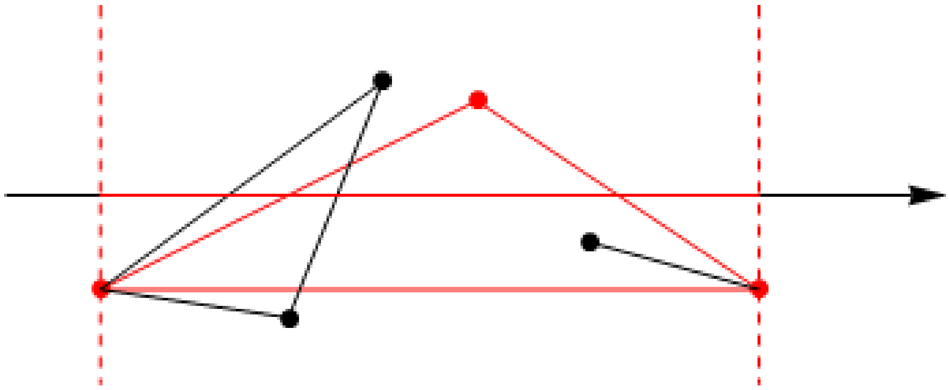

Figure 6: A compressed generic one-dimensional realization of a graph with two cut-vertices.

realization in $\mathbb{R}^1$, see e.g. [4]. The same holds for a complete graph K_2 on two vertices. Thus we may assume that G has at least three vertices and has at least one cut-vertex.

It is easy to see that if G has at most two cut-vertices then there exists a maximal 2-connected subgraph H of G, or possibly a subgraph H isomorphic to K_2, such that every cut-vertex is part of H. Construct a generic realization (H, p) of H for which every vertex, which is a cut-vertex of G, is extreme. This realization is globally rigid. Then extend this to a realization of G by mapping the remaining vertices to generic points in the interior of $conv(H, p)$, see Figure 6. The extended framework is compressed. $\square$

6 Compressible graphs in $\mathbb{R}^2$

The equivalent realizations of a generic rigid framework (G, p) in $\mathbb{R}^2$ can be obtained by a sequence of partial reflections. In $\mathbb{R}^2$ it is much more difficult to find a description of the equivalent realizations for rigid (but not globally rigid) graphs. There exist 3-connected rigid graphs which are not globally rigid (e.g. the prism), showing that one needs more complex moves than partial reflections along lines determined by cut-pairs of G. In this section we shall build on the following "exact" result and consider compressed generic realizations of M-connected graphs in $\mathbb{R}^2$.

Theorem 18. *[5] Let (G, p) be a generic realization of an M-connected graph G in $\mathbb{R}^2$. Then we can obtain a representative of each distinct congruence class of frameworks which are equivalent to (G, p) by iteratively applying the following operation to (G, p): choose a cut-pair $\{u, v\}$ of G and reflect some, but not all, of the components of $G - \{u, v\}$ in the line through the points $p(u), p(v)$.*

The definition of M-connected graphs[5] is based on the so-called two-dimensional rigidity matroid of a graph, see [5]. Here we use a theorem from [5] and provide an

[5]We also remark that in $\mathbb{R}^2$ every globally rigid graph is M-connected and every M-connected graph is rigid [5]. Furthermore, a graph G is globally rigid if and only if it is 3-connected and M-connected [5].

equivalent definition which is based on the decomposition of 2-connected graphs into cleavage units (or maximal 3-connected subgraphs). We need the following definitions.

Let $H = (V, E)$ be a 2-connected graph and let a, b be a cut-pair in H. Suppose that H is the union of two subgraphs H_1, H_2 with $V(H_1) \cap V(H_2) = \{a, b\}$. For $1 \le i \le 2$ let $H'_i = H_i + ab$ if ab is not an edge of H_i and otherwise put $H'_i = H_i$. We say that H'_1, H'_2 are the *cleavage graphs* obtained by *cleaving* H along $\{a, b\}$. We say that a cut-pair $\{x_1, x_2\}$ *crosses* another cut-pair $\{y_1, y_2\}$ in a 2-connected graph G, if x_1 and x_2 are in different components of $G - \{y_1, y_2\}$. It is easy to see that if $\{x_1, x_2\}$ crosses $\{y_1, y_2\}$ then $\{y_1, y_2\}$ crosses $\{x_1, x_2\}$. Thus, we can say that these cut-pairs are *crossing*.

Let $G = (V, E)$ be a 2-connected graph. The *cleavage units* of G are the graphs obtained by recursively cleaving G along each of its cut-pairs. If G has no crossing cut-pairs, this sequence of operations is uniquely defined and results in a unique set of cleavage units each of which is 3-connected or isomorphic to K_3.

Since M-connected graphs have no crossing cut-pairs or K_3 cleavage units (see [5]), an equivalent definition for the cleavage units in M-connected graphs is to first construct the *augmented graph* $\hat{G}$ from G by adding all edges uv for which $\{u, v\}$ is a cut-pair of G and $uv \notin E$, and then take the cleavage units to be the maximal 3-connected subgraphs of $\hat{G}$. It can also be shown that the decomposition into cleavage units has a natural decomposition tree. (These definitions are a special case of a general decomposition theory for 2-connected graphs due to Tutte [11].)

By combining Theorems 3.7 and 7.1 of [5], we have the following result that we can use to define M-connectivity.

Theorem 19. *[5] A graph G is M-connected if and only if it is 2-connected and each of its cleavage units is globally rigid.*

We need one more structural observation before we can verify the main result of this section.

Lemma 20. *Let G be a 2-connected graph. Suppose that $\bar{G}$ is a cycle. Then there is a cleavage unit H of G with $V(\bar{G}) \subseteq V(H)$.*

Proof. Let $\{u, v\}$ be a cut-pair of G and let $C = \bar{G}$. Consider two components X, Y of $G - \{u, v\}$ and suppose, for a contradiction, that some vertex a of C is in X and some other vertex b of C is in Y. Since C is a cycle, it contains a path from a to b that avoids u, v. Thus there exists an edge xy of C that connects a pair of vertices that belong to different components of $G - \{u, v\}$. This implies that $\{u, v\}$ and $\{x, y\}$ are crossing cut-pairs, a contradiction.

Therefore the vertex set of C intersects exactly one component of $G - \{u, v\}$. Hence we may perform the cleaving operations so that we end up with a cleavage unit H with $V(C) \subseteq V(H)$. $\square$

Theorem 21. *Let G be an M-connected graph for which $\overline{G}$ is a cycle of length t. Then G is generically compressible in $\mathbb{R}^2$.*

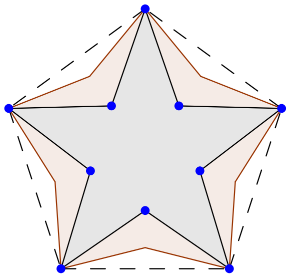

Figure 7: A compressed two-dimensional realization of graph G. The dashed edges represent $\bar{G}$.

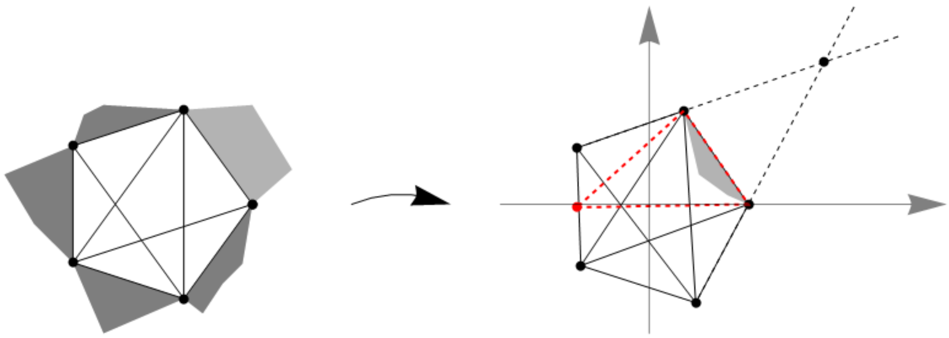

Figure 8: The feasible area for the vertices of X.

Proof. By Lemma 20 there is a cleavage unit H of G which contains the vertices of all cut-pairs $\{u_i, v_i\}$, $1 \leq i \leq t$ and hence each component of $G - V(H)$ corresponds to a component of $G - \{u_i, v_i\}$ for some $1 \leq i \leq t$.

Let $S = V(\bar{G})$ denote the union of the cut-pairs of G. Construct a realization (G, p) of G as follows. First map S to generic points in $\mathbb{R}^2$ so that $conv(S, p)$ is a (close to) regular t-gon in which the cyclic ordering of the vertices coincides with that in $\bar{G}$, see Figure 7. Then map the vertices of each component X of $G - \{u_i, v_i\}$ (except for the one that contains vertices from H), for all $1 \leq i \leq t$, to generic points in the interior of $conv(S, p)$ independently, making sure that by reflecting the vertices of X about the line through $p(u_i), p(v_i)$, the reflected points stay on the same side of each line that passes through $p(u_j), p(v_j)$, for all $1 \leq j \leq t$, $j \neq i$. See Figure 8.

This realization is compressed by Claim 5 and Theorem 18. □

We believe that the conditions of Lemma 7 together lead to a complete characterization of generically compressible M-connected graphs:

Conjecture 22. *Let G be M-connected in $\mathbb{R}^2$. Then G is generically compressible in $\mathbb{R}^2$ if and only if $\bar{G}$ is a cycle or a collection of pairwise vertex-disjoint paths.*

7 Concluding remarks

We have a long list of open problems motivated by our new notions and results. For example, as we noted earlier, we may reverse the inequality in the definition of compressed frameworks and introduce the notion of expanded frameworks. Then it is natural to call a graph G *(generically) expansible* in $\mathbb{R}^d$ if it has an expanded d-dimensional (generic) realization (G, p). Our initial investigations show that it is not hard to characterize the (generically) expansible graphs in $\mathbb{R}^1$, but the higher dimensional questions remain open.

We may also consider frameworks (G, p) for which *there exists* a non-congruent equivalent realization (G, q) in which all pairwise distances are greater than or equal to the original ones. This happens to be a question about tensegrity frameworks, in which, on top of the fixed length bars, cables and struts - that give rise to upper or lower bounds for the distance of their endvertices - may also be present [9]. It is not hard to observe that (G, p) has this property if and only if the tensegrity framework (T, p), obtained from (G, p) by adding a strut uv for each non-adjacent pair u, v is not globally rigid.

If this bar-strut structure (T, p) is not even rigid, the framework (G, p) has a so-called expansive motion [2]. In this context the notion of compressed framework can be viewed as a kind of discrete version of expansive motions.

8 Acknowledgements

The first steps of this research project were made in the framework of the Budapest Semesters in Mathematics Undergraduate Research Experience Program.

This work was supported by the Hungarian Scientific Research Fund grant no. K 109240.

References

[1] T. G. Abbot: Generalizations of Kempe's universality theorem, MSc thesis, MIT (2008). `http://web.mit.edu/tabbott/www/papers/mthesis.pdf`

[2] R. Connelly, E.D. Demaine, and G. Rote: Straightening polygonal arcs and convexifying polygonal cycles, *Discrete Comp. Geometry*, Vol. 30, 205-239 (2003).

[3] S. Gortler, A. Healy, and D. Thurston: Characterizing generic global rigidity, *American Journal of Mathematics*, Volume 132, Number 4, August 2010, pp. 897-939.

[4] B. Jackson: Notes on the Rigidity of Graphs, Levico Conference Notes, 2007.

[5] B. Jackson and T. Jordán: Connected rigidity matroids and unique realizations of graphs, *J. Combin. Theory Ser. B* 94, 2005, pp 1-29.

[6] B. Jackson, T. Jordán, and Z. Szabadka: Globally linked pairs of vertices in equivalent realizations of graphs, *Discrete Comp. Geometry*, Vol. 35, 493-512, 2006.

[7] B. Jackson, T. Jordán, and Z. Szabadka: Globally linked pairs of vertices in rigid frameworks, in: Rigidity and Symmetry, *Fields Institute Communications*, Vol. 70, R. Connelly, A. Ivic Weiss, W. Whiteley (Eds.) 2014, pp. 177-203.

[8] T. Jordán: Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks, in: Discrete Geometric Analysis, MSJ Memoirs, vol. 34, pp. 33-112, 2016.

[9] B. Roth and W. Whiteley: Tensegrity frameworks, *Trans. Amer. Math. Soc.* Vol. 265, No. 2. 1981, pp. 419-446.

[10] J.B. Saxe: Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[11] W.T. Tutte: *Connectivity in Graphs*, University of Toronto Press (1966).

[12] W. Whiteley: Some matroids from discrete applied geometry, in Matroid theory (J.E. Bonin, J.G. Oxley and B. Servatius eds., Seattle, WA, 1995), Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.

[13] H. Whitney: Nonseparable and planar graphs, *Trans. Amer. Math. Soc.*, 34 (1932), pp. 339-362.