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On some special cases of Ryser's conjecture

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Abstract

A famous conjecture (usually called Ryser's conjecture), appeared in the Ph.D thesis of his student, J. R. Henderson [6], states that for an r-uniform r-partite hypergraph $\mathcal{H}$ the inequality $\tau(\mathcal{H}) \le (r-1)\cdot\nu(\mathcal{H})$ always holds.

This conjecture is widely open, except in the case of $r = 2$, when it is equivalent to Kőnig's theorem [8], and in the case of $r = 3$, which was proved by Aharoni in 2001 [1], using topological results from [2].

Here we study some special cases of Ryser's conjecture. First of all the most studied special case is when $\nu = 1$, i.e., when $\mathcal{H}$ is intersecting. Even for this special case, not too much is known. The case $r = 2$ is an observation of Erdős and Rado. The cases $r = 3, 4$ were proved by Gyárfás, and the case $r = 5$ by Tuza [11]. For $r > 5$ this conjecture is also widely open. Some recent papers study this special case, e.g., see [3, 10].

We strengthen the conjecture for intersecting hypergraphs by conjecturing the following. If an r-uniform r-partite hypergraph $\mathcal{H}$ is t-intersecting (i.e., every two hyperedges meet in at least t vertices), then $\tau(\mathcal{H}) \le r - t$. We prove this conjecture for the case $t > r/4$.

Gyárfás [5] showed that Ryser's conjecture for intersecting hypergraphs is equivalent to saying that the vertices of an r-edge-colored complete graph can be covered by $r - 1$ monochromatic components.

Motivated by this formulation, we examine how much fraction of the vertices can be covered by $r - 1$ monochromatic components of *different* colors in an r-edge-colored complete graph. We prove a sharp bound for this problem.

Finally we prove Ryser's conjecture for the very special case when the maximum degree of the hypergraph is two.

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1 Introduction

A hypergraph is r-partite if its vertex set has a partition to r nonempty classes such that no edge contains two vertices from the same set. A hypergraph is r-uniform if all of its edges have cardinality r. A hypergraph is d-regular if every vertex is contained in exactly d edges. A hypergraph is t-intersecting if every pair of edges have at least t common vertices. A hypergraph is intersecting if it is 1-intersecting.

Let us introduce some more standard notations. For a hypergraph $\mathcal{H}$ with vertex-set $V = V(\mathcal{H})$ and edge-set $E = E(\mathcal{H})$

$$\tau(\mathcal{H}) = \min\{|T| : T \subseteq V,\ T \cap f \neq \emptyset \ \ \forall f \in E\}$$

$$\varrho(\mathcal{H}) = \min\{|F| : F \subseteq E,\ \bigcup F = V\}$$

$$\nu(\mathcal{H}) = \max\{|F| : F \subseteq E,\ f_1 \cap f_2 = \emptyset \ \ \forall f_1 \neq f_2 \in F\}$$

$$\Delta(\mathcal{H}) = \max\{|F| :\ F \subseteq E,\ \bigcap F \neq \emptyset\}$$

$$\alpha(\mathcal{H}) = \max\{|X| : X \subseteq V,\ f \not\subseteq X \ \ \forall f \in E\}$$

$$\alpha'(\mathcal{H}) = \max\{|X| : X \subseteq V,\ |f \cap X| \leq 1 \quad \forall f \in E\}$$

The famous Ryser's conjecture (that appeared in the Ph.D thesis of his student, J.R. Henderson [6]) states that for an r-uniform r-partite hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leq (r-1)\cdot\nu(\mathcal{H})$.

This conjecture is widely open, except in the special case of $r = 2$, when it is equivalent to Kőnig's theorem [8], and when $r = 3$, which was proved by Aharoni in 2001 [1], using topological results from [2]. We mention also some related results. Henderson [6] showed that the conjecture cannot be improved, if $r-1$ is a prime power. Füredi [4] proved that the fractional covering number is always at most $(r-1)\cdot\nu(\mathcal{H})$, and Lovász [9] proved that the fractional matching number is always at least $\frac{2}{r}\cdot\tau(\mathcal{H})$.

Here we study some special cases of this conjecture. First of all the most known special case is when $\nu = 1$, i.e., when $\mathcal{H}$ is intersecting. Even for this special case, not too much is known. Gyárfás [5] showed that this special case of the conjecture is equivalent to saying that the vertices of an r-edge-colored complete graph can be covered by $r-1$ monochromatic components (see below). He also proved this conjecture for $r = 3, 4$ (for $r = 2$ this is an observation of Erdős and Rado), and later Tuza [11] proved it for $r = 5$. For $r > 5$ this conjecture is also widely open. Some recent papers study this special case, e.g., see [3, 10]. For intersecting hypergraphs, we strengthen Ryser's conjecture by conjecturing the following. If an r-uniform r-partite hypergraph $\mathcal{H}$ is t-intersecting (i.e., every two hyperedges meet in at least t vertices), then $\tau(\mathcal{H}) \leq r - t$. We prove this conjecture for the case $t > r/4$.

The construction of Gyárfás [5] (see also in [7]) is the following. We associate a multi edge-colored graph to an r-partite r-uniform hypergraph.

Definition 1.1. For an r-partite r-uniform hypergraph $\mathcal{H}$, let $G = G(\mathcal{H})$ be the following multi edge-colored graph:

The vertex set of G is $V(G) = E(\mathcal{H})$. Two vertices $u, v \in V(G)$ are connected by an edge if the corresponding hyperedges $E_u, E_v \in E(\mathcal{H})$ have a nonempty intersection.

The edge uv is colored by the colors $\{i : E_u$ and E_v share a vertex from the i^{th} class$\}$. We denote the set of colors of edge uv by $\mathrm{Col}(uv)$. If $i \in \mathrm{Col}(u, v)$, then we say that the edge uv *uses* the color i.

Remark 1.2. The original construction of Gyárfás colored each edge uv by only one color, chosen arbitrarily from $\mathrm{Col}(uv)$.

The edges of G having color i form a partition of $V(G)$ to i-colored cliques. Each vertex of $\mathcal{H}$ in the class i corresponds to one clique of color i, which is also a monochromatic (i-colored) connected component of G. A set of vertices $T \subseteq V(\mathcal{H})$ covers the hyperedges of $\mathcal{H}$ (as in the definition of τ), if and only if the monochromatic components corresponding to its elements cover $V(G)$. If $\mathcal{H}$ is intersecting, then G is complete.

Ryser's conjecture for intersecting hypergraphs is equivalent to the statement that $r - 1$ monochromatic components can cover $V(G(\mathcal{H}))$. The stronger conjecture for t-intersecting hypergraphs is equivalent to the statement that if in an r-edge-colored complete graph where each color is a union of cliques, each edge has at least t colors, then the vertices can be covered by $r - t$ monochromatic components.

For the case of r-edge-colored complete graphs, we also study the following problem: How much fraction of the vertices can be covered by $r-1$ monochromatic components of *different* colors? We prove a sharp bound for this problem, namely $(1-\varepsilon(r))\cdot|V(G)|$ where $\varepsilon(r) = \frac{r-2}{(r-1)^2}$.

Finally we prove Ryser's conjecture for the very special case when the maximum degree of the hypergraph is two.

2 The t-intersecting case

Conjecture 2.1. *Let $\mathcal{H}$ be an r-uniform r-partite t-intersecting hypergraph* ($t \geq 1$). *Then $\tau(\mathcal{H}) \leq r - t$.*

Theorem 2.2. *If $\mathcal{H}$ is an r-uniform r-partite t-intersecting hypergraph and $t > \frac{r}{4}$, then $\tau(\mathcal{H}) \leq r - t$.*

Using Gyárfás' construction (Definition 1.1), Theorem 2.2 follows from the following statement:

Theorem 2.3. *Let G be a multi edge-colored complete graph colored by r colors, where each edge has at least t different colors. If $t > \frac{r}{4}$, then $V(G)$ can be covered by at most $r - t$ monochromatic components.*

We can suppose that each monochromatic component is a clique. Indeed, if a monochromatic component is not a clique, we can add its color to the color set of those edges that connect two vertices from the same monochromatic component but do not have this color. This way the condition that each edge has at least t colors is not violated. Also, the set of vertices of each monochromatic component remains the same. Hence throughout the proof we suppose that the monochromatic components are cliques.

We assume $[r] = \{1, 2, \dots, r\}$ is the set of colors, and if x is a vertex and $I \subseteq [r]$ is a set of colors, then we denote by $\mathcal{C}(x, I)$ the set of monochromatic components containing x and having a color in I.

First we prove the statement of the theorem for $r \leq 4t - 2$.

Lemma 2.4. *Let G be a multi edge-colored complete graph colored by r colors, where each edge has at least t different colors. If $r \leq 4t - 2$, then $V(G)$ can be covered by at most $r - t$ monochromatic components.*

Proof. Take an edge xy such that $|\mathrm{Col}(xy)|$ is maximal. By the assumption $\ell := |\mathrm{Col}(xy)| \geq t$. We may suppose that $\mathrm{Col}(xy) = I = \{r - \ell + 1, \dots, r\}$. Let $J = \{1, \dots, \lfloor \frac{r-t-\ell}{2} \rfloor\}$ if $r - t - \ell \geq 2$ and $J = \emptyset$ otherwise.

First we consider the case $\ell \leq r - t$. Take $\mathcal{C}(x, I) \cup \mathcal{C}(x, J) \cup \mathcal{C}(y, J)$. We claim that these (at most $r - t$) monochromatic components cover the vertices of G.

Let z be an arbitrary vertex. If $\mathrm{Col}(xz) \cap I \neq \emptyset$ or $\mathrm{Col}(yz) \cap I \neq \emptyset$, then z is covered. Otherwise $\mathrm{Col}(xz) \cup \mathrm{Col}(yz) \subseteq \{1, \dots, r - \ell\}$. In this case $\mathrm{Col}(xz) \cap \mathrm{Col}(yz) = \emptyset$, since if z is connected to both x and y by a color $i \leq r - \ell$, then xy would also use i (since the monochromatic components are cliques), but $i \notin I$. By our assumption $|\mathrm{Col}(xz)| \geq t$ and $|\mathrm{Col}(yz)| \geq t$. Hence the edges xz and yz use at least $2t$ different colors.

If z is not covered, all these at least $2t$ colors need to be from the set $\{\lfloor \frac{r-t-\ell}{2} \rfloor + 1, \dots r - \ell\}$, hence $2t \leq r - \ell - \lfloor \frac{r-t-\ell}{2} \rfloor \leq \lceil \frac{r+t-\ell}{2} \rceil \leq \lceil \frac{4t-2+t-t}{2} \rceil = 2t - 1$, since $r \leq 4t - 2$ and $\ell \geq t$. This is a contradiction, hence z is covered.

Now consider the case $\ell \geq r - t + 1$. Let $I' = \{t + 1, \dots, r\}$. Since now $\ell \geq r - t + 1$, $I' \subseteq I$. We claim that $\mathcal{C}(x, I') = \mathcal{C}(y, I')$ covers $V(G)$. If a vertex z is not covered, then $\mathrm{Col}(xz) \cup \mathrm{Col}(yz) \subseteq \{1, \dots, t\}$. Moreover, $\mathrm{Col}(xz) \cap \mathrm{Col}(yz) \subseteq \{r - l + 1, \dots t\}$, hence $|\mathrm{Col}(xz) \cup \mathrm{Col}(yz)| \geq 2t - (t - r + \ell - 1) = t + r - \ell + 1 \geq t + 1$ as $\ell \leq r$. This is a contradiction. $\square$

With the same proof we also get the following.

Lemma 2.5. *Let G be a multi edge-colored complete graph colored by $r \leq 4t - 1$ colors, where each edge has at least t different colors. If there is an edge xy with $|\mathrm{Col}(xy)| > t$, then $V(G)$ can be covered by at most $r - t$ monochromatic components.*

It remains to prove the case $r = 4t - 1$ and $\ell = t$. Let k be the largest number j such that there is a triangle in G with j colors occurring on all three edges. Let xyz be a triangle with k common colors on its edges. Let us introduce some further notations.

Let $K = \mathrm{Col}(xy) \cap \mathrm{Col}(yz) \cap \mathrm{Col}(zx)$ and $X = \mathrm{Col}(yz) \setminus K$ and $Y = \mathrm{Col}(xz) \setminus K$ and $Z = \mathrm{Col}(xy) \setminus K$, finally let $S = [r] \setminus (K \cup X \cup Y \cup Z)$.
For a set A, A' always denotes a subset of A. Moreover, we denote $A \setminus A'$ by A''. Note that $|K| = k$ and $|X| = |Y| = |Z| = t - k$.

Case 0: $k = 0$

Now $|V(G)| \leq r + 1$ as no two incident edges may use the same color. Let $V(G) = \{u_1, \dots, u_n\}$ where $n \leq r + 1$ and let $c_i \in \mathrm{Col}(u_{2i-1}u_{2i})$ for $1 \leq i \leq n/2$. If n

is even, then consider $\mathcal{C}(u_1, c_1) \cup \mathcal{C}(u_3, c_2) \cup \ldots \cup \mathcal{C}(u_{n-1}, c_{n/2})$, otherwise consider $\mathcal{C}(u_1, c_1) \cup \mathcal{C}(u_3, c_2) \cup \ldots \cup \mathcal{C}(u_{n-2}, c_{(n-1)/2}) \cup \mathcal{C}(u_n, 1)$. These (at most $\lfloor (n+1)/2 \rfloor \leq \lfloor (r+2)/2 \rfloor$) monochromatic components obviously cover $V(G)$, and $\lfloor (r+2)/2 \rfloor \leq r-t$ as $r = 4t-1$.

Case 1: $0 < 3k \leq t$.

Choose $Y' \subseteq Y$ and $Z' \subseteq Z$ so that $|Y'| + |Z'| = t + k - 1$. This is possible, since $|Y| + |Z| = 2t - 2k \geq t + k$ because $t \geq 3k$.

Take the following monochromatic components: $\mathcal{C}(x, K \cup Y \cup Z) \cup \mathcal{C}(y, Y') \cup \mathcal{C}(z, Z')$. The number of components chosen is at most $(2t-k) + (t+k-1) = 3t-1 = r-t$.

We claim that these components cover each vertex. Suppose that a vertex w is not covered. Then we claim that $\mathrm{Col}(xw) \subseteq X \cup S$ and $\mathrm{Col}(yw) \subseteq X \cup Y'' \cup S$ and $\mathrm{Col}(zw) \subseteq X \cup Z'' \cup S$.

As w is not covered, $\big(\mathrm{Col}(xw) \cup \mathrm{Col}(yw) \cup \mathrm{Col}(zw)\big) \cap K = \emptyset$. We also have $\mathrm{Col}(xw) \cap Y = \emptyset$, otherwise w would be covered. Similarly, $\mathrm{Col}(yw) \cap Y' = \emptyset$. We claim that also $\mathrm{Col}(zw) \cap Y = \emptyset$. Indeed, as $Y \subseteq \mathrm{Col}(xz)$, if zw had a color from Y, then xw would also have that color (since the monochromatic components are cliques), a contradiction.

By the same reasoning, $\mathrm{Col}(xw) \cap Z = \emptyset$, $\mathrm{Col}(yw) \cap Z = \emptyset$ and $\mathrm{Col}(zw) \cap Z' = \emptyset$.

Next we claim that the colors in X can occur altogether (counting with multiplicity) at most t times on the edges xw, yw and zw. Let $c \in X$ be a color. If it occurs more than once on edges xw, yw and zw, then it is in $\mathrm{Col}(yw) \cap \mathrm{Col}(zw)$ but $c \notin \mathrm{Col}(xw)$. If $c \in \mathrm{Col}(xw) \cap \mathrm{Col}(yw)$, then $c \in \mathrm{Col}(xy)$ contradicting to $X \cap \mathrm{Col}(xy) = \emptyset$. Similarly $c \notin \mathrm{Col}(xw) \cap \mathrm{Col}(zw)$. By the choice of k, $|\mathrm{Col}(yw) \cap \mathrm{Col}(zw)| \leq k$. Hence the colors in X occur at most $|X| + k \leq t$ times on the edges xw, yw and zw.

Each color in S can only occur once on xw, yw and zw, since otherwise it would also occur on one of the edges xy, yz and zx, and that would contradict the definition of S.

Hence counting the colors of the edges xw, yw and zw: $3t \leq |S| + |Z''| + |Y''| + t = |S| + (|Y| + |Z| - (|Y'| + |Z'|)) + t = (4t - 1 - (3t - 2k)) + (2t - 2k - (t + k - 1)) + t = (t + 2k - 1) + (t - 3k + 1) + t = 3t - k$, which is a contradiction.

Case 2: $3k > t$.

If $|X| + |Y| + |Z| = 3t - 3k \geq 2k - 1$, then choose $X' \subseteq X$, $Y' \subseteq Y$ and $Z' \subseteq Z$ so that $|X'| + |Y'| + |Z'| = 2k - 1$. If $|X| + |Y| + |Z| = 3t - 3k < 2k - 1$, then let $X' = X$, $Y' = Y$ and $Z' = Z$.

Take the following monochromatic components: $\mathcal{C}(x, K \cup X' \cup Y \cup Z) \cup \mathcal{C}(y, Y') \cup \mathcal{C}(z, X \cup Z')$. The number of components chosen is at most $|K| + |X| + |Y| + |Z| + |X'| + |Y'| + |Z'| \leq k + 3(t-k) + 2k - 1 = 3t - 1 = r - t$.

We claim that the components chosen cover each vertex. Suppose that there is a vertex w which is not covered. Similarly to the previous case, it is easy to prove that the colors of xw, yw and zw are all from $S \cup X'' \cup Y'' \cup Z''$, and each color is used at most once altogether on these three edges. Hence $3t \leq |S| + |X''| + |Y''| + |Z''|$.

If $3t-3k \geq 2k-1$, then $3t \leq |S|+|X''|+|Y''|+|Z''| = 4t-1-(|K|+|X'|+|Y'|+|Z'|) = 4t-1-(k+2k-1) = 4t-3k < 3t$ since $t < 3k$. This is a contradiction.

If $3t-3k < 2k-1$, then $3t \leq |S \cup X'' \cup Y'' \cup Z''| = |S| = 4t-1-(3t-2k) = t+2k-1$. But this implies $2t \leq 2k-1$, hence $k > t$, which contradicts to the assumption that each edge has at most t colors.

Remark 2.6. We think that with a more diversified case analysis Theorem 2.3 can be extended to the case $t \geq r/5$. Note however, that for example the case $t = r/6$ would include the first unsolved case of Ryser's conjecture for intersecting hypergraphs.

3 Covering large fraction by few monochromatic components

In this section, we give a sharp bound for the ratio of vertices that can be covered by $r-1$ monochromatic components of pairwise different colors in an r-edge colored complete graph.

Theorem 3.1. *Let G be a multi edge-colored complete graph colored with r colors. Then at least $\left(1-\frac{r-2}{(r-1)^2}\right)\cdot|V(G)|$ vertices of G can be covered by $r-1$ monochromatic components of pairwise different colors, and this bound is sharp for infinitely many r values.*

Applying the construction of Gyárfás (Definition 1.1), we get the following statement for hypergraphs.

Theorem 3.2. *If $\mathcal{H}$ is an r-partite r-uniform intersecting hypergraph, then at least $\left(1-\frac{r-2}{(r-1)^2}\right)\cdot|E(\mathcal{H})|$ edges of $\mathcal{H}$ can be covered by $r-1$ points from pairwise different color classes, and this bound is sharp for infinitely many r values.*

Proof of Theorem 3.1. We call an edge-coloring of G *spanning* if for every color c and vertex u there is an edge uv of G such that $c \in \mathrm{Col}(u,v)$. If the edge-coloring of G is not spanning, then we claim that we can cover all the vertices of G by $r-1$ monochromatic components of pairwise different colors. Indeed, if there is a vertex v and a color i such that no edge incident to v has color i, then $\mathcal{C}(x,[r]\setminus\{i\})$ cover the vertices of G.

Now suppose that the coloring of G is spanning. For $r=2$ we can cover the vertex set by one monochromatic component by a well-known result of Erdős and Rado, so we may assume $r \geq 3$. Let the number of monochromatic components of color i be k_i. Let us denote the set of monochromatic components of color i by $\mathcal{C}_i$. Let $|V(G)| = n$. We may suppose that $k_1 \geq k_2 \geq \ldots \geq k_r \geq 2$, otherwise (if $k_r = 1$) we are done.

Case 1: $k_1 \geq r-1$. We have

$$\sum_{C\in\mathcal{C}_1,\, C'\in\mathcal{C}_r} |C - C'| = (k_r - 1)\cdot n, \tag{1}$$

since each vertex occurs in exactly one component of color r and one component of color 1. Hence each vertex is counted $k_r - 1$ times for the $k_r - 1$ components of color r that does not contain it.

From (1) it follows that among the $k_1 \cdot k_r$ sets $\{C - C' : C \in \mathcal{C}_1, C' \in \mathcal{C}_r\}$, there is one which has size at most $\frac{k_r-1}{k_1 \cdot k_r} \cdot n$. Let this be the set $C_1 - C'_r$. As $k_1 \geq k_r$ we have $\frac{k_r-1}{k_r} \leq \frac{k_1-1}{k_1}$, so $\frac{k_r-1}{k_1 \cdot k_r} \cdot n \leq \frac{k_1-1}{k_1^2} \cdot n$. Using $2 \leq r-1 \leq k_1$ we also have $\frac{k_1-1}{k_1^2} \leq \frac{r-2}{(r-1)^2}$, so $\frac{k_r-1}{k_1 \cdot k_r} \cdot n \leq \frac{r-2}{(r-1)^2} \cdot n$.

Let x be a vertex in $C_1 \cap C'_r$. Take $\mathcal{C}(x, [r] \setminus \{1\})$. These components cover each vertex outside $C_1 - C'_r$, hence at least $(1 - \frac{r-2}{(r-1)^2}) \cdot n$ vertices.

Case 2: $k_1 \leq r - 1$ (i.e., $k_i \leq r - 1$ for all i).

Notice that Case 1 and Case 2 overlap. However, this overlapping categorization will be more convenient when examining sharpness.

For a vertex v and a color $i \in \{1, \dots, r\}$, let $d_i(v) = |\{u \in V(G) : \mathrm{Col}(uv) = \{i\}\}|$, i.e., the number of neighbors of v that are connected to v by an edge having only color i. It is enough to show that there exists $v \in V$ and $i \in \{1, \dots, r\}$ such that $d_i(v) \leq \frac{r-2}{(r-1)^2} \cdot n$. Indeed, in this case $\mathcal{C}(x, [r] \setminus \{i\})$ cover each vertex except for those that are connected to v by an edge of unique color i, that is, at most $\frac{r-2}{(r-1)^2} \cdot n$ vertices are uncovered.

Let $m_i = |\{uv \in E(G) : \mathrm{Col}(uv) = \{i\}\}|$, and $M_i = |\{uv \in E(G) : i \in \mathrm{Col}(uv)\}|$. Since $\sum_{v \in V} d_i(v) = 2m_i$, it is enough to show that there exists a color i such that $m_i \leq \frac{r-2}{2(r-1)^2} \cdot n^2$. For this, it is enough to show that $\sum_{i=1}^{r} m_i \leq \frac{r(r-2)}{2(r-1)^2} \cdot n^2$. We have $\sum_{i=1}^{r} m_i = \binom{n}{2} - t$ where t denotes the number of edges having multiple colors.

It is not hard to see that $t \geq \frac{1}{r-1} \cdot \left[\sum_{i=1}^{r} M_i - \binom{n}{2}\right]$, since each edge has at most r colors.

Claim 3.3. *If $\ell = k_i \leq r - 1$, then $M_i \geq \frac{n^2}{2\ell} - \frac{n}{2} \geq \frac{n^2}{2(r-1)} - \frac{n}{2}$.*

Proof. Let the cardinalities of the components of color i be $\gamma_1, \dots, \gamma_\ell$. $M_i = \binom{\gamma_1}{2} + \dots + \binom{\gamma_\ell}{2} = \frac{\gamma_1^2 + \dots + \gamma_\ell^2}{2} - \frac{\gamma_1 + \dots + \gamma_\ell}{2} = \frac{\gamma_1^2 + \dots + \gamma_\ell^2}{2} - \frac{n}{2}$.

Now it is enough to show that $\frac{\gamma_1^2 + \dots + \gamma_\ell^2}{2} \geq \frac{n^2}{2\ell}$ but this follows from the Arithmetic Mean–Quadratic Mean Inequality. □

Using the claim, we get that $t \geq \frac{1}{r-1} \cdot \left[\sum_{i=1}^{r} M_i - \binom{n}{2}\right] \geq \frac{1}{r-1} \cdot \left[\frac{r(n^2-(r-1)n)}{2(r-1)} - \binom{n}{2}\right] = \frac{rn^2 - r(r-1)n - (r-1)n^2 + (r-1)n}{2(r-1)^2} = \frac{n^2}{2(r-1)^2} - \frac{n}{2}$.

So $\sum_{i=1}^{r} m_i = \binom{n}{2} - t \leq \binom{n}{2} - \frac{n^2}{2(r-1)^2} + \frac{n}{2} = \frac{(r-1)^2 n^2 - (r-1)^2 n - n^2 + (r-1)^2 n}{2(r-1)^2} = \frac{r(r-2)n^2}{2(r-1)^2}$.

For the proof of sharpness see Theorem 3.5. □

3.1 Characterization of sharp examples

In this subsection we characterize the sharp examples for Theorem 3.1. Let us start with a definition.

Definition 3.4. We call a multi edge-colored graph G the blowup of an affine plane, if there is an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L})$, whose lines are colored such that two lines have the same color if and only if they are disjoint (i.e., parallel), and a positive integer b, such that to every point $p \in \mathcal{P}$ of the affine plane, b vertices correspond in $V(G)$, and two vertices are connected by an edge having color i if and only if the corresponding points in $\mathcal{A}$ are incident to a common line of color i (this includes also the case if the two points correspond to the same point of $\mathcal{A}$).

Theorem 3.5. *For a multi edge-colored complete graph G colored with r colors, the maximum number of vertices coverable by $r-1$ monochromatic components of pairwise different colors equals $\left(1-\frac{r-2}{(r-1)^2}\right)\cdot|V(G(\mathcal{H}))|$ if and only if G is a blowup of an affine plane.*

Proof. As noted in the beginning of the proof of Theorem 3.1, if the edge-coloring of G is not spanning, or $r = 2$, then all the vertices of G can be colored by $r-1$ monochromatic components of pairwise different colors, hence in these cases, there is no sharp example.

Now suppose that the coloring of G is spanning, and $r \geq 3$. We examine the proof of Theorem 3.1 to see how the inequalities can be equalities. In Case 1, $k_1 = \cdots = k_r = r-1$ for a sharp example, since otherwise $\frac{k_r-1}{k_1 \cdot k_r}\cdot n$ would be strictly smaller than $\frac{r-2}{(r-1)^2}\cdot n$.

Also in Case 2, $k_1 = \cdots = k_r = r-1$ for a sharp example, since we need $M_i = \frac{n^2}{2(r-1)} - \frac{n}{2}$ for each i. But if $k_i < r-1$ for some i, then $M_i \geq \frac{n^2}{2k_i} - \frac{n}{2} > \frac{n^2}{2(r-1)} - \frac{n}{2}$.

Hence a sharp example is necessarily in the intersection of Case 1 and Case 2, and the bounds of both cases are sharp for it.

Now, knowing that $k_1 = \cdots = k_r = r-1$, we could repeat the computation of Case 1 for two arbitrary colors i and j instead of 1 and r, and obtain that in a sharp example, the intersection of any two components of different colors must have size $\frac{n}{(r-1)^2}$.

Moreover, from $t = \frac{1}{r-1}\cdot\left[\sum_{i=1}^{r} M_i - \binom{n}{2}\right]$, for each edge either $|\mathrm{Col}(uv)| = 1$ or $|\mathrm{Col}(uv)| = r$. From this, the following useful property follows:

Claim 3.6. *If $C_1 \cap \cdots \cap C_r \neq \emptyset$ where $C_1 \in \mathcal{C}_1, \ldots, C_r \in \mathcal{C}_r$, then for arbitrary $1 \leq i < j \leq r$, $C_i \cap C_j = C_1 \cap \cdots \cap C_r$.*

Proof. If there were a vertex $x \in C_1 \cap \cdots \cap C_r$ and a vertex $y \in C_i \cap C_j \setminus C_\ell$ for some ℓ, then the edge xy would have color i and j but not color ℓ, which would contradict the fact that either $|\mathrm{Col}(uv)| = 1$ or $|\mathrm{Col}(uv)| = r$. □

Now let us take the following incidence structure $\mathcal{A}$: Let the points of $\mathcal{A}$ be the the nonempty intersections $C_1 \cap \cdots \cap C_r \neq \emptyset$, where $C_1 \in \mathcal{C}_1, \ldots, C_r \in \mathcal{C}_r$. Let the lines of $\mathcal{A}$ be the monochromatic components of G. Let a point corresponding to $C_1 \cap \cdots \cap C_r \neq \emptyset$ be incident with the lines corresponding to $C_1, \ldots, C_r$. Since each vertex of G is incident with edges of each color, this way each vertex of G is mapped to a point of $\mathcal{A}$. Also, for a nonempty intersection, $C_1 \cap \cdots \cap C_r = C_1 \cap C_2$. Since $|C_1 \cap C_2| = \frac{n}{(r-1)^2}$, each point of $\mathcal{A}$ corresponds exactly to $\frac{n}{(r-1)^2} =: b$ vertices of G.

We claim that $\mathcal{A}$ is an affine plane of order r-1. Moreover, we claim that two lines are disjoint if and only if the corresponding monochromatic components have the same color. Note that if we prove these statements, it follows that G is the blowup of an affine plane.

We have already proved that two components of G of different colors have a nonempty intersection. On the other hand, two monochromatic components of the same color are disjoint by the definition of component. Hence indeed two lines in $\mathcal{A}$ are disjoint if and only if the corresponding monochromatic components have the same color. To prove that $\mathcal{A}$ is an affine plane of order $r-1$, we need to check the following five conditions:

(i) Each two points are connected by exactly one line.

(ii) For each point x and line L such that $x \notin L$, there exists exactly one line L' such that $x \in L'$, but L' is disjoint from L.

(iii) Each line contains at least 2 points.

(iv) Each point is incident with at least 3 lines.

(v) The maximum number of pairwise parallel lines is $r-1$.

(i) We claim that the points corresponding to $C_1 \cap \dots \cap C_r \neq \emptyset$ and $C'_1 \cap \dots \cap C'_r \neq \emptyset$ where $C_1, C'_1 \in \mathcal{C}_1, \dots, C_r, C'_r \in \mathcal{C}_r$ have at least one common monochromatic component. Indeed, take $x \in C_1 \cap \dots \cap C_r$ and $y \in C'_1 \cap \dots \cap C'_r$. x and y are connected by an edge, since G is complete. This edge has at least one color, hence x and y has a common monochromatic component.

Now we claim that these two points have at most one common monochromatic component. Indeed, by Claim 3.6, if $C_i = C'_i$ and $C_j = C'_j$ for some $i \neq j$, then $C_1 \cap \dots \cap C_r = C_i \cap C_j = C'_i \cap C'_j = C'_1 \cap \dots \cap C'_r$.

(ii) Let C be the monochromatic component of G corresponding to the line L. As we noted before, two monochromatic components in G are disjoint if and only if they have the same color. Suppose that C has color i. Let C' be the component of color i that contains x. The line corresponding to C' satisfies the requirements of (ii).

(iii) If there is a line containing only one point, let the monochromatic component of G corresponding to the line be $C_i \in \mathcal{C}_i$ and the intersection corresponding to the point be $C_1 \cap \dots \cap C_r \neq \emptyset$ where $C_1 \in \mathcal{C}_1, \dots, C_r \in \mathcal{C}_r$. From the fact that the line has only one point, $C_i \subseteq C_1 \cap \dots \cap C_{i-1} \cap C_{i+1} \cap \dots \cap C_r$. But then $C_1, \dots C_{i-1}, C_{i+1}, \dots, C_r$ cover all the vertices of G since G is complete. Thus, the example is not sharp.

(iv) It can be seen from the definition that each point of $\mathcal{A}$ is incident with $r \geq 3$ lines.

(v) This follows from the fact that two lines are parallel if and only if they correspond to monochromatic components of the same color, and for each color, there are exactly $r-1$ monochromatic components.

With this, we have proved that any sharp example needs to be a blowup of an affine plane. We claim that the blowup of an affine plane is always a sharp example.

Indeed, we claim that $r-1$ monochromatic components of pairwise different colors cover at most $\left(1 - \frac{r-2}{(r-1)^2}\right) \cdot n = \left(1 - \frac{r-2}{(r-1)^2}\right) \cdot b(r-1)^2 = b((r-1)^2 - r - 2)$ vertices. Indeed, take the first component. This covers $b(r-1)$ vertices. The second component has different color from the first, hence they have an intersection of size $\frac{n}{(r-1)^2} = b$. Hence the two components together cover at most $b(r-1+r-2)$ vertices. And so on, each subsequent component needs to have an intersection of size at least b with the union of the previous ones, hence altogether, they cover at most $b((r-1)^2 - r - 2)$ vertices. We can also see, that for covering so many vertices, we need to take $r-1$ monochromatic components having a common intersection of b points. □

Remark 3.7. In the case if $\left(1 - \frac{r-2}{(r-1)^2}\right) \cdot |V(G)|$ is not an integer, it would be reasonable to call the multi edge-colored G sharp if the number of vertices coverable by $r-1$ monochromatic components of pairwise different colors is the minimum possible, i.e.,

$$\left\lceil \left(1 - \frac{r-2}{(r-1)^2}\right) \cdot |V(G)| \right\rceil.$$

We do not know the structure of the sharp examples in this sense.

4 Ryser's conjecture in the case $\Delta(\mathcal{H}) = 2$

For $r = 2$, Ryser's conjecture follows from Kőnig's theorem. In this section, we prove Ryser's conjecture for the very special case $\Delta(\mathcal{H}) = 2$ and $r \geq 3$. We note, that in this special case, the hypergraph does not even need to be r-partite for Ryser's bound to hold.

Theorem 4.1. *Let $\mathcal{H}$ be an r-uniform hypergraph with $r \geq 3$ and $\Delta(\mathcal{H}) = 2$. Then $\tau(\mathcal{H}) \leq (r-1) \cdot \nu(\mathcal{H})$.*

Proof. Let the dual of a hypergraph $\mathcal{H}$ be the following hypergraph $\mathcal{H}^*$, with multiple edges possible:

$$V(\mathcal{H}^*) = E(\mathcal{H})$$
$$E(\mathcal{H}^*) = \{\{e \in E(\mathcal{H}) : e \ni v\} : v \in V(\mathcal{H})\} \quad \text{taken as a multiset.}$$

We have $\mathcal{H}^{**} = \mathcal{H}$, hence vertices of $\mathcal{H}$ correspond exactly to hyperedges in $\mathcal{H}^*$ and hyperedges of $\mathcal{H}$ correspond exactly to vertices in $\mathcal{H}^*$.

Note that a set of vertices $T \subseteq V(\mathcal{H})$ covers the hyperedges of $\mathcal{H}$ if and only if the corresponding hyperedge set in $\mathcal{H}^*$ covers the vertices of $\mathcal{H}^*$, so $\tau(\mathcal{H}) = \varrho(\mathcal{H}^*)$.

The degree of a vertex of $\mathcal{H}^*$ is the cardinality of the corresponding hyperedge of $\mathcal{H}$. Hence $\mathcal{H}$ is r-uniform if and only if $\mathcal{H}^*$ is r-regular, consequently $\Delta(\mathcal{H}^*) = r$. By definition, $\alpha'(\mathcal{H}^*) = \nu(\mathcal{H})$.

If $\Delta(\mathcal{H}) = 2$, then $\mathcal{H}^*$ is a hypergraph with hyperedge cardinalities one or two, and the statement of the proposition is equivalent to $\varrho(\mathcal{H}^*) \leq (\Delta(\mathcal{H}^*) - 1)\alpha'(\mathcal{H}^*)$.

We can suppose that there are no hyperedges of cardinality one in $\mathcal{H}^*$. Indeed, if an hyperedge of cardinality one is contained by an hyperedge of cardinality two, then we can leave the hyperedge of cardinality one. This does not change the value of α', and $\Delta = \Delta(\mathcal{H}^*)$ can only decrease. If the statement is true to this hypergraph, then the same covering hyperedge set will be good for the original one.

If an hyperedge of cardinality one is not contained by an hyperedge of cardinality two, then this hyperedge (or a parallel copy of it) needs to occur in each hyperedge cover. Hence leaving this vertex and the cardinality one hyperedges incident to it, ϱ decreases by one. On the other hand, α' also decreases by one and Δ can only decrease. Hence if the statement is true to the modified hypergraph, it was also true to the original one.

Hence it is enough to prove, that for a graph G, $\varrho(G) \leq (\Delta(G) - 1) \cdot \alpha(G)$, as for graphs $\alpha'(G) = \alpha(G)$.

Lemma 4.2. *If G is not a cycle, then $\varrho(G) \leq (\Delta(G) - 1) \cdot \alpha(G)$.*

Proof. The statement is easily seen to be true for complete graphs with at least four vertices, hence we can suppose that G is not complete.

Let $|V(G)| = n$. Since G is not a cycle, using Brooks' theorem, G is colorable by $\Delta(G)$ colors. As consequence, $\alpha(G) \geq \frac{n}{\Delta}$.

Take an independent vertex set $I \subseteq V(G)$ of maximum size, and take a maximum matching M in $G[V(G) - I]$. Let $X = V(M)$ and $Y = V - I - X$. Since M is a maximum matching in $G[V(G) - I]$, it follows that Y is an independent set. Hence $G[Y \cup I]$ is a bipartite graph.

We show that in $G[Y \cup I]$ there is a matching covering Y. Suppose for contradiction that the condition of Hall's theorem is not satisfied, i.e., $\exists U \subseteq Y$ such that $|\Gamma(U)| < |U|$. then $(I \setminus \Gamma(U)) \cup U$ is an independent set, whose size is greater than $|I|$, which contradicts the choice of I.

Now take the following set of edges: The edges of M, the edges of a matching covering Y in $G[Y \cup I]$, and for each thus uncovered vertex in I, an edge covering it. This is an edge cover by definition. On the other hand, its cardinality is at most $|M| + |Y| + (|I| - |Y|) = |M| + |I|$, thus $\varrho \leq |M| + |I|$.

We show that $|M| + |I| \leq (\Delta(G) - 1)\alpha(G)$. Indeed, since $|X| \leq n - |I| = n - \alpha(G) \leq n(1 - 1/\Delta)$, we have $|M| \leq \lfloor \frac{n(1-1/\Delta)}{2} \rfloor = \lfloor (\Delta - 1)\frac{n}{2\Delta} \rfloor \leq \lfloor \frac{(\Delta-1)\alpha}{2} \rfloor$. Thus $\varrho \leq |M| + |I| \leq \lfloor \frac{(\Delta-1)\alpha}{2} \rfloor + \alpha \leq (\Delta - 1)\alpha$. □

Now the only remaining case is if the cardinality two edges of H^* form a cycle, that is, H^* is a cycle with some additional cardinality one edges. Suppose that the cycle has l vertices, plus there are k isolated vertices. Then the vertex set of H^* can be covered by $\lceil \frac{l}{2} \rceil + k$ edges, and $\alpha'(H^*) = \lfloor \frac{l}{2} \rfloor + k$. Since $r = \Delta(H^*) > 2$, this means $\varrho(\mathcal{H}^*) \leq (\Delta(\mathcal{H}^*) - 1)\alpha'(\mathcal{H}^*)$. □

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