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The directed disjoint shortest paths problem

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Abstract

In the k disjoint shortest paths problem, we are given a graph and its vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$, and the objective is to find k pairwise disjoint paths $P_1, \ldots, P_k$ such that each path P_i is a shortest path from s_i to t_i, if they exist. If the length of each edge is equal to zero, then this problem amounts to the disjoint paths problem, which is one of the well-studied problems in algorithmic graph theory and combinatorial optimization. Eilam-Tzoreff [5] focused on the case when the length of each edge is positive, and showed that the undirected version of the 2 disjoint shortest paths problem can be solved in polynomial time. Polynomial solvability of the directed version was posed as an open problem in [5]. In this paper, we solve this problem affirmatively, that is, we give a first polynomial time algorithm for the directed version of the 2 Disjoint Shortest Paths Problem when the length of each edge is positive. Note that the 2 disjoint paths problem in digraphs is NP-hard, which implies that the directed 2 disjoint shortest paths problem is NP-hard if the length of each edge can be zero. We extend our result to the case when the instance has two terminal pairs and the number of paths is a fixed constant greater than two. We also show that the undirected k disjoint shortest paths problem and the vertex-disjoint version of the directed k disjoint shortest paths problem can be solved in polynomial time if the input graph is planar and k is a fixed constant.

1 Introduction

1.1 Disjoint paths problem and disjoint shortest paths problem

The *vertex-disjoint paths problem* is one of the classic and well-studied problems in algorithmic graph theory and combinatorial optimization. In the problem, the input is a graph (or a digraph) $G = (V, E)$ and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, and the objective is to find k pairwise vertex-disjoint paths from s_i to t_i, if they exist. If k is part of the input, the vertex-disjoint paths problem is NP-hard [9], and it remains NP-hard even if the input graph is constrained to be planar [12]. The vertex-disjoint paths problem in undirected graphs can be solved in polynomial time when $k = 2$ [17, 18, 20],

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and Robertson and Seymour's graph minor theory gives an $O(|V|^3)$-time algorithm for the problem when k is a fixed constant [15]. The running time of this algorithm is improved to $O(|V|^2)$ in [10]. The vertex-disjoint paths problem in digraphs is much harder than the undirected version. Indeed, the directed version is NP-hard even when $k = 2$ [6]. The vertex-disjoint paths problem in planar digraphs can be solved in polynomial time for fixed k [16], and it is fixed parameter tractable with respect to parameter k [3].

The vertex-disjoint paths problem has many applications, for example in transportation networks, VLSI-design [7, 14], or routing in networks [13, 19]. When we deal with such practical applications, it is natural to generalize the problem to finding *short* or *cheep* vertex-disjoint paths. There are many results on the problem to find disjoint paths minimizing a given objective function such as the total length of the paths or the length of the longest path (see Section 1.2). In this paper, we consider the disjoint shortest paths problem introduced in [5], in which each path has to be a shortest path from s_i to t_i. Note that, in contrast to the other problems, the length of each path appears in the constraint of the problem. For an integer k, our problem is formally described as follows.

Directed (Undirected) k Disjoint Shortest Paths Problem

Input. A digraph (or a graph) $G = (V, E)$ with a length function $\ell : E \to \mathbb{R}_+$ and k pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$ in G.

Find. Pairwise disjoint (vertex-disjoint or edge-disjoint) paths $P_1, \dots, P_k$ such that P_i is a shortest path from s_i to t_i for $i = 1, 2, \dots, k$, if they exist.

Note that $\mathbb{R}_+$ denotes the set of non-negative real numbers. We can consider both directed and undirected variants of this problem, which we call the *directed k disjoint shortest paths problem* and the *undirected k disjoint shortest paths problem*, respectively. For each problem, we can consider vertex-disjoint and edge-disjoint versions. If the length of each edge is equal to zero, then these problems amount to the directed or the undirected version of the k disjoint paths problem. With this observation, most hardness results on the k disjoint paths problem can be extended to the directed (or undirected) k disjoint shortest paths problem. In particular, since the k disjoint paths problem in digraphs is NP-hard even when $k = 2$ [6], almost all variants of the directed k disjoint shortest paths problem are hard.

Only few positive results are known for the k disjoint shortest paths problem. An important positive result is a polynomial time algorithm of Eilam-Tzoreff [5] for the undirected 2 disjoint shortest paths problem, in which the length of each edge is positive. It is interesting to note that the algorithm in [5] is completely different from the algorithms for the 2 disjoint paths problem in [17, 18, 20]. This means that properties or tractability of the k disjoint shortest paths problem will be different from those of the k disjoint paths problem by assuming that the length of each edge is positive. This fact motivates us to study polynomial solvability of the directed k disjoint shortest paths problem under this assumption. Indeed, for the case when k is a fixed constant and the length of each edge is positive, polynomial solvability of the directed k disjoint shortest paths problem was posed as an open problem in [5].

1.2 Related work

There are many results on the problem in which we find k disjoint paths minimizing a given objective function. Such a problem is sometimes called the *shortest disjoint paths problem*. A natural objective function is the total length of the paths. That is, the aim of the problem is to find disjoint paths $P_1, \dots, P_k$ that minimizes $\sum_i \ell(P_i)$ when we are given a length function $\ell : E \to \mathbb{R}_+$, which we call the *min-sum k disjoint paths problem*. Here, $\ell(P_i)$ denotes the length of P_i. We note that a solution of the k disjoint shortest paths problem must be an optimal solution of the corresponding min-sum k disjoint paths problem, which shows that if we can solve the min-sum k disjoint paths problem, then we can also solve the k disjoint shortest paths problem. Another objective function is the length of the longest path. That is, the aim of the problem is to find disjoint paths $P_1, \dots, P_k$ that minimizes $\max_i \ell(P_i)$, which we call the *min-max k disjoint paths problem*.

Since the min-sum or min-max k disjoint paths problem is a generalization of the k disjoint paths problem, hardness results on the k disjoint paths problem can be extended to the optimization problem. See [11] for classical results on the min-sum and min-max k disjoint paths problems. We now describe several positive results on the min-sum k disjoint paths problem. Colin de Verdière and Schrijver [4] presented a polynomial time algorithm for the case when the input digraph (or graph) is planar, $s_1, \dots, s_k$ are on the boundary of a common face, and $t_1, \dots, t_k$ are on the boundary of another face. Kobayashi and Sommer [11] gave a polynomial time algorithm for the case when the graph is planar, $k = 2$, and the terminals are on at most two faces. Borradaile et al. [2] gave a polynomial time algorithm for the case when the graph is planar, the terminals are ordered nicely on a common face. Björklund and Husfeldt [1] gave a randomized polynomial time algorithm for the case when $k = 2$ and each edge has a unit length, which is based on interesting algebraic techniques. This result was recently generalized to the case with two terminal pairs by Hirai and Namba [8].

1.3 Our results

In this subsection, we describe our results, which are summarized in Table 1.

As mentioned in Section 1.1, it is not difficult see that the directed k disjoint shortest paths problem is NP-hard even when $k = 2$ if the length of each edge can be zero.

Proposition 1.1. Both vertex-disjoint and edge-disjoint versions of the directed k disjoint shortest paths problem are NP-hard even when $k = 2$.

Proof. Suppose that the length of each edge is equal to zero. In this case, since any path is a shortest path, the directed k disjoint shortest paths problem is equivalent to finding two vertex-disjoint (or edge-disjoint) paths P_1 and P_2 such that P_i is from s_i to t_i. This problem is known to be NP-hard [6], and hence the directed k disjoint shortest paths problem is NP-hard even when $k = 2$. $\square$

The main result of this paper is to show that the directed k disjoint shortest paths

	Conditions	Disjoint Paths	Disjoint Shortest Paths
$k = 2$	undirected	P [17, 18, 20]	P [5] $(*)$
	directed	NP-hard [6]	NP-hard (Proposition 1.1)
			P (Theorem 1.2) $(*)$
k: fixed	undirected	P [14]	OPEN
	planar, vertex-disjoint		P (Corollary 5.1)
	planar, edge-disjoint		P (Theorem 1.5)
	directed	NP-hard [6]	OPEN $(*)$ / NP-hard
	planar, vertex-disjoint	P [16]	P (Theorem 1.4)
	planar, edge-disjoint	OPEN	OPEN
k: general	undirected/directed	NP-hard [9]	NP-hard

Table 1: Results on the k disjoint paths problem and the k disjoint shortest paths problem. In the results with $(*)$, we assume that the length of each edge is positive.

problem can be solved in polynomial time when the length of each dicycle (directed cycle) is positive and $k = 2$.

Theorem 1.2. *If the length of each dicycle is positive, both vertex-disjoint and edge-disjoint versions of the directed* 2 *disjoint shortest paths problem can be solved in polynomial time. In particular, the directed* 2 *disjoint shortest paths problem can be solved in polynomial time if each edge has a positive length.*

The proof of this theorem is given in Section 3. It is posed as an open problem by Eilam-Tzoreff [5] to determine whether or not the directed k disjoint shortest paths problem can be solved in polynomial time when each edge has a positive length and k is a fixed constant. Theorem 1.2 answers this problem affirmatively for the case of $k = 2$. It is interesting to note that the assumption on the edge length affect the polynomial solvability of the problem as we can see in Proposition 1.1 and Theorem 1.2. We also note that a polynomial time algorithm for the undirected version can be derived from Theorem 1.2, that is, we obtain an alternative proof for the following result.

Corollary 1.3 (Eilam-Tzoreff [5])**.** If each edge has a positive length, both vertex-disjoint and edge-disjoint versions of the undirected 2 disjoint shortest paths problem can be solved in polynomial time.

Proof. Suppose we are given an instance of the undirected 2 disjoint shortest paths problem, where $\ell(e) > 0$ for every $e \in E$. Replace each edge $e = uv$ with two new vertices x_e, y_e and five new directed edges $ux_e, vx_e, x_ey_e, y_eu, y_ev$ (see Fig. 1). Define a new length function ℓ' by $\ell'(ux_e) = \ell'(vx_e) = \ell'(x_ey_e) = \ell'(y_eu) = \ell'(y_ev) = \frac{\ell(x_ey_e)}{3}$. Then, each edge has a positive length in the obtained digraph. In this way, we can reduce the undirected 2 disjoint shortest paths problem to the directed 2 disjoint shortest paths problem, which shows the corollary by Theorem 1.2. □

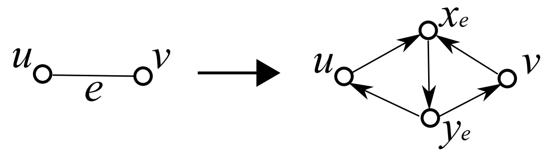

Figure 1: Reduction to the directed case

Theorem 1.2 can be extended to the case when the input graph contains two terminal pairs and k is a fixed constant, which is discussed in Section 4.

We also discuss the case when the input (di)graph is restricted to be planar in Section 5. We first show that the vertex-disjoint version of the directed k disjoint shortest paths problem can be solved in polynomial time in planar digraphs.

Theorem 1.4. *If k is a fixed constant and the input digraph is planar, the vertex-disjoint version of the directed k disjoint shortest paths problem can be solved in polynomial time.*

The proof is given in Section 5. Our proof is based on the reduction technique used in the proof of Theorem 1.2 and the algorithm for the disjoint paths problem in planar digraphs proposed in [16]. Note that this result implies that we can also solve the undirected version in polynomial time. Since Schrijver's algorithm for the disjoint paths problem [16] works only for the vertex-disjoint case, the proof of Theorem 1.4 cannot be extended to the edge-disjoint case directly. However, when the graph is undirected, we can show the following theorem, whose proof is given in Section 5.

Theorem 1.5. *If k is a fixed constant and the input graph is planar, the edge-disjoint version of the undirected k disjoint shortest paths problem can be solved in polynomial time.*

2 Preliminary

For a digraph $G = (V, E)$, a directed edge from u to v is denoted by uv. For a directed edge e in G, the head and the tail of e are denoted by $\mathrm{head}_G(e)$ and $\mathrm{tail}_G(e)$, respectively, that is, e is a directed edge from $\mathrm{tail}_G(e)$ to $\mathrm{head}_G(e)$. A *dipath* (or a *directed path*) is a sequence $(v_0, e_1, v_1, e_2, \dots, e_p, v_p)$ such that $v_0, v_1, \dots, v_p \in V$ are distinct vertices and $e_i = v_{i-1}v_i \in E$ for each i. If $v_0 = v_p$ in the definition of a dipath, the sequence is called a *dicycle* (or a *directed cycle*). If no confusion may arise, a dicycle, a dipath, and a directed edge are simply called a cycle, a path, and an edge, respectively. For a dipath, a dicycle, or a subgraph Q, its vertex set and edge set are denoted by $V(Q)$ and $E(Q)$, respectively. For a length function $\ell : E \to \mathbb{R}_+$ and for an edge set $F \subseteq E$, we denote $\ell(F) = \sum_{e \in F} \ell(e)$. For a dipath or a dicycle Q, we identify Q with its edge set, and $\ell(E(Q))$ is simply denoted by $\ell(Q)$.

3 Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2, that is, we show that the directed 2 disjoint shortest paths problem can be solved in polynomial time if the length of each dicycle is positive. We first note that the vertex-disjoint version of the directed 2 disjoint shortest paths problem can be reduced to the edge-disjoint version of the directed 2 disjoint shortest paths problem by the following procedure: replace each vertex v with two vertices v^+ and v^-, replace each edge uv with an edge u^+v^- of the same length, and add an edge v^-v^+ of length zero for each v (see Fig. 2). Therefore, it suffices to give a polynomial-time algorithm for the edge-disjoint version of the problem.

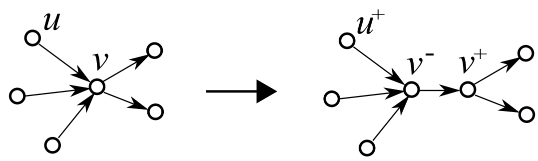

Figure 2: Reduction to the edge-disjoint version

Suppose we have an instance of the edge-disjoint version of the directed 2 disjoint shortest paths problem, in which each dicycle is of positive length. For $i = 1, 2$, let $E_i \subseteq E$ be the set of edges that are contained in some shortest path from s_i to t_i. By the definition, an s_i-t_i path is a shortest s_i-t_i path if and only if it consists of edges in E_i. Note that we can compute E_i in polynomial time as follows. We first apply a shortest path algorithm (e.g., Dijkstra's algorithm) and obtain the distance $d_i(v)$ from s_i to v for every $v \in V$. Let $E'_i \subseteq E$ be the set of all the edges uv with $d_i(v) - d_i(u) = \ell(uv)$. Then, $\{uv \in E'_i \mid E'_i \text{ contains a } v\text{-}t_i \text{ path}\}$ is the desired set E_i. With this observation, the edge-disjoint version of the directed 2 disjoint shortest paths problem can be reduced to the following problem: given a digraph $G = (V, E)$, subsets $E_1, E_2 \subseteq E$, and two pairs of vertices (s_1, t_1) and (s_2, t_2) in G, find edge-disjoint paths P_1 and P_2 such that $E(P_i) \subseteq E_i$ and P_i is a path from s_i to t_i for $i = 1, 2$. We now show some properties of E_i.

Claim 3.1. *The edge set E_i forms no dicycle for $i = 1, 2$.*

Proof of the claim. Assume that E_i forms a dicycle C. By the definition of d_i and E_i, $d_i(v) - d_i(u) = \ell(uv)$ for each $uv \in E(C)$. This shows that $\ell(C) = \sum_{uv \in E(C)} \ell(uv) = \sum_{uv \in E(C)} (d_i(v) - d_i(u)) = 0$, which contradicts that the length of each dicycle is positive. □

For a set F of directed edges, let $\overline{F}$ be the set of directed edges obtained from F by reversing all the edges, that is, $\overline{F} = \{vu \mid uv \in F\}$. Then, we have the following claim.

Claim 3.2. *Suppose that C is a dicycle in $E_1 \cup \overline{E}_2$. Then, $E_1 \cap E(C) \subseteq E_2$ and $E_2 \cap \overline{E(C)} \subseteq E_1$.*

Proof of the claim. Since C is a dicycle in $E_1 \cup \overline{E}_2$, it can be decomposed into subpaths $P_1, \overline{Q_1}$, $P_2, \overline{Q_2}$, $\dots$, $P_r, \overline{Q_r}$ such that P_i is a dipath from u_i to v_i with $E(P_i) \subseteq E_1$ and Q_i

is a dipath from u_{i+1} to v_i with $E(Q_i) \subseteq E_2$ for $i = 1, \dots, r$, where we denote $u_{r+1} = u_1$. By the definition of d_1 and E_1, $d_1(v_i) - d_1(u_i) = \ell(P_i)$ and $d_1(v_i) - d_1(u_{i+1}) \geq \ell(Q_i)$ for $i = 1, \dots, r$. By combining them, we obtain $\sum_{i=1}^r \ell(P_i) \geq \sum_{i=1}^r \ell(Q_i)$. Similarly, by the definition of d_2 and E_2, $d_2(v_i) - d_2(u_i) \geq \ell(P_i)$ and $d_2(v_i) - d_2(u_{i+1}) = \ell(Q_i)$ for $i = 1, \dots, r$, which shows that $\sum_{i=1}^r \ell(P_i) \leq \sum_{i=1}^r \ell(Q_i)$. Therefore, $\sum_{i=1}^r \ell(P_i) = \sum_{i=1}^r \ell(Q_i)$ and all the above inequalities are tight. That is, $d_1(v_i) - d_1(u_{i+1}) = \ell(Q_i)$ and $d_2(v_i) - d_2(u_i) = \ell(P_i)$ for $i = 1, \dots, r$, which shows that $E(Q_i) \subseteq E'_1$ and $E(P_i) \subseteq E'_2$. Since $E(P_i) \subseteq E_1$ for $i = 1, \dots, r$, there is a v_i-t_1 path in E'_1. This implies that E'_1 contains a v-t_1 path for any $v \in V(Q_i)$, and hence $E(Q_i) \subseteq E_1$. Similarly, since $E(Q_i) \subseteq E_2$ for $i = 1, \dots, r$, there is a v_i-t_2 path in E'_2, which shows that $E(P_i) \subseteq E_2$. □

We add four vertices s'_1, s'_2, t'_1, and t'_2, and four edges $s'_1 s_1$, $s'_2 s_2$, $t_1 t'_1$, and $t_2 t'_2$. We update $E_i \leftarrow E_i \cup \{s'_i s_i, t_i t'_i\}$ for $i = 1, 2$. Then, a path from s_i to t_i is corresponding to a path whose first and last edges are $s'_i s_i$ and $t_i t'_i$, respectively. By using this correspondence, we can rephrase the problem to the following: find edge-disjoint paths P_1 and P_2 such that $E(P_i) \subseteq E_i$ and P_i is a path whose first and last edges are $s'_i s_i$ and $t_i t'_i$, respectively.

Let $E_0 := E_1 \cap E_2$, $E^*_1 = E_1 \setminus E_0$, and $E^*_2 = E_2 \setminus E_0$. We remove all the edges in $E \setminus (E_1 \cup E_2)$ from G, contract all the edges in E_0, and reverse all the edges in E^*_2. Then, we obtain a digraph $G^* = (V^*, E^*)$. Let $V_0 \subseteq V^*$ be the set of all the vertices in V^* that are newly created by contracting E_0. In other words, $V^* \setminus V_0 \subseteq V$ is the set of all original vertices. For $v \in V_0$, let G_v be the subgraph of $G - (E \setminus (E_1 \cup E_2))$ induced by the vertex set corresponding to v. For any edge e in G_v, by the definition of G_v, either $e \in E_0$ or there exist edges $f_1, f_2, \dots, f_r \in E_0$ such that $e, f_1, f_2, \dots, f_r$ form a cycle when we ignore the direction of the edges. In the latter case, these edges induce a dicycle C in $E_1 \cup \overline{E}_2$, which shows that $e \in E_0$ by Claim 3.2. Thus, every edge in G_v is in E_0, which implies that we can identify E^* with $E^*_1 \cup \overline{E^*_2}$. Furthermore, since every edge in G_v is in E_0, G_v is an acyclic digraph by Claim 3.1.

We can also see that, by Claim 3.2, G^* is an acyclic digraph. In what follows, roughly, we find two disjoint paths in G^* such that one is from s'_1 to t'_1 and the other is from t'_2 to s'_2. Our algorithm is based on the algorithm for finding disjoint paths in digraphs proposed in [6].

We define a new digraph $\mathcal{G}$ whose vertex set is $W = E^*_1 \times \overline{E^*_2}$ as follows. For $(e_1, e_2), (e'_1, e'_2) \in W$, $\mathcal{G}$ has a directed edge from (e_1, e_2) to (e'_1, e'_2) if one of the following holds.

- $e'_1 = e_1$, $\mathrm{head}_{G^*}(e_2) = \mathrm{tail}_{G^*}(e'_2) =: v$, and there is no path in G^* from $\mathrm{head}_{G^*}(e_1)$ to v. Furthermore, if $v \in V_0$, then G_v contains a path from $\mathrm{tail}_G(e'_2)$ to $\mathrm{head}_G(e_2)$.
- $e'_2 = e_2$, $\mathrm{head}_{G^*}(e_1) = \mathrm{tail}_{G^*}(e'_1) =: v$, and there is no path in G^* from $\mathrm{head}_{G^*}(e_2)$ to v. Furthermore, if $v \in V_0$, then G_v contains a path from $\mathrm{head}_G(e_1)$ to $\mathrm{tail}_G(e'_1)$.
- $\mathrm{head}_{G^*}(e_1) = \mathrm{head}_{G^*}(e_2) = \mathrm{tail}_{G^*}(e'_1) = \mathrm{tail}_{G^*}(e'_2) =: v$. Furthermore, if $v \in V_0$, then G_v contains two edge-disjoint paths such that one is from $\mathrm{head}_G(e_1)$ to $\mathrm{tail}_G(e'_1)$

and the other is from $\text{tail}_G(e'_2)$ to $\text{head}_G(e_2)$.

To construct $\mathcal{G}$, it suffices to solve the two disjoint paths problem in each acyclic digraph G_v, which can be done in polynomial time by [6]. We now show that we can solve the edge-disjoint version of the directed 2 disjoint shortest paths problem by finding a path in $\mathcal{G}$ from $(s'_1 s_1, t'_2 t_2)$ to $(t_1 t'_1, s_2 s'_2)$.

Claim 3.3. *There is a path in $\mathcal{G}$ from $(s'_1 s_1, t'_2 t_2)$ to $(t_1 t'_1, s_2 s'_2)$ if and only if G has two edge-disjoint paths P_1 and P_2 such that P_i is from s_i to t_i and $E(P_i) \subseteq E_i$ for $i = 1, 2$.*

Proof. Sufficiency ("if" part). Suppose that G has two edge-disjoint paths P_1 and P_2 such that P_i is from s_i to t_i and $E(P_i) \subseteq E_i$ for $i = 1, 2$. $E(P_1) \setminus E_0$ forms a path P^*_1 from s_1 to t_1 in G^*, and $\overline{E(P_2) \setminus E_0}$ forms a path P^*_2 from t_2 to s_2 in G^*. Suppose that P^*_1 traverses edges $e^1_1, e^2_1, \dots, e^p_1$ in this order, and let $e^0_1 := s'_1 s_1$ and $e^{p+1}_1 := t_1 t'_1$. Similarly, suppose that P^*_2 traverses edges $e^1_2, e^2_2, \dots, e^q_2$ in this order, and let $e^0_2 := t'_2 t_2$ and $e^{q+1}_2 := s_2 s'_2$. It is obvious that $e^i_1 \in E^*_1$ for $i = 0, 1, \dots, p+1$ and $e^j_2 \in \overline{E^*_2}$ for $j = 0, 1, \dots, q+1$. Since G^* is acyclic, for any $i = 0, 1, \dots, p+1$ and for any $j = 0, 1, \dots, q+1$, at least one of the following holds.

(1) There is no dipath in G^* from $\text{head}_{G^*}(e^i_1)$ to $\text{head}_{G^*}(e^j_2)$.

(2) There is no dipath in G^* from $\text{head}_{G^*}(e^j_2)$ to $\text{head}_{G^*}(e^i_1)$.

(3) $\text{head}_{G^*}(e^i_1) = \text{head}_{G^*}(e^j_2)$.

For each case, we obtain the following by the definition of the edge set of $\mathcal{G}$.

- If (1) holds and $j \neq q+1$, then $\mathcal{G}$ has an edge from (e^i_1, e^j_2) to (e^i_1, e^{j+1}_2). Note that if $v := \text{head}_{G^*}(e^j_2) \in V_0$, then $E(P_2) \cap E(G_v)$ forms a path in G_v from $\text{tail}_G(e^{j+1}_2)$ to $\text{head}_G(e^j_2)$.
- If (2) holds and $i \neq p+1$, then $\mathcal{G}$ has an edge from (e^i_1, e^j_2) to (e^{i+1}_1, e^j_2). Note that if $v := \text{head}_{G^*}(e^i_1) \in V_0$, then $E(P_1) \cap E(G_v)$ forms a path in G_v from $\text{head}_G(e^i_1)$ to $\text{tail}_G(e^{i+1}_1)$.
- If (3) holds, then $\mathcal{G}$ has an edge from (e^i_1, e^j_2) to (e^{i+1}_1, e^{j+1}_2). Note that if $v := \text{head}_{G^*}(e^i_1) = \text{head}_{G^*}(e^j_2) \in V_0$, then $E(P_1) \cap E(G_v)$ and $E(P_2) \cap E(G_v)$ form two edge-disjoint paths in G_v such that one is from $\text{head}_G(e^i_1)$ to $\text{tail}_G(e^{i+1}_1)$ and the other is from $\text{tail}_G(e^{j+1}_2)$ to $\text{head}_G(e^j_2)$.

By observing that (1) holds if $i = p+1$ and (2) holds if $j = q+1$, we can see that $\mathcal{G}$ has an edge from (e^i_1, e^j_2) to (e^i_1, e^{j+1}_2), (e^{i+1}_1, e^j_2), or (e^{i+1}_1, e^{j+1}_2) unless $(i, j) = (p+1, q+1)$. We begin with $(i, j) = (0, 0)$ and find an edge leaving (e^i_1, e^j_2) in $\mathcal{G}$ as above, repeatedly. Then, we obtain a path in $\mathcal{G}$ from $(e^0_1, e^0_2) = (s'_1 s_1, t'_2 t_2)$ to $(e^{p+1}_1, e^{q+1}_2) = (t_1 t'_1, s_2 s'_2)$, which shows the sufficiency of the claim.

Necessity ("only if" part). Suppose that there is a path in $\mathcal{G}$ from $(f^0_1, f^0_2) := (s'_1 s_1, t'_2 t_2)$ to $(f^r_1, f^r_2) := (t_1 t'_1, s_2 s'_2)$ that traverses vertices $(f^0_1, f^0_2), (f^1_1, f^1_2), \dots, (f^r_1, f^r_2)$ of $\mathcal{G}$ in this order. In this proof, we regard a path in G as a sequence of edges, and the concatenation of two paths P and Q is denoted by $P \cdot Q$. We define two paths P_1 and P_2 as follows.

1. Set $P_1 = P_2 = \emptyset$.
2. For $i = 0, 1, 2, \dots, r$, we update P_i as follows.
   - Suppose that $f_1^{i+1} = f_1^i$, $\mathrm{head}_{G^*}(f_2^i) = \mathrm{tail}_{G^*}(f_2^{i+1}) =: v$, and there is no dipath in G^* from $\mathrm{head}_{G^*}(f_1^i)$ to v. In this case, let Q be the path in G_v from $\mathrm{tail}_G(f_2^{i+1})$ to $\mathrm{head}_G(f_2^i)$ if $v \in V_0$ and let $Q = \emptyset$ if $v \notin V_0$. Then, update P_2 as $P_2 \leftarrow f_2^{i+1} \cdot Q \cdot P_2$.
   - Suppose that $f_2^{i+1} = f_2^i$, $\mathrm{head}_{G^*}(f_1^i) = \mathrm{tail}_{G^*}(f_1^{i+1}) =: v$, and there is no dipath in G^* from $\mathrm{head}_{G^*}(f_2^i)$ to v. In this case, let Q be the path in G_v from $\mathrm{head}_G(f_1^i)$ to $\mathrm{tail}_G(f_1^{i+1})$ if $v \in V_0$ and let $Q = \emptyset$ if $v \notin V_0$. Then, update P_1 as $P_1 \leftarrow P_1 \cdot Q \cdot f_1^{i+1}$.
   - Suppose that $\mathrm{head}_{G^*}(f_1^i) = \mathrm{head}_{G^*}(f_2^i) = \mathrm{tail}_{G^*}(f_1^{i+1}) = \mathrm{tail}_{G^*}(f_2^{i+1}) =: v$. In this case, if $v \in V_0$, then G_v contains two edge-disjoint paths Q_1 and Q_2 such that Q_1 is from $\mathrm{head}_G(f_1^i)$ to $\mathrm{tail}_G(f_1^{i+1})$ and Q_2 is from $\mathrm{tail}_G(f_2^{i+1})$ to $\mathrm{head}_G(f_2^i)$. Let $Q_1 = Q_2 = \emptyset$ if $v \notin V_0$. Then, update P_1 and P_2 as $P_1 \leftarrow P_1 \cdot Q_1 \cdot f_1^{i+1}$ and $P_2 \leftarrow f_2^{i+1} \cdot Q_2 \cdot P_2$.

Then, P_1 and P_2 are edge-disjoint paths in G such that P_i is from s_i to t_i and $E(P_i) \subseteq E_i$ for $i = 1, 2$, which shows the necessity of the claim. □

Since $\mathcal{G}$ contains at most $|E|^2$ vertices, we can detect a path in $\mathcal{G}$ in polynomial time. Thus, Claim 3.3 shows that the directed 2 disjoint shortest paths problem can be solved in polynomial time. □

4 Extension to the Case with Two Pairs

In this section, we extend Theorem 1.2 to the case when the digraph has two terminal pairs. More precisely, for fixed integers k_1 and k_2, we consider the following problem and give a polynomial time algorithm for it.

Directed Disjoint Shortest Paths Problem with Two Terminal Pairs.

Input. A digraph $G = (V, E)$ with a length function $l : E \to \mathbb{R}_+$, two pairs of vertices (s_1, t_1) and (s_2, t_2) in G.

Find. Pairwise disjoint (internally-vertex-disjoint or edge-disjoint) paths $P_1^1, \dots, P_{k_1}^1$, $P_1^2, \dots, P_{k_2}^2$ such that P_j^i is a shortest path from s_i to t_i for $i = 1, 2$ and $j = 1, 2, \dots, k_i$.

Our result is formally stated as follows.

Theorem 4.1. *Let k_1 and k_2 be fixed integers. If the length of each dicycle is positive, both internally-vertex-disjoint and edge-disjoint versions of the directed disjoint shortest paths problem with two terminal pairs can be solved in polynomial time.*

Proof. In the same way as the proof of Theorem 1.2, it suffices to give an algorithm for the edge-disjoint version. For $i = 1, 2$, let $E_i \subseteq E$ be the set of all the edges that are contained in some shortest path from s_i to t_i, which satisfy Claims 3.1 and 3.2. Then, an s_i-t_i path is a shortest s_i-t_i path if and only if it consists of edges in E_i.

We add $2(k_1 + k_2)$ vertices $s'_{1,1}, \dots, s'_{1,k_1}, s'_{2,1}, \dots, s'_{2,k_2}, t'_{1,1}, \dots, t'_{1,k_1}, t'_{2,1}, \dots, t'_{2,k_2}$, and $2(k_1 + k_2)$ edges $s'_{1,j}s_1$ and $t_1, t'_{1,j}$ for $j = 1, \dots, k_1$, and $s'_{2,j}s_2$ and $t_2, t'_{2,j}$ for $j = 1, \dots, k_2$. We update $E_i \leftarrow E_i \cup \{s'_{i,j}s_i, t_it'_{i,j} \mid j = 1, \dots, k_i\}$ for $i = 1, 2$. Then, we can rephrase the problem to the following: find edge-disjoint paths $P^1_1, \dots, P^1_{k_1}, P^2_1, \dots, P^2_{k_2}$ such that $E(P^i_j) \subseteq E_i$ and P^i_j is a path whose first and last edges are $s'_{i,j}s_i$ and $t_it'_{i,j}$ for each i and j.

Define E_0, E^*_1, E^*_2, G^*, V_0, and G_v for $v \in V_0$ in the same way as the proof of Theorem 1.2. Let $S_0 := \{(i, j) \mid i = 1, 2,\ j = 1, \dots, k_i\}$. We define a digraph $\mathcal{G}$ whose vertex set is $W = (E^*_1)^{k_1} \times (\overline{E^*_2})^{k_2}$ as follows. For $(e^1_1, \dots, e^1_{k_1}, e^2_1, \dots, e^2_{k_2}), (f^1_1, \dots, f^1_{k_1}, f^2_1, \dots, f^2_{k_2}) \in W$, $\mathcal{G}$ has an edge from $(e^1_1, \dots, e^1_{k_1}, e^2_1, \dots, e^2_{k_2})$ to $(f^1_1, \dots, f^1_{k_1}, f^2_1, \dots, f^2_{k_2})$ if there exists a non-empty set $S \subseteq S_0$ and a vertex $v \in V^*$ such that

> $\mathrm{head}_{G^*}(e^i_j) = \mathrm{tail}_{G^*}(f^i_j) = v$ for $(i, j) \in S$, and $e^i_j = f^i_j$ and there is no path in G^* from $\mathrm{head}_{G^*}(e^i_j)$ to v for $(i, j) \in S_0 \setminus S$. Furthermore, if $v \in V_0$, then G_v contains $|S|$ edges disjoint paths such that each path is from $\mathrm{head}_G(e^1_j)$ to $\mathrm{tail}_G(f^1_j)$ with $(1, j) \in S$ or from $\mathrm{tail}_G(f^2_j)$ to $\mathrm{head}_G(e^2_j)$ with $(2, j) \in S$.

Note that this is a generalization of the construction in the proof of Theorem 1.2. To construct $\mathcal{G}$, it suffices to solve the disjoint paths problem with at most k terminal pairs in each acyclic digraph G_v, which can be done in polynomial time by [6].

In the same way as Claim 3.3, there is a path in $\mathcal{G}$ from $(s'_{1,1}s_1, \dots, s'_{1,k_1}s_1, t'_{2,1}t_2, \dots, t'_{2,k_2}t_2)$ to $(t_1t'_{1,1}, \dots, t_1t'_{1,k_1}, s_2s'_{2,1}, \dots, s_2s'_{2,k_2})$ if and only if G has $k_1 + k_2$ edge-disjoint paths $P^1_1, \dots, P^1_{k_1}$, $P^2_1, \dots, P^2_{k_2}$ such that $E(P^i_j) \subseteq E_i$ and P^i_j is a path whose first and last edges are $s'_{i,j}s_i$ and $t_it'_{i,j}$ for each i and j. Since $\mathcal{G}$ has a polynomial size in $|V|$, we can detect a path in $\mathcal{G}$ from $(s'_{1,1}s_1, \dots, s'_{1,k_1}s_1, t'_{2,1}t_2, \dots, t'_{2,k_2}t_2)$ to $(t_1t'_{1,1}, \dots, t_1t'_{1,k_1}, s_2s'_{2,1}, \dots, s_2s'_{2,k_2})$ in polynomial time. Hence, we can solve the directed disjoint shortest paths problem with two terminal pairs in polynomial time. $\square$

5 Planar Cases

In this section, we discuss the case when the input (di)graph is planar. We first give a proof of Theorem 1.4, which we restate here.

Theorem. If k is a fixed constant and the input digraph is planar, the vertex-disjoint version of the directed k disjoint shortest paths problem can be solved in polynomial time.

Proof. For $i = 1, \ldots, k$, let $E_i \subseteq E$ be the set of all the edges that are contained in some shortest path from s_i to t_i. Since an s_i-t_i path is a shortest s_i-t_i path if and only if it consists of edges in E_i, the directed k disjoint shortest paths problem in a planar digraph can be reduced to the following problem: given a planar digraph $G = (V, E)$, subsets $E_1, \ldots, E_k \subseteq E$, and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ in G, find vertex-disjoint paths $P_1, \ldots, P_k$ such that $E(P_i) \subseteq E_i$ and P_i is a path from s_i to t_i for $i = 1, \ldots, k$. It is shown in [16] that this problem can be solved in polynomial time for fixed k if G is planar. Therefore, for fixed k, the directed k disjoint shortest paths problem can be solved in polynomial time if the input digraph is planar. □

By replacing each edge with two parallel edges in opposite directions, we can reduce the undirected version to the directed version. Hence, Theorem 1.4 implies the following as a corollary.

Corollary 5.1. If k is a fixed constant and the input graph is planar, the vertex-disjoint version of the undirected k disjoint shortest paths problem can be solved in polynomial time.

We note that Schrijver's algorithm for finding disjoint paths $P_1, \ldots, P_k$ with $E(P_i) \subseteq E_i$ [16] works only for the vertex-disjoint case, and no polynomial time algorithm is known for the edge-disjoint version of this problem. However, when the graph is undirected, the edge-disjoint version of the k disjoint shortest paths problem can be solved in polynomial time (Theorem 1.5). We restate Theorem 1.5 here.

Theorem. If k is a fixed constant and the input graph is planar, the edge-disjoint version of the undirected k disjoint shortest paths problem can be solved in polynomial time.

To prove Theorem 1.5, we first give a polynomial time algorithm for the case when the obtained paths do not cross each other. Here, we say that two edge-disjoint paths P and Q in a planar graph *cross* at a vertex v if P contains two edges e_1 and e_2 and Q contains two edges f_1 and f_2 such that e_1, f_1, e_2, and f_2 are incident to v clockwise in this order. The problem is formally described as follows.

Undirected k Edge-disjoint Non-crossing Shortest Paths Problem

Input. A planar graph $G = (V, E)$ with a length function $\ell : E \to \mathbb{R}_+$ and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$ in G.

Find. Pairwise edge-disjoint paths $P_1, \ldots, P_k$ such that P_i is a shortest path from s_i to t_i for $i = 1, 2, \ldots, k$ and they do not cross each other, if they exist.

Proposition 5.2. If k is a fixed constant, then the undirected k edge-disjoint non-crossing shortest paths problem can be solved in polynomial time.

Proof. We first reduce the problem to the case when each terminal is of degree one. Suppose we are given an instance of the the undirected k edge-disjoint non-crossing shortest paths problem. For $i = 1, \dots, k$, we guess the first and last edges of P_i, say $s_i u_i$ and $v_i t_i$. Then, replace edge $s_i u_i$ with a new vertex u'_i and a new edge $u'_i u_i$, and define a new terminal $s'_i = u'_i$. Similarly, replace edge $v_i t_i$ with a new vertex v'_i and a new edge $v_i v'_i$, and define a new terminal $t'_i = v'_i$. In the obtained graph, we consider the undirected k edge-disjoint non-crossing shortest paths problem with terminal pairs $(s'_1, t'_1), \dots, (s'_k, t'_k)$. Note that each terminal is of degree one in the obtained instance. Since the number of choices of $s_i u_i$ and $v_i t_i$ is at most $|V|^{O(k)}$, which is polynomial in $|V|$, in order to solve the original instance, it suffices to solve $|V|^{O(k)}$ instances in which each terminal is of degree one.

In what follows, we give an algorithm for the case when each terminal is of degree one by using a reduction to the vertex-disjoint version of the undirected k disjoint shortest paths problem. Suppose that we are given an instance $G = (V, E)$, $\ell : E \to \mathbb{R}_+$, and $(s_1, t_1), \dots, (s_k, t_k)$ of the undirected k edge-disjoint non-crossing shortest paths problem in which each terminal is of degree one. For a vertex $v \in V$ of degree at least four, let $e_1, \dots, e_r$ be the edges that are incident to v clockwise in this order. We replace v with r vertices $w_1, \dots, w_r$ so that each edge e_i is incident to w_i, and add r edges $w_1 w_2, w_2 w_3, \dots, w_{r-1} w_r, w_r w_1$ (see Fig. 3). Note that this transformation keeps the planarity of the graph. By applying this transformation to every vertex $v \in V$ of degree at least four, we obtain a new planar graph $G' = (V', E')$ whose maximum degree is at most three. We can easily see that the undirected k edge-disjoint non-crossing shortest paths problem in G is equivalent to that in G'. Since the maximum degree of G' is at most three and the degree of each terminal is one, edge-disjoint paths in G' have to be vertex-disjoint, and hence it suffices to solve the vertex-disjoint version of the undirected k disjoint shortest paths problem in G'. This can be done in polynomial time by Corollary 5.1, which completes the proof.

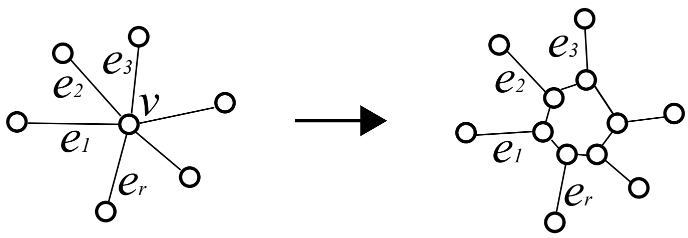

Figure 3: Reduction to the vertex-disjoint version

□

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose we are given an instance of the edge-disjoint version of the undirected k disjoint shortest paths problem in a planar graph. We begin with the following claim.

Claim 5.3. *If there exists a solution of the edge-disjoint version of the undirected k disjoint shortest paths problem in a planar graph, then there exists a solution $P_1, \ldots, P_k$ such that P_i and P_j cross at most once for every pair $i, j \in \{1, \ldots, k\}$.*

Proof. Let $P_1, \ldots, P_k$ be a solution of the edge-disjoint version of the undirected k disjoint shortest paths problem that minimizes the total number of crossings of the paths. We show that this solution satisfies the condition in the claim. Assume to the contrary that P_i and P_j cross at two distinct vertices u and v. Then, there exists a subpath Q_i of P_i and a subpath Q_j of P_j such that both Q_i and Q_j are paths from u to v. Since P_i is a shortest path from s_i to t_i and P_j is a shortest path from s_j to t_j, we have $\ell(Q_i) = \ell(Q_j)$. This shows that, we can obtain another solution of the edge-disjoint version of the undirected k disjoint shortest paths problem by replacing P_i and P_j with two paths P'_i and P'_j such that $E(P'_i) = (E(P_i) \setminus E(Q_i)) \cup E(Q_j)$ and $E(P'_j) = (E(P_j) \setminus E(Q_j)) \cup E(Q_i)$. We can see that the number of crossings of P'_i and P'_j is strictly smaller than that of P_i and P_j. We can also see that, for any $h \in \{1, \ldots, k\} \setminus \{i, j\}$, the number of crossings of P_h and $\{P'_i, P'_j\}$ is at most that of P_h and $\{P_i, P_j\}$. Therefore, the total number of crossings of the obtained solution is smaller than the original solution, which is a contradiction. $\square$

Let $P_1, \ldots, P_k$ be a solution of the edge-disjoint version of the undirected k disjoint shortest paths problem satisfying the condition in the above claim. For $i = 1, \ldots, k$, by Claim 5.3, there exist at most $k-1$ vertices $u^i_1, u^i_2, \ldots, u^i_{r_i}$ such that P_i crosses another path at some u^i_j and $s_i =: u^i_0, u^i_1, u^i_2, \ldots, u^i_{r_i}, u^i_{r_i+1} := t_i$ appear in this order along P_i. Then, P_i can be divided into $r_i + 1 \leq k$ subpaths $Q^i_1, \ldots Q^i_{r_i+1}$, where Q^i_j is a shortest path from u^i_{j-1} to u^i_j. By the definition of Q^i_j, we can see that Q^i_j $(i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1)$ are edge-disjoint paths and they do not cross each other.

With this observation, we can solve the edge-disjoint version of the undirected k disjoint shortest paths problem as follows.

Step 1. For $i = 1, \ldots, k$, guess an integer $r_i \leq k - 1$ and vertices $u^i_1, u^i_2, \ldots, u^i_{r_i}$.

Step 2. Find pairwise edge-disjoint paths Q^i_j $(i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1)$ such that they do not cross each other and Q^i_j is a shortest path from u^i_{j-1} to u^i_j, where $u^i_0 = s_i$ and $u^i_{r_i+1} = t_i$.

Step 3. For each i, define P_i as the concatenation of $Q^i_1, \ldots, Q^i_{r_i+1}$. Check whether or not $P_1, \ldots, P_k$ are a solution of the original instance.

In Step 1, the number of choices of r_i and $u^i_1, u^i_2, \ldots, u^i_{r_i}$ is at most $|V|^{O(k^2)}$, which is polynomial in $|V|$. In Step 2, we can find desired edge-disjoint paths Q^i_j $(i = 1, \ldots, k,\ j = 1, \ldots, r_i + 1)$ if they exist in polynomial time by Proposition 5.2. Note that the number of terminals is at most $O(k^2)$, which is a fixed constant. In Step 3, we can easily check whether or not $P_1, \ldots, P_k$ are a solution of the original problem in polynomial time. Therefore, the edge-disjoint version of the undirected k disjoint shortest paths problem can be solved in polynomial time if k is a fixed constant and the input graph is planar. $\square$

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