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Possible and necessary allocations under serial dictatorship with incomplete preference lists

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Abstract

Abstract. We study assignment problems in a model where agents have strict preferences over objects, allowing preference lists to be incomplete. We investigate the questions whether an agent can obtain or necessarily obtains a given object under serial dictatorship. We prove that both problems are computationally hard even if agents have preference lists of length at most 3; by contrast, we give linear-time algorithms for the case where preference lists are of length at most 2. We also study a capacitated version of these problems where objects come in several copies.

1 Introduction

We study assignment problems that involve a set of agents and a set of objects. Agents have strict ordinal preferences over objects, but not vice versa. We assume that the preference lists can be *incomplete*: an agent might find a given object *unacceptable*. In such situations, there is a very natural and intuitive mechanism called *serial dictatorship* (SD): agents are ordered into a *picking sequence* (which is in our model simply a permutation of the agents) and everybody who has her turn picks her most preferred object out of those that are still available.

Variants of this mechanism are often used in practice. As described by Sönmez and Switzer [23], the United States Military Academy determines a strict priority ranking of its cadets based on a weighted average of their academic performance, physical fitness test scores, and military performance; the Military Academy then uses serial dictatorship with this ranking as the picking sequence to assign cadets to slots at different specialties. Another example of serial dictatorship is the drafting system

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used in football, basketball and other professional sports in the United States, where teams pick new players in the draft without the active participation of the players. The first team to pick a new player is the one with the worst win-loss record of the previous season, the one with the second worse record continues, and so on [11].

We study the following questions for a given agent a and a given object o:

(i) Can agent a receive object o under SD with *some* picking sequence?

(ii) Is it true that agent a receives object o under SD with *any* picking sequence?

Saban and Sethuraman [22] proved that (i) is NP-complete and gave a polynomial-time algorithm for (ii). In their model, the number of agents equals the number of objects, and each agent finds all objects acceptable. The authors also expressed their belief that these results hold even if these assumptions are omitted.

Our contributions. In this note we assume that agents can consider some objects unacceptable. Hence, some agents and also some objects can stay unassigned. In this setting, NP-completeness of (i) follows from the results of Saban and Sethuraman [22]; we complement this by proving that (ii) is coNP-complete. Then we deal with instances where the lengths of preference lists are restricted. If each agent finds at most two objects acceptable, we provide polynomial-time algorithms for both problems, based on searching appropriate digraphs. By contrast, we show intractability already for the case where preference lists can have length 3.

We also study an extension of our model where objects come in several identical copies. We prove that these capacitated versions of (i) and (ii) are computationally hard already if each object has capacity at most 2 *and* all preference lists are of length at most 2.

2 Related work

The serial dictatorship mechanism appears under various names in the literature: besides "serial dictatorship" [1, 21] it is also dubbed as "queue allocation" [24], "Greedy-POM" [2], "sequential mechanism" [8, 7], etc. Its importance is stressed by the fact that the assignment it produces is Pareto-optimal (or, in the economic terminology, efficient); moreover, if each agent may receive at most one object and there is only one copy of each object (called the *one-to-one* case), each Pareto-optimal assignment can be produced by SD with a suitable picking sequence [24, 1, 2, 10].

Recently, two lines of research have emerged. One deals primarily with many-to-many extensions of the basic assignment problem, additionally accompanied by constraints imposed either on the structure of the sets of objects that an individual agent can receive [13, 15] or on the whole set of allocated objects [18], or by lower quotas on the number of agents assigned to individual objects [19, 17, 14] and with possible extensions of serial dictatorship to such settings.

Another line of research explores in detail the properties of serial dictatorship and different types of sequential allocation mechanisms, often with a focus on manipulability [8, 9, 4]. Asinowski et al. [3] study the sets of objects that can or have to be allocated in some or all Pareto-optimal matchings (without specifying to which agents those objects are allocated).

Saban and Sethuraman [22] consider a randomized setting and prove that computing the proportion of the picking sequences under which agent a receives object o is #P-complete; this was independently obtained also by Aziz, Brandt and Brill [5]. Questions (i) and (ii) are equivalent to asking if the probability of an agent obtaining a given object in the randomized model is greater than 0 or equal to 1, resp. Saban and Sethuraman [22] determine the complexity of these problems assuming that the number of agents equals the number of objects, and agents find all objects acceptable; they find (i) to be NP-complete and (ii) polynomial-time solvable.

Motivated by the intractability results in [22], Aziz and Mestre [6] compute the probability of an agent getting an object in time that is FPT if the parameter is the number of objects, and polynomial if the number of agent types is fixed. Notably, they allow incomplete preferences; up to our knowledge, the only other work considering such a setting is by Asinowski et al. [3].

Aziz, Walsh and Xia [7] examine the complexity of deciding whether an agent can get or necessarily gets some object or set of objects under serial dictatorship with different classes of picking sequences. Their algorithmic results rely on the assumptions that agents have complete preferences and may obtain more than one objects.

3 Definitions and notation

There is a set A of n *agents* and a set $\mathcal{O}$ of m *objects*. Each agent can consume at most one object and each object is available in only one copy. Agents have strict *preferences* over objects and they are allowed to declare some objects *unacceptable*. The n-tuple of agents' preferences is called a *preference profile* and it is denoted by $\mathcal{P}$. The triple $I = (A, \mathcal{O}, \mathcal{P})$ is a *matching profile*.

We consider serial dictatorship, where agents are ordered into a *picking sequence* σ, which is a permutation of A. Agents have their turn successively according to σ, and everybody on her turn picks her most preferred object among those that are still available.

Obviously, different sequences can lead to different assignments. In this context, we study the following problems associated with a matching profile I, agent a and object o:

> **Problem** POSSIBLE OBJECT POS(I, a, o).
> **Question:** Is it true in I that a receives o under SD with *some* picking sequence?

> **Problem** NECESSARY OBJECT NEC(I, a, o).
> **Question:** Is it true in I that a receives o under SD with *any* picking sequence?

Let us remark that in the one-to-one case, Pareto-optimal matchings (POMs) are exactly those that can be obtained by serial dictatorship, so our questions are equivalent to asking whether a given agent can be allocated a given object in some POM and whether a given agent is allocated a given object in every POM, resp.

4 Incomplete preference lists of unbounded length

Here we show that if the preference lists are not complete, then both problems are hard. We remark that the NP-completeness of $\textsc{pos}(I, a, o)$ follows from the NP-hardness result of Saban and Sethuraman [22] obtained for the case of complete preference lists. However, the coNP-completeness of $\textsc{nec}(I, a, o)$ may seem somewhat surprising, sharply contrasting the polynomial-time algorithm given by Saban and Sethuraman [22] for complete preference lists.

Theorem 4.1. $\textsc{pos}(I, a, o)$ *is* NP*-complete and* $\textsc{nec}(I, a, o)$ *is* coNP*-complete.*

Proof. $\textsc{pos}(I, a, o)$ belongs to NP and $\textsc{nec}(I, a, o)$ belongs to coNP, since in both cases it suffices to give a picking sequence σ and check whether a gets (does not get) o under SD with σ.

To prove NP-hardness, we give a polynomial reduction from VERTEX COVER. Let our input be a graph $G = (V, E)$ with $|V| = p$ and $|E| = q$ and some $k \in \mathbb{N}$. We construct a matching profile $I(G)$ involving two agents a, b and object o in a way that G has a vertex cover of size k if and only if agent b gets o in SD under some picking sequence (i.e., $\textsc{pos}(I(G), b, o)$ is true), which in turn happens exactly if it is not true that a gets o in all picking sequences (i.e., $\textsc{nec}(I(G), a, o)$ is false).

We define the set A of agents and $\mathcal{O}$ of objects in $I(G)$ as

$$A = \{a(v) \mid v \in V\} \cup \{a(e,u), a(e,v) \mid e = \{u,v\} \in E\} \cup \{a, b\} \text{ and}$$
$$\mathcal{O} = \{s_1, \ldots, s_{p+q-k}\} \cup \{o(v) \mid v \in V\} \cup \{o(e) \mid e \in E\} \cup \{o\}.$$

Thus $|A| = p + 2q + 2$, and there are $p + q - k$ *special s-objects*, one *vertex-object* for each vertex, one *edge-object* for each edge, and a distinguished object o. Preferences are as follows:

$$\begin{array}{rll} P(a(v)): & s_1, s_2, \ldots, s_{p+q-k}, o(v) & \text{for each } v \in V, \\ P(a(e,v)): & s_1, s_2, \ldots, s_{p+q-k}, o(v), o(e) & \text{for each } e \in E \text{ and } v \in e, \\ P(b): & o(e_1), \ldots, o(e_q), o, & \\ P(a): & o. & \end{array}$$

It is easy to see that for any picking sequence, b gets o in SD if and only if a does not get o.

Now suppose that G admits a vertex cover $U \subseteq V$ of size k. Let us orient the edges of G in a way that each edge points toward a vertex of U, and for each edge $e \in E(G)$ let $a_1(e)$ and $a_2(e)$ denote the tail and the head vertex, resp., of e in this orientation. Then order the agents according to sequence σ given in (1) (agents within square brackets are ordered arbitrarily).

$$\underbrace{[a(v) \mid v \notin U]}_{A_1}, \underbrace{[a_1(e) \mid e \in E]}_{A_2}, \underbrace{[a(v) \mid v \in U]}_{A_3}, \underbrace{[a_2(e) \mid e \in E]}_{A_4}, b, a \qquad (1)$$

How are the objects picked under σ? First, the agents in $A_1 \cup A_2$ take all the special s-objects. Then the agents in A_3 pick vertex-objects corresponding to the vertex cover U. Notice that some vertex-objects stay unassigned, but when the agents in A_4 have their turn, all the vertex-objects they are interested in are exhausted. This means that these agents use up all the edge-objects. So when agent b comes to choose, all the edge-objects are gone and b has to pick o. This leaves a with no object assigned. Hence $\textsc{pos}(I, b, o)$ is true and $\textsc{nec}(I, a, o)$ is false.

Conversely, suppose that there exists a picking sequence σ where b gets o, i.e., $\textsc{pos}(I, b, o)$ is true and $\textsc{nec}(I, a, o)$ is false. This means that when it was b's turn, all the edge-objects were gone. In other words, for each edge $e = \{u, v\}$, one of the pair of agents $a(e, u), a(e, v)$ picked the edge-object $o(e)$. Hence, the remaining q agents in these pairs and all agents in $\{a(v) \mid v \in V\}$ (a total of $p + q$ agents) must have picked all special s-objects and some vertex-objects. As they all prefer the $p + q - k$ special s-objects, exactly k of these agents could have picked vertex-objects. Thus, the k picked vertex-objects define a vertex cover for G of size k. □

5 Bounded lengths of preference lists

Given the intractability result of Section 4, here we shall concentrate on the problems with preference lists of restricted length, and refine the boundary between polynomial-time solvable and intractable cases. Let us call a matching profile a *length-k matching profile*, if each agent finds at most k objects acceptable, and let k-$\textsc{pos}(I, a, o)$ and k-$\textsc{nec}(I, a, o)$ be the restrictions of the studied problems to instances with length-k matching profiles.

5.1 Preference lists of length 3

Here we strengthen Thm. 4.1 for the case where all preference lists have length at most 3.

Given a matching profile with arbitrary preference lists, we eliminate all agents having preference lists longer than 3, while preserving certain crucial properties of I. Our strategy is to eliminate such agents one-by-one. To this end, for any matching profile $I = (A, \mathcal{O}, \mathcal{P})$ we define a matching profile $J(I) = (A', \mathcal{O}', \mathcal{P}')$, called a *substitute for* I, as follows.

Take any agent $x \in A$ whose preference list in I has length $\ell > 3$; let this list be $o_1, o_2, ..., o_\ell$. We introduce $\ell - 3$ *chain objects* $y_1, y_2, \dots, y_{\ell-3}$, and we replace agent x with new agents $x_1, \dots, x_{\ell-2}$, called the *brothers* of x. Hence $A' = A \setminus \{x\} \cup \{x_1, \dots, x_{\ell-2}\}$ and $\mathcal{O}' = \mathcal{O} \cup \{y_1, \dots, y_{\ell-3}\}$. We set $P'(a) = P(a)$ for each agent $a \in A \setminus \{x\}$; the preferences of the brothers of x in $J(I)$ are as follows:

$$\begin{aligned} P'(x_1): &\quad o_1, o_2, y_1, \\ P'(x_i): &\quad y_{i-1}, o_{i+1}, y_i \qquad \text{for } i = 2, \dots, \ell - 3, \\ P'(x_{\ell-2}): &\quad y_{\ell-3}, o_{\ell-1}, o_\ell. \end{aligned}$$

To state the crucial property of a substitute, for each agent-object pair (a, o) in I we define a corresponding agent-object pair $J(a, o)$ in $J(I)$ as follows. If $a = x$

and $o = o_i$ for some i, then we let $J(a, o) = J(x, o_i) = (x_{i-1}, o_i)$; here $x_0 := x_1$ and $x_{\ell-1} := x_{\ell-2}$. Otherwise, we let $J(a, o) = (a, o)$.

Lemma 5.1. *Let $J(a, o) = (a', o)$ for any $a \in A$ and $o \in \mathcal{O}$. Then agent a can obtain o under some picking sequence in I if and only if a' can obtain o under some picking sequence in J'.*

Proof. First, for any picking sequence φ for I we define a picking sequence φ' for $J(I)$ by replacing x in φ with her brothers $x_1, \ldots, x_{\ell-2}$ in that order. We claim that agent a receives object o under φ if and only if a' receives o under φ'.

Clearly, all the agents preceding x in φ receive the same object under φ' as under φ. Suppose that x picks o_i under φ; this means that all the objects o_j for $j < i$ were gone before her turn, so these objects are no longer available for the brothers of x under φ' when they get their turn.

So, how do the agents pick under φ'? If $i = 1$ or $i = 2$, then x_1 picks o_1 (or o_2, of o_1 is no longer available), and all the brothers $x_2, \ldots, x_{\ell-2}$ pick their first choice chain objects $y_1, \ldots, y_{\ell-3}$, resp. If $i \geq 3$, then each of the brothers $x_1, \ldots, x_{i-2}$ has to pick his last object $y_1, \ldots, y_{i-2}$, resp., and x_{i-1} can pick o_i. If $i < \ell - 2$, then the remaining brothers $x_i, x_{i+1}, \ldots, x_{\ell-2}$ receive their first choice chain objects $y_{i-1}, y_i, \ldots, y_{\ell-3}$, resp. In any case, the set of available objects is the same after x's turn under φ in I as after the last brother $x_{\ell-2}$'s turn under φ' in $J(I)$, proving our claim.

For the converse direction, let σ' be a picking sequence in $J(I)$ under which a' receives o.

Case I. If x_i is the brother of x that first picks an object from $\{o_1, \ldots, o_\ell\}$ under σ', then we replace x_i by x and delete all other brothers of x; let σ be the resulting picking sequence in I.

We claim that x_i can receive some object o_{i+1} only if at her turn, all the objects $o_1, \ldots, o_i$ had already been taken, and the brothers $x_1, \ldots, x_{i-1}$ have all picked chain objects before x_i's turn. This is clear if $i = 1$; if $i > 2$, then y_{i-1} must have been already taken by x_{i-1}, which in turn implies that both o_i and y_{i-2} were taken by that point, and so on; repeating this argument we get the claim. (The proof goes similarly for the cases where x_1 picks o_1 or $x_{\ell-2}$ picks x_ℓ.) Hence, agents of A picking before x_i receive the same object under σ in I as under σ' in $J(I)$, and x picks under σ the object picked by x_i under σ'. Furthermore, let us observe that each of the brothers picking after x_i in σ' (that is, the brothers $x_{i+1}, \ldots, x_{\ell-2}$) can pick their first choice chain objects under σ'. Hence, all agents of A picking after x in σ receive the same object as they do under σ'.

Case II. If every brother of x receives either a chain object or nothing under σ', then we delete all of them from σ' and append x as the last agent to obtain σ. Observe that the deletion of the brothers of x does not affect what the remaining agents in $A \setminus \{x\}$ obtain. Note also that a' is not a brother of x (as we assume a' to get o), and hence, $a \neq x$. Thus, a gets the same object in I under σ as a' in $J(I)$ under σ'. $\square$

We next apply the above construction iteratively. Clearly, the number of agents with preference list longer than 3 is one less in $J(I)$ than in I. Hence, repeatedly constructing a substitute for the current matching profile (i.e., taking $J(I)$, then

$J(J(I))$, and so on), we finally end up with a length-3 matching profile $J^\star(I)$. We also define, for any given agent-object pair (a, o) in I, the corresponding agent-object pair $J^\star(a, o)$ obtained in this process (first taking $J(a, o)$), then $J(J(a, o))$, and so on). Applying Lemma 5.1 repeatedly, we obtain Cor. 5.2.

Corollary 5.2. *Let $J^\star(a, o) = (a^\star, o)$ for any $a \in A$ and $o \in \mathcal{O}$. Then agent a can obtain o under some picking sequence in I if and only if $a^\star$ can obtain o under some picking sequence in $J^\star(I)$.*

Theorem 5.3. *3-*$\textsc{pos}(I, a, o)$ *is* NP*-complete and 3-*$\textsc{nec}(I, a, o)$ *is* coNP*-complete.*

Proof. We prove our theorem by modifying the reductions and the matching profile I described in the proof of Thm. 4.1; recall that $\textsc{pos}(I, b, o)$ is true if and only if $\textsc{nec}(I, a, o)$ is false.

By the definition of a substitute, we can observe that $J^\star(b, o) = (b^\star, o)$ where $b^\star$ is either the last brother of b introduced when eliminating agent b or (if b does not get eliminated) $b^\star = b$. By Cor. 5.2, it is immediate that 3-$\textsc{pos}(J^\star(I), b, o)$ is equivalent to $\textsc{pos}(I, b^\star, o)$. Furthermore, our construction for $J^\star(I)$ ensures that object o is only contained in the preference list of agents $b^\star$ and a; the preference list of a remains the same in $J^\star(I)$ as in I, containing only o. Therefore, it should be clear that agent a receives o under all picking sequences in $J^\star(I)$ if and only if there is no picking sequence in $J^\star(I)$ under which agent b receives o. In other words, 3-$\textsc{pos}(J^\star(I), b, o)$ is true if and only if 3-$\textsc{nec}(J^\star(I), a, o)$ is false.

Observe that $J^\star(I)$ can be computed in time polynomial in $|I|$. Thus, replacing the instance $\textsc{pos}(I, b, o)$ and $\textsc{nec}(I, a, o)$ constructed in the proof of Thm. 4.1 with 3-$\textsc{pos}(J^\star(I), b, o)$ and 3-$\textsc{nec}(J^\star(I), a, o)$, resp., yields polynomial-time reductions that prove our theorem. $\square$

5.2 Preference lists of length 2

Given a length-2 matching profile I and an agent a in I, there are in fact six different questions that can be asked: POSSIBLE OBJECT and NECESSARY OBJECT for the first and second object in the preference list of a (denoted by $f(a)$ and $s(a)$, resp.), and for the special object $\emptyset$ representing the situation when agent a gets nothing. Some of these questions are trivial, see the following assertion.

Lemma 5.4. *For any $k \in \mathbb{N}$ and $a \in A$, k-$\textsc{nec}(I, a, s(a))$ and k-$\textsc{nec}(I, a, \emptyset)$ are false, and k-$\textsc{pos}(I, a, f(a))$ is true. Also, 2-$\textsc{nec}(I, a, f(a))$ is true exactly if both 2-$\textsc{pos}(I, a, s(a))$ and 2-$\textsc{pos}(I, a, \emptyset)$ are false.*

Hence, for length-2 matching profiles, it suffices to solve the two problems of the form 2-$\textsc{pos}(I, a, s(a))$ and 2-$\textsc{pos}(I, a, \emptyset)$.

How can it happen that agent a does not pick $o = f(a)$? Clearly, if o is the first object in the preference list of some other agent a', then it suffices to place a' before a in the picking sequence. If o is the second object in the preference list of a', then a' may still pick o, if her first choice object was picked before it was her turn, say by an agent a'', etc. To be able to discover such chains of agents, we construct for any

$P(a_1): \quad o_1, o_2$
$P(a_2): \quad o_1, o_3$
$P(a_3): \quad o_2, o_1$
$P(a_4): \quad o_3, o_2$

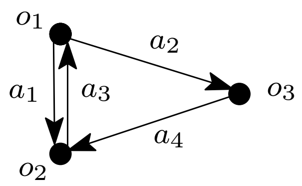

Figure 1: Example for a length-2 matching profile and its digraph.

length-2 matching profile I the following directed multigraph $G(I)$. Its vertex set is $\mathcal{O}$ and its arc set is A, where each $x \in A$ leads from $f(x)$ to $s(x)$ or, if x finds only one object acceptable, to $f(x)$; see Fig. 1 for an illustration. We denote by $\delta^+_H(p)$ the out-degree of any object p in a subgraph H of $G(I)$; for $H = G(I)$ we might omit the subscript.

Theorem 5.5. *Let $a \in A$ with $|P(a)| = 2$. Then 2-$\textsc{pos}(I, a, s(a))$ is true if and only if $G(I)$ contains a directed path starting from a vertex $p \in \mathcal{O}$ with $\delta^+(p) > 1$ and ending with the arc a.*

Proof. Suppose that $a = a_0$ picks her second object $s(a)$ under some picking sequence σ. This means that $f(a)$ was picked before a by some agent a_1. If $f(a) = f(a_1)$, then $\delta^+(f(a)) > 1$ as it is the tail of both a and a_1, so the arc a in itself is a path as required. If $f(a) = s(a_1)$, then the first object of a_1 must have been picked earlier in σ by some other agent a_2. Continuing the same argument, we arrive at a sequence of agents $a_0, a_1, \ldots, a_k, a_{k+1}$ appearing in the order $a_{k+1}, a_k, \ldots, a_1, a_0$ in σ, and picking the objects $f(a_{k+1}) = f(a_k), s(a_k) = f(a_{k-1}), \ldots, s(a_1) = f(a_0), s(a_0) = s(a)$, resp. Thus, the arcs $a_k, a_{k-1}, \ldots, a_0$ induce a directed path P in $G(I)$; by $f(a_{k+1}) = f(a_k)$ we get $\delta^+(f(a_{k+1})) > 1$, so P is as required.

Conversely, suppose that $G(I)$ contains a directed path P consisting of arcs $a_k, a_{k-1}, \ldots, a_1$ and vertices $o_k, o_{k-1}, \ldots, o_1, o_0$, appearing on P in this order, with $a_1 = a$ and $\delta^+(o_k) > 1$. By this latter fact, there exists an agent x, $x \notin \{a_1, \ldots, a_k\}$, whose first choice is o_k. Under SD with a picking sequence σ starting with $x, a_k, a_{k-1}, \ldots, a_1 = a$, agent x picks her first object, while each agent a_i with $k \geq i \geq 1$ picks her second object. Hence 2-$\textsc{pos}(I, a, s(a))$ is true. $\square$

To see how to use Thm. 5.5, let us consider the example shown in Fig. 1. 2-$\textsc{pos}(I, a_4, s(a_4))$=2-$\textsc{pos}(I, a_4, o_2)$ is true, since $G(I)$ contains the path $P = (a_2, a_4)$; note also $\delta^+(f(a_2)) = 2$. This path shows that for a picking sequence starting with a_1, a_2, a_4, agent a_1 picks object o_1, agent a_2 gets object o_3, and agent a_4 picks her second object a_2. Similarly, to decide whether 2-$\textsc{pos}(I, a_3, s(a_3))$=2-$\textsc{pos}(I, a_3, o_1)$ is true, we need to check the existence of a path ending with a_3 and starting from a vertex with out-degree at least 2. However, the only such vertex is o_1, but no path of $G(I)$ can start from o_1 and end with a_3, so 2-$\textsc{pos}(I, a_3, o_1)$ is false.

To decide whether an agent whose preference list contains only one object can end up with nothing, we can use similar arguments as in the proof of Thm. 5.5. In fact, the following statement can be viewed as a simplified version of Thm. 5.5.

Theorem 5.6. *Let $a \in A$ with $|P(a)| = 1$. Then* 2-POS$(I, a, \emptyset)$ *is true if and only if $G(I)$ contains a directed path (possibly of length 0) leading from a vertex $p \in \mathcal{O}$ with $\delta^+(p) > 1$ to $f(a)$.*

Using similar a method, we obtain a somewhat more complex condition that can be used to decide whether 2-POS$(I, a, \emptyset)$ is true for an agent a who finds two objects acceptable.

Theorem 5.7. *Let $a \in A$ with $|P(a)| = 2$. Then* 2-POS$(I, a, \emptyset)$ *is true if and only if $G(I)$ contains directed paths P_1 and P_2 such that*
(1) P_1 leads from some $p_1 \in \mathcal{O}$ to $f(a)$, and P_2 leads from some $p_2 \in \mathcal{O}$ to $s(a)$, allowing $p_1 = p_2$; (2) neither P_1 nor P_2 contains the arc a;
(3) $\delta^+_H(p_i) > 0$ for $i = 1, 2$ where H is obtained from $G(I)$ by deleting all arcs of P_1, P_2, and a.

Proof. Suppose a gets nothing under a picking sequence σ. As in the proof of Thm. 5.5, either $f(a)$ was picked by some agent whose first choice is $f(a)$, or we can find a sequence $a_{k+1}, a_k, \ldots, a_1$ of agents in σ such that a_{k+1} picks $f(a_{k+1}) = f(a_k)$, agent a_i picks $s(a_i) = f(a_{i-1})$ for each $i = k, \ldots, 2$, and agent a_1 picks $s(a_1) = f(a)$ under SD with σ. Let P_1 be the path $(a_k, \ldots, a_1)$; we allow P_1 to contain only the vertex $f(a)$. Similarly, let $b_{\ell+1}, b_\ell, \ldots, b_1$ be the sequence of agents in σ that explains how $s(a)$ was picked before a got her turn under SD with σ, and let P_2 be the path $(b_\ell, \ldots, b_1)$; again, P_2 might only consist of the vertex $s(a)$. Naturally, P_1 and P_2 satisfy (1).

Clearly, neither P_1 nor P_2 contains a, implying (2). Note that P_1 and P_2 may not be arc-disjoint, and $a_{k+1} = b_{\ell+1}$ is possible. However, $a \notin \{a_{k+1}, b_{\ell+1}\}$, and both P_1 and P_2 must be disjoint from $\{a_{k+1}, b_{\ell+1}\}$, because a_{k+1} and $b_{\ell+1}$ obtain their first choice under SD with σ, while all agents on P_1 and P_2 obtain their second choice. Thus a_{k+1} and $b_{\ell+1}$ witness that P_1 and P_2 satisfy (3) too.

Conversely, let P_1 and P_2 be paths in $G(I)$ satisfying (1)–(3), and let $a_k, \ldots, a_1$ and $b_\ell, \ldots, b_1$ be the agents corresponding to the sequence of arcs in P_1 and P_2, resp. By (3), there is a set Q of one or two agents, disjoint from P_1, P_2 and not containing a, for which $\{f(q) \mid q \in Q\} = \{f(a_k), f(b_\ell)\}$. Let us construct a picking sequence σ starting first with the agents in Q, followed by $a_k, \ldots, a_1$ and $b_\ell, \ldots, b_1$; repetitions are ignored (so each agent picks when it first appears in this sequence). Clearly, the agents in Q pick $f(a_k)$ and $f(b_\ell)$ (which may coincide), and then every agent x in $\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_\ell\}$ picks either her second choice or, if that is already gone by the time x gets her turn, gets nothing under SD with σ; during this process, both $f(a)$ and $s(a)$ gets picked, at latest by a_1 and b_1, resp., leaving nothing for a to pick. □

Let us again illustrate Thm. 5.7 on the instance of Fig. 1. To see that 2-POS$(I, a_4, \emptyset)$ is true, consider the path $P_1 = (a_2)$ leading to $o_3 = f(a_4)$, and the length-0 path P_2 containing only $o_2 = s(a_4)$; note that condition (3) is witnessed by a_1 leaving o_1 and a_3 leaving o_2. A corresponding picking sequence is thus a_1, a_3 followed by a_2, and ending with a_4. The first three agents pick the objects o_1, o_2, o_3, leaving nothing for a_4 at her turn.

Let us now discuss the complexity of the algorithms implied by Thms. 5.5, 5.6 and 5.7. We can construct $G(I)$ in time $O(|A| + |\mathcal{O}|) = O(n)$. Searching for the relevant paths in $G(I)$ can also be performed by, e.g., DFS in $O(n)$ time. This implies that 2-$\textsc{pos}(I, a, s(a))$ and 2-$\textsc{pos}(I, a, \emptyset)$ can be decided in $O(n)$ time for any agent a in I. By Lemma 5.4, we get Cor. 5.8.

Corollary 5.8. *Problems* 2-$\textsc{pos}(I, a, o)$ *and* 2-$\textsc{nec}(I, a, o)$ *are solvable in* $O(n)$ *time (even for the case* $o = \emptyset$*), where* n *is the number of agents in* I*.*

6 Multiple copies of objects

In this section we allow multiple identical copies for each object. The number of copies available for an object o is its *capacity*, determined by a capacity function $c : \mathcal{O} \to \mathbb{N}$. Given a *capacitated matching profile* $I = (A, \mathcal{O}, \mathcal{P}, c)$, we refer to the capacitated versions of the studied problems as $\textsc{cpos}(I, a, o)$ and $\textsc{cnec}(I, a, o)$.

Since $\textsc{cpos}(I, a, o)$ and $\textsc{cnec}(I, a, o)$ are generalizations of $\textsc{pos}(I, a, o)$ and $\textsc{nec}(I, a, o)$, resp., by Thm. 5.3 it is immediate that they are NP-complete and coNP-complete, resp., already if the maximum length of preference lists is 3. Hence, we focus on length-2 matching profiles. We adopt the notation k-$\textsc{cpos}(I, a, o)$ and k-$\textsc{cnec}(I, a, o)$ to refer to the corresponding problems restricted to capacitated length-k matching profiles. The following statement is trivial.

Lemma 6.1. *For any* $k \in \mathbb{N}$*,* k-$\textsc{cpos}(I, a, f(a))$ *is true, while* k-$\textsc{cnec}(I, a, s(a))$ *and* k-$\textsc{cnec}(I, a, \emptyset)$ *are false for any* $a \in A$*.*

Next, we show that both 2-$\textsc{cpos}(I, a, o)$ and 2-$\textsc{cnec}(I, a, o)$ are computationally intractable in every case not covered by Lemma 6.1.

Theorem 6.2. *Problems* 2-$\textsc{cpos}(I, a, s(a))$ *and* 2-$\textsc{cpos}(I, a, \emptyset)$ *are* NP*-complete, while* 2-$\textsc{cnec}(I, a, f(a))$ *is* coNP*-complete.*

Proof. Containment in NP or in coNP for the respective problems is trivial. We first provide a reduction from the $\textsc{exact 3-cover}$ problem to 2-$\textsc{cpos}(I, a, s(a))$. An instance of $\textsc{exact 3-cover}$ consists of a $3n$-element set $X = \{x_1, x_2, \dots, x_{3n}\}$ and a family $\mathcal{T}$ of 3-element subsets of X. The question is whether there exists a subfamily $\mathcal{T}' \subseteq \mathcal{T}$ containing exactly n sets whose union covers X. $\textsc{exact 3-cover}$ is NP-complete also in the case when each element $x \in X$ is contained in at most three sets from $\mathcal{T}$ [16]. We shall denote by $\ell(x)$ the number of sets in $\mathcal{T}$ that contain x.

Given an instance H of $\textsc{exact 3-cover}$, we define a capacitated length-2 matching profile I as follows. The set A of agents in I contains a special agent a, one agent for each set, and one agent for each element-set pair: $A = \{a\} \cup \{a(T) \mid T \in \mathcal{T}\} \cup \{a(x, T) \mid T \in \mathcal{T}, x \in T\}$. There are four types of objects in I: an object $o(x)$ for each element $x \in X$ with capacity $\ell(x) - 1$, an object $o(T)$ for each set $T \in \mathcal{T}$ with capacity 3, and two special objects: o_1 with capacity n and o_2 with capacity 1. The preferences are as follows:

$$\begin{aligned} P(a(T)) :&\quad o(T), o_1 && \text{for each } T \in \mathcal{T}, \\ P(a(x,T)) :&\quad o(x), o(T) && \text{for each } x \in X \text{ and } T \in \mathcal{T} \text{ such that } x \in T, \\ P(a) :&\quad o_1, o_2. \end{aligned}$$

Clearly, the construction is polynomial in the size of H. We claim that agent a can obtain object o_2 in I (that is, 2-CPOS$(I, a, s(a))$ is true) if and only if there is an exact cover in H.

Assume first that there is an exact cover $\mathcal{T}'$ in H consisting of sets $T_1, T_2, \ldots, T_n$. Let those agents $a(x, T)$ pick first for which $T \notin \mathcal{T}'$. These agents exhaust all the element-objects, i.e., all objects $o(x)$, $x \in X$. Now agents of type $a(x, T_i)$ for $i = 1, 2, \ldots, n$ follow. They all get their second choices and thus completely exhaust all set-objects belonging to $\mathcal{T}'$, i.e. all objects $o(T)$, $T \in \mathcal{T}'$. Next come agents $a(T_i)$ for $i = 1, 2, \ldots, n$. They again get their second choices and exhaust all copies of object o_1. Hence, if agent a gets her turn after this point, she gets her second choice, o_2.

Conversely, assume now that agent a gets o_2 under some picking sequence. This means that o_1 was already exhausted when a got her turn, implying that n set-agents received their second choice. Let these agents be $a(T_1), \ldots, a(T_n)$. To finish the proof, we have to show that sets $T_1, \ldots, T_n$ form an exact cover, or equivalently, that these sets are pairwise disjoint. Assume for the contrary that an element x belongs both to T_r and T_s. As both object $o(T_r)$ and object $o(T_s)$ were exhausted before $a(T_r)$ and $a(T_s)$ pick, this means that both agents $a(x, T_r)$ and $a(x, T_s)$ must have received their second object, $o(T_r)$ and $o(T_s)$, resp. So their first choice, object $o(x)$ was already exhausted by the time they picked. But this could not happen as $o(x)$ has capacity $\ell(x) - 1$ and the number of agents interested in $o(x)$ is only $\ell(x)$, proving our claim.

The above reduction can be easily modified to show that the problem 2-CPOS$(I, a, \emptyset)$ is NP-complete and 2-CNEC$(I, a, f(a))$ is coNP-complete: we simply need to add a new agent b whose preference list contains only o_2. In this modified instance the following statements are equivalent: (i) a can obtain o_2, (ii) b does not necessarily obtain o_2, and (iii) b might end up with no object assigned to her. Furthermore, from the correctness of the above reduction, these hold exactly if H admits an exact cover, proving our theorem. □

Observe that in the proof of Thm. 6.2 each object with capacity $c \geq 3$ has the following property: it is the first choice of a unique agent p, and it is the second choice of several agents $q_1, \ldots, q_k$ for some $k \geq c$; let us call such objects with capacity at least 3 *counter objects*.

Lemma 6.3. *Given an instance $(I, a, s(a))$ of* 2-CPOS *where only counter objects have capacity greater than* 2 *and $s(a)$ is not a counter object, we can in quadratic time construct an equivalent instance $(I', a, s(a))$ of* 2-CPOS *where all capacities are at most* 2*.*

Proof. Let o be a counter object with capacity $c \geq 3$ in I, let p be the unique agent whose first choice is o, and let $q_1, \ldots, q_k$ $(k \geq c)$ be those agents whose second choice is o. We describe a method to replace o with a gadget containing only objects with capacities at most 2, while not changing the answer to our instance.

In what follows, we will denote the modified capacitated matching profile by I', and we will make sure that the agent set A' of I' is a superset of the agent set A of I. We say that a picking sequence φ of I and a picking sequence φ' of I' are *A-equivalent*, if the set of those agents in A that are assigned their second choice is the same under φ and under φ'.

Case I. First assume $k = c$. We start by replacing o in q_i's preference list with a newly introduced object o_i that has capacity 1, for each $i \in \{1, \dots, c\}$. Next, we fix any rooted binary tree T with c leaves. We identify the leaves of T with the objects $o_1, \dots, o_c$, and we identify its root with object o. We add a new object $o(t)$ with capacity 2 for each vertex t of T that is neither a leaf nor root, and we also change the capacity of object o to 2. Furthermore, for any edge $t_1 t_2$ of T where t_1 is the child of t_2, we add a new agent $a(t_1, t_2)$ whose first choice and second choice is $o(t_1)$ and $o(t_2)$, resp. This finishes the construction.

Now we show that $\textsc{cpos}(I', a, s(a))$ is true if and only if $\textsc{cpos}(I, a, s(a))$ is true. To this end, we prove that for any picking sequence φ in I there is a picking sequence in I' that is A-equivalent to φ, and conversely. By $o \neq s(a)$, this guarantees the equivalence of our two instances.

"$\Leftarrow$": Suppose that φ is a picking sequence in I. Note that if p is assigned its first choice o under φ, then the newly added agents of I' do not "interfere" with the agents of A; simply letting all agents in $A' \setminus A$ pick after agents of A yields a picking sequence A-equivalent to φ. On the other hand, if p is assigned its second choice in I, then by the definition of a counter object and by $k = c$, we know that all agents q_i, $i \in \{1, \dots, c\}$, must also be assigned their second choice (that is, o) under φ. Let us create a picking sequence φ' from φ by inserting the agents $A' \setminus A$ immediately before p in a consecutive, bottom-up way: an agent $a(t_1, t_2)$ corresponding to an edge of T is allowed to pick only after her first choice t_1 is already exhausted. Thus, when p picks in φ', its first choice o is already exhausted (by the two agents corresponding to the two edges connecting o to its children in T). Hence, in the remainder of φ', all agents are assigned the same objects as in φ, showing that φ and φ' are A-equivalent.

"$\Rightarrow$": For the other direction, let φ' be a picking sequence in I'. We prove that the restriction of φ' to A (let us call this picking sequence φ) is A-equivalent to φ'. If p gets o under φ', then this is trivial. If, by contrast, p gets his second choice under φ', then the capacities of the newly introduced objects imply that each agent $a(t_1, t_2) \in A'$ must be assigned its second choice by φ', from which follows also that each agent q_i is assigned her second choice under φ'. This, however, ensures that p gets her second choice in I under φ, proving our claim.

Case II. Assume now $k > c$. First, we create $k - c + 1$ layers of new objects: for each $j \in \{0, \dots, k - c\}$, layer j contains the objects $o_{j,1}, \dots, o_{j,k-j}$; notice that each layer contains one object less than the previous layer. Next, for each $i \in \{1, \dots, k\}$, we replace o in the preference list of agent q_i with $o_{0,i}$. We let all objects in layer $k-c$ have capacity 1. Within some layer j with $0 \leq j < k-c$, we let the two "outermost" objects, that is, $o_{j,1}$ and $o_{j,k-j}$, have capacity 1, and all the remaining objects have capacity 2. Next, we create $k-c$ layers of new agents: for each $j \in \{1, \dots, k-c\}$, layer j contains $2(k-j)$ agents, namely agents $a(j, i, \uparrow)$ and $a(j, i, \nwarrow)$ for each $i \in \{1, \dots, k-j\}$. We define the preference list of these agents as follows: both $a(j, i, \uparrow)$ and $a(j, i, \nwarrow)$ have $o(j, i)$ as their second choice, but the first choice of $a(j, i, \uparrow)$ is $o(j-1, i)$, while the first choice of $a(j, i, \nwarrow)$ is $o(j-1, i+1)$. We finish the construction by adding the gadget described in Case I, with the only difference that we choose the c objects of

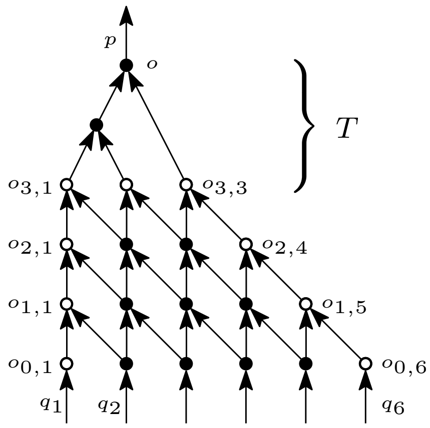

Figure 2: Illustration for the construction in Lemma 6.3 for $k = 6$ and $c = 3$, depicting (a part of) the corresponding digraph. Objects with capacity 2 and 1 are black and white circles, resp.

layer $k - c$ as the leaves of the binary tree T (and, as in Case I, we again set the capacity of o to 2). See Fig. 2 for an illustration.

Let I' be the obtained instance of 2-POS. We call an agent *active* in a picking sequence, if it is assigned its second choice. Let us now prove the equivalence of CPOS$(I, a, s(a))$ and CPOS$(I', a, s(a))$.

"$\Rightarrow$": Let φ' be a picking sequence in I'. By the capacities of the newly added objects, for any $j \in \{1, \dots, k - c\}$ it holds that layer j contains at most as many active agents as layer $j - 1$ (we let layer 0 contain the agents $q_1, \dots, q_k$). Recalling the properties of the gadget constructed in Case I, it is not hard to verify that p can become active only if layer $k - c$, and hence each of the previous layers too, contains at least c active agents. Hence, at least c agents among $q_1, \dots, q_k$ are active under φ', implying that the restriction of φ' to A is A-equivalent to φ'.

"$\Leftarrow$": Suppose that φ is a picking sequence in I where p is active, that is, where at least c agents among $q_1, \dots, q_k$ are active. We can easily construct a picking sequence φ' in I' that is A-equivalent to φ such that under φ' exactly c agents become active in each layer and all the objects in layer $k - c$ get exhausted by agents of layer $k - c$, thus implying that each agent corresponding to an edge of our binary tree T, and therefore also agent p, becomes active in φ'. To determine such a picking sequence, we need, roughly speaking, to find c object-disjoint paths from the active agents in layer 0 to agents of layer $k - c$ (note that such paths always exist).

The replacement described above takes $O(k^2)$ time; replacing all counter objects therefore takes $O(|I|^2)$ time, proving our lemma. □

Applying Lemma 6.3 to the instance constructed in Thm. 6.2, we get the following consequence.[1]

[1] Note that the modifications in the reduction given in the proof of Thm. 6.2 that prove the hardness of 2-CPOS$(I, a, \emptyset)$ and 2-CNEC$(I, a, f(a))$ still work after the application of Lemma 6.3.

Corollary 6.4. *Problems* 2-CPOS$(I, a, s(a))$ *and* 2-CPOS$(I, a, \emptyset)$ *are* NP*-complete and* 2-CNEC$(I, a, f(a))$ *is* coNP*-complete, even if all capacities are at most* 2*.*

7 Conclusion

We showed that if we enable agents to declare certain objects unacceptable, both the problems to decide whether a given agent can get a given object or whether a given agent always gets a given object in serial dictatorship are intractable, unless in the very special case when the lengths of preference lists are bounded by 2, and each object comes in a single copy. These results have direct consequences for manipulation possibilities of serial dictatorship: if it is difficult to compute which objects can an agent achieve then it is even the more difficult to compute a successful manipulation.

A possible direction of further research is to investigate a model where preference lists may contain ties. It is known that simply applying serial dictatorship is not enough to find a Pareto-optimal matching (POM) if ties can occur; recently Krysta et al. [20] and Cechlárová et al. [12] provided polynomial-time algorithms combining the greedy approach of serial dictatorship with network flow to find POMs in such situations. Up to our knowledge, the question of possible and necessary allocations has not yet been investigated in the presence of ties.

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