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Finding strongly popular b-matchings in bipartite graphs

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Abstract

The computational complexity of the bipartite popular matching problem depends on how indifference appears in the preference lists. If one side has strict preferences while nodes on the other side are all indifferent (but prefer to be matched), then a popular matching can be found in polynomial time [Cseh, Huang, Kavitha, 2015]. However, as the same paper points out, the problem becomes NP-complete if nodes with strict preferences are allowed on both sides and indifferent nodes are allowed on one side. We show that the problem of finding a *strongly popular* matching is polynomial-time solvable even in the latter case. More generally, we give a polynomial-time algorithm for the many-to-many version, i.e. the strongly popular b-matching problem, in the setting where one side has strict preferences while agents on the other side may have one tie of arbitrary length at the end of their preference list.

1 Introduction

A **bipartite preference system with ties** consists of a bipartite multigraph $G = (S, T; E)$ and, for every node $v \in S \cup T$, a partial order $\preceq_v$ on the edges incident to v. This partial order is usually called the **preference list** of v. Given a bipartite preference system with ties, a node **prefers** a matching M_1 to a matching M_2 if it is either matched in M_1 but unmatched in M_2, or matched by a better edge in M_1 than in M_2. A matching M_1 is **more popular** than a matching M_2 if the number of nodes preferring M_1 to M_2 is strictly larger than the number of nodes preferring M_2 to M_1. This relation is not transitive; it is possible that M_1 is more popular than M_2, M_2 is more popular than M_3, and M_3 is more popular than M_1 [2]. A matching M is **popular** if no matching is more popular than M, and it is **strongly popular** if M is more popular than any other matching. These notions were first introduced by Gärdenfors [8], who showed that *a)* every strongly popular matching is stable and *b)* in case of no ties, all stable matchings are popular.

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Informally, the existence of a strongly popular matching means that the 'popularity contest' among matchings has an undisputed winner. There obviously cannot be two distinct strongly popular matchings, because both of them would have to be more popular than the other, which is impossible. Furthermore, a strongly popular matching must be a unique popular matching. However, there are instances where the popular matching is unique but it is not strongly popular; we refer the reader to the full version of [2] for an example.

Algorithmic questions about popular matchings have generated a lot of interest lately, see Section 1.1 for a short summary of recent results. Here we just mention that for any preference system with ties (even non-bipartite), it can be decided in polynomial time if a given matching is popular or strongly popular [2]. This means that the decision problem for popular matchings is in the complexity class NP, while the decision problem for strongly popular matchings is in the lesser-known complexity class UP (Unambiguous Polynomial-time). The latter class, introduced by Valiant [20], consists of the decision problems solvable by an NP machine such that all witnesses are rejected in a "no" instance, while exactly one witness is accepted in a "yes" instance. The strongly popular matching problem belongs to this class because each "yes" instance has a single strongly popular matching and it can be verified in polynomial time. In contrast to the abundance of NP-complete problems, no complete problem is known for the class UP.

For a node $v \in S \cup T$, let $\delta_G(v)$ denote the set of edges incident to v in G. In the most general setting, the preference list of v can be an arbitrary partial order on $\delta_G(v)$. However, we will also consider three types of nodes with restrictions on their preferences:

- **nodes with strict preferences**, where the preference order $\preceq_v$ is a total order on $\delta_G(v)$,
- **indifferent nodes**, where every incident edge is equally good (and better than being unmatched),
- **nodes with restricted ties**, whose preference list contains a single tie of arbitrary length at the end of the list, i.e. who have a set of *least preferred edges*, and a total order on the rest of $\delta_G(v)$.

Notice that the first two types are included in the third. Our main result concerns preference systems where nodes in S have strict preferences while nodes in T have restricted ties. This kind of preference system is well-known in the stable matching literature: Irving and Manlove [14] studied it in the context of the Hospitals/Residents problem, motivated by practical applications like the Scottish Foundation Allocation Scheme. It also has theoretical interest: Huang et al. [9] gave a 1.25-approximation algorithm for the maximum stable matching problem under such preferences, which is the only known case that matches the lower bound of approximability under the Unique Games Conjecture.

Let us describe the known results on the popular and strongly popular matching problem under various preferences. If all nodes have strict preferences, then every stable matching is popular [8]. On one hand, this implies that there always exists a popular matching and one can be found using the well-known Gale-Shapley algorithm [7]. On the other hand, we can decide if a strongly popular matching exists by finding an arbitrary stable matching and checking whether it is strongly popular (this also works in non-bipartite preference systems without ties [2]).

The problems become considerably harder when indifferent nodes are also allowed on one of the sides. If nodes in S have strict preferences while those in T are all indifferent, then the existence of a popular matching can still be decided in polynomial time, as shown by Cseh, Huang, and Kavitha [4]. However, they also showed that the problem becomes NP-complete if we allow nodes with strict preferences in both S and T; see the full version of [4] and [5] for proofs.

In this paper, we prove that the existence of a *strongly* popular matching can be decided in polynomial time in the latter setting, and even when nodes with restricted ties are allowed in T. This is the first result that demonstrates that the strongly popular matching problem is considerably easier than the popular matching problem.

Theorem 1.1. *Given a bipartite preference system* $(G = (S, T; E), \preceq)$ *where nodes in* S *have strict preferences and nodes in* T *have restricted ties, it can be decided in polynomial time if there is a strongly popular matching.*

Our algorithm successively finds edges that *cannot* be in a strongly popular matching or *must* be in any strongly popular matching, and also maintains a directed graph related to the possible structure of the strongly popular matching. The set of possible candidates keeps shrinking until, at the end, we can either conclude that there is no strongly popular matching, or exactly one candidate matching remains. In the latter case, we can check in polynomial time whether this matching is strongly popular.

Our result also extends to the **strongly popular b-matching problem** in bipartite preference systems. In the b-matching problem, each node v has a quota $b(v) \in \mathbb{Z}_+$. An edge set $M \subseteq E$ is a **b-matching** if $d_M(v) \leq b(v)$ for every v, where $d_M(v)$ denotes the number of edges of M incident to v. There are various ways to define popularity for b-matchings, and in this paper we follow the definition used by Brandl and Kavitha [3] and Kamiyama [15] (we note that a different definition of popularity is used in [19]). The precise definition of popularity in the many-to-many setting is presented in Section 2, where we also prove the following result, which has not yet been published in the literature.

Theorem 1.2. *In arbitrary bipartite preference systems with ties, popularity and strong popularity of a given* b*-matching can be decided in polynomial time using all-pairs shortest paths in an auxiliary digraph.*

For the problem of deciding the existence of a strongly popular b-matching, we again consider the setting where one side has strict preferences while the other side has restricted ties. All nodes prefer to fill as much of their quota as possible. As mentioned by Manlove et al. [17] and Irving and Manlove [14], this setting is relevant to several real-world applications: for example, in a Hospitals/Residents allocation problem, hospitals may not be willing to rank all admissible residents, but may opt instead to rank only the best ones and put the rest in a tie at the end of their preference list. Similar partial ranking of admissible students is natural in student allocation and project allocation problems. We prove the following extension of Theorem 1.1.

Theorem 1.3. *Given a bipartite preference system* $(G = (S, T; E), \preceq)$ *with quotas* $b(v) \in \mathbb{Z}_+$ $(v \in S \cup T)$*, where nodes in* S *have strict preferences and nodes in* T *have restricted ties, it can be decided in polynomial time if there is a strongly popular* b*-matching.*

The proof is presented in Section 3. Theorem 1.1 is obtained by setting $b \equiv 1$.

1.1 Related work on popular matchings

There are several fascinating questions about the computational complexity of the popular matching problem. For bipartite preference systems with no ties, Huang and Kavitha [10] showed that a maximum size popular matching can be found in polynomial time, and Cseh and Kavitha [6] gave an algorithm for deciding if a given edge belongs to a popular matching. On the other hand, the complexity of deciding the existence of a popular matching in a non-bipartite preference system without ties is still open. Huang and Kavitha [11] introduced the notion of unpopularity factor, and showed that, for any positive ε, it is NP-hard to compute a matching with unpopularity factor within $\frac{4}{3} - \varepsilon$ of optimal. In another paper, they showed that the problem of finding a maximum-weight popular matching is NP-hard, while a maximum-weight popular half-integral matching can be found in polynomial time. The complexity of the bipartite maximum-weight popular matching problem is open.

Several recent results concern a slightly different, one-sided model (also known as the House Allocation model), where one side has preference lists, while nodes on the other side do not vote at all and do not prefer to be matched. Abraham et al. [1] gave a polynomial-time algorithm for finding a popular matching in this model. If the preferences are strict, then optimal popular matchings can also be found for various notions of optimality [16, 18].

2 Popular b-matchings

The popular matching problem can be extended to the many-to-one setting (the so-called Hospitals-Residents problem), and also to the many-to-many setting (the bipartite b-matching problem). Two models have been proposed, one by Nasre and Rawat [19], and another by Brandl and Kavitha [3]. In this paper we use the latter model, but introduce it in a slightly different way. Both papers proved that, in case of strict preferences, a maximum size popular b-matching in the respective models can be found in polynomial time. Kamiyama [15] extended the second model with matroid constraints, and showed how to find a maximum size popular b-matching if the constraints are defined by weakly base orderable matroids.

Let $G = (S, T; E)$ be a bipartite graph, and let $b : S \cup T \to \mathbb{Z}_+$ denote the quota function. We define the notion of popularity in b-matchings for general bipartite preference systems with ties, where the partial orders $\preceq_v$ ($v \in S \cup T$) can be arbitrary. For edges $e, f \in \delta_G(v)$, let

$$\mathrm{vote}_v(e,f) = \begin{cases} 1 & \text{if } e \succ_v f \\ -1 & \text{if } e \prec_v f \\ 0 & \text{otherwise.} \end{cases}$$

For technical reasons, we need to allow the empty set as an argument, so we extend the definition by $\mathrm{vote}_v(e, \emptyset) = 1$, $\mathrm{vote}_v(\emptyset, f) = -1$, and $\mathrm{vote}_v(\emptyset, \emptyset) = 0$. Let M_1 and M_2 be b-matchings, and let $v \in S \cup T$. We say that $(e_1, \dots, e_{b(v)}; f_1, \dots, f_{b(v)})$ is a **valid enumeration** of (M_1, M_2) at v if

- each e_i is either an edge in $M_1 \cap \delta_G(v)$ or the empty set
- each f_i is either an edge in $M_2 \cap \delta_G(v)$ or the empty set
- each edge of $M_1 \cap \delta_G(v)$ appears exactly once among the e_is
- each edge of $M_2 \cap \delta_G(v)$ appears exactly once among the f_is
- if $e_i = f_j \in M_1 \cap M_2 \cap \delta_G(v)$, then $i = j$
- if $e_i = \emptyset$ and $f_i \neq \emptyset$ for some i, then there is no j such that $e_i \neq \emptyset$ and $f_i = \emptyset$.

The last property implies that the number of indices i where $e_i \neq \emptyset$ and $f_i \neq \emptyset$ is $\min\{|M_1 \cap \delta_G(v)|, |M_2 \cap \delta_G(v)|\}$. We define

$$\mathrm{vote}_v(M_1, M_2) = \min\{\sum_{i=1}^{b(v)} \mathrm{vote}_v(e_i, f_i) : (e_1, \dots, e_{b(v)}; f_1, \dots, f_{b(v)}) \text{ is a valid enumeration of } (M_1, M_2) \text{ at } v\}.$$

Observe that we take the valid enumeration that is worst from the point of view of M_1. This implies that $\mathrm{vote}_v(M_1, M_2) + \mathrm{vote}_v(M_2, M_1) \leq 0$, but equality does not necessarily hold. Define

$$\mathrm{vote}(M_1, M_2) = \sum_{v \in S \cup T} \mathrm{vote}_v(M_1, M_2).$$

A b-matching M is **popular** if $\mathrm{vote}(M, M') \geq 0$ for every b-matching M', and it is **strongly popular** if $\mathrm{vote}(M, M') > 0$ for every b-matching M' distinct from M. If $b \equiv 1$, then these definitions coincide with the standard definitions for matchings. Since $\mathrm{vote}(M_1, M_2) + \mathrm{vote}(M_2, M_1) \leq 0$ for any two b-matchings (M_1, M_2), there can be at most one strongly popular b-matching.

In the remainder of the section, we show that popularity and strong popularity of a given b-matching M can be decided in polynomial time. We define an auxiliary bipartite graph $\hat{G}_M = (\hat{S}, \hat{T}; \hat{E}_M)$ where every node $v \in S \cup T$ is replaced by $b(v)$ nodes $\hat{v}_1, \dots, \hat{v}_{b(v)}$. For $st \in E \setminus M$, we introduce edges $\hat{s}_i \hat{t}_j$ for every $i \in [b(s)]$ and $j \in [b(t)]$.

We also define edges corresponding to M that form a matching $\hat{M}$: for every edge $st \in M$, we add a single edge $\widehat{st} = \hat{s}_i\hat{t}_j$, in such a way that $\hat{M} = \{\widehat{st} : st \in M\}$ is a matching in $\hat{G}_M$. This is possible because $d_M(v) \le b(v)$ for every $v \in S \cup T$. Thus, an edge $e = st \in E$ has a single corresponding edge in $\hat{E}_M$ if $e \in M$, and $b(s)b(t)$ corresponding edges if $e \in E \setminus M$.

The preference system on G induces a preference system on $\hat{G}_M$, with the additional definition that each node is indifferent between edges in $\hat{E}_M$ corresponding to the same edge in $E \setminus M$. Note that it is *not* true that M is popular if and only if $\hat{M}$ is popular. However, we can characterize the popularity of M using alternating paths with respect to $\hat{M}$ in $\hat{G}_M$, and this leads to a polynomial-time algorithm for deciding if M is popular / strongly popular.

We define a weight function w on $\hat{E}_M$. Let $w(e) = 0$ if $e \in \hat{M}$. For given $\hat{s}_i\hat{t}_j \in \hat{E}_M \setminus \hat{M}$, let e_1 be the edge of $\hat{M}$ incident to $\hat{s}_i$ if $\hat{s}_i$ is covered by $\hat{M}$, otherwise let e_1 be the empty set. Similarly, let e_2 be the edge of $\hat{M}$ incident to $\hat{t}_j$ if $\hat{t}_j$ is covered by $\hat{M}$, otherwise let $e_2 = \emptyset$. Let

$$w(\hat{s}_i\hat{t}_j) = \mathrm{vote}_{\hat{s}_i}(e_1, \hat{s}_i\hat{t}_j) + \mathrm{vote}_{\hat{t}_j}(e_2, \hat{s}_i\hat{t}_j).$$

An **alternating cycle** with respect to $\hat{M}$ is a cycle with edges alternating between $\hat{M}$ and $\hat{E}_M \setminus \hat{M}$. An **alternating path** is a path alternating between $\hat{M}$ and $\hat{E}_M \setminus \hat{M}$, such that if the first or last edge is in $\hat{E}_M \setminus \hat{M}$, then the corresponding end-node of the path is not covered by $\hat{M}$. The **modifier** of an alternating path P, denoted by $\mathrm{mod}(P) \in \{0, 1, 2\}$, is the number of its end-nodes covered by $\hat{M}$. An alternating path P is **invalid** if $\mathrm{mod}(P) = 1$ and its end-nodes are $\hat{v}_i$ and $\hat{v}_j$ for the same v, otherwise it is **valid**.

Theorem 2.1. *A b-matching M is popular if and only if $w(C) \ge 0$ for every alternating cycle C and $w(P) + \mathrm{mod}(P) \ge 0$ for every valid alternating path P in the auxiliary graph $\hat{G}_M$. A b-matching M is strongly popular if and only if $w(C) > 0$ for every alternating cycle C and $w(P) + \mathrm{mod}(P) > 0$ for every valid alternating path P in the auxiliary graph $\hat{G}_M$.*

Proof. We first prove that if M is not popular, then the alternating cycle or path described in the theorem exists (the same proof works for strong popularity). Since M is not popular, there exists a b-matching M' such that $\mathrm{vote}(M, M') < 0$. We may assume that at least one endpoint of every edge of M is covered by M', since otherwise that edge can be added to M'.

We define a matching $\hat{M}'$ in $\hat{G}_M$ based on the voting at the individual nodes. If $st \in M \cap M'$, then we include $\widehat{st}$ in $\hat{M}'$. For every $v \in S \cup T$, we fix a valid enumeration $(e^v_1, \dots, e^v_{b(v)}; f^v_1, \dots, f^v_{b(v)})$ of (M, M') at v that is a minimizer of $\mathrm{vote}_v(M, M')$ and satisfies the following property: if e^v_i is an edge vw of M, then $\widehat{vw} = \hat{v}_i\hat{w}_j$ for some j. It is easy to see that there is an enumeration with this property, since we can permute the indices arbitrarily. If $f = st$ is an edge in $M' \setminus M$, then there are indices i and j such that $f = f^s_i$ and $f = f^t_j$. We include $\hat{s}_i\hat{t}_j$ in $\hat{M}'$, and denote it by $\hat{f}$.

Consider $\hat{M}\Delta\hat{M}'$, i.e. the symmetric difference of $\hat{M}$ and $\hat{M}'$. Since its degrees are at most 2, $\hat{M}\Delta\hat{M}'$ is the disjoint union of cycles $C_1, \dots, C_k$ and paths $P_1, \dots, P_l$.

Observe that $C_1,\dots,C_k$ are alternating cycles and $P_1,\dots,P_l$ are alternating paths. Furthermore, each P_i is valid, because the validity of the enumeration at v means that if some v_i is covered by $\hat{M}'$ but not by $\hat{M}$, then any v_j covered by $\hat{M}$ is also covered by $\hat{M}'$.

Claim 2.2. $\text{vote}(M,M') = \sum_{i=1}^k w(C_i) + \sum_{j=1}^l (w(P_j) + \text{mod}(P_j))$.

Proof. Consider an edge $f = st$ in $M' \setminus M$, and the corresponding edge $\hat{f} = \hat{s}_i\hat{t}_j$ in $\hat{M}'$. Let e_1 (e_2) be the edge of $\hat{M}$ incident to $\hat{s}_i$ ($\hat{t}_j$) if it is covered by $\hat{M}$, and the empty set otherwise. By definition, $f = f_i^s$ and $f = f_j^t$, so

$$w(\hat{f}) = \text{vote}_{\hat{s}_i}(e_1,\hat{f}) + \text{vote}_{\hat{t}_j}(e_2,\hat{f}) = \text{vote}_s(e_i^s, f_i^s) + \text{vote}_t(e_j^t, f_j^t).$$

This means that $\sum_{i=1}^k w(C_i) + \sum_{j=1}^l w(P_j)$ counts all votes except those of type $\text{vote}_v(e_i^v, f_i^v)$ where $e_i^v \in M$ and $f_i^v = \emptyset$. The number of these votes is exactly the number of nodes covered by $\hat{M}$ but not by $\hat{M}'$, which equals $\sum_{j=1}^l \text{mod}(P_j)$. □

We obtained that $\sum_{i=1}^k w(C_i) + \sum_{j=1}^l (w(P_j) + \text{mod}(P_j)) = \text{vote}(M,M') < 0$, so either there is a cycle C_i for which $w(C_i) < 0$, or there is a path P_j for which $w(P_j) + \text{mod}(P_j) < 0$. This proves the "if" direction of the theorem.

To prove the "only if" direction, we examine two cases.

Case 1: there is an alternating cycle C with $w(C) < 0$. We cannot directly use the cycle to construct a b-matching M' and a valid enumeration of (M,M') that shows $\text{vote}(M,M') < 0$, because the enumeration determined by the cycle may violate the last property in the definition of valid enumerations. However, this may happen only if C contains distinct edges $\hat{s}_{i_1}\hat{t}_{j_1}$ and $\hat{s}_{i_2}\hat{t}_{j_2}$ for some $st \in E \setminus M$. The following claim shows how to avoid this problem.

Claim 2.3. *If an alternating cycle C contains distinct edges $\hat{s}_{i_1}\hat{t}_{j_1}$ and $\hat{s}_{i_2}\hat{t}_{j_2}$ corresponding to the same edge $st \in E \setminus M$, then, by removing these edges and adding $\hat{s}_{i_1}\hat{t}_{j_2}$ and $\hat{s}_{i_2}\hat{t}_{j_1}$, we get two alternating cycles C_1, C_2 with $w(C_1) + w(C_2) = w(C)$.*

Proof. The exchange results in 2 cycles because the graph is bipartite. The definition of w implies that $w(\hat{s}_{i_1}\hat{t}_{j_1}) + w(\hat{s}_{i_2}\hat{t}_{j_2}) = w(\hat{s}_{i_1}\hat{t}_{j_2}) + w(\hat{s}_{i_2}\hat{t}_{j_1})$, so $w(C_1) + w(C_2) = w(C)$. □

Since $w(C) < 0$, one of the resulting cycles has negative weight. We can repeat similar operations until we get a cycle C' with $w(C') < 0$ which in addition does not contain distinct edges corresponding to the same edge of $E \setminus M$. By exchanging along the cycle, we obtain a matching $\hat{M}'$, which determines a b-matching M' and a valid enumeration of (M,M') at each node. This implies $\text{vote}(M,M') \le w(C') < 0$.

Case 2: there is a valid alternating path P with $w(P) + \text{mod}(P) < 0$. As in Case 1, some modifications are needed in order to obtain valid enumerations. The counterpart of Claim 2.3 for this case is the following.

Claim 2.4. *If a valid alternating path P contains distinct edges $\hat{s}_{i_1}\hat{t}_{j_1}$ and $\hat{s}_{i_2}\hat{t}_{j_2}$ corresponding to the same edge $st \in E \setminus M$, then, by removing these edges and adding $\hat{s}_{i_1}\hat{t}_{j_2}$ and $\hat{s}_{i_2}\hat{t}_{j_1}$, we get an alternating cycle C' and a valid alternating path P' with $w(C') + w(P') = w(P)$ and $\mathrm{mod}(P') = \mathrm{mod}(P)$.*

Proof. The proof is the same as the proof of Claim 2.3, with the additional observation that P' has the same end-nodes as P, so P' is also valid and $\mathrm{mod}(P') = \mathrm{mod}(P)$. □

The claim implies that, by repeating this operation as many times as needed, we either get a a cycle C' with $w(C') < 0$ or a valid path P' with $w(P') + \mathrm{mod}(P') < 0$, having the additional property that it does not use distinct edges corresponding to the same edge of $E \setminus M$. By exchanging along the path or cycle, we get a matching $\hat{M}'$, which determines a b-matching M' and a valid enumeration of (M, M') at each node. In the cycle case we have $\mathrm{vote}(M, M') \leq w(C') < 0$. In the path case, voting according to this valid enumeration gives the result $w(P') + \mathrm{mod}(P')$, because the number of nodes covered by $\hat{M}$ but not by $\hat{M}'$ is $\mathrm{mod}(P')$. This implies $\mathrm{vote}(M, M') \leq w(P') + \mathrm{mod}(P') < 0$. This concludes the proof for popularity. The proof for strong popularity is analogous, with $\mathrm{vote}(M, M') \leq 0$ in place of $\mathrm{vote}(M, M') < 0$. □

Using Theorem 2.1, it is straightforward to check the popularity or strong popularity of a b-matching M.

Proof of Theorem 1.2. Given a b-matching M, we construct the auxiliary bipartite graph $\hat{G}_M$ and the weight function w as specified in the first part of the section. Let D_M be the directed graph obtained from $\hat{G}_M$ by orienting the edges of $\hat{M}$ from $\hat{S}$ to $\hat{T}$ and the edges of $\hat{E} \setminus \hat{M}$ from $\hat{T}$ to $\hat{S}$. We can use the Bellman-Ford algorithm to check if there is a directed cycle C with $w(C) < 0$ or $w(C) = 0$. If there is no negative cycle, then we can compute the minimum weight paths between all pairs of nodes. Based on this, we can decide if there is a valid alternating path P with $w(P) + \mathrm{mod}(P) < 0$ or $w(P) + \mathrm{mod}(P) = 0$. □

3 Proof of the main theorem

In this section we prove Theorem 1.3. We are given a bipartite multigraph $G = (S, T; E)$ and a quota function $b : S \cup T \to \mathbb{Z}_+$. Every node $v \in S$ has a strict preference order $\preceq_v$ over its incident edges, while the preference list of a node in T may contain one tie of arbitrary length at the end of the list. We give a polynomial-time algorithm which decides if the instance admits a strongly popular b-matching (SPbM for short), as defined in Section 2.

3.1 Preliminaries

Before going into the details, we give an overview of the main ideas of the proof. We may assume, without loss of generality, that G has no isolated nodes and that $b(v) \leq d_G(v)$ for every $v \in S \cup T$. During the algorithm, we modify the instance using the following two **reduction operations**.

1. We remove edges that cannot appear in an SPbM of the current instance. We remove any isolated nodes that arise, and we also decrease quotas if they exceed the degree.

2. We fix edges that are guaranteed to belong to the SPbM of the current instance if it has one. Fixed edges are removed, and the quotas of their two endnodes are decreased by one. Nodes that become isolated are also removed.

Let $G^k = (S^k, T^k; E^k)$ be the instance after performing k of the above reduction operations, let b^k be the corresponding quota function, and let F be the set of edges fixed so far.

Lemma 3.1. *If the original instance has an SPbM M, then $F \subseteq M$, and $M \setminus F$ is an SPb^kM in G^k.*

Proof. The proof is by induction on k; the claim is obviously true for $k = 0$. Let G^{k-1} be the instance before the last operation. If the last operation was the removal of an edge st, then, by induction, M contains F, $M \setminus F$ is an SPb^{k-1}M of G^{k-1}, and $st \notin M$. Thus $M \setminus F$ is an SPb^kM of G^k.

If we fixed an edge st in the last operation, then $M \setminus (F - st)$ is an SPb^{k-1}M of G^{k-1} by induction, and $st \in M \setminus (F - st)$ because we only fix edges that are guaranteed to belong to the SPb^{k-1}M. This implies that $st \in M$. Since b^k is obtained from b^{k-1} by decreasing the quota of s and t by one, $M \setminus F$ is an SPb^kM of G^k. □

Note that it is possible that G^k has an SPb^kM even though G does not have an SPbM. This is not a problem however: if we eventually obtain an empty graph by reduction operations, then F is the only candidate for an SPbM by Lemma 3.1, and we can check in polynomial time if it is an SPbM of G or not by Theorem 1.2. On the other hand, if we obtain a graph G^k that has no SPb^kM, then G has no SPbM by Lemma 3.1.

To summarize, our strategy is to do reduction operations until we get to the empty graph or we get a certificate that there is no SPbM. The question is how to identify edges that can be removed or fixed. In the proof, we will present a sequence of *claims* about certain edges being removable or fixable. Each claim is constructive in the sense that the specified edges can be found in linear time.

Each claim is followed by the description of a *property* satisfied by any instance that cannot be further reduced by the reduction operations described in the claim. At any point in the proof, we assume (without explicitly stating it) that our instance G has all the properties described previously. This assumption is valid because the algorithm can make reductions until all properties are satisfied.

Additional definitions. An edge $st \in E$ is called a **blocking edge** with respect to a b-matching M if it satisfies the following three conditions: *i)* $st \notin M$, *ii)* s does not fill its quota in M or there is a node t' such that $st' \in M$ and $t \succ_s t'$, and *iii)* t does not fill its quota in M or there is a node s' such that $s't \in M$ and $s \succ_t s'$. If M is an SPbM, then there is no blocking edge with respect to M. Indeed, if M' is the

b-matching obtained from M by adding a blocking edge st and removing st' if s fills its quota in M and removing $s't$ if t fills its quota in M, then vote$(M, M') \leq 0$, which contradicts that M is an SPbM.

In addition to blocking edges, we will use *alternating paths and cycles* to show that certain b-matchings cannot be strongly popular. Note that these are paths and cycles in G, not in the auxiliary graph $\hat{G}_M$ defined in Section 2 (we do not use $\hat{G}_M$ in this section). Given a b-matching M and an alternating path or cycle w.r.t. M, let M' be the b-matching obtained from M by exchanging along the path or cycle – if we exchange along a path whose first or last edge is not in M and the endpoint fills its quota, then we also remove (one of) the worst edge(s) of M covering the corresponding endpoint of the path. If we can show that vote$(M, M') \leq 0$, then M is not an SPbM.

At some points in the proof (proofs of Claim 3.7, Lemmas 3.10 and 3.11), we append an extra edge ts' to an alternating path ending in $st \in M$. We call the result an **alternating quasi-path** if s' appears earlier on the path. If we exchange along the quasi-path and s' fills its quota in M, then, as above, we have to remove the worst edge $s't'$ of M at s'. This is a problem if $s't'$ belongs to the quasi-path, because we cannot remove it twice. However, in the situations where we use this technique, ts' is guaranteed to be better than $s't'$ at s', and if $s't'$ is in the quasi-path, then we can exchange along the cycle part of the quasi-path to get a b-matching M' with vote$(M, M') \leq 0$.

The following *auxiliary digraph D* plays an important role in the proof: for every node $v \in S \cup T$, we orient the first $b(v)$ strictly ordered edges on v's preference list towards their other endpoint. (Note that there may be less than $b(v)$ outgoing edges from a node $v \in T$ because the edges in the tie at the end of the list are not directed outwards.) Some edges may be bidirected; however, as our first claim (Claim 3.2) will show, bidirected edges can always be fixed. The digraph D has to be recomputed after each reduction.

3.2 Description of the algorithm

We assume that G has no isolated nodes and that $b(v) \leq d_G(v)$ for every $v \in S \cup T$.

Orientation. For every node $v \in S \cup T$, we orient the first $b(v)$ strictly ordered edges on v's preference list towards their other endpoint. The digraph obtained this way is denoted by D, and it is recomputed after every reduction.

Claim 3.2. *If an edge $st \in E$ is bidirected in D, then it belongs to the SPbM if there is one.*

Proof. If M is a b-matching and $st \notin M$, then st is blocking with respect to M. □

Property 1. *After reductions according to Claim 3.2, D does not contain bidirected edges.*

Claim 3.3. *If there are $b(v)$ directed edges to a node $v \in S \cup T$: $w_1v, w_2v, \ldots, w_{b(v)}v$, and uv is an edge such that $uv \prec_v w_iv$ for every $1 \leq i \leq b(v)$, then uv cannot belong to the SPbM.*

Proof. Suppose that M is an SPbM and $uv \in M$. Then there is an index $1 \leq i \leq b(v)$ such that $w_i v \notin M$, but then $w_i v$ is a blocking edge with respect to M. □

Property 2. *After reductions according to the previous claims, every node $s \in S$ is entered by at most $b(s)$ directed edges in D, and if there are more than $b(t)$ directed edges entering a node $t \in T$, then t is indifferent between at least two of them.*

Definition 3.4. Suppose the current instance admits an SPbM M. Let T_1 be the set of nodes $t \in T$ for which there is an edge in M which is one of the least preferred edges by t, and there is a directed edge not in M entering t.

Claim 3.5. *If there is a directed edge entering a node $t \in T$ and the number of directed edges incident to t is at most $b(t)$, then the edges entering t belong to the SPbM if there is one.*

Proof. Suppose that the SPbM M does not contain one of these edges st; then t has to fill its quota, otherwise st would be blocking. Since at most $b(t)$ directed edges are incident to t, there has to be an edge ut that belongs to M and is not directed. There are $b(u)$ directed edges from u, therefore there is a directed edge ut_2 which does not belong to M. Consider the path that starts with st, tu and then alternates between directed edges that do not belong to M and edges of M that are not directed backwards in the path. If at some point the path returns to a previous node, then we get an alternating cycle, and exchanging along the cycle yields a matching M' such that $\mathrm{vote}(M, M') \leq 0$. Otherwise we can continue the path, until we reach a node $t' \in T$ such that the edges of M incident to t' are all directed towards t'. We reached t' in a directed edge not in M, therefore t' fills its quota, which means that there are $b(t')$ directed edges in M entering t', so by Property 2, $t' \in T_1$. By exchanging along the path we get a matching M' such that $\mathrm{vote}(M, M') \leq 0$. Indeed, the vote of each node of S on the path is -1, while only the following nodes may have a positive vote: the nodes of T in the path except for t and t', and the other endpoint of the edge of M incident to s. (t' has a nonpositive vote because $t' \in T_1$, and therefore one of the least preferred edges by t' is in M. This means that the edge that we remove from M at t' is also a least preferred edge.) This contradicts the assumption that M is an SPbM.

See Figure 1 for an illustration of both cases. □

Property 3. *After reductions according to the previous claims, if there is a directed edge entering a node $t \in T$, the number of directed edges incident to t is more than $b(t)$.*

Definition. Let T_2 be the set of nodes in T with out-degree less than $b(t)$ in D and with no incoming directed edges.

Claim 3.6. *Suppose that the number of directed edges in $D[S \cup (T \setminus T_2)]$ entering a node $s \in S$ plus the number of edges between s and T_2 is more than $b(s)$. Let sv be the least preferred by s among these edges. Then sv cannot belong to the SPbM.*

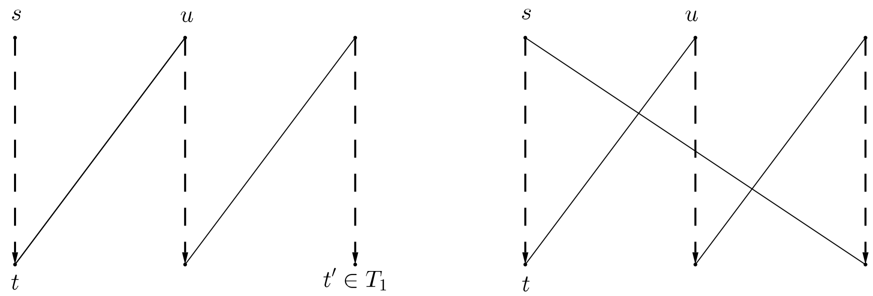

Figure 1: The solid edges belong to the SPbM.

Proof. Suppose that sv belongs to the SPbM M. Every directed edge entering s has to belong to M too, because these are preferred over sv and therefore they would be blocking. Thus, there has to be an edge $st \notin M$ with $t \in T_2$. There is an edge $ts' \in M$ because t fills its quota (otherwise st would be blocking). There are $b(s')$ directed edges leaving s', therefore there is one which is not in M. Consider the alternating path that starts with vs, st, ts' and then alternates between directed edges that do not belong to M and edges of M that are not directed backwards in the path. If at some point this path returns to a previous node of the path, then we get an alternating cycle, otherwise we reach a node $t' \in T$ such that the edges of M incident to t' are all directed towards t'. This means that $t' \in T_1$. Let M' be the b-matching we get from M by exchanging along the cycle or path; this satisfies $\text{vote}(M, M') \leq 0$ similarly as in the proof of Claim 3.5. □

Property 4. *After reductions according to the previous claims, the number of directed edges in $D[S \cup (T \setminus T_2)]$ entering a node $s \in S$ plus the number of edges between s and T_2 is at most $b(s)$.*

Claim 3.7. *If uv is an edge in $G[S \cup (T \setminus T_2)]$ and it is not a directed edge in any direction, then uv cannot belong to the SPbM.*

Proof. Suppose that uv is in the SPbM M, $u \in T \setminus T_2$, and $v \in S$. The number of directed edges incident to u is at least $b(u)$ by Property 3. If such an edge has head or tail v, then it is also in M, otherwise it would be blocking. We can conclude that there is a directed edge su or us for some $s \neq v$, which is not in M.

Case 1: There is a directed edge su entering u for some $s \neq v$, which is not in M. There are $b(v)$ directed edges leaving v, therefore there is a directed edge $vt \notin M$. Consider the path starting with su, uv and then alternating between directed edges that are not in M and edges of M that are not directed backwards in the path. Similarly to the proof of Claim 3.5, we either reach a node $t' \in T_1$, or return to a previous node of the path, and exchanging along the obtained path or cycle yields a matching M' such that $\text{vote}(M, M') \leq 0$.

Case 2: There is a directed edge us leaving u for some $s \neq v$, which is not in M. First we build a path starting with vu and then alternating between directed edges that are not in M and edges of M in $G[S \cup (T \setminus T_2)]$ that are not directed backwards until we can. The latter can be achieved because a node $s \in S$ entered by a directed edge not in M fills its quota, so by Property 4 there is an incident edge in M not directed towards s whose other endpoint is not in T_2. If we return to a previous node of the path, then exchanging along the cycle yields a matching M' and vote$(M, M') \leq 0$. If we reach a node $t \in T \setminus T_2$ such that there is no $s' \in S$ such that ts' is a directed edge not in M, then there is a directed edge not in M pointing to this node t by Claim Property 3, which we add to the path. Let this (quasi-)path be denoted by P. We continue P from v with edges alternating between directed edges not in M and edges of M that are not directed backwards. If we reach a node $t' \in T_1$, then exchanging along the (quasi-)path yields a matching M' and $vote(M, M') \leq 0$; see Figure 2 for an illustration of this case. If we return to a previous node of the path, then, again, exchanging along the obtained cycle yields a matching M' and $vote(M, M') \leq 0$. □

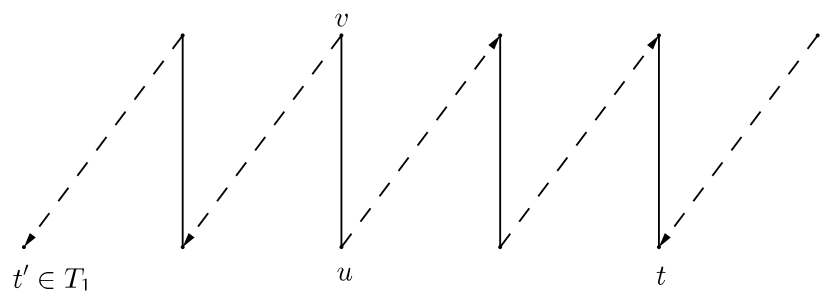

Figure 2: The solid edges belong to the SPbM.

Property 5. *After reductions according to the previous claims, every edge of G is either directed in D or has one endpoint in T_2.*

Lemma 3.8. *Suppose there is an SPbM M, and a directed edge $st \notin M$. Then there is a directed path from s to a node in T_1 that starts with st and which is alternating with respect to M.*

Proof. We claim that if $t \in T \setminus T_1$, then there is a directed edge ts_2 in M. Indeed, t fills its quota (otherwise st would be blocking), and $t \in T \setminus T_2$ because st enters t. Therefore, by Property 5, only edges of D are incident to t, and if all edges of M incident to t are directed towards t, then $t \in T_1$ by Properties 2 and 3. We can conclude that there is at least one outgoing directed edge ts_2 that belongs to M.

There are $b(s_2)$ directed edges leaving s_2, so there is a directed edge $s_2t_2 \notin M$. Continuing the alternating directed path like this we either reach a node in T_1 or we get an alternating directed cycle which contradicts the strong popularity of M. □

Lemma 3.9. *Suppose there is an SPbM M, and a directed edge $st \in M$. Then there is a directed path from s to a node in T_1 that starts with st and which is alternating with respect to M.*

Proof. If $t \in T \setminus T_1$, then there is a directed edge $ts_2 \notin M$, because if not, then there are less than $b(t)$ directed edges leaving t, therefore all edges entering t are in the tie at the end of t's preference list, and there is a directed edge not in M entering t by Property 3, which means that $t \in T_1$. The node s_2 fills its quota since otherwise ts_2 would be blocking; therefore, there is a directed edge $s_2t_2 \in M$ because of Property 4. Continuing the alternating directed path like this we either reach a node in T_1 or we get an alternating directed cycle which contradicts that M is strongly popular. □

Lemma 3.10. *If M is an SPbM, then $d_M(v) = b(v)$ for every $v \in S \cup T$.*

Proof. Let M be an SPbM, and suppose that a node $s \in S$ does not fill its quota. Then there is a directed edge leaving s that is not in M. By Lemma 3.8, there is an alternating directed path from s to a node $t \in T_1$ that starts with this edge. Exchanging along this alternating path yields a b-matching M' with $\text{vote}(M, M') \leq 0$.

Now suppose that a node $t \in T \setminus T_2$ does not fill its quota. There are no directed edges not in M entering t, because they would be blocking. Thus by Property 3 there is a directed edge $ts \notin M$. We have already seen that nodes in S fill their quota. By Property 4, there is a directed edge $st' \in M$ with $t' \neq t$, and by Lemma 3.9 there is an alternating directed path P from s to a node $t_1 \in T_1$ that starts with st'. If this path reaches t, then we get an alternating directed cycle which contradicts the strong popularity of M. If not, let $s't_1$ be a directed edge which does not belong to M (such an edge exists by the definition of T_1). Let M' be the b-matching we get from M by exchanging along the alternating (quasi-)path $ts \cup P \cup t_1s'$. This satisfies $\text{vote}(M, M') \leq 0$.

Finally, suppose that a node $t \in T_2$ does not fill its quota. Let st be an edge that is not in M. It follows from Property 4 that there is a directed edge $st' \in M$ since s fills its quota. By Lemma 3.9, there is a directed alternating path P from s to a node $t_1 \in T_1$ that starts with st'. Let $s't_1$ be a directed edge that does not belong to M. Let M' be the b-matching we get from M by exchanging along the alternating (quasi-)path $ts \cup P \cup t_1s'$. This satisfies $\text{vote}(M, M') \leq 0$. □

Lemma 3.11. *Let M be an SPbM and $st \in M$ a directed edge. If s prefers st' over st, then $st' \in M$.*

Proof. Suppose $st' \notin M$. By Lemma 3.9, there is an alternating directed path P from s to a node $t_1 \in T_1$ which starts with st, and by Lemma 3.8 there is an alternating directed path P' from s to a node $t'_1 \in T_1$ which starts with st'. If P' intersects P, then we get an alternating cycle, and exchanging along this cycle yields a b-matching M' such that $\text{vote}(M, M') \leq 0$. Otherwise, let $s't_1$ be a directed edge that does not belong to M, and let M' be the b-matching we get from M by exchanging along the alternating (quasi-)path $s't_1 \cup P \cup P'$. This satisfies $\text{vote}(M, M') \leq 0$. □

Claim 3.12. *If the number of directed edges in $D[T \setminus T_2 \cup S]$ entering a node $s \in S$ plus the number of edges between s and T_2 are less than $b(s)$, then s's most preferred edge has to belong to the SPbM if there is one.*

Proof. This follows from Lemmas 3.10 and 3.11. □

Property 6. *After reductions according to the previous claims, the number of directed edges in $D[T \setminus T_2 \cup S]$ entering a node $s \in S$ plus the number of edges between s and T_2 is exactly $b(s)$.*

Lemma 3.13. *If G is not the empty graph (and all the listed properties hold), then there is no SPbM.*

Proof. Suppose G is not empty and M is an SPbM. If a directed edge st is in M, then there is either a directed edge entering s that does not belong to M, or there is an edge $st' \notin M$ with $t' \in T_2$ because of Property 6. If a directed edge ts is not in M and $t \in T \setminus T_2$, then there is a directed edge in M entering t since t fills its quota.

It follows that given a directed edge $st \in M$, we can go on a backwards directed alternating path from s until we reach a node s' such that there is an edge $s'u \notin M$ with $u \in T_2$ (the backwards walk cannot intersect itself because that would give a directed alternating cycle showing that M is not strongly popular). Thus, for every directed edge $st \in M$ with $t \in T_1$, there is a node $s' \in S$ such that there is a directed alternating path P from s' to t with last edge st, and there is an edge $s'u \notin M$ with $u \in T_2$. We call the path $us \cup P$ a **type 1 path** from t to u. From every node in T_1 there is a type 1 path to a node in T_2.

Let us be an edge in M with $u \in T_2$. By Property 6, there is a directed edge $st \notin M$. By Lemma 3.8, there is a directed alternating path P from s to a node $t' \in T_1$ which starts with the edge st. We call the path $us \cup P$ a **type 2 path** from u to t'. From every node in T_2 there is a type 2 path to a node in T_1.

We go back and forth between T_1 and T_2 on type 1 and type 2 paths, such that we choose the first edge of a type 1 path from a node $t \in T_1$ to be one of the least preferred edges by t (we can do this because of the definition of T_1), and we choose the first edge of a type 2 path from $u \in T_2$ to be one of the least preferred edges by u (we can do this because there are less than $b(u)$ directed edges leaving u, and u fills its quota). We stop if we return to a node in T_1 or T_2 that we have already touched, or the first time a type 2 path P intersects a previous type 2 path P' at a node $t \in T \setminus (T_1 \cup T_2)$. If this happens we can forget about the paths before P' and P', and we can replace the part of P after t with the part of P' after t and still get a type 2 path.

We can conclude that if G is not the empty graph, then there are nodes $t_1, t_2, \ldots, t_k$ in T_1 and $u_1, u_2, \ldots, u_k$ in T_2 such that there is a type 1 path P_i from t_i to u_i and a type 2 path P'_i from u_i to t_{i+1} for $i = 1, \ldots, k$ where $t_{k+1} := t_1$, the first edge of P_i is one of the least preferred edges by t_i and the first edge of P'_i is one of the least preferred edges by u_i. Type 2 paths do not intersect at a node in $T \setminus (T_1 \cup T_2)$. Next we show that these paths can be assumed to be edge-disjoint.

We may choose the paths to be edge-disjoint (although not necessarily node-disjoint), because when we reach a node in $S \cup T \setminus (T_1 \cup T_2)$ that we have already used, we can always choose to proceed on an edge not yet used. Indeed, if there are ℓ non-outgoing edges in M incident to a node $s \in S$, then there are at least ℓ edges not in M leaving s, and if there are ℓ edges in M leaving s, then there are at least ℓ non-outgoing edges not in M incident to s (because of Lemma 3.10 and Property 6). Similarly, if there are ℓ edges not in M leaving a node $t \in T \setminus (T_1 \cup T_2)$, then there are at least ℓ edges in M entering t (because of Lemma 3.10).

Let C denote the closed tour $P_1 \cup P_1' \cup P_2 \cup P_2' \cup \cdots \cup P_k \cup P_k'$. The nodes of C in order are $v_1 v_2 \ldots v_n$, where $v_1 = v_n$. These nodes are not necessarily distinct; let $V(C)$ denote the node set of C without multiplicities. The tour C is alternating with respect to M. Let M' be the b-matching we get from M by exchanging the edges along C.

For each v_j, let $(e_1^j, \ldots, e_{b(v_j)}^j; f_1^j, \ldots, f_{b(v_j)}^j)$ be the valid enumeration of (M, M') at v_j where e_i^j and f_i^j are consecutive edges of C whenever $e_i^j \neq f_i^j$. By definition, we have

$$\mathrm{vote}_{v_j}(M, M') \leq \sum_{i=1}^{b(v_j)} \mathrm{vote}_{v_j}(e_i^j, f_i^j).$$

Observe that $v_i v_{i+1} \in M$ if i is odd and $v_i v_{i+1} \in M'$ if i is even. Therefore

$$\begin{aligned}\mathrm{vote}(M, M') &= \sum_{v_j \in V(C)} \mathrm{vote}_{v_j}(M, M') \\ &\leq \sum_{j \text{ is odd}} \mathrm{vote}_{v_j}(v_j v_{j+1}, v_{j-1} v_j) + \sum_{j \text{ is even}} \mathrm{vote}_{v_j}(v_{j-1} v_j, v_j v_{j+1}).\end{aligned}$$

The right hand side is equal to 0, because

- if we sum up the votes of the nodes in $S \cup T \setminus (T_1 \cup T_2)$ of a type 1 path between the two edges of the path incident to that node, we get $+1$
- if we sum up the votes of the nodes in $S \cup T \setminus (T_1 \cup T_2)$ of a type 2 path between the two edges of the path incident to that node, we get -1
- the vote of each node in $T_1 \cup T_2$ is at most 0.

We obtained that $\mathrm{vote}(M, M') \leq 0$, thus M is not strongly popular. □

Lemma 3.13 shows that by performing the reduction operations defined in the claims, we either get a certificate that the graph has no SPbM (which implies that the original graph also has none), or we eventually reach the empty graph. The latter means that the only possible candidate for an SPbM in the original graph is the set of fixed edges F. We can check in polynomial time if F is an SPbM of the original preference system or not.

4 Conclusion

We proved that in case of strict preferences on one side and restricted ties on the other side, the existence of a strongly popular b-matching can be decided in polynomial time. This is a clear indication that the strongly popular matching problem is significantly easier than the popular matching problem. It seems to be difficult to complement this with hardness results; as mentioned in the introduction, the strongly popular matching problem is in the complexity class UP, for which no complete problems are known. Therefore, the more promising direction is to attempt to show polynomial-time solvability for other types of preference systems. In particular, the decision problem for strongly popular matchings is open in the following two cases:

- bipartite preference systems with strict preference and indifference allowed on both sides,
- bipartite preference systems with strict preferences on one side, and arbitrary preferences on the other side.

Our techniques do not seem to extend easily to these problems.

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