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Quentin Fortier*, Csaba Király**, Zoltán Szigeti***, and Shin-ichi Tanigawa‡

Abstract

Let $D = (V + s, A)$ be a digraph with a designated root vertex s. Edmonds' seminal result [4] implies that D has a packing of k spanning s-arborescences if and only if D has a packing of k (s, t)-paths for all $t \in V$, where a packing means arc-disjoint subgraphs.

Let $\mathcal{M}$ be a matroid on the set of arcs leaving s. A packing of (s, t)-paths is called $\mathcal{M}$-based if their arcs leaving s form a base of $\mathcal{M}$ while a packing of s-arborescences is called $\mathcal{M}$-based if, for all $t \in V$, the packing of (s, t)-paths provided by the arborescences is $\mathcal{M}$-based. Durand de Gevigney, Nguyen and Szigeti proved in [3] that D has an $\mathcal{M}$-based packing of s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s, t)-paths for all $t \in V$. Bérczi and Frank conjectured that this statement can be strengthened in the sense of Edmonds' theorem such that each s-arborescence is required to be spanning. Specifically, they conjectured that D has an $\mathcal{M}$-based packing of spanning s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s, t)-paths for all $t \in V$.

We disprove this conjecture in its general form and we prove that the corresponding decision problem is NP-complete. However, we prove that the conjecture holds for several fundamental classes of matroids, such as graphic matroids and transversal matroids. For all the results presented in this paper, the undirected counterpart also holds.

1 Introduction

The packing problem in digraphs is one of the fundamental topics in graph theory and combinatorial optimization, where the goal is to find the largest family of disjoint subgraphs satisfying a specified property in a given digraph. In this paper, by packing subgraphs, we always mean a set of arc-disjoint subgraphs.

Suppose that we are given a **rooted digraph**, that is, a digraph $D = (V + s, A)$ with a designated root vertex s. An **s-arborescence** is a directed tree rooted at s, i.e., the underlying undirected graph is a tree and every vertex except s has in-degree one. The celebrated Edmonds theorem gives an exact relation between arborescence packings and path packings as follows.

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Revision 1
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Theorem 1.1 ([4]). *There exists a packing of k spanning s-arborescences in a rooted digraph $D = (V + s, A)$ if and only if there exists a packing of k (s, t)-paths in D for every $t \in V$.* □

The problem of packing k (s, t)-paths is equivalent to asking whether one can send k distinct commodities from s to t by assuming that each arc can transmit at most one commodity. Then what happens if commodities have an involved independence structure? Here we are interested in a situation that each commodity c_i is assigned to some vertex s_i at the beginning, and we would like to know whether every vertex can receive a sufficient amount of independent commodities to understand the whole structure. By adding an auxiliary root vertex s and arcs from s to s_i for each i, we may convert the situation such that all commodities are assigned to the root s and each arc from the root can be used to transmit only a particular commodity.

More formally, suppose that we are given a **matroid-rooted digraph** $(D = (V + s, A), \mathcal{M})$, that is, a matroid $\mathcal{M}$ is given on the set of arcs leaving the root s that we call **root arcs**. We are interested in a packing of (s, t)-paths whose root arcs form a base of $\mathcal{M}$. Such a packing is said to be an **$\mathcal{M}$-based packing** of (s, t)-paths. A packing of s-arborescences is called **$\mathcal{M}$-based** if, for all $t \in V$, the packing of (s, t)-paths provided by the arborescences is $\mathcal{M}$-based. A natural question is whether Edmonds' theorem can be extended for $\mathcal{M}$-based packings. The result of Durand de Gevigney, Nguyen and Szigeti [3] gives a partial answer to this question.

Theorem 1.2 ([3]). *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. Then there exists an $\mathcal{M}$-based packing of s-arborescences in D if and only if there exists an $\mathcal{M}$-based packing of (s, t)-paths in D for every $t \in V$.* □

Notice that at the quantitative level, Theorem 1.1 always guarantees the existence of k spanning s-arborescences while the number of s-arborescences in Theorem 1.2 may be more than the rank of $\mathcal{M}$ since these arborescences are not necessarily spanning.

1.1 Contribution and key ideas

K. Bérczi and A. Frank [8] conjectured that Theorem 1.2 can be strengthened in the sense of Edmonds' theorem. This conjecture appeared also in a paper of Bérczi, T. Király and Kobayashi [2]. More formally, the conjecture is the following.

Conjecture 1.3 ([2]). *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. There exists an $\mathcal{M}$-based packing of* ***spanning*** *s-arborescences in D if and only if there exists an $\mathcal{M}$-based packing of (s, t)-paths in D for every $t \in V$.*

The main result of this paper is that Conjecture 1.3 is false in its general form. We will even prove that the following decision problem is NP-complete, which was conjectured by E.R. Bérczi-Kovács [8].

Problem 1.4. Given a matroid-rooted digraph $(D = (V + s, A), \mathcal{M})$, decide whether there exists an $\mathcal{M}$-based packing of spanning s-arborescences in D.

As positive results, we will prove that Conjecture 1.3 is true for several fundamental classes of matroids such as graphic and transversal matroids.

We present the main ideas of the proofs below.

Graphic matroids. Let $(D, \mathcal{M})$ be a matroid-rooted digraph where $\mathcal{M}$ is a graphic matroid of rank k. Let $G = (\{0, 1, \dots, k\}, E)$ be a connected undirected graph representing $\mathcal{M}$, so the edges of G corresponds to the root arcs of D. The idea is to restrict an admissible packing such that each root arc belong to T_i only if its corresponding edge in G is incident to the vertex i

of G. This condition gives an extra property for the packing obtained by induction, based on which we show how to extend the packing.

Transversal matroids. Let $(D, \mathcal{M})$ be a matroid-rooted digraph where $\mathcal{M}$ is a transversal matroid of rank k. Let $G = (S, T; E)$ be a bipartite graph representing $\mathcal{M}$ where S corresponds to the set of root arcs of D and $T = \{1, \ldots, k\}$. The plan is to replace the matroid-based condition by the following new condition: a root arc e may belong to T_i only if its corresponding vertex is connected to $i \in T$ in G. The key observation is that if a packing of arborescences satisfies this new condition then any set of k root arcs belonging to different arborescences of the packing forms a base of $\mathcal{M}$. Thus the packing is automatically $\mathcal{M}$-based.

Counterexample and NP-completeness. One of the simplest non-graphic and non-transversal matroids is the Fano matroid. A simple proof shows that Conjecture 1.3 is true for the Fano matroid when the digraph is acyclic. However, it turns out that Conjecture 1.3 is false when we allow to extend the Fano matroid by parallel elements. The symmetry of the Fano matroid will be widely explored in the proof. We will construct our acyclic digraph step by step by adding sink vertices of in-degree 3. This construction will ensure not only the existence of the required $\mathcal{M}$-based path packings but also that every $\mathcal{M}$-based arborescence packing is an extension of the previous instance. We design each construction step so that possible extensions are restricted.

1.2 Related works

Connectivity is one of the most well-studied properties of graphs. The earliest results related to our main interest on packing problems concerning connectivity are the papers of Nash-Williams [17] and Tutte [20] on packing trees in undirected graphs from 1961. The topic of packing arborescences has been extensively studied in the seventies by Edmonds and Frank [4, 6]. The connection between these problems was pointed out in a work of Frank [7] on orientations of graphs.

The hypergraphic counterparts of the above packing results were discovered by Frank, T. Király, Z. Király and Kriesell [9, 10]. A surprising extension of Edmonds' result was given by Katoh, Kamiyama and Takizawa [13] and Fujishige [11] for the case when no spanning arborescences exist. Szegő [19] gave an abstract version of Edmonds' result that was extended to an abstract version of the result of [13] in a paper of Bérczi and Frank [1].

Investigations in rigidity theory inspired an extensive research on possible extensions of Nash-Williams' and Tutte's result. Katoh and Tanigawa [14] introduced the concept of matroid-based packing of rooted trees and presented several applications of this result in rigidity theory. Durand de Gevigney, Nguyen and Szigeti [3] used the techniques of Frank to show that, by an extension of Edmonds' result, an alternative proof of the packing result of [14] can be obtained. These breakthrough results inspired an intensive research in the last few years on this topic to extend the above mentioned results, see [2, 5, 15, 16].

2 Definitions

We will use some basics from matroid theory listed below. For details, we refer to [18]. Recall that, for a set function $r\colon 2^{\mathsf{S}} \to \mathbb{Z}_+$, $\mathcal{M} = (\mathsf{S}, r)$ is called a **matroid** if r is 0 on the $\emptyset$, monotone non-decreasing, subcardinal ($r(\mathsf{Q}) \le |\mathsf{Q}|$) and submodular ($r(\mathsf{P}) + r(\mathsf{Q}) \ge r(\mathsf{P} \cap \mathsf{Q}) + r(\mathsf{P} \cup \mathsf{Q})$). The members of $\mathcal{I} = \{\mathsf{Q} \subseteq \mathsf{S} : r(\mathsf{Q}) = |\mathsf{Q}|\}$ are called **independent** sets of the matroid and r is called the **rank function** of the matroid. It is well known that a matroid can also be defined by its independent sets. Let $\mathsf{Q} \subseteq \mathsf{S}$. The maximal independent sets in Q are called **bases** of Q.

Note that all bases are of the same size. The bases of S are called the bases of $\mathcal{M}$. We define **Span(Q)**$:= \{\mathsf{s} \in \mathsf{S} : r(\mathsf{Q} \cup \{\mathsf{s}\}) = r(\mathsf{Q})\}$. Note that Span is monotone. Two elements $\mathsf{a}, \mathsf{a}' \in \mathsf{S}$ are said to be **parallel** in $\mathcal{M} = (\mathsf{S}, r)$ (in notation, $\mathbf{a} \parallel \mathbf{a}'$) if $r(\{\mathsf{a}\}) = r(\{\mathsf{a}'\}) = r(\{\mathsf{a}, \mathsf{a}'\}) = 1$.

The following classes of matroids will be discussed in this paper:

1. **graphic matroid** : given a graph $G = (V, E)$ with a bijection $\pi : E \to \mathsf{S}$, $\mathcal{I} := \{\pi(F) : F$ is the edge set of a forest of $G\}$;
2. **Fano matroid** : a rank-3 matroid derived from the Fano plane (the smallest projective plane with 7 points) on a 7-element ground set (the points of the Fano plane) where every set of cardinality 3 is a base except the lines of the Fano plane;
3. **transversal matroid** : given a bipartite graph $G = (S, T; E)$ with a bijection $\pi : S \to \mathsf{S}$, $\mathcal{I} := \{\pi(X) : X \subseteq S$ that can be covered by a matching in $G\}$.

A special class of the transversal matroids where G is the complete bipartite graph $K_{n,k}$ is called the **uniform matroid $\boldsymbol{U_{k,n}}$**. It is well known that a graphic matroid is always representable by a connected graph and a transversal matroid is always representable by a bipartite graph where $|T|$ is equal to the rank. It is also well known that a matroid of rank at most 3 is not graphic if and only if it has a minor isomorphic to the Fano matroid or $U_{2,4}$ (see e.g., [18]).

An ***s*-arborescence** is a directed tree on a vertex-set containing the **root** vertex s in which each vertex has in-degree 1 except s. An s-arborescence in a digraph $D = (V + s, A)$ is **spanning** if its vertex set is $V + s$. For an s-arborescence T and a vertex $v \neq s$ of T, we denote the unique arc of T entering v by $\boldsymbol{T(v)}$, the unique path from s to v by $\boldsymbol{T[s, v]}$, and its first arc by $\boldsymbol{e_{T[s,v]}}$. For disjoint sets $X, Y \subseteq V + s$, we denote by $\boldsymbol{\partial^D_X(Y)}$ the subset of arcs in D with tail in X and head in Y. The superscript D will be omitted, when it is clear from the context. The in-degree of a set $X \subseteq V + s$ is denoted by $\boldsymbol{\varrho_D(X)}:= |\partial^D_{V+s-X}(X)|$.

We say that a matroid-rooted digraph $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ is **rooted $\mathcal{M}$-arc-connected** if there exists an $\mathcal{M}$-based packing of (s, t)-paths for all vertices t in V. One can easily prove a Menger-type theorem saying that D is rooted $\mathcal{M}$-arc-connected if and only if

$$r(\partial_s(X)) + \varrho_{D-s}(X) \geq r(\mathcal{M}) \text{ for all } X \subseteq V, \tag{1}$$

where $\boldsymbol{r(\mathcal{M})}$ denotes the rank of $\mathcal{M}$. For simplicity, we will call an $\mathcal{M}$-based packing of spanning s-arborescences in D that covers $\partial_s(V)$ a **feasible packing**.

3 Positive results

In this section, we prove Conjecture 1.3 for several special cases. The necessity of Conjecture 1.3 is always true by Theorem 1.2 (and is easy to prove anyway), so we will only prove the sufficiency in each case.

3.1 Overview of the proof of Theorem 1.2

Some of our positive results are obtained by extending the proof of Theorem 1.2 given by [3], and hence we shall first review it by introducing several key ingredients used later. In [3], Theorem 1.2 was proved in a slightly stronger form by imposing an extra technical condition as follows. Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph. D is called **$\mathcal{M}$-independent** if $\partial_s(v)$ is independent in $\mathcal{M}$ for every $v \in V$. This condition ensures that each root arc can be used in an $\mathcal{M}$-based packing of s-arborescences in D, as follows.

Theorem 3.1 ([3]). *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph. There exists an $\mathcal{M}$-based packing of s-arborescences in D that covers $\partial_s(V)$ if and only if D is rooted $\mathcal{M}$-arc-connected and $\mathcal{M}$-independent.*

Let $(D, \mathcal{M})$ be as in Theorem 1.2. Observe that, by omitting some root arcs of a rooted $\mathcal{M}$-arc-connected digraph, one can get a rooted $\mathcal{M}'$-arc-connected and $\mathcal{M}'$-independent digraph, where $\mathcal{M}'$ is a submatroid of $\mathcal{M}$ with the same rank. Therefore, Theorem 1.2 follows from Theorem 3.1. Observe also that $\mathcal{M}$-independence is a trivial necessary condition for an $\mathcal{M}$-based packing that covers $\partial_s(V)$.

Let $(D, \mathcal{M})$ be as in Theorem 3.1. We call $X \subseteq V$ **tight** if (1) holds with equality. We say that a non-root arc uv is **good** if $\partial_s(u) \not\subseteq \mathrm{Span}_{\mathcal{M}}(\partial_s(v))$. A pair (uv, x) of a good arc uv in $D - s$ and $x \in \partial_s(u) - \mathrm{Span}_{\mathcal{M}}(\partial_s(v))$ is said to be **admissible** if there is no tight set X with $v \in X$ and $u \notin X$ such that x is in the span of $\partial_s(X)$. The **shifting** (of $(D, \mathcal{M})$) along (uv, x) is a new instance $(D', \mathcal{M}')$ obtained from $(D, \mathcal{M})$ by removing uv and inserting a new root arc sv such that sv is a parallel element to x in the underlying matroid. Note that the shifting satisfies $\mathcal{M}$-independence (resp. rooted $\mathcal{M}$-arc-connectivity) if and only if uv is good (resp., (uv, x) is admissible). A key observation in [3] is the following. (See Case 2 in the proof of Theorem 1.6 in [3].)

Lemma 3.2 ([3]). *Suppose that D has a good arc. Then D has an admissible pair (uv, x).*

The proof of the sufficiency of Theorem 3.1 is done by induction on the number of non-root arcs. If no good arc exists, then the set of root arcs form an $\mathcal{M}$-based packing of s-arborescences. Otherwise, by Lemma 3.2, there exists an admissible pair (e, x), and hence the shifting $(D', \mathcal{M}')$ along (e, x) is $\mathcal{M}'$-independent and rooted $\mathcal{M}'$-arc-connected. By induction, there exists an $\mathcal{M}'$-based packing $\mathcal{T}$ of s-arborescences in D' such that it covers $\partial'_s(V)$. We can suppose that each s-arborescence in $\mathcal{T}$ has only one root arc since otherwise we can split it into several s-arborescences to satisfy this condition. Let $T \in \mathcal{T}$ be the arborescence covering x and $T' \in \mathcal{T}$ be the arborescence covering the new root arc f in D'. Then $(\mathcal{T} - \{T, T'\}) \cup \{T \cup (T' - f) + e\}$ is a desired $\mathcal{M}$-based packing of s-arborescences in D that covers $\partial_s(V)$, and this completes the proof of Theorem 3.1.

Now consider applying the proof to Conjecture 1.3. In the same manner, by induction, one gets an $\mathcal{M}'$-based packing $\mathcal{T}$ of *spanning* s-arborescences in D'. Our goal is to construct a feasible packing in D based on $\mathcal{T}$. Let $T \in \mathcal{T}$ be an arborescence that covers the new root arc f of D'. If T also contains x, then $(\mathcal{T} - \{T\}) \cup \{T - f + e\}$ is an $\mathcal{M}$-based packing of spanning s-arborescences in D, and we are done. The difficult case is when T does not contain x. We will show how to overcome this difficulty by new ideas if $\mathcal{M}$ has rank at most 2 or is graphic.

3.2 Matroids of rank at most 2

In this section we prove that Conjecture 1.3 is true when $r(\mathcal{M}) \leq 2$. We first prove the following technical lemma.

Lemma 3.3. *Let T_1 and T_2 be arc-disjoint spanning s-arborescences on $V + s$. Let T'_1 (resp. T'_2) be an s-subarborescence of T_1 (resp. T_2) such that no non-root arc of T_1 (resp. T_2) leaves its vertex set, and let $X = V(T'_1) \cap V(T'_2)$. Let T^*_1 and T^*_2 be obtained from T_1 and T_2 by exchanging for every vertex v in $X - s$ the arcs $T_1(v)$ and $T_2(v)$. Then T^*_1 and T^*_2 are spanning s-arborescences on $V + s$.*

Proof. We prove the result for T^*_1. Suppose that T^*_1 is not an s-arborescence. Since $\varrho_{T^*_1}(v) = \varrho_{T_1}(v) = 1$ for every $v \in V$, there exists a circuit C in T^*_1. Since neither T_1 nor T_2 contains a circuit, C contains at least one arc from each arborescence T_1 and T_2. It follows that there exist not necessarily distinct arcs uv and wz of C such that uv and wz belong to T_2 and the path of C from z to u belongs to T_1. Note then that $T_1(u) = T_1^*(u)$ as T_1^* contains C and C contains $T_1(u)$.

Since uv belongs to T_2 and to T_1^*, v is in X and hence in T_2', and thus u is also in T_2'. Since wz belongs to T_2 and to T_1^*, z is in X and hence in T_1', and thus, since the path of C from z to u belongs to T_1, u is also in T_1'. It follows that u is in X, and so we have a contradiction, $T_1(u) \neq T_1^*(u) = T_1(u)$. □

Theorem 3.4. *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph with $r(\mathcal{M}) \leq 2$. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D that covers $\partial_s(V)$ if and only if D is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected.*

Proof. The proof is done by induction on the number of non-root arcs. As we remarked above, if no good arc exists, then we can form $r(\mathcal{M})(= 1 \text{ or } 2)$ spanning s-stars that gives a feasible packing in D. Hence we assume that D has a good arc. Then, by Lemma 3.2, there exists an admissible pair (uv, x) along which the shifting $(D', \mathcal{M}')$ satisfies the conditions of the theorem. Now, by induction, we get that there exists a feasible packing in D'. Let f be the new root arc in D' from s to v. We have the following two cases.

Case 1. If x and f are contained in the same arborescence T of the packing, then substituting T with $T - f + uv$ in the packing one gets a feasible packing in D.

Case 2. Otherwise, the packing consists of two arborescences T_1 and T_2 (thus the rank of $\mathcal{M}'$ is 2), and we can assume that $x \in T_1$ and $f \in T_2$. Let $V_f \subseteq V$ be the set of vertices which is reachable from s in T_2 by a path starting with the arc f or an arc parallel to f in $\mathcal{M}$. Let $\{T_1^*, T_2^*\}$ be the packing that arises from $\{T_1, T_2\}$ by using Lemma 3.3 with $T_1' := T_1$, and $T_2' := T_2[V_f + s]$. We claim the following.

Claim 3.5. *$\{T_1^*, T_2^*\}$ is an $\mathcal{M}'$-based packing of spanning s-arborescences covering the root arcs in D'.*

Proof. By Lemma 3.3, T_1^* and T_2^* are spanning s-arborescences. Let $V^* \subseteq V$ denote the set of vertices $v \in V$ for which $V(T_1^*[s, v]) \cap V_f \neq \emptyset$. Observe that, for every $v \in V^*$ and $u \in V - V^*$, $e_{T_1^*[s,v]}$ is parallel to f in $\mathcal{M}'$ and $e_{T_1^*[s,u]} = e_{T_1[s,u]}$. On the other hand, for every $w \in V - V_f$, $e_{T_2^*[s,w]} = e_{T_1[s,w]}$; moreover, for every $w \in V$, $e_{T_2^*[s,w]}$ is not parallel to f in $\mathcal{M}'$ as T_2^* has no root arcs parallel to f by the definition of V_f. Finally, $\{T_1, T_2\}$ and $\{T_1^*, T_2^*\}$ cover the same set of root arcs. These imply the claim. □

By Claim 3.5, $\{T_1^*, T_2^*\}$ is also a feasible packing in D' where x and f are in T_1^*. Thus we are in Case 1. This completes the proof of Theorem 3.4. □

3.3 Graphic matroids

We prove that Conjecture 1.3 is true for graphic matroids.

Theorem 3.6. *Let $(D = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph where $\mathcal{M} = (\partial_s(V), r)$ is a graphic matroid of rank k. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D covering $\partial_s(V)$ if and only if D is rooted $\mathcal{M}$-arc-connected and $\mathcal{M}$-independent.*

Proof. Let $\boldsymbol{G = (\{0, 1, \dots, k\}, E)}$ be a connected undirected graph with a bijection $\boldsymbol{\pi}\colon E \to \partial_s(V)$ representing $\mathcal{M}$. From now on, we will refer to the matroid-rooted digraph $(D, \mathcal{M})$ as (D, G, π). For an edge $e \in E$, let $\boldsymbol{x_e} = \pi(e)$. For $X \subseteq V$, let $\boldsymbol{E_X} = \pi^{-1}(\partial_s(X))$. For each $v \in V$, let $\boldsymbol{C_v}$ be the vertex set of the connected component $\boldsymbol{Q_v}$ of $(V(G), E_v)$ that contains 0. Note that, since D is $\mathcal{M}$-independent, $k - |E_v| \geq 0$ and Q_v is a tree. For $v \in V$, let $\vec{\boldsymbol{Q}}_{\boldsymbol{v}}$ be the arborescence rooted at 0 that arises by orienting each edge e of Q_v to $\vec{\boldsymbol{e}}$.

We prove the theorem by imposing the following extra property for the packing $\{T_1, \dots, T_k\}$:

$$\text{for } \vec{e} = ij \text{ belonging to } \vec{Q}_v \text{ for some } v \in V,\ x_e \text{ belongs to } T_j. \tag{2}$$

Let (D, G, π) be a counterexample minimizing $k|V| - \sum_{v \in V} |E_v| \geq 0$. We take $\boldsymbol{v^*}$ such that $|C_{v^*}|$ is as small as possible. If $C_{v^*} = V(G)$, then Q_v is a spanning tree of G for every $v \in V$. In this case, using only the root arcs, the 0-arborescences $\vec{Q}_v$ show how to define a feasible packing satisfying (2).

From now on, we suppose that C_{v^*} is a proper subset of $V(G)$. Let $\boldsymbol{W} = \{v \in V : C_v = C_{v^*}\}$. Then the vertex set C_W of the connected component that contains 0 in $(V(G), E_W)$ is equal to C_{v^*}. For $p \in V - W$, an element $e \in E_p$ is called **critical** if $\vec{e}$ belongs to $\vec{Q}_p$ and $\vec{e}$ leaves C_W. By the minimality of $|C_{v^*}|$ and $p \in V - W$, we have $C_p - C_W \neq \emptyset$. Hence the following claim follows from the fact that $\vec{Q}_p$ is a spanning 0-arborescence on C_p.

Claim 3.7. *For $p \in V - W$, E_p contains a critical element.* □

Claim 3.8. *Let pq be an arc in D with $p \in V - W$ and $q \in W$ and e a critical element in E_p. Then (pq, x_e) is not admissible.*

Proof. Since e is critical, $\vec{e}$ leaves $C_W = C_q$, so x_e is not spanned by $\pi(E_q)$, that is the arc pq is good. Suppose that (pq, x_e) is admissible. Then the shifting (D', G', π') of (D, G, π) along (pq, x_e) satisfies the $\mathcal{M}'$-independence and rooted $\mathcal{M}'$-arc-connectivity conditions. Since (D, G, π) is a minimum counterexample, we have a feasible packing $T'_1, \dots, T'_k$ for (D', G', π') satisfying (2). Let e' be the new edge parallel to e assigned to the new arc $x_{e'} := sq$ in the shifting. As e is critical, (2) implies that x_e and $x_{e'}$ belong to the same spanning s-arborescences T_j of D. Therefore, by setting T_ℓ $(1 \leq \ell \leq k)$ with $T_\ell = T'_\ell$ for $\ell \neq j$ and $T_j = T'_j - x_{e'} + pq$, we obtain a feasible packing $T_1, \dots, T_k$ for (D, G, π) satisfying (2). This contradicts that (D, G, π) is a counterexample. □

Since C_W is a proper subset of $V(G)$, $r(\pi(E_W)) < k$. Therefore, by the rooted $\mathcal{M}$-arc-connectivity of D, D has an arc pq with $p \in V - W$ and $q \in W$. By Claim 3.7, E_p contains a critical element e, and then Claim 3.8 says that (pq, x_e) is not admissible. In other words, there exists a tight set $X \subseteq V$ with $q \in X$ and $p \notin X$ such that x_e is contained in the span of $\pi(E_X)$.

We shall take such a pair (pq, x_e) such that X is minimal. Since $\pi(E_X)$ spans x_e while, as e is critical, $\pi(E_W)$ does not span x_e, we have $r(\pi(E_{X \cap W})) < r(\pi(E_X))$. Hence, by the rooted $\mathcal{M}$-arc-connectivity of D and the tightness of X, $\varrho_{D-s}(X \cap W) \geq k - r(\pi(E_{X \cap W})) > k - r(\pi(E_X)) = \varrho_{D-s}(X)$. Hence $D - s$ has an arc $p'q'$ with $p' \in X - W$ and $q' \in X \cap W$. Since $E_{p'}$ contains a critical element e' by Claim 3.7, $(p'q', x_{e'})$ is not admissible by Claim 3.8, that is, there exists a tight set $X' \subseteq V$ with $q' \in X'$ and $p' \notin X'$ such that $x_{e'} \in \mathrm{Span}(\pi(E_{X'}))$. Since $p' \in X - W$, $E_{p'} \subseteq E_X$ and hence $e' \in E_X$. [3, Claim 2.1(a)] says that $X \cap X'$ is tight and $x_{e'} \in \mathrm{Span}(\pi(E_{X \cap X'}))$. Furthermore, $q' \in X \cap X'$, $p' \notin X \cap X'$, and $e' \in E_{p'}$ is critical, contradicting the minimal choice of X, since $p' \in X - X'$. □

3.4 Transversal matroids

The case when $\mathcal{M}$ is transversal can be solved by a completely different idea, by reducing the problem to a packing problem of reachability branchings. For a non-empty set $R \subseteq U$, the subdigraph $T = (U, A')$ of a digraph $D^* = (V^*, A)$ is said to be an R-**branching** if it consists of $|R|$ vertex-disjoint arborescences in D^* whose roots are in R. We say that T is a **reachability R-branching** in D^* if U is the set of reachable vertices from a vertex in R in D^*. The following surprising generalization of Edmonds' theorem was discovered by Kamiyama, Katoh and Takizawa [13].

Theorem 3.9 ([13]). *Let $D^* = (V^*, A^*)$ be a digraph and $\mathcal{R} := \{R_1, ..., R_k\}$ a family of non-empty subsets of V^*. There exits a packing of reachability $\mathcal{R}$-branchings in D^* if and only if*

$$\varrho_{D^*}(X) \geq p_{\mathcal{R}}(X) \text{ for every } \emptyset \neq X \subseteq V^* \tag{3}$$

where $p_{\mathcal{R}}(X)$ denotes the number of R_i's for which $R_i \cap X = \emptyset$ and there exits a path from a vertex in R_i to a vertex in X. □

We prove now that Conjecture 1.3 is true for transversal matroids.

Theorem 3.10. *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph, where $\mathcal{M}$ is a transversal matroid. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D if and only if D is rooted $\mathcal{M}$-arc-connected.*

Proof. Let $G = (S, T; E)$ be a bipartite graph representing $\mathcal{M}$ such that $T = \{1, \ldots, k\}$, where $k = r(\mathcal{M})$, and $\pi : S \to \partial_s(V)$ a bijection. Let $D^* = (V^*, A^*)$ be the digraph that arises from D by splitting s into $|S|$ new vertices of out-degree one. Let r_e denote the tail of e in D^* for each $e \in \partial_s^D(V)$, R^* the set of new vertices r_e and $R_i = \{r_e \in R^* : \pi^{-1}(e) \text{ is adjacent to } i \text{ in } G\}$ for $i \in T$.

Claim 3.11. *Every vertex $v \in V^* - R^* (= V - s)$ is reachable from each R_i in D^*.*

Proof. By rooted $\mathcal{M}$-arc-connectivity, there exist k arc-disjoint paths in D from s to any other vertex v such that the set of their first arcs $\{e_1, \ldots, e_k\}$ is a base of $\mathcal{M}$. As G has a matching covering $\{\pi^{-1}(e_1), \ldots, \pi^{-1}(e_k)\}$ and T, the set $\{r_{e_1}, \ldots, r_{e_k}\}$ intersects R_i for $i = 1, \ldots, k$. □

Claim 3.12. *Condition* (3) *of Theorem 3.9 holds.*

Proof. Let X be a set of vertices in D^*. If X is a subset of R^* then the claim is obvious. Otherwise, let v be a vertex of $X - R^*$. By rooted $\mathcal{M}$-arc-connectivity, there exist an $\mathcal{M}$-based packing of (s, v)-paths $\{P_1, \ldots, P_k\}$ in D. Hence, for every i with $R_i \cap X = \emptyset$, there exists an arc of P_i that enters X in D^*, so by the arc-disjointness of the paths, (3) is satisfied. □

By Claim 3.12 and Theorem 3.9, there exists a packing of reachability $\{R_1, \ldots, R_k\}$-branchings in D^*. By Claim 3.11, each reachability R_i-branching B_i covers $V - s$. By contracting R^* into s, we obtain k pairwise arc-disjoint spanning s-arborescences $T_i = B_i/R^*$ in D. The construction implies that, for each root arc e in T_i, G has an edge between $\pi^{-1}(e)$ and i. Therefore, for each $v \in V$ and for each $i \in \{1, \ldots, k\}$, the root arc in $T_i[s, v]$ is connected to i in G, implying that these root arcs over all i form a base of $\mathcal{M}$. Hence $T_1, \ldots, T_k$ indeed form an $\mathcal{M}$-based packing of spanning s-arborescences. □

3.5 Fano matroid – when D is acyclic

If D is acyclic, the condition (1) for rooted $\mathcal{M}$-arc-connectivity can be significantly simplified as follows.

Lemma 3.13. *Let $(D = (V + s, A), \mathcal{M} = (\partial_s(V), r))$ be a matroid-rooted digraph, where D is acyclic. Then D is rooted $\mathcal{M}$-arc-connected if and only if*

$$\varrho_{D-s}(v) + r(\partial_s(v)) \geq r(\mathcal{M}) \text{ for all } v \in V. \tag{4}$$

Proof. As (4) follows from (1) when $X = \{v\}$, we only prove the sufficiency. Let $X \subseteq V$. As D is acyclic, there exists a vertex v_0 of $D[X]$ with $\varrho_{D[X]}(v_0) = 0$. By the monotonicity of the in-degree and the rank function r and (4) we get

$$\varrho_{D-s}(X) + r(\partial_s(X)) \geq \varrho_{D-s}(v_0) + r(\partial_s(v_0)) \geq r(\mathcal{M})$$

thus (1) follows. □

In view of Lemma 3.13 one can consider the following strategy to prove Conjecture 1.3 for acyclic digraphs. Consider proving Conjecture 1.3 by induction on $|V|$. Without loss of generality we may assume that D is $\mathcal{M}$-independent. Note that in this case (4) is equivalent to saying that each vertex v is of in-degree at least $r(\mathcal{M})$. Since the claim is obvious when $|V| = 0$, we also assume $|V| \geq 1$. As D is acyclic, it has a vertex $v \in V$ with out-degree 0. Let $k = r(\mathcal{M})$. By Lemma 3.13, $D - v$ is rooted $\mathcal{M}|_{\partial_s(V-v)}$-arc-connected and there exist ℓ arcs entering v in $D - s$ for some $0 \leq \ell \leq k$ along with $k - \ell$ root-arcs entering v which are independent in $\mathcal{M}$. By induction, there exists an $\mathcal{M}|_{\partial_s(V-v)}$-based packing of spanning s-arborescences $\{T_1, \ldots, T_k\}$ in $D - v$. Consider extending this packing in $D - v$ to a packing of D. For each non-root-arc $e = uv$ entering v, let $B_e = \{e_{T_i[s,u]} | u \in V(T_i), 1 \leq i \leq k\}$. To extend the packing of $D - v$ to an $\mathcal{M}$-based packing of D, we need to choose one element from B_e for each non-root-arc e entering v such that the chosen elements form a base of $\mathcal{M}$ with the $k - \ell$ root-arcs entering v. The following lemma claims that this is always possible in the Fano matroid.

Lemma 3.14. *Let $B_1, \ldots, B_\ell$ be at most 3 bases of the Fano matroid with a 3-coloring of $\bigcup_{i=1}^{\ell} B_i$ such that each base B_i is colorful for $i = 1, \ldots, \ell$, and let $a_{\ell+1}, \ldots, a_3$ be $3-\ell$ independent elements of the Fano matroid that are not elements of $\bigcup_{i=1}^{\ell} B_i$. Then there is a doubly colorful base of the Fano matroid, that is, one can choose elements $a_i \in B_i$ for $i \in \{1, \ldots, \ell\}$ of different colors such that $\{a_1, a_2, a_3\}$ is a base of the Fano matroid.*

Proof. The statement is obvious when $\ell = 0$ and also when $\ell = 1$ as in the latter case there exists an element of B_1 which is not on the a_2a_3-line of the Fano plane. Similarly, when $\ell = 2$ then we can take any element $a_2 \in B_2$ and an element a_1 of B_1 which is not on the a_2a_3-line. If we have at least two such choice for a_1, then we can chose it to have different color than a_2. Otherwise, the other two elements of B_1 are the elements on the a_2a_3-line different from a_3. Hence a_2 is an element of B_1 and a_1 has a different color than a_2 by the colorfulness of B_1.

Let now $\ell = 3$. The three basis cannot be disjoint, otherwise the Fano matroid should contain 9 distinct elements and it has just has 7 elements. By relabeling the bases, we can assume that $B_1 \cap B_2 \neq \emptyset$. Assume that $B_1 = B_2$. Since B_3 is a base, $B_3 \neq \bigcup\{\mathrm{Span}_{\mathcal{M}}(B_1 - b) - B_1 : b \in B_1\} =: L$ as L is a line. Take $a_3 \in B_3 - L$ and $a_1, a_2 \in B_1 = B_2$ with 3 different colors. Then, as $a_3 \notin L$ and $a_3 \neq a_1$ nor a_2, $\{a_1, a_2, a_3\}$ is a colorful base. Therefore, we can assume $B_1 \neq B_2$. Let $y_1 \in B_1 - B_2$ and $y_2 \in B_2 - B_1$ with the same color and let $x \in B_1 \cap B_2$. Take $a_3 \in B_3$ with the third color. As the line xa_3 may contain only one of y_1 and y_2, we can assume that, say, y_2 is not on this line. Therefore, we can take $a_1 := x$ and $a_2 := y_2$ such that $\{a_1, a_2, a_3\}$ is a colorful base. □

Thus we have the following for the Fano matroid.

Theorem 3.15. *Let $(D, \mathcal{M})$ be a matroid-rooted digraph where $D = (V + s, A)$ is acyclic and $\mathcal{M} = (\partial_s(V), r)$ is a submatroid of the Fano matroid. There exists an $\mathcal{M}$-based packing of spanning s-arborescences in D if and only if D is rooted $\mathcal{M}$-arc-connected.*

4 Negative results

In this section, we will give a counterexample to Conjecture 1.3 and prove that Problem 1.4 is NP-complete for acyclic digraphs and a certain class of matroids. The precise statements are given as follows.

Theorem 4.1. *There exist an acyclic digraph $D = (V + s, A)$ and a matroid $\mathcal{M}$ of rank three such that $(D, \mathcal{M})$ is a counterexample to Conjecture 1.3.*

Theorem 4.2. *Problem 1.4 is NP-complete even if $D = (V + s, A)$ is acyclic and $\mathcal{M}$ is a linear matroid of rank three with a given linear representation.*

As we noted before, the matroid $\mathcal{M}$ used in the construction, that we call a **parallel extension of the Fano matroid**, will arise from the Fano matroid by adding some parallel copies of its elements.

The proof is done by defining several gadget constructions, each of which restricts possible packings. Each construction step is referred to as an **operation** below. In each construction, we insert new vertices one by one together with three new arcs entering it and no arc leaving it. A new root arc will always be added keeping the $\mathcal{M}$-independence as well as the fact that $\mathcal{M}$ is a parallel extension of the Fano matroid (or its submatroid). Thus, an instance $(D = (V + s, A), \mathcal{M})$ constructed by a sequence of operations always satisfies the following properties:

(i) D is acyclic and, by Lemma 3.13, $(D, \mathcal{M})$ is rooted $\mathcal{M}$-arc-connected;

(ii) If $(D, \mathcal{M})$ is constructed from $(D', \mathcal{M}')$ by an operation, then every feasible packing for $(D, \mathcal{M})$ is an extension of a feasible packing for $(D', \mathcal{M}')$.

By the property (i), in the subsequent discussion we omit to mention that $(D, \mathcal{M})$ is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected. By using the property (ii), we shall be able to control possible extensions of feasible packings.

We say that a vertex $v \in V$ **gets** a base B in a feasible packing $\{T_1, T_2, T_3\}$ if $B = \{e_{T_1[s,v]}, e_{T_2[s,v]}, e_{T_3[s,v]}\}$. We also say that v **gets** $e_{T_i[s,v]}$ **from** u if u is on the path $T_i[s, v]$ $(i = 1, 2, 3)$. T_1, T_2 and T_3 will be called the red, blue and black arborescences, respectively. We say that an element of $\mathcal{M}$ is **colored** by λ if it is in the arborescence of color λ.

In the following, the elements of $\mathcal{M}$ will be denoted by the first 7 letters of the alphabet (see Figure 1) and apostrophes, superscripts (when we would need too many apostrophes) or subscripts will be used when we consider a parallel element of a previously used one (that may be also an identical element to this previous one). It is well-known that the Fano plane have automorphisms moving arbitrary 3 points in general position (that is not lying on a line) to any 3 points in general position.

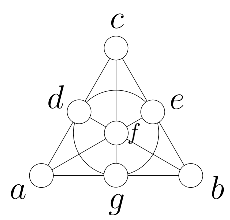
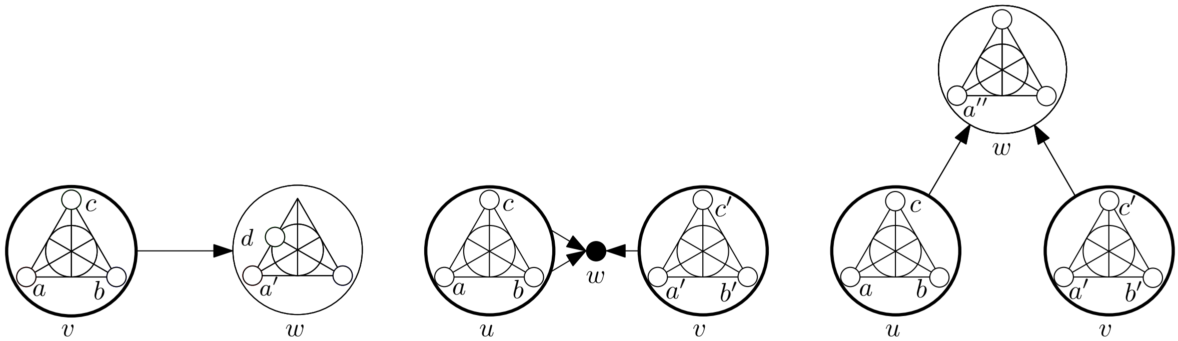


Figure 1: The elements of the Fano matroid.

Each operation is best described with figures, which are illustrated by the following rule. (See, e.g., Figure 2a.) The root vertex s is not represented in the figures. A vertex will be represented



(a) $FC_{(a,b,c)}(v)$. (b) $ADC(u,v)$. (c) $AF_a(u,v)$.

Figure 2: The three elementary operations.

as a big circle in which Fano plane is illustrated with three particular elements (empty circles) which represent the base that the vertex will get in every feasible packing. Existing vertices in the original digraph will be denoted by thicker circles, in which the elements of the bases that they get in every feasible packing will be signed by their letters. For a vertex v which is added in an operation, a letter x may be assigned to a point in the Fano plane, which means that a new root arc sv is added with a new element x in the underlying matroid. Sometimes a new vertex will be represented by just a point for simplicity.

Operation 4.3. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing.* **Force-color $\boldsymbol{FC_{(a,b,c)}(v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with 2 incoming root arcs a' and d and one non-root arc vw, where $a' \parallel a$ and $\{a, c, d\}$ is a line of the Fano plane. See Figure 2a.*

Note that, by the automorphisms of the Fano plane, $FC_{(x,y,z)}(v)$ is also defined for any base $\{x, y, z\}$ (and the same remark is applied for other operations given below).

Lemma 4.4. *With the notation as in Operation 4.3, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$. Moreover, in every feasible packing of $(D', \mathcal{M}')$, w gets the base $\{a', b, d\}$, that is, the arc vw will be in the same arborescence as the root arc b.*

Proof. Consider any possible extension of a feasible packing of $(D, \mathcal{M})$, where we distribute the three arcs entering w among the three arborescences. From the construction, w always gets a' and d from the root. Also, by the assumption of the lemma, v gets $\{a, b, c\}$, and w gets one of them from v. Now, in a feasible extension, w cannot get a from v as $a' \parallel a$ and cannot get c as $\{a', c, d\}$ is a line. Hence w gets b from v and the packing is feasible as $\{a', b, d\}$ is a base. □

For simplicity, we also use $FC_{(a,b,c)}(v)$ to denote the new vertex w in Operation 4.3.

Operation 4.5. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Avoid-different-coloring $\boldsymbol{ADC(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with two parallel arcs from u to w and an arc from v to w. See Figure 2b.*

Lemma 4.6. *With the notation as in Operation 4.5, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where all the parallel pairs (a, a'), (b, b') and (c, c') have different colors.*

Proof. By symmetry, we may assume without loss of generality, that w gets a and b from u in a feasible packing in $(D', \mathcal{M}')$. Then w should get c' from v, which is possible if and only if the color of c' is equal to that of c. Thus the claim follows as any feasible packing in $(D', \mathcal{M}')$ is an extension of that in $(D, \mathcal{M})$. $\square$

For simplicity, we use $ADC(u, v)$ to denote the new vertex w in Operation 4.5.

Operation 4.7. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Avoid-flip $\boldsymbol{AF_a(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w along with an incoming root arc a'', an arc from u to w and an arc from v to w to D. See Figure 2c.*

Lemma 4.8. *With the notation as in Operation 4.7, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where a and a' have the same color and the colors of the pairs (b, b') and (c, c') are different.*

Proof. First we prove that feasible packings in the exceptional case of AF cannot be extended. The vertex w can get either the base $\{a'', b, c'\}$ or the base $\{a'', b', c\}$. However, both contain two elements of the same color, which is impossible.

Next observe that in the non-exceptional case of AF, either b and c' or b' and c are of different colors, say b and c'. Let us color a'' by the color not used by b and c'. Then w can get the base $\{a'', b, c'\}$ that uses the three colors. $\square$

By the previous operations, we define the following operation. For bases $\{a, b, c\}$ and $\{x, y, z\}$ in the Fano plane with parallel extension, we denote by $\{a, b, c\} \parallel \{x, y, z\}$ if each element in $\{a, b, c\}$ is parallel to some element in $\{x, y, z\}$.

Operation 4.9. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{x, y, z\}$ in every feasible packing, respectively, such that $a \nparallel x$ and $\{a, b, e\} \parallel \{x, y, k\}$, where $\{b, e, c\}$ and $\{y, k, z\}$ are lines of the Fano plane.* **Forbid-same-color $\boldsymbol{FSC_{(a,x)}(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 4 new vertices to D and 5 new elements to $\mathcal{M}$ as follows. Add $w_1 := FC_{(b,a,c)}(u)$ with new root arcs b'' and e, $w_2 := FC_{(y,x,z)}(v)$, $w_3 := ADC(w_1, w_2)$, and $w_4 := AF_t(w_1, w_2)$, where t denotes the element with $t \in \{b'', e\}$ and $t \nparallel x$. See Figure 3 for a possible configuration of FSC.*

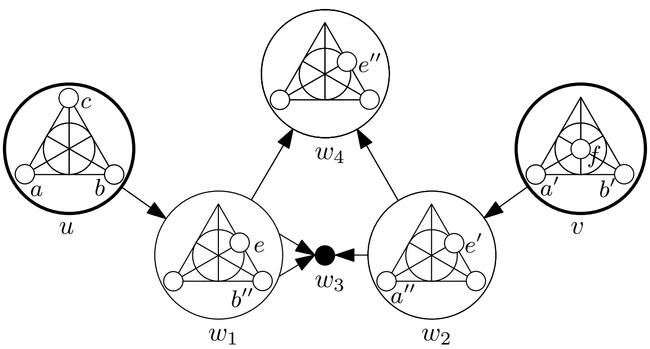


Figure 3: The operation $FSC_{(a,b')}(u, v)$, where $(x, y, z) = (b', a', f)$ and $t = e$.

One can see other examples of FSC in Figure 4, $FSC_{(d,g)}(v, w_2)$ and $FSC_{(f,a''')}(w_1, w_2)$, where (a, b, c, x, y, z, t) in Operation 4.9 corresponds to $(d, a', b', g, a''', c, a^{(4)})$, and $(f, a', b'', a''', g, c, g'')$, respectively.

Lemma 4.10. *With the notation as in Operation 4.9, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where the colors of a and x are the same.*

Proof. By $a \nparallel x$ and $\{x, y, k\} \parallel \{a, b, e\}$, we have $\{a, b, e\} \parallel \{x, y, k\} \parallel \{a, x, t\}$.

First we prove that feasible packings in the exceptional case of FSC cannot be extended. Suppose that a and x are of the same color. By Lemma 4.4, each of w_1 and w_2 gets a base parallel to $\{a, x, t\}$. Since a in w_1 and x in w_2 are of the same color, by Lemma 4.6, the element parallel to t should be of the same color at w_1 and w_2. However, by Lemma 4.8, the element parallel to t should be of different colors at w_1 and w_2, which is a contradiction.

Now in the non-exceptional case a and x have different colors, say red and black. Then, by Lemma 4.4, each of w_1 and w_2 may get a base parallel to (a, x, t) with colors (red, black, blue). By Lemmas 4.6 and 4.8, the packing extends to w_3 and w_4. □

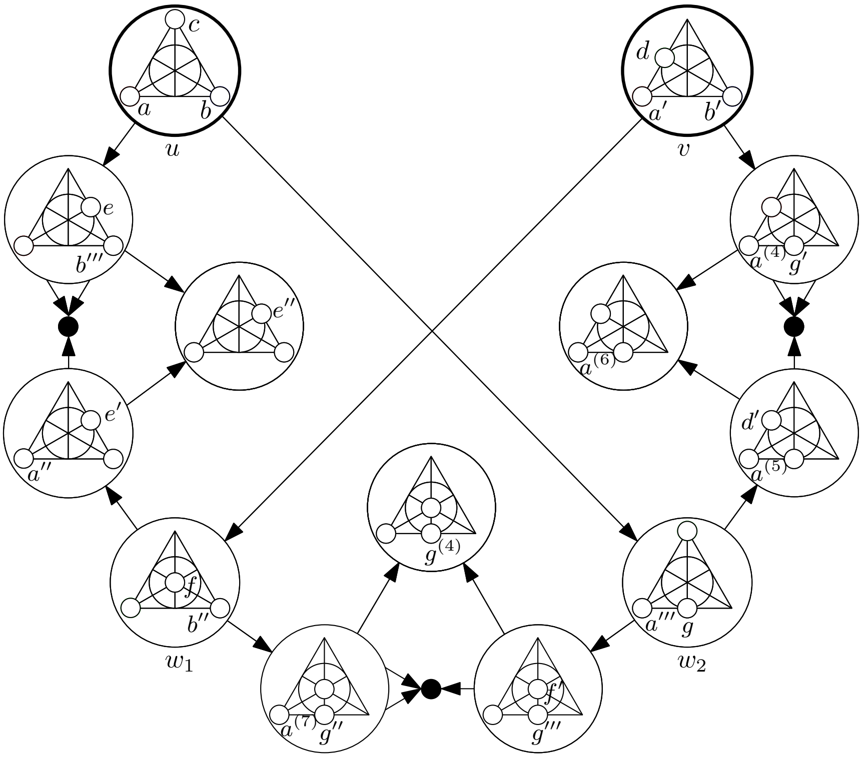


Figure 4: The operation $SAF_{(a,b,c)}(u, v)$.

The main operation is the following.

Operation 4.11. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', d\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $\{a, d, c\}$ is a line of the Fano plane.* **Strong-avoid-flip $\boldsymbol{SAF_{(a,b,c)}(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 14 new vertices to D and 19 new elements to $\mathcal{M}$ as follows. First, add 2 new vertices to D and 4 new elements to $\mathcal{M}$ by $w_1 := FC_{(b',a',d)}(v)$ (with new root arcs b'' and f) and $w_2 := FC_{(a,c,b)}(u)$ (with new root arcs a''' and g). Then add the remaining new vertices of D' and new elements of $\mathcal{M}'$ by the operations $FSC_{(a,b'')}(u, w_1)$, $FSC_{(d,g)}(v, w_2)$ and $FSC_{(f,a''')}(w_1, w_2)$. See Figure 4.*

Lemma 4.12. *With the notation as in Operation 4.11, a feasible packing in $(D, \mathcal{M})$, where b and b' have the same color, extends to a feasible packing in $(D', \mathcal{M}')$ if and only if the colors of the pairs (a, a') and (c, d) are the same.*

Proof. By relabeling the colors we may assume that the base (a, b, c) that v gets is colored by $(\text{red}, \text{blue}, \text{black})$.

First, suppose that (a', b', d) is colored by $(\text{black}, \text{blue}, \text{red})$. w_1 gets the base (a', b'', f) that, by Lemma 4.10 applied for $FSC_{(a,b'')}(u, w_1)$, cannot be colored by (black, red, blue), so, by Lemma 4.4, it is colored by (black, blue, red). Similarly, w_2 gets the base (a''', c, g) that, by Lemma 4.10 applied for $FSC_{(d,g)}(v, w_2)$, cannot be colored by (blue, black, red), so, by Lemma 4.4, it is colored by (red, black, blue). Finally, since f and a''' are red, Lemma 4.10 applied for $FSC_{(f,a''')}(w_1, w_2)$ shows that the packing cannot be extended.

Second, suppose that (a', b', d) is colored by $(\text{red}, \text{blue}, \text{black})$. w_1 gets the base (a', b'', f) and w_2 gets the base (a''', g, c) and both can be colored, by Lemma 4.4, by $(\text{red}, \text{blue}, \text{black})$. Lemma 4.10 applied for $FSC_{(a,b'')}(u, w_1)$, $FSC_{(d,g)}(v, w_2)$ and $FSC_{(f,a''')}(w_1, w_2)$ shows that the packing can be extended. $\square$

We are now ready to prove Theorem 4.1.

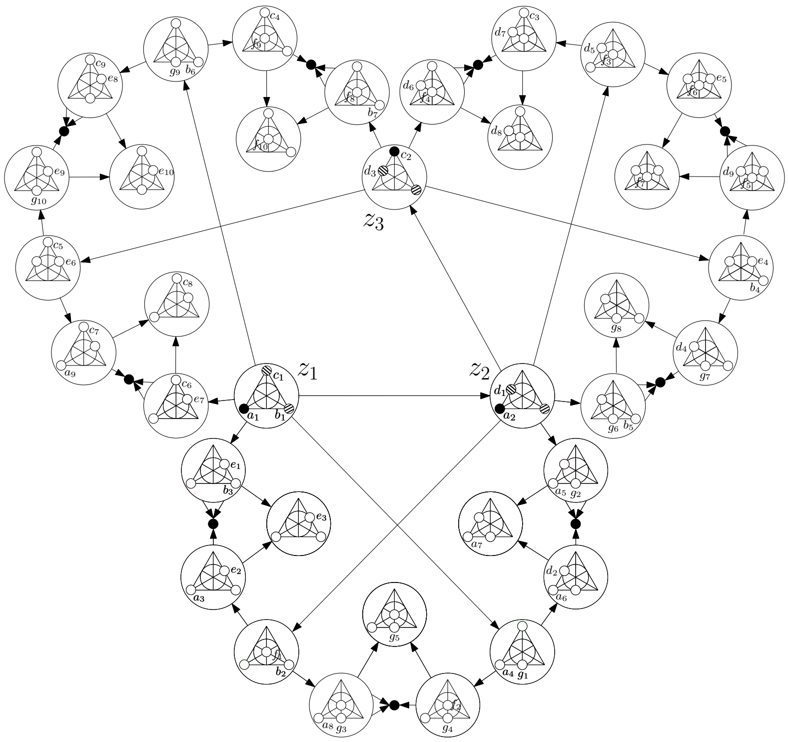


Figure 5: The counterexample shown in the proof of Theorem 4.1.

Proof of Theorem 4.1. We start with a digraph on two vertices, a root s and the other vertex z_1, along with 3 parallel arcs a_1, b_1 and c_1 from s to v. The underlying matroid is the free matroid on $\partial_s(v)$. In the following, the arborescences containing a_1, b_1 and c_1 will be called red, blue and black, respectively. First, add new vertices $z_2 := FC_{(a_1,b_1,c_1)}(v)$ (which gets $\{a_2, b_1, d_1\}$), $z_3 := FC_{(d_1,b_1,a_2)}(w_1)$ (which gets $\{b_1, c_2, d_3\}$). Then apply the operations $SAF_{(a_1,b_1,c_1)}(z_1, z_2)$, $SAF_{(d_1,b_1,a_2)}(z_2, z_3)$ and $SAF_{(c_2,b_1,d_3)}(z_3, z_1)$. See Figure 5. Applying Lemmas 4.4 and 4.12 twice shows that the base (a_2, b_1, d_1) that z_2 gets is colored by $(\text{red}, \text{blue}, \text{black})$, the base (b_1, c_2, d_3) that z_3 gets is colored by $(\text{blue}, \text{red}, \text{black})$. Finally, by Lemma 4.12, no feasible packing exists in the resulting instance. By Lemma 3.13, the resulting instance is rooted $\mathcal{M}$-arc-connected, and hence is a counterexample to Conjecture 1.3. This completes the proof of Theorem 4.1. $\square$

Now we turn to the proof of Theorem 4.2. Problem 1.4 is in NP in the case where a linear representation of the matroid is given as input since the packing itself is a witness for the problem that can be checked in polynomial time. We will use the well-known 3-SAT (see [12]) to prove the NP-completeness of our problem.

Let us take a 3-CNF formula. Using the previous operations (and a new one) we will construct a matroid-rooted digraph that has a feasible packing if and only if the formula is satisfiable. In order to express each clause, our idea is to represent it as a concatenation of majority functions and implement each majority function by using our operations. We first remark the following lemma. Recall that the majority function $\mathsf{maj}(\alpha, \beta, \gamma)$ is a Boolean function that has a value 1 if and only if at least two among α, β, γ have value 1.

Lemma 4.13. *Let $\alpha, \beta, \gamma \in \{0, 1\}$. Then*

$$\alpha \vee \beta \vee \gamma = \mathsf{maj}(\mathsf{maj}(\alpha, \beta, 1), \mathsf{maj}(\alpha, \gamma, 1), \mathsf{maj}(\beta, \gamma, 1)). \tag{5}$$

Proof. $\alpha \vee \beta \vee \gamma = 1$ if and only if at least one of α, β and γ is 1. If, say, $\alpha = 1$, then $\mathsf{maj}(\alpha, \beta, 1) = 1$ and $\mathsf{maj}(\alpha, \gamma, 1) = 1$ hence the right hand side of (5) is 1. If $\alpha = \beta = \gamma = 0$, then $\mathsf{maj}(\alpha, \beta, 1) = \mathsf{maj}(\alpha, \gamma, 1) = \mathsf{maj}(\beta, \gamma, 1) = 0$ hence the right hand side of (5) is 0. □

Operation 4.14. *Given $(D, \mathcal{M})$, suppose that $v_1, v_2, v_3 \in V$ get the bases $\{a, b, c\}$, $\{a', b', c'\}$ and $\{a'', b'', c''\}$, respectively, in every feasible packing where $a \parallel a' \parallel a''$, $b \parallel b' \parallel b''$ and $c \parallel c' \parallel c''$.* **Majority $\boldsymbol{MAJ(v_1, v_2, v_3)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w with 3 incoming arcs v_1w, v_2w and v_3w. See Figure 6.*

Lemma 4.15. *With the notation as in Operation 4.14, consider a feasible packing of $(D, \mathcal{M})$ such that all of b, b' and b'' are colored by λ (and hence there are only two types of possible coloring schemes on each v_i). Then the packing extends to a feasible packing of $(D', \mathcal{M}')$. Moreover, in every such extension w gets a base formed by parallel copies of a, b, and c with a coloring of the same type as the majority among the three on v_1, v_2 and v_3. See Figure 6.*

Proof. Without loss of generality, we can assume that the colorings of (a, b, c) and (a', b', c') coincide, say, they are colored by (red, blue, black). As w has an entering arc from each v_i, w always gets a parallel copy of b colored by blue. Moreover, as w has in-arcs from v_1 and v_2 too, w gets a parallel copy of a or c from v_1 or v_2. Hence w gets a parallel copy of a colored by red or a parallel copy of c colored by black. These two facts already determine the coloring scheme on w as stated in the lemma. □

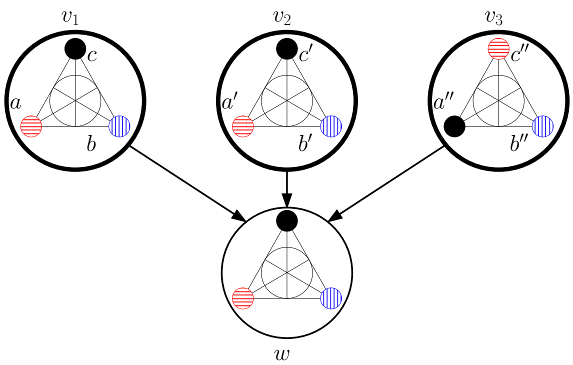


Figure 6: $MAJ(v_1, v_2, v_3)$.

Two more operations are needed in the NP-completeness proof.

Operation 4.16. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Copy-one-color $\boldsymbol{COC_b(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 3 new vertices to D and 2 new elements to $\mathcal{M}$ by operations $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$.*

Lemma 4.17. *With the notation as in Operation 4.16, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$ except those where the colors of b and b' are different.*

Proof. Note that any feasible packing of $(D, \mathcal{M})$ satisfies either one of the following: (i) each pair in (a, a'), (b, b'), and (c, c') has the same color; (ii) all the pairs (a, a'), (b, b'), and (c, c') have different colors; (iii) only (a, a') has the same color; (iv) only (b, b') has the same color; (v) only (c, c') has the same color. By $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$, the packing is extendable if and only if (i) or (iv) holds, meaning that (b, b') has the same color. $\square$

Operation 4.18. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing.* **Change-colors $\boldsymbol{CC_{(a,c)}(v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 45 new vertices to D and 63 new elements to $\mathcal{M}$ as follows. First, add new vertices $w_1 := FC_{(a,b,c)}(v)$ (which gets $\{a', b, d\}$), $w_2 := FC_{(d,b,a')}(w_1)$ (which gets $\{b, c', d'\}$) and $w := FC_{(c',b,d')}(w_2)$ (which gets $\{a'', b, c''\}$). Then add the remaining new vertices of D' and new elements of $\mathcal{M}'$ by the operations $SAF_{(a,b,c)}(v, w_1)$, $SAF_{(d,b,a')}(w_1, w_2)$ and $SAF_{(c',b,d')}(w_2, w)$.*

Lemma 4.19. *With the notation as in Operation 4.18, every feasible packing in $(D, \mathcal{M})$ extends to a feasible packing in $(D', \mathcal{M}')$. Moreover, if the base $\{a, b, c\}$ that v gets is colored by $(\lambda_1, \lambda_2, \lambda_3)$, then the base $\{a'', b, c''\}$ that w gets is colored by $(\lambda_3, \lambda_2, \lambda_1)$.*

Proof. By relabeling the colors we may assume that the base (a, b, c) that v gets is colored by (red, blue, black). Applying Lemmas 4.4 and 4.12 three times shows that the base (a', b, d) that w_1 gets is colored by (red, blue, black), the base (b, c', d') that w_2 gets is colored by (blue, red, black) and the base (a'', b, c'') that w gets is colored by (black, blue, red). $\square$

For simplicity, we denote $w = CC_{(a,c)}(v)$ if w is as in Operation 4.18.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We have seen that the problem is in NP, hence we only prove the completeness. Let us take a 3-CNF formula on variables $x_1, x_2, \ldots, x_n$. First, let $V := \{v_0, \ldots, v_n\}$ and take a digraph D on $V + s$ whose arc set consists of only root arcs sv_i ($i = 0, \ldots, n$), three copies of each. Take a base $\{a, b, c\}$ of the Fano matroid and define a parallel extension of the Fano matroid $\mathcal{M}$ on $\partial_r(V)$ such that, for each $i \in \{0, \ldots, n\}$, the three arc sv_i form a parallel copy $\{a_i, b_i, c_i\}$ of $\{a, b, c\}$. Next use operation $COC_{b_{i-1}}(v_{i-1}, v_i)$ for $i = 1, \ldots, n$. This ensures that in every feasible packing the parallel copies of b got by $v_0, \ldots, v_n$ are colored by the same color, say, blue.

Add $v'_1, \ldots, v'_n$ by using operations $CC_{(a_i,c_i)}(v_i)$ for $i = 1, \ldots, n$. Hence, in every feasible packing, v'_i gets the colored base (a'_i, b_i, c'_i) with the same coloring as (c_i, b_i, a_i) for $i = 1, \ldots, n$. In the following construction, v_i will represent the variable x_i and v'_i its negate $\bar{x}_i$ for $i = 1, \ldots, n$. Moreover, v_0 will represent 1.

For each clause ψ of the formula, we first add 4 new vertices w_1^ψ, w_2^ψ, w_3^ψ and w_4^ψ using operation MAJ so that it represents ψ according to the equation in Lemma 4.13. (In other words, for a clause, say, for $\psi = x_1 \vee \bar{x}_2 \vee x_3$ we add w_1^ψ with arcs $v_1 w_1^\psi$, $v'_2 w_1^\psi$ and $v_0 w_1^\psi$, w_2^ψ with arcs $v_1 w_2^\psi$, $v_3 w_2^\psi$ and $v_0 w_2^\psi$, w_3^ψ with arcs $v'_2 w_3^\psi$, $v_3 w_3^\psi$ and $v_0 w_3^\psi$, and w_4^ψ with arcs $w_1^\psi w_4^\psi$, $w_2^\psi w_4^\psi$ and $w_3^\psi w_4^\psi$.) Finally, to ensure the truth of each clause ψ, we further use operation $AF_{b_0}(v_0, w_4^\psi)$. See Figure 7.

We claim that the formula is satisfiable if and only if $(D, \mathcal{M})$ admits a feasible packing. Note that v_0 always gets the base $\{a_0, b_0, c_0\}$, and without loss of generality we may always suppose that (a_0, b_0, c_0) is colored by (red, blue, black). Then the claim follows by identifying the coloring scheme (red, blue, black) (resp., (black, blue, red)) for a parallel copy of (a, b, c) with a true assignment (resp., a false assignment).

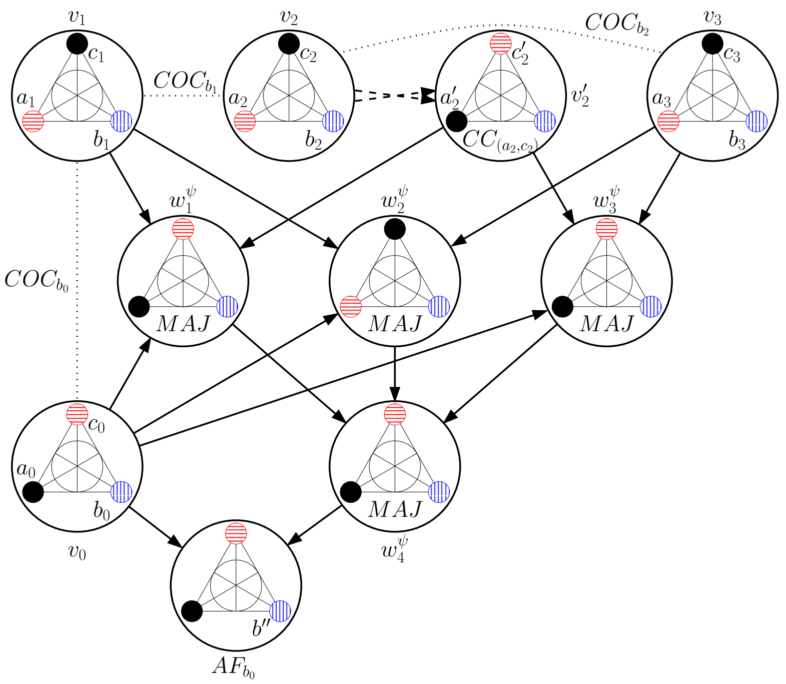


Figure 7: A part of the construction in the proof of Theorem 4.2.This demonstrates how the assignment $x_1 = x_2 = x_3 = 0$ makes the clause $\psi = x_1 \vee \bar{x}_2 \vee x_3$ true in the corresponding feasible packing. The crossing dashed arcs represent the operation CC and the dotted edges represent the operation COC.

More formally, suppose that the formula has a true assignment. Then, we first construct a feasible packing restricted on $\{s, v_0, v_1, \ldots, v_n\}$ such that v_0 gets the base (a_0, b_0, c_0) colored by $(\text{red}, \text{blue}, \text{black})$ and each v_i $(1 \le i \le n)$ gets the base (a_i, b_i, c_i) colored by $(\text{red}, \text{blue}, \text{black})$ if $x_i = 1$ and by $(\text{black}, \text{blue}, \text{red})$ if $x_i = 0$. By Lemma 4.19, this packing always extends on $\{v_1', \ldots, v_n'\}$ such that each v_i' gets a base formed by parallel copies of a, b, and c colored by black, blue, and red, respectively, if $x_i = 1$ and by red, blue, and black, respectively, if $x_i = 0$. Since the assignment satisfies the formula, Lemmas 4.15 and 4.8 imply that the packing is extendable to a feasible packing on the whole vertex set of D.

Conversely, if $(D, \mathcal{M})$ has a feasible packing, then by $COC_{b_{i-1}}(v_{i-1}, v_i)$, b_i is colored by blue on each v_i. We set x_i in such a way that $x_i = 1$ if and only if (a_i, b_i, c_i) is colored by $(\text{red}, \text{blue}, \text{black})$ (as in (a_0, b_0, c_0)). By $CC_{(a_i,c_i)}(v_i)$, coloring of (a_i', c_i') is the reverse of the coloring of (a_i, c_i). Moreover, since $AF_{b_0}(v_0, w_4^\psi)$ is used for each clause ψ, the base on w_4^ψ has the same coloring scheme as that of $\{a_0, b_0, c_0\}$ on v_0 by Lemma 4.8. Thus, by Lemma 4.15, the formula is satisfied for this assignment. □

5 Concluding remarks

All the results presented here have undirected and hypergraphic counterparts. To get an undirected counterpart of our positive results for rank-2, graphic or transversal matroids, one can use [3, Corollary 1.1] and the proof after that. This extends a result of Katoh and Tanigawa [14] on these fundamental matroid classes. Moreover, with the techniques of [5], we also have extensions of these results for dypergraphs (that is, oriented hypergraphs), hypergraphs and mixed hypergraphs.

On the other hand, Problem 1.3 is NP-complete for dypergraphs as it is NP-complete for digraphs. Also, the proof of the NP-completeness can be applied even for the undirected case.

This is because that in the construction of the NP-completeness we only add vertices with in-degree 3 one by one, and hence the ordering of the vertex addition prescribes the orientation of each edge in a rooted-tree packing.

A challenging open problem is to give a complete characterization of the class of matroids for which Conjecture 1.3 is true. A much easier but still interesting question is whether one can abstract our proof technique for graph matroids to solve wider classes of matroids such as regular matroids.

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6 APPENDIX – Negative results (original version)

In Revision 1, we changed the proof of the negative results hence we also present the original proof. Note that several operations with the same name will be defined differently here.

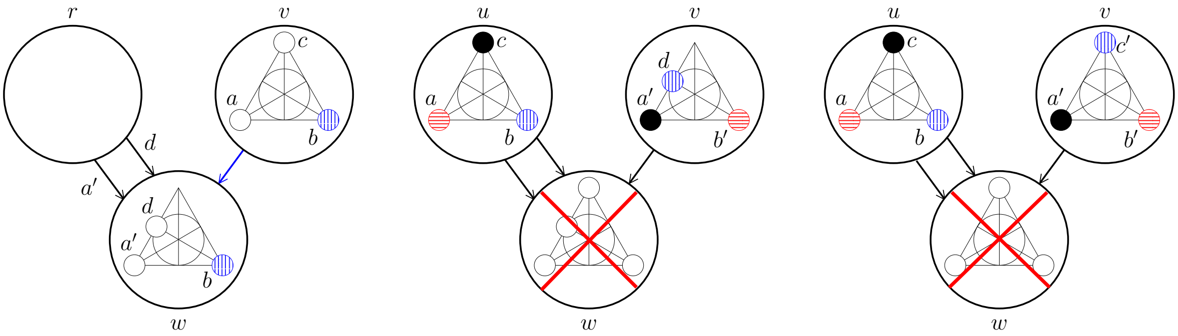
In this section, we will give a counterexample to Conjecture 1.3 and prove that Problem 1.4 is NP-complete for acyclic digraphs and a certain class of matroids. The precise statements are given as follows.

Theorem 6.1. *There exist an acyclic digraph $D = (V + s, A)$ and a matroid $\mathcal{M}$ of rank three such that $(D, \mathcal{M})$ is a counterexample to Conjecture 1.3.*

Theorem 6.2. *Problem 1.4 is NP-complete even if $D = (V + s, A)$ is acyclic and $\mathcal{M}$ is a linear matroid of rank three with a given linear representation.*

As we noted before, the matroid $\mathcal{M}$ used in the construction, that we call a **parallel extension of the Fano matroid**, will arise from the Fano matroid by adding some parallel copies of its elements.

The proof is done by defining several gadget constructions, each of which restricts possible packings. Each construction step is referred to as an **operation** below, and we shall define several distinct operations. In each construction, we insert new vertices one by one together with three new arcs entering it. A new root arc will always be added keeping the $\mathcal{M}$-independence as well as the fact that $\mathcal{M}$ is a parallel extension of the Fano matroid (or its submatroid). Thus, $D = (V + s, A)$ is always acyclic and, by Lemma 3.13, the resulting instance $(D, \mathcal{M})$ will be rooted $\mathcal{M}$-arc-connected. Hence in the subsequent discussion we omit to mention that $(D, \mathcal{M})$ is $\mathcal{M}$-independent and rooted $\mathcal{M}$-arc-connected.



(a) $FC_{(a,b,c)}(v)$. (b) $AC(u,v)$. (c) $ADC(u,v)$.

Figure 8: The three elementary operations.

We say that a vertex $v \in V$ **gets** a base B in a feasible packing $\{T_1, T_2, T_3\}$ if $B = \{e_{T_1[s,v]}, e_{T_2[s,v]}, e_{T_3[s,v]}\}$. We also say that v **gets** $e_{T_i[s,v]}$ **from** u if u is on the path $T_i[s,v]$ $(i = 1, 2, 3)$. T_1, T_2 and T_3 will be called the red, blue and black arborescences, respectively. We say that an element of $\mathcal{M}$ is **colored** by λ if it is in the arborescence of color λ. In the following, the elements of $\mathcal{M}$ will be denoted by the first 7 letters of the alphabet and apostrophes will be used when we consider a parallel element of a previously used one (that may be also an identical element to this previous one).

We also remark that, as we will always extend a digraph by adding a vertex of out-degree zero one by one, every feasible packing of the resulting digraph is an extension of a feasible packing of the original digraph. By using the following operations, we shall control possible extensions of packings.

Operation 6.3. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing.* **Force-color** $\boldsymbol{FC_{(a,b,c)}(v)}$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with 2 incoming root arcs a' and d and one non-root arc vw where $a' \parallel a$ and $\{a, c, d\}$ is a line of the Fano plane. See Figure 8a.*

Lemma 6.4. *With the notation as in Operation 6.3, every feasible packing of D extends to a feasible packing of D' such that w gets the base $\{a', b, d\}$, that is, the arc vw will be in the same arborescence as the root arc b.*

Proof. Consider any possible extension of a feasible packing of D, where we distribute the three arcs entering w among the three arborescences. From the construction, w always gets a' and d from the root. Also, by the assumption of the lemma, v gets $\{a, b, c\}$, and w gets one of them from v. Now, in a feasible extension, w cannot get a from v as $a' \parallel a$ and cannot get c as $\{a', c, d\}$ is a line. Hence w gets b from v and the packing is feasible as $\{a', b, d\}$ is a base. □

For simplicity, we also use $FC_{(a,b,c)}(v)$ to denote the new vertex w in Operation 6.3.

Operation 6.5. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', d\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $\{a, c, d\}$ is a line.* **Avoid-coloring** $\boldsymbol{AC(u,v)}$ *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with two parallel arcs from u to w and an arc from v to w. See Figure 8b.*

Lemma 6.6. *With the notation as in Operation 6.5, every feasible packing of D extends to a feasible packing of D' except those where (a, b, c) is colored by $(\lambda_1, \lambda_2, \lambda_3)$ and (a', b', d) is colored by $(\lambda_3, \lambda_1, \lambda_2)$ for some distinct three colors $\lambda_1, \lambda_2, \lambda_3$.*

Proof. Suppose that (a, b, c) is colored by $(\lambda_1, \lambda_2, \lambda_3)$ in a feasible packing of D, and consider extending it. If the set of colors of $\{a', d\}$ is not equal to $\{\lambda_2, \lambda_3\}$, the packing is extendable since w can get b and c from u, and a' or d from v. If the color of d is equal to λ_3, then the packing is extendable since w can get a and b from u, and d from v. Combining these two facts, the packing is extendable if (a', b', d) is not colored by $(\lambda_3, \lambda_1, \lambda_2)$. Conversely, if (a', b', d) is colored by $(\lambda_3, \lambda_1, \lambda_2)$, then the packing is not extendable as w cannot get a base formed by three differently colored elements as neither $\{a, b, a'\}$, $\{a, c, d\}$, nor $\{b, c, b'\}$ is a base. See Figure 8b. □

We use $AC(u, v)$ to denote the new vertex w in Operation 6.5.

Operation 6.7. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Avoid-different-coloring $\boldsymbol{ADC(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w to D along with two parallel arcs from u to w and an arc from v to w. See Figure 8c.*

Lemma 6.8. *With the notation as in Operation 6.7, every feasible packing in D extends to a feasible packing in D' except those where all the parallel pairs (a, a'), (b, b') and (c, c') have different colors.*

Proof. By symmetry, we may assume without loss of generality, that w gets a and b from u in a feasible packing in D'. Then w should get c' from v, which is possible if and only if the color of c' is equal to that of c. Thus the claim follows as any feasible packing in D' is an extension of that in D. □

For simplicity, we use $ADC(u, v)$ to denote the new vertex w in Operation 6.7.

Operation 6.9. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Avoid-flip $\boldsymbol{AF_a(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding 5 new vertices $w_1, \dots, w_5$ to D and 4 new elements to $\mathcal{M}$ by $w_1 := FC_{(a',b',c')}(v)$ (with new root arcs a'' and d), $w_2 := AC(w_1, u)$, $w_3 := FC_{(c,b,a)}(u)$ (with new root arcs d' and c''), $w_4 := AC(v, w_3)$, $w_5 := AC(w_1, w_3)$. See Figure 9.*

Lemma 6.10. *With the notation as in Operation 6.9, every feasible packing in D extends to a feasible packing in D' except those where a and a' have the same color and the colors of the pairs (b, b') and (c, c') are different.*

Proof. First we prove that feasible packings in the exceptional case cannot be extended. Assume for a contradiction that, say, a and a' are colored by red, b and c' are colored by blue, and b' and c are colored by black. Then by FC, in the base that w_1 gets, b' is also black. Moreover, as w_2 is an AC-vertex, (a'', b', d) cannot be colored by $(\text{blue}, \text{black}, \text{red})$ hence it is colored by $(\text{red}, \text{black}, \text{blue})$. By FC, we know that b is colored by blue in the base that w_3 gets. However, by the two AC-vertices w_4 and w_5, we know that this base (b, c'', d') can neither be colored by $(\text{blue}, \text{red}, \text{black})$ nor by $(\text{blue}, \text{black}, \text{red})$, a contradiction. See Figure 9.

Next observe that it is obvious that the feasible packing can be extended to D' when b and b' have the same color as in the exceptional case of AC the colors of the parallel elements, that the two input vertices get, are different. The remaining cases are solved in Figure 10. □

By the previous two operations, we get the following.

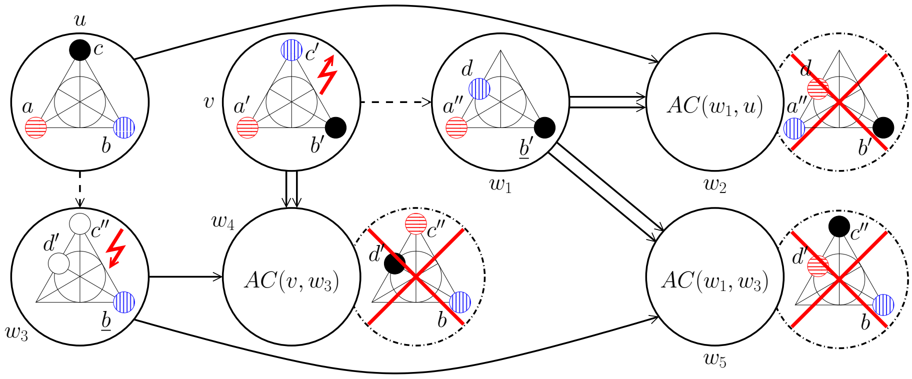


Figure 9: The proof of $AF_a(u, v)$ for the exceptional case. A dashed arc represents an FC operation where the forced element is underlined.

Operation 6.11. *Given $(D, \mathcal{M})$, suppose that vertices $u, v \in V$ get the bases $\{a, b, c\}$ and $\{a', b', c'\}$ in every feasible packing, respectively, where $a' \parallel a$, $b' \parallel b$ and $c' \parallel c$.* **Copy-one-color $\boldsymbol{COC_b(u, v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding $1 + 2 \cdot 5 = 11$ new vertices to D and $2 \cdot 4 = 8$ new elements to $\mathcal{M}$ by operations $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$.*

Lemma 6.12. *With the notation as in Operation 6.11, every feasible packing in D extends to D' except those where the colors of b and b' are different.*

Proof. Note that any feasible packing of D satisfies either one of the following: (i) each pair in (a, a'), (b, b'), and (c, c') has the same color; (ii) all the pairs (a, a'), (b, b'), and (c, c') have different colors; (iii) only (a, a') has the same color; (iv) only (b, b') has the same color; (v) only (c, c') has the same color. By $ADC(u, v)$, $AF_a(u, v)$, and $AF_c(u, v)$, the packing is extendable if and only if (i) or (iv) holds, meaning that (b, b') has the same color. □

The main operation is the following.

Operation 6.13. *Given $(D, \mathcal{M})$, suppose that a vertex $v \in V$ gets the base $\{a, b, c\}$ in every feasible packing.* **Strong-force-coloring $\boldsymbol{SFC_{(a,b,c)}(v)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding $9 + 2 \cdot 11 + 5 + 1 = 37$ new vertices to D and $9 \cdot 2 + 2 \cdot 8 + 4 = 38$ new elements to $\mathcal{M}$ as follows. First, add 9 new vertices to D and $9 \cdot 2$ new elements to $\mathcal{M}$ by $w_1 := FC_{(a,b,c)}(v)$ (with new root arcs a' and d), $w_2 := FC_{(b,a',d)}(w_1)$ (with new root arcs b' and f), $w_3 := FC_{(b,a,c)}(v)$ (with new root arcs b'' and e), $w_4 := FC_{(a,b'',e)}(w_3)$ (with new root arcs a'' and f'), $w_5 := FC_{(a,c,b)}(v)$ (with new root arcs a''' and g), $w_6 := FC_{(a''',g,c)}(w_5)$ (with new root arcs $a^{(4)}$ and d'), $w_7 := FC_{(a',d,b)}(w_1)$ (with new root arcs $a^{(5)}$ and g'), $w_8 := FC_{(a',f,b')}(w_2)$ (with new root arcs $a^{(6)}$ and g''), and $w_9 := FC_{(g,a''',c)}(w_5)$ (with new root arcs g''' and f''). Then add the remaining $2 \cdot 11 + 5 + 1$ new vertices and $2 \cdot 8 + 4$ new elements of $\mathcal{M}'$ by the operations $COC_{b'}(w_2, w_4)$, $COC_g(w_6, w_7)$, $ADC(w_8, w_9)$, and $AF_{g''}(w_8, w_9)$. See Figure 11.*

Lemma 6.14. *With the notation as in Operation 6.13, every feasible packing of D extends to a feasible packing of D'. Moreover, if the base (a, b, c) that v gets is colored by $(\lambda_1, \lambda_2, \lambda_3)$, then the base (a', b, d) that w_1 gets is colored by $(\lambda_1, \lambda_2, \lambda_3)$.*

Proof. By relabeling the colors we can assume that (a, b, c) is colored by (red, blue, black). It is straightforward to check that such an extension always exists, by coloring the parallel copies of a by red, the parallel copies of b and g by blue and the parallel copies of d, e and f by black (see Figure 11). Hence we only prove that no other extension is possible.

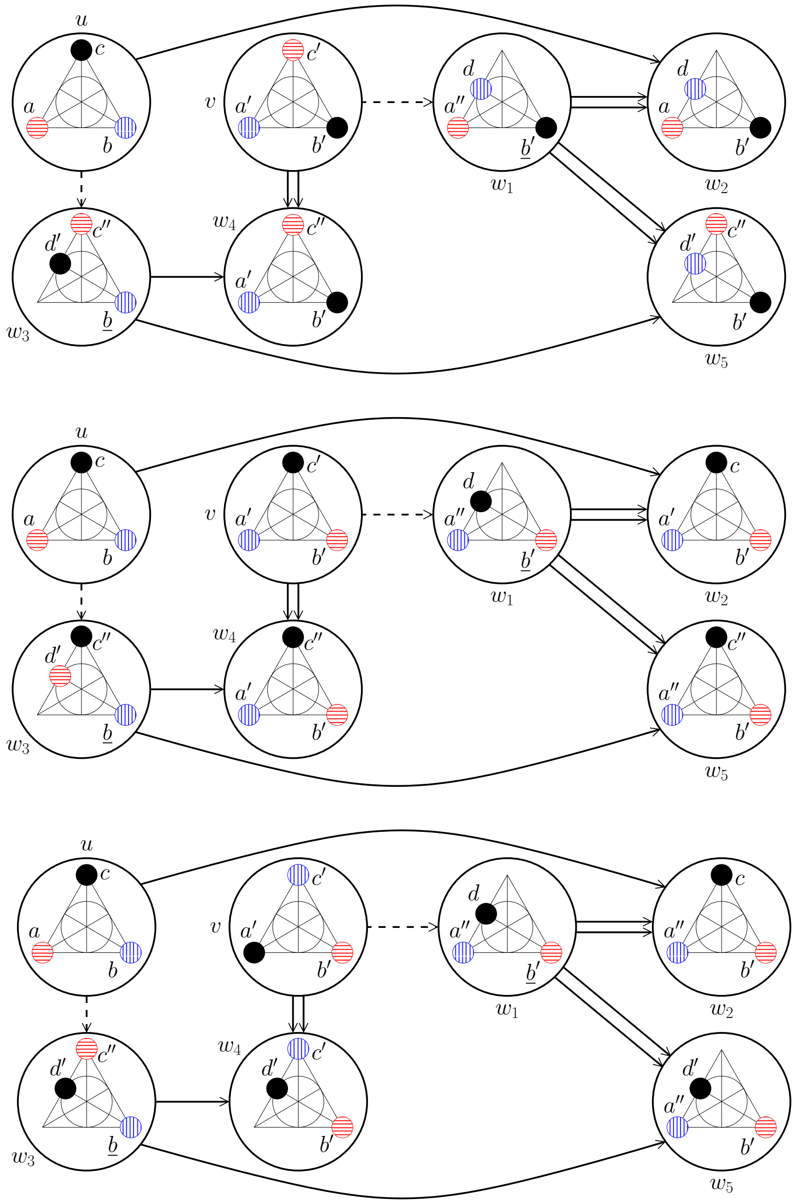


Figure 10: The proof of $AF_a(u,v)$ for the non-exceptional cases where b' has different color as b.

Assume for a contradiction that (a', b, d) is colored by (black, blue, red). (As the color of b is forced this is the only other possible coloring of (a', b, d).) Hence, in the base $\{a', b', f\}$ that w_2 gets, a' is forced to be black and hence b' cannot be black. On the other hand, b'' cannot be red as a is forced to be red in the base $\{a, b'', e\}$ got by w_3. Since the colors of b' and b'' coincide by $COC_{b'}(w_2, w_4)$, b' is blue as it is neither black nor red. Therefore,

$$w_2 \text{ gets } (a', b', f) \text{ colored by } (\text{black}, \text{blue}, \text{red}). \tag{6}$$

Next we turn to determine the coloring of the base $\{a''', c, g\}$ got by w_5. We know that c is forced to be black hence g is not black. Thus g is not black in the base $\{a^{(4)}, d', g\}$ got by w_6. On the other hand, in the base $\{a^{(5)}, d, g'\}$ got by w_7, d is forced to be red by our assumption hence g' is not red. By $COC_g(w_6, w_7)$, the colors of g and g' coincide. Thus the color of g is blue, as it is neither black nor blue. Therefore,

$$w_5 \text{ gets } (a''', c, g) \text{ colored by } (\text{red}, \text{black}, \text{blue}). \tag{7}$$

Finally, f is colored by red in the base got by w_8 by (6) and a''' is colored by red in the base got by w_9 by (7). Hence the colors of g'' and g''' must coincide by $ADC(w_8, w_9)$. Therefore, we get a contradiction with $AF_{g''}(w_8, w_9)$. See Figure 12. □

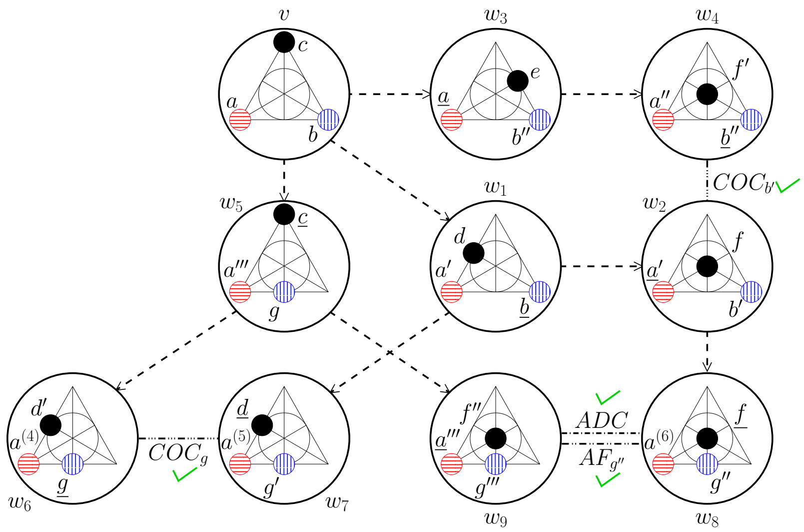


Figure 11: There always exists an extension of a feasible packing in $SFC_{a,b,c}(v)$.

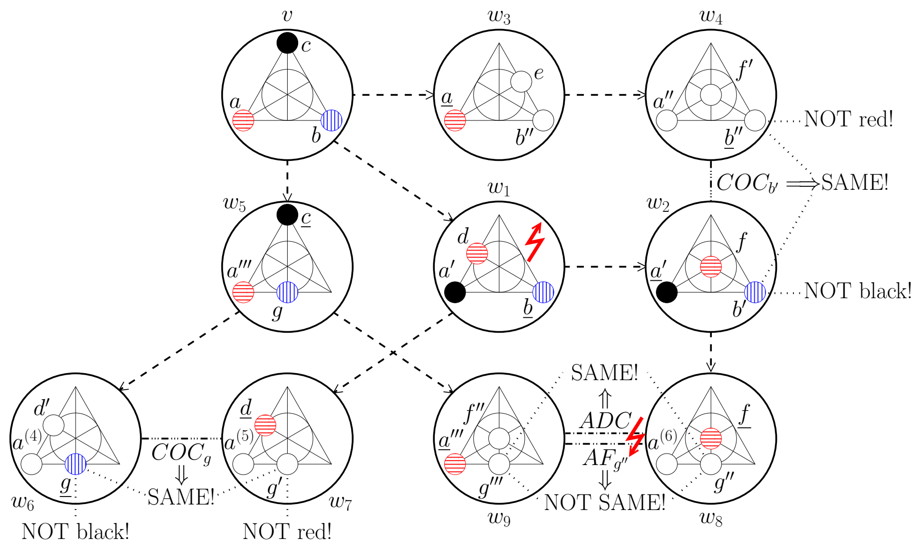


Figure 12: The proof that shows the color of the base that w_1 gets cannot be the other possibility in $SFC_{a,b,c}(v)$.

Operation 6.15. *Given* $(D, \mathcal{M})$*, suppose that a vertex* $v \in V$ *gets the base* $\{a, b, c\}$ *in every feasible packing.* **Change-colors** $\boldsymbol{CC_{a,c}(v)}$ *extends* $(D, \mathcal{M})$ *to* $(D', \mathcal{M}')$ *by adding* $3 \cdot 37 = 111$ *new vertices to* D *and* $3 \cdot 38 = 114$ *new elements to* $\mathcal{M}$ *as follows. First, construct a new vertex* w_1 *by* $SFC_{a,b,c}(v)$*, which gets* $\{a', b, d\}$*. Next construct a new vertex* w_2 *by* $SFC_{d,b,a'}(w_1)$*, which gets* $\{b, c', d'\}$*. Finally construct a new vertex* w *by* $SFC_{c',b,d'}(w_2)$ *which gets* $\{a'', b, c''\}$*. See Figure 13.*

Lemma 6.16. *With the notation as in Operation 6.15, every feasible packing in D extends to a feasible packing in D'. Moreover, if the base (a, b, c) that v gets is colored by $(\lambda_1, \lambda_2, \lambda_3)$, then the base (a'', b, c'') that w gets is colored by $(\lambda_3, \lambda_2, \lambda_1)$.*

Proof. Since each step is done by SFC, the coloring is determined as shown in Figure 13 by Lemma 6.14. □

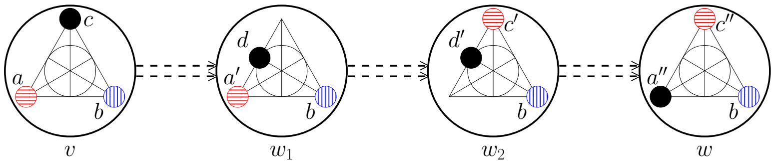


Figure 13: $CC_{a,c}(v)$. The parallel dashed arcs represent the operation SFC.

For simplicity, we denote $w = CC_{a,c}(v)$ if w is as in Operation 6.15. We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. We start with a digraph on two vertices, a root s and the other vertex v, along with 3 parallel arcs a, b and c from s to v. The underlying matroid is the free matroid on $\partial_s(v)$. We extend this by using the operations defined above. In the following, the arborescences covering a, b and c will be called red, blue and black, respectively. By using $CC_{a,b}(v)$, the instance is extended such that $w = CC_{a,b}(v)$ gets a base (a'', b, c'') with elements parallel to the elements of a, b and c and colors $(\text{black}, \text{blue}, \text{red})$ by Lemma 6.16. We further extend the instance by $AF_b(v, w)$. Then, by Lemma 6.10, no feasible packing exists in the resulting instance. By Lemma 3.13, the resulting instance is rooted $\mathcal{M}$-arc-connected, and hence is a counterexample to Conjecture 1.3. This completes the proof of Theorem 6.1. □

Now we turn to the proof of Theorem 6.2. Problem 1.4 is in NP in the case where a linear representation of the matroid is given as input since the packing itself is a witness for the problem that can be checked in polynomial time. We will use the well-known 3-SAT (see [12]) to prove the NP-completeness of our problem.

Let us take a 3-CNF formula. Using the previous operations (and a new one) we will construct a matroid-rooted digraph that has a feasible packing if and only if the formula is satisfiable. In order to express each clause, our idea is to represent it as a concatenation of majority functions and implement each majority function by using our operations. We first remark the following lemma. Recall that the majority function $\mathsf{maj}(\alpha, \beta, \gamma)$ is a Boolean function that has a value 1 if and only if at least two among α, β, γ have value 1.

Lemma 6.17. *Let $\alpha, \beta, \gamma \in \{0, 1\}$. Then*

$$\alpha \vee \beta \vee \gamma = \mathsf{maj}(\mathsf{maj}(\alpha, \beta, 1), \mathsf{maj}(\alpha, \gamma, 1), \mathsf{maj}(\beta, \gamma, 1)). \tag{8}$$

Proof. $\alpha \vee \beta \vee \gamma = 1$ if and only if at least one of α, β and γ is 1. If, say, $\alpha = 1$, then $\mathsf{maj}(\alpha, \beta, 1) = 1$ and $\mathsf{maj}(\alpha, \gamma, 1) = 1$ hence the right hand side of (8) is 1. If $\alpha = \beta = \gamma = 0$, then $\mathsf{maj}(\alpha, \beta, 1) = \mathsf{maj}(\alpha, \gamma, 1) = \mathsf{maj}(\beta, \gamma, 1) = 0$ hence the right hand side of (8) is 0. □

Operation 6.18. *Given $(D, \mathcal{M})$, suppose that $v_1, v_2, v_3 \in V$ get the bases $\{a, b, c\}$, $\{a', b', c'\}$ and $\{a'', b'', c''\}$, respectively, in every feasible packing where $a \parallel a' \parallel a''$, $b \parallel b' \parallel b''$ and $c \parallel c' \parallel c''$.* **Majority $\boldsymbol{MAJ(v_1, v_2, v_3)}$** *extends $(D, \mathcal{M})$ to $(D', \mathcal{M}')$ by adding a new vertex w with 3 incoming arcs v_1w, v_2w and v_3w. See Figure 14.*

Lemma 6.19. *With the notation as in Operation 6.18, consider a feasible packing of D such that all of b, b' and b'' are colored by λ (and hence there are only two types of possible coloring schemes on each v_i). Then the packing extends to a feasible packing of D'. Moreover, in every such extension w gets a base formed by parallel copies of a, b, and c with a coloring of the same type as the majority among the three on v_1, v_2 and v_3. See Figure 14.*

Proof. Without loss of generality, we can assume that the colorings of (a, b, c) and (a', b', c') coincide, say, they are colored by (red, blue, black). As w has an entering arc from each v_i, w always gets a parallel copy of b colored by blue. Moreover, as w has in-arcs from v_1 and v_2 too, w gets a parallel copy of a or c from v_1 or v_2. Hence w gets a parallel copy of a colored by red or a parallel copy of c colored by black. These two facts already determine the coloring scheme on w as stated in the lemma. □

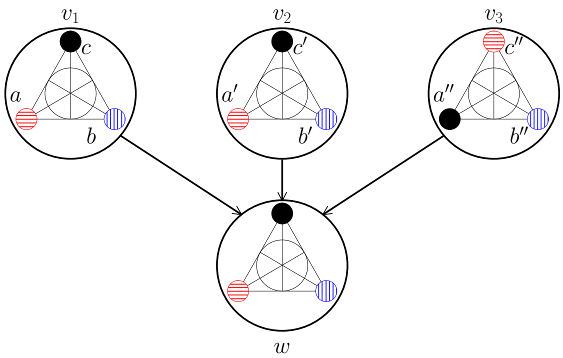


Figure 14: $MAJ(v_1, v_2, v_3)$.

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. We have seen that the problem is in NP, hence we only prove the completeness. Let us take a 3-CNF formula on variables $x_1, x_2, \dots, x_n$. First, let $V := \{v_0, \dots, v_n\}$ and take a digraph D on $V + s$ whose arc set consists of only root arcs sv_i $(i = 0, \dots, n)$, three copy of each. Take a base $\{a, b, c\}$ of the Fano matroid and define a parallel extension of the Fano matroid $\mathcal{M}$ on $\partial_r(V)$ such that, for each $i \in \{0, \dots, n\}$, the three arc sv_i form a parallel copy $\{a_i, b_i, c_i\}$ of $\{a, b, c\}$. Next use operation $COC_{b_{i-1}}(v_{i-1}, v_i)$ for $i = 1, \dots, n$. This ensures that in every feasible packing the parallel copies of b got by $v_0, \dots, v_n$ are colored by the same color, say, blue.

Add $v'_1, \dots, v'_n$ by using operations $CC_{a^i,c^i}(v_i)$ for $i = 1, \dots, n$. Hence, in every feasible packing, v'_i gets the colored base (a'_i, b'_i, c'_i) with the same coloring as (c_i, b_i, a_i) for $i = 1, \dots, n$. In the following construction, v_i will represent the variable x_i and v'_i its negate $\bar{x}_i$ for $i = 1, \dots, n$. Moreover, v_0 will represent 1.

For each clause ψ of the formula, we first add 4 new vertices w_1^ψ, w_2^ψ, w_3^ψ and w_4^ψ using operation MAJ so that it represents ψ according to the equation in Lemma 6.17. (In other words, for a clause, say, for $\psi = x_1 \vee \bar{x}_2 \vee x_3$ we add w_1^ψ with arcs $v_1w_1^\psi$, $v'_2w_1^\psi$ and $v_0w_1^\psi$, w_2^ψ with arcs $v_1w_2^\psi$, $v_3w_2^\psi$ and $v_0w_2^\psi$, w_3^ψ with arcs $v'_2w_3^\psi$, $v_3w_3^\psi$ and $v_0w_3^\psi$, and w_4^ψ with arcs $w_1^\psi w_4^\psi$, $w_2^\psi w_4^\psi$ and $w_3^\psi w_4^\psi$.) Finally, to ensure the truth of each clause ψ, we further use operation $AF_{b_0}(v_0, w_4^\psi)$. See Figure 15.

We claim that the formula is satisfiable if and only if $(D, \mathcal{M})$ admits a feasible packing. Note that v_0 always gets the base $\{a_0, b_0, c_0\}$, and without loss of generality we may always suppose that (a_0, b_0, c_0) is colored by (red, blue, black). Then the claim follows by identifying the coloring scheme (red, blue, black) (resp., (black, blue, red)) for a parallel copy of (a, b, c) with a true assignment (resp., a false assignment).

More formally, suppose that the formula has a true assignment. Then, we first construct a feasible packing restricted on $\{s, v_0, v_1, \dots, v_n\}$ such that v_0 gets the base (a_0, b_0, c_0) colored by (red, blue, black) and each v_i $(1 \leq i \leq n)$ gets the base (a_i, b_i, c_i) colored by (red, blue, black) if $x_i = 1$ and by (black, blue, red) if $x_i = 0$. By Lemma 6.16, this packing always extends on $\{v'_1, \dots, v'_n\}$ such that each v'_i gets a base formed by parallel copies of a, b, and c colored by black, blue, and red, respectively, if $x_i = 1$ and by red, blue, and black, respectively, if $x_i = 0$. Since the assignment satisfies the formula, Lemmas 6.19 and 6.10 imply that the packing is extendable to a feasible packing on the whole vertex set of D.

Conversely, if $(D, \mathcal{M})$ has a feasible packing, then by $COC_{b_{i-1}}(v_{i-1}, v_i)$, b_i is colored by blue on each v_i. We set x_i in such a way that $x_i = 1$ if and only if (a_i, b_i, c_i) is colored by (red, blue, black) (as in (a_0, b_0, c_0)). By $CC_{a^i,c^i}(v_i)$, each b'_i is colored by blue and the coloring of (a'_i, c'_i) is different from that of (a_i, c_i). Moreover, since $AF_{b_0}(v_0, w_4^\psi)$ is used for each clause ψ, the base on w_4^ψ has the same coloring scheme as that of $\{a_0, b_0, c_0\}$ on v_0 by Lemma 6.10. Thus by Lemma 6.19 the formula is satisfied for this assignment. □

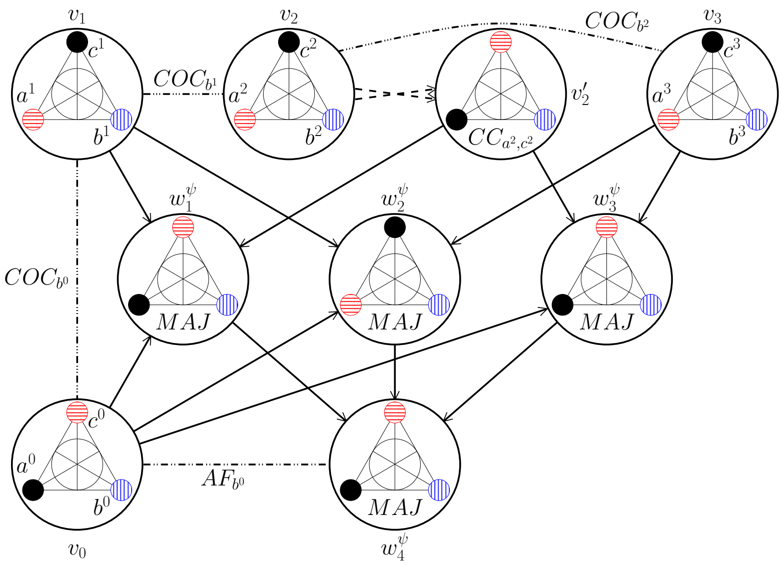


Figure 15: A part of the construction in the proof of Theorem 6.2. This demonstrates how the assignment $x_1 = x_2 = x_3 = 0$ makes the clause $\psi = x_1 \vee \bar{x}_2 \vee x_3$ true in the corresponding feasible packing. The crossing dashed arcs represent the operation CC.