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Supermodularity in Unweighted Graph Optimization II: Matroidal Term Rank Augmentation

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Abstract

Ryser's max term rank formula with graph theoretic terminology is equivalent to a characterization of degree sequences of simple bipartite graphs with matching number at least ℓ. In a previous paper [1] by the authors, a generalization was developed for the case when the degrees are constrained by upper and lower bounds. Here two other extensions of Ryser's theorem are discussed. The first one is a matroidal model, while the second one settles the augmentation version. In fact, the two directions shall be integrated into one single framework.

1 Introduction

Ryser [16] derived a formula for the maximum term rank of a $(0,1)$-matrix with specified row- and column-sums. In graph theoretic terms, his theorem is equivalent to a characterization for the existence of a degree-specified simple bipartite graph (bigraph for short) with matching number at least ℓ. Several natural extension, like the min-cost and the subgraph version, turned out to be **NP**-complete, but in a previous paper [1], we could extend Ryser's theorem to the degree-constrained case when, instead of exact degree-specifications, lower and upper bounds are imposed on the degrees of the bigraph. An even more general problem was also solved when, in addition, lower and upper bounds are imposed on the number of edges. The main tool in [1] for proving these extensions was a general framework for covering an intersecting supermodular function by degree-constrained simple bipartite graphs.

In the present paper we consider two other extensions of Ryser's theorem: the augmentation and the matroidal version. In the first one, a given initial bigraph is to be augmented to get a simple degree-specified bigraph with matching number at least ℓ. In original matrix terms, this means that some of the entries of the $(0,1)$-matrix are specified to be 1. The solvability of this version is in sharp contrast with the

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NP-completeness of another variation when some entries of the matrix are specified to be 0. (This follows from the **NP**-completeness of the problem that seeks to decide whether an initial bigraph G_0 has a perfectly matchable degree-specified subgraph, see [11], [13], [14].)

In the matroidal extension of Ryser's theorem, there are matroids on S and on T and the goal is to find a degree-specified simple bigraph including a matching that covers bases in both matroids. These results will be consequences of a general framework including both the augmentation and the matroidal cases.

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1.1 Notions and notations

We use the notation of [1]. Here we briefly repeat the most important notions. For a family $\mathcal{T}$ of sets, let $\cup\mathcal{T}$ denote the union of the members of $\mathcal{T}$. For a subpartition $\mathcal{T} = \{T_1, \dots, T_q\}$, we always assume that its members T_i are non-empty but $\mathcal{T}$ is allowed to be empty (that is, $q = 0$).

An arc st **enters** or **covers** a set X if $s \notin X$, $t \in X$. A digraph **covers** X if it contains an arc covering X. Let S and T be two non-empty subsets of a ground-set V. By an ST**-arc** we mean an arc st with $s \in S$ and $t \in T$. A set-function p is called **positively** T**-intersecting (**ST**-crossing) supermodular** if the supermodular inequality

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

holds whenever $p(X) > 0$, $p(Y) > 0$, and $X \cap Y \cap T \neq \emptyset$ (respectively, $X \cap Y \cap T \neq \emptyset$ and $S - (X \cup Y) \neq \emptyset$). Two sets X and Y are ST**-independent** if $X \cap Y \cap T = \emptyset$ and $S - (X \cup Y) = \emptyset$, that is, no ST-arc enters both sets.

For a function $m : V \to \mathbf{R}$, the set-function $\widetilde{m}$ is defined by $\widetilde{m}(X) = \sum[m(v) : v \in X]$. A set-function p can analogously extended to families $\mathcal{F}$ of sets by $\widetilde{p}(\mathcal{F}) = \sum[p(X) : X \in \mathcal{F}]$.

The following min-max theorem of Frank and Jordán [8] will be a basic tool in the proof of the main theorem.

Theorem 1.1 (Supermodular arc-covering, set-function version)**.** A positively ST-crossing supermodular function p for which $p(V') \leq 0$ holds when no ST-arc enters V' can be covered by γ (possibly parallel) ST-arcs if and only if $\widetilde{p}(\mathcal{I}) \leq \gamma$ holds for every ST-independent family $\mathcal{I}$ of subsets of V.

Henceforth we assume that S and T are two disjoint non-empty sets and $V := S \cup T$. Let $G^* = (S, T; E^*)$ denote the complete bipartite graph on bipartition (S, T). Let $D^* = (S, T; A^*)$ be the digraph arising from G^* by orienting each of its edges from S to T, that is, A^* consists of all ST-arcs. More generally, for a bigraph $H = (S, T; F)$, let $\overrightarrow{H} = (S, T; \overrightarrow{F})$ denote the digraph arising from H by orienting each of its edges from S toward T.

Throughout we are given a simple bigraph $H_0 = (S, T; F_0)$ serving as an initial bigraph to be augmented. For $E_0 := E^* - F_0$, the bigraph $G_0 = (S, T; E_0)$ is called the **bipartite complement** of H_0, that is, F_0 and E_0 partition E^*. Note that a bigraph $G = (S, T; E)$ is a subgraph of G_0 precisely if the augmented bigraph $G^+ = (S, T; F_0 + E)$ is simple. For $X \subseteq S$ and $Y \subseteq T$, let $d_{G_0}(X, Y)$ denote the number of edges of G_0 connecting X and Y.

2 Matroidal covering and augmentation

Let p_T be a positively intersecting supermodular function on T. In [1], we studied the problem of finding a simple degree-specified bigraph $G = (S, T; E)$ covering p_T in the sense that

$$|\Gamma_G(Y)| \geq p_T(Y) \text{ for every subset } Y \subseteq T,$$

where $\Gamma_G(Y)$ denotes the set of neighbours of Y. Here we consider a framework which is more general in two directions. First, for a given initial simple bigraph $H_0 = (S, T; F_0)$ we want to find a degree-specified bigraph G in such a way that $G^+ := G + H_0$ is simple and covers p_T. This kind of problems is often referred to as augmentation problem to be distinguished form the synthesis problem where F_0 is empty. If $p_T \equiv 0$, the augmentation problem is equivalent to finding a degree-specified subgraph of the bipartite complement of H_0.

Second, we extend the notion of covering to matroidal covering in the following sense. Let $M_S = (S, r_S)$ be a matroid on S with rank function r_S. A bigraph G is said to M_S**-cover** p_T if

$$r_S(\Gamma_G(Y)) \geq p_T(Y) \text{ for every subset } Y \subseteq T. \tag{1}$$

Clearly, when M_S is the free matroid, we are back at the original notion of covering by a bigraph.

2.1 Degree-specified matroidal augmentation

Let $m_V = (m_S, m_T)$ be a degree-specification. Our main goal is to describe a characterization for the existence of a simple bigraph G fitting m_V so that $G + H_0$ is simple and M_S-covers p_T. The more general problem, when there are upper and lower bounds on V, is the subject of an ongoing research. The general degree-constrained case was solved in [1] in the special case when H_0 has no edges and M_S is the ℓ-uniform matroid on S.

THEOREM 2.1. We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a positively intersecting supermodular function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph

$G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if

$$\begin{cases} \widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X,Y) + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}| r_S(X) \leq \gamma \\ \text{for } Y \subseteq T,\ X \subseteq S,\ \ \mathcal{T} \text{ a subpartition of } T - Y \text{ with } d_{H_0}(S - X, \cup\mathcal{T}) = 0, \end{cases} \tag{2}$$

where G_0 is the bipartite complement of H_0 and $d_{G_0}(X, Y)$ is the number of edges of G_0 connecting X and Y.

Proof. Necessity. Suppose that there is a requested graph G and let $G^+ = (S, T; E \cup F_0)$. Let $\mathcal{T} = \{T_1, \dots, T_q\}$ be a subpartition occurring in (2). Since $d_{H_0}(S - X, T_i) = 0$, we have

$$p_T(T_i) \leq r_S(\Gamma_{G^+}(T_i)) \leq r_S(\Gamma_{G^+}(T_i) \cap X)) + r_S(\Gamma_{G^+}(T_i) - X) =$$

$$r_S(\Gamma_{G^+}(T_i) \cap X)) + r_S(\Gamma_G(T_i) - X) \leq r_S(X) + |\Gamma_G(T_i) - X| \leq r_S(X) + d_G(T_i, S - X),$$

from which $d_G(T_i, S - X) \geq p_T(T_i) - r_S(X)$. Therefore G has at least $\sum_{i=1}^{q} [p_T(T_i) - r_S(X)]$ edges connecting $T - Y$ and $S - X$, and has at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y)$ edges with end-nodes in X or in Y, from which the inequality in (2) follows.

Let $\mathcal{H}_0 = \{V' \subseteq V : \text{no arc of } \overrightarrow{H_0} \text{ enters } V'\}$. Then $\mathcal{H}_0$ is closed under taking union and intersection. Define a set-function p_0 on V as follows.

$$p_0(V') := \begin{cases} \max\{p_T(y) - r_S(X), m_T(y) - |X| + d_{H_0}(y)\} & \text{if } V' = X + y \in \mathcal{H}_0\text{:} \\ & X \subseteq S,\ \ y \in T \\ p_T(Y) - r_S(X) & \text{if } V' = X \cup Y \in \mathcal{H}_0\text{:} \\ & X \subseteq S,\ Y \subseteq T,\ |Y| \geq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Note that $p_0(V')$ can be positive only if $V' \in \mathcal{H}_0$ and $\overrightarrow{G_0}$ covers V'.

Lemma 2.2. The set-function p_0 is positively T-intersecting supermodular.

Proof. Let X_1, X_2 be subsets of S and let Y_1, Y_2 be subsets of T for which $Y_1 \cap Y_2 \neq \emptyset$. Suppose that $p_0(V_i) > 0$ for $V_i = X_i \cup Y_i$ $(i = 1, 2)$. Then each of the sets V_1, V_2, $V_1 \cap V_2$, and $V_1 \cup V_2$ belongs to $\mathcal{H}_0$. We distinguish three cases.

Case 1 $p_0(V_i) = p_T(Y_i) - r_S(X_i)$ for $i = 1, 2$. If $|Y_1 \cap Y_2| \geq 2$, then

$$p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [p_T(Y_2) - r_S(X_2)] \leq$$

$$p_T(Y_1 \cap Y_2) - r_S(X_1 \cap X_2) + p_T(Y_1 \cup Y_2) - r_S(X_1 \cap X_2) = p_0(V_1 \cap V_2) + p_0(V_1 \cup V_1).$$

If $Y_1 \cap Y_2 = \{y\}$ for some element $y \in T$, then

$$p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [p_T(Y_2) - r_S(X_2)] \leq$$

$$p_T(Y_1 \cap Y_2) - r_S(X_1 \cap X_2) + p_T(Y_1 \cup Y_2) - r_S(X_1 \cap X_2) =$$

$$[p_T(y) - r_S(X_1 \cap X_2)] + p_0(V_1 \cup V_2) \le p_0((X_1 \cap X_2) + y) + p_0(V_1 \cup V_2) = p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2).$$

Case 2 $p_0(V_i) > p_T(Y_i) - r_S(X_i)$ for $i = 1, 2$. Then $Y_1 = Y_2 = \{y\}$ for some $y \in T$, and $p_0(V_i) = m_T(y) - |X_i| + d_{H_0}(y)$. We have

$$p_0(V_1) + p_T(V_2) = m_T(y) - |X_1| + d_{H_0}(y) + m_T(y) - |X_2| + d_{H_0}(y) =$$

$$m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) + m_T(y) - |X_1 \cup X_2| + d_{H_0}(y) \le p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2).$$

Case 3 $p_0(V_1) = p_T(Y_1) - r_S(X_1)$ and $p_0(V_2) > p_T(Y_2) - r_S(X_2)$. (The situation is analogous when the indices $i = 1, 2$ are interchanged.) Then $Y_2 = \{y\}$ for some $y \in T$ and $y \in Y_1$. Since

$$r_S(X_1 \cup X_2) - r_S(X_1) \le |(X_1 \cup X_2) - X_1| = |X_2| - |X_1 \cap X_2|,$$

we have $-r_S(X_1) - |X_2| \le -r_S(X_1 \cup X_2) - |X_1 \cap X_2|$ and hence

$$\begin{gathered} p_0(V_1) + p_0(V_2) = [p_T(Y_1) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] = \\ [p_T(Y_1 \cup Y_2) - r_S(X_1)] + [m_T(y) - |X_2| + d_{H_0}(y)] \le \\ p_T(Y_1 \cup Y_2) - r_S(X_1 \cup X_2) + m_T(y) - |X_1 \cap X_2| + d_{H_0}(y) \le \\ p_0(V_1 \cup V_2) + p_0(V_1 \cap V_2), \end{gathered}$$

as required. •

Claim 2.3. $m_S(s) \le d_{G_0}(s)$ for each $s \in S$.

Proof. By applying (2) to $Y = T, X = \{s\}$, and $\mathcal{T} = \emptyset$, the claim follows. •

For $s \in S$, let $V_s = \{v \in V - s : \ sv \notin F_0\}$, and let a set-function p_1 on V be defined as follows.

$$p_1(V') := \begin{cases} m_S(s) & \text{if } V' = V_s \text{ for some } s \in S \\ p_0(V') & \text{otherwise.} \end{cases} \tag{4}$$

Note that $p_1(V')$ can be positive only if $V' \in \mathcal{H}_0$ and $\overrightarrow{G_0}$ covers V'.

Claim 2.4. $p_1(V_s) \ge p_0(V_s)$ holds for every $s \in S$.

Proof. Consider first the case when $V_s \cap T = \{y\}$ for some $y \in T$. By applying (2) to $X = S - s$, to $Y = \{y\}$, and to $\mathcal{T} = \emptyset$, we get

$$m_T(y) - |S - s| + d_{H_0}(y) = m_T(y) - d_{G_0}(S - s, y) \le m_S(s).$$

By applying (2) to $X = S - s$, to $Y = \emptyset$, and to $\mathcal{T} = \{y\}$, we get $p_T(y) - r_S(S - s) \le m_S(s)$ from which

$$m_S(s) \ge \max\{p_T(y) - r_S(S - s),\ m_T(y) - |S - s| + d_{H_0}(y)\} = p_0(V_s).$$

Second, assume that $|V_s \cap T| \ge 2$. By applying (2) to $X = S - s$, to $Y = \emptyset$, and to $\mathcal{T} = \{V_s \cap T\}$ we get

$$p_0(V_s) = p_T(V_s \cap T) - r_S(S - s) \le m_S(s). \text{ •}$$

Claim 2.5. The set-function p_1 is positively ST-crossing supermodular.

Proof. It follows from Claim 2.4 that p_1 arises from p_0 by increasing its values on sets V_s $(s \in S)$. Let $V' \subset V$ be a set with ST-crossing V_s. Then $S \not\subseteq V_s \cup V'$ and hence $V' \cap S \subseteq V_s \cap S$. Therefore $V' \cap T \not\subseteq V_s \cap T$, that is, there is an element $t \in (V' - V_s) \cap T$. Since st is an arc of $\overrightarrow{H_0}$ entering V', we conclude that $p_1(V') = 0$, implying that p_1 is indeed positively ST-crossing supermodular. •

Let ν denote the maximum total p_1-value of ST-independent sets.

Lemma 2.6. $\nu = \gamma$.

Proof. Since the family $\mathcal{L} = \{V_s : s \in S\}$ is ST-independent, $\nu \geq \widetilde{p}_1(\mathcal{L}) = \widetilde{m}_S(S) = \gamma$. Suppose indirectly that $\nu > \gamma$ and let $\mathcal{I}$ be an ST-independent family for which $\widetilde{p}_1(\mathcal{I}) = \nu$. We can assume that $|\mathcal{I}|$ is minimal in which case $p_1(V') > 0$ for each $V' \in \mathcal{I}$.

Claim 2.7. There are no two T-intersecting members $V_1 = X_1 \cup Y_1$ and $V_2 = X_2 \cup Y_2$ of $\mathcal{I}$ for which $p_1(V_i) = p_0(V_i)$ $(i = 1, 2)$.

Proof. Suppose indirectly the existence of such V_1 and V_2. Since $\mathcal{I}$ is ST-independent, we must have $X_1 \cup X_2 = S$ and hence $p_0(V_1 \cup V_2) = 0$. Since p_0 is T-intersecting,

$$p_1(V_1) + p_1(V_2) = p_0(V_1) + p_0(V_2) \leq$$
$$p_0(V_1 \cap V_2) + p_0(V_1 \cup V_2) = p_0(V_1 \cap V_2) \leq p_1(V_1 \cap V_2).$$

Now $\mathcal{I}' = \mathcal{I} - \{V_1, V_2\} + \{V_1 \cap V_2\}$ is also ST-independent and $\widetilde{p}_1(\mathcal{I}') \geq \widetilde{p}_1(\mathcal{I})$, but we must have equality by the optimality of $\mathcal{I}$, contradicting the minimality of $|\mathcal{I}|$. •

We say that a member $V' \in \mathcal{I}$ is of Type I if $V' = X_t + t$ for some $t \in T$ and $X_t \subseteq S$ and

$$p_1(X_t + t) = p_0(X_t + t) = m_T(t) - |X_t| + d_{H_0}(t) > p_T(t) - r_S(X_t).$$

Let $\mathcal{I}_1$ $(\subseteq \mathcal{I})$ denote the family of sets of Type I. Claim 2.7 implies that if $X_1 + t_1 \in \mathcal{I}_1$ and $X_2 + t_2 \in \mathcal{I}_1$ $(X_i \subseteq S, t_i \in T)$ then $t_1 \neq t_2$. Let

$$Y := \{t \in T : \text{there is a member } X_t + t \in \mathcal{I}_1\}.$$

Note that $|Y| = |\mathcal{I}_1|$.

We say that a member $V' \in \mathcal{I}$ is of Type II if

$$p_1(V') = p_0(V') = p_T(V' \cap T) - r_S(V' \cap S).$$

Let $\mathcal{I}_2 = \{V_1, V_2, \cdots, V_q\}$ $(\subseteq \mathcal{I})$ denote the family of set of Type II. Let

$$\mathcal{T} := \{T_1, \ldots, T_q\} \text{ where } T_i := V_i \cap T \text{ for } i = 1, \ldots, q.$$

Since $p_1(V_i) > 0$, the members of $\mathcal{T}$ are non-empty. Furthermore, Claim 2.7 implies that $\mathcal{T}$ is a subpartition.

Let $\mathcal{I}_3 := \mathcal{I} - (\mathcal{I}_1 \cup \mathcal{I}_2)$. The members of $\mathcal{I}_3$ are called of Type III. Then each member V' of $\mathcal{I}_3$ is of form $V' = V_s$ for some $s \in S$ such that $m_S(s) = p_1(V') > p_0(V')$. Let

$$X := \{s \in S : V_s \in \mathcal{I}_3\}.$$

It follows from the definitions that $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ form a partition of $\mathcal{I}$.

Claim 2.8. Let $t \in Y$ and $X_t + t \in \mathcal{I}_1$. Then $X \subseteq X_t$.

Proof. Suppose indirectly that there is an element $s \in X - X_t$. By the ST-independence of the sets $X_t + t$ and V_s, the element t cannot be in V_s. Therefore the arc st belongs to $\overrightarrow{F_0}$. Since st enters $X_t + t$, we have $p_1(X_t + t) \leq 0$, a contradiction. •

Claim 2.9. $\sum[|X_t| - d_{H_0}(t) : t \in Y] \geq d_{G_0}(X, Y)$.

Proof. What we prove is that $|X_t| - d_{H_0}(t) \geq d_{G_0}(X, t)$ for $t \in Y$ and $X_t + t \in \mathcal{I}_1$. Since no arc of $\overrightarrow{H_0}$ enters $X_t + t$ and since $X \subseteq X_t$ by Claim 2.8, we have

$$|X_t| - d_{H_0}(t) = |X_t| - d_{H_0}(X_t, t) = d_{G_0}(X_t, t) \geq d_{G_0}(X, t),$$

as required. •

Claim 2.10. $X \subseteq V_i \cap S$ holds for each $i = 1, \ldots, q$.

Proof. If, indirectly, there is an $s \in X - V_i$, then the ST-independence of V_s and V_i implies that $V_s \cap V_i \cap T = \emptyset$. In this case, an element $t \in V_i \cap T$ cannot be in V_s implying that $st \in \overrightarrow{F_0}$. But in this case $p_0(V_i) = p_1(V_i) = 0$, contradicting the property $p_0(V') > 0$ for each $V' \in \mathcal{I}$. •

By the ST-independence of $\mathcal{I}$, family $\mathcal{T}$ forms a subpartition of T whose members are disjoint from Y. This and the last two claims imply

$$\gamma < \nu = \widetilde{p}_1(\mathcal{I}) = \widetilde{p}_1(\mathcal{I}_1) + \widetilde{p}_1(\mathcal{I}_2) + \widetilde{p}_1(\mathcal{I}_3) =$$

$$\sum[m_T(t) - |X_t| + d_{H_0}(t) : X_t + t \in \mathcal{I}_1] + \sum_{i=1}^{q}[p_T(T_i) - r_S(V_i \cap S)] + \sum[m_S(s) : V - s \in \mathcal{I}_3] \leq$$

$$\sum[m_T(t) : X_t + t \in \mathcal{I}_1] - d_{G_0}(X, Y) + \sum_{i=1}^{q}[p_T(T_i) - r_S(X)] + \widetilde{m}_S(X) =$$

$$\widetilde{m}_T(Y) - d_{G_0}(X, Y) + \sum_{i=1}^{q}[p_T(T_i) - r_S(X)] + \widetilde{m}_S(X),$$

in a contradiction with (2), completing the proof of the lemma. • •

By Theorem 1.1, there is a digraph $D = (V, A)$ on V with $\nu = \gamma$ ST-arcs that covers p_1, that is, $\varrho_D(V') \geq p_1(V')$ for every subset $V' \subseteq V$. Let $G = (S, T; E)$ denote the undirected bipartite graph underlying D.

Claim 2.11. $d_G(s) = m_S(s)$ for every $s \in S$ and $d_G(t) = m_T(t)$ for every $t \in T$.

Proof. Since $d_G(s) = \delta_D(s) \geq \varrho_D(V_s) \geq p_1(V_s) = m_S(s)$ for every $s \in S$, we have $\widetilde{m}_S(S) = |E| = \sum[d_G(s) : s \in S] \geq \widetilde{m}_S(S)$, from which $d_G(s) = m_S(s)$ follows for every $s \in S$.

Since $d_G(t) = \varrho_D(t) = \varrho_D(\Gamma_{H_0}(t) + t) \geq p_0(\Gamma_{H_0}(t) + t) \geq m_T(t)$ for every $t \in T$, we have $\widetilde{m}_S(S) = |E| = \sum[d_G(t) : t \in T] \geq \widetilde{m}_T(T) = \widetilde{m}_S(S)$, from which $d_G(t) = m_T(t)$ follows for every $t \in T$. •

Claim 2.12. The bigraph $G^+ = (S, T; E + F_0)$ is simple.

Proof. The minimality of D implies that each arc of D enters a subset V' with $p_1(V') > 0$. Since $p_1(V')$ can be positive only if no arc of $\overrightarrow{H_0}$ enters V', we can conclude that no edge of G is parallel to an edge of H_0.

Suppose indirectly that there are two parallel edges e and e' of G connecting s and t for some $s \in S$ and $t \in T$. Let $X := \{u \in S : ut \in F_0\}$. Then $p_1(X + t) \geq m_T(t) = \varrho_D(t)$. For $V' = X + s + t$, we have $\varrho_D(t) - 2 \geq \varrho_D(V') \geq p_1(V') \geq p_1(X + t) - 1 \geq m_T(t) - 1 = \varrho_D(t) - 1$, a contradiction. •

Claim 2.13. $r_S(\Gamma_{G^+}(Y)) \geq p_T(Y)$ for every subset $Y \subseteq T$.

Proof. Let $X := \Gamma_{G^+}(Y)$ and $V' := Y \cup X$. Then $0 = \varrho_D(V') \geq p_1(V') \geq p_T(Y) - r_S(X)$, from which the claim follows. •

We conclude that G meets all the requirements of the theorem, and the proof is complete. • • •

2.2 Variations

2.2.1 Degree-specification only on S

With the proof technique used above, one can derive the following variation where the degrees are specified only for the nodes in S.

THEOREM 2.14. We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a positively intersecting supermodular function p_T on T, and a degree-specification m_S on S for which $\widetilde{m}_S(S) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_S for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if

$$m_S(s) + d_{H_0}(s) \leq |T| \text{ for every } s \in S \tag{5}$$

and

$$\begin{cases} \widetilde{m}_S(X) + \widetilde{p}_T(\mathcal{T}) - |\mathcal{T}| r_S(X) \leq \gamma \\ \text{for } X \subseteq S,\ \mathcal{T} \text{ a subpartition of } T \text{ with } d_{H_0}(S - X, \cup\mathcal{T}) = 0. \ \bullet \end{cases} \tag{6}$$

2.2.2 Fully supermodular p_T

In the special case when $p_T \equiv 0$, it suffices to require the inequality in (2) only for the empty $\mathcal{T}$, in which case Theorem 2.1 reduces to the following classic result (which actually holds for non-simple bigraphs, too).

Theorem 2.15 (Ore [12]). A simple bigraph $G_0 = (S, T; E_0)$ has a subgraph fitting a degree-specification (m_S, m_T) with $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$ if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) \leq \gamma \text{ for } X \subseteq S, \ Y \subseteq T. \tag{7}$$

The content of the next result is that the condition in Theorem 2.1 can also be simplified when p_T is fully supermodular.

THEOREM 2.16. We are given a simple bigraph $H_0 = (S, T; F_0)$, a matroid $M_S = (S, r_S)$, a fully supermodular function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T if and only if *(7)* holds and

$$\begin{cases} \widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + p_T(T_0) - r_S(X) \leq \gamma \\ \text{for } Y \subseteq T, \ X \subseteq S, \ T_0 \subseteq T - Y, \ \text{ with } d_{H_0}(S - X, T_0) = 0, \end{cases} \tag{8}$$

where G_0 is the bipartite complement of H_0 and $d_{G_0}(X, Y)$ is the number of edges of G_0 connecting X and Y. If, in addition, p_T is monotone non-decreasing, then T_0 in *(8)* can be chosen to by $T_0 = T - (Y \cup \Gamma_{H_0}(S - X))$, that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + p_T(T - (Y \cup \Gamma_{H_0}(S - X))) - r_S(X) \leq \gamma \text{ for } X \subseteq S, \ Y \subseteq T. \tag{9}$$

Proof. Conditions (7) and (8) correspond the special cases of Condition (2) when $|\mathcal{T}| = 0$ and $|\mathcal{T}| = 1$, respectively. Therefore their necessity was proved earlier. To see sufficiency, by Theorem 2.1 it suffices to show that (2) holds in general. Suppose, indirectly, that there are X, Y, and $\mathcal{T}$ violating (2). Assume that $|\mathcal{T}|$ is minimal. Then (7) and (8) imply that $|\mathcal{T}| \geq 2$. Let T_1, T_2 be two members of $\mathcal{T}$. Since

$$p_T(T_1 \cup T_2) - r_S(X) \geq p_T(T_1) + p_T(T_2) - 2r_S(X),$$

the unchanged sets X, Y and the partition $\mathcal{T}'$ obtained from $\mathcal{T}$ by replacing T_1 and T_2 with the single set $T_1 \cup T_2$ also violate (2), contradicting the minimal choice of $\mathcal{T}$.

When p_T, in addition, is monotone non-decreasing, we can choose T_0 in (8) as large as possible, that is, T_0 is a maximal subset of $T - Y$ for which $d_{H_0}(S - X, T_0) = 0$. But then $T_0 = T - (Y \cup \Gamma_{H_0}(S - X)$. •

It is worth formulating Theorem 2.16 in the special cases when H_0 has no edges.

Corollary 2.17. We are given a matroid $M_S = (S, r_S)$, a fully supermodular function p_T on T, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$.

There is a simple bigraph $G = (S, T; E)$ fitting m_V and M_S-covering p_T if and only if ***(7)*** holds and

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + p_T(T_0) - r_S(X) \leq \gamma \text{ for } Y \subseteq T,\ X \subseteq S,\ T_0 \subseteq T - Y, \tag{10}$$

If, in addition, p_T is monotone non-decreasing, then T_0 in ***(10)*** can be chosen to by $T_0 = T - Y$, that is,

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + p_T(T - Y) - r_S(X) \leq \gamma \text{ for } X \subseteq S,\ Y \subseteq T. \bullet \tag{11}$$

3 Matroidal max term rank

Let $\mathcal{G}(m_S, m_T)$ denote the set of simple bigraphs $G = (S, T; E)$ fitting a degree-specification (m_S, m_T) with $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. It follows from Theorem 2.15 that $\mathcal{G}(m_S, m_T)$ is non-empty if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| \leq \gamma \text{ for } X \subseteq S,\ Y \subseteq T. \tag{12}$$

In [1] (Theorem 7.1), we reformulated Ryser's classic max term rank formula in graph theoretic language.

Theorem 3.1 (Ryser)**.** Let $\ell \leq |T|$ be an integer. Suppose that $\mathcal{G}(m_S, m_T)$ is non-empty. Then $\mathcal{G}(m_S, m_T)$ has a member G with matching number $\nu(G) \geq \ell$ if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + (\ell - |X| - |Y|) \leq \gamma \ \text{ whenever } X \subseteq S,\ Y \subseteq T. \tag{13}$$

Moreover, ***(13)*** holds if the inequality in it is required only when X consists of the i largest values of m_S and Y consists of the j largest values of m_T $(i = 0, 1, \dots, |S|,\ j = 0, 1, \dots, |T|)$.

We keep using graph terminology, but the original expression max term rank of Ryser is retained. Our present goal is to extend Ryser's theorem in two directions. In the augmentation version an initial bigraph is to be augmented while in the matroidal form the matching is expected to cover a basis of a matroid M_S on S and a basis of matroid M_T on T. Actually, we shall integrate the two generalizations into one single framework.

In what follows, $M_S = (S, r_S)$ and $M_T = (T, r_T)$ will be matroids of rank ℓ. In [1], the complementary set-function p of a set-function b was defined by $p(Y) := b(S) - b(S - Y)$. Clearly, b is submodular if and only if p is supermodular, and p is monotone non-decreasing if and only if b is monotone non-decreasing. The complementary function p_T of the rank function r_T of M_T is called the **co-rank function** of M_T. It can easily be shown that $p_T(Y) = \min\{|Y \cap B| : B \text{ a basis of } M_T\}$.

The following extension of Edmonds' matroid intersection theorem [5] will be used. For notational convenience, the bipartite graph in the theorem is denoted by G^+.

Theorem 3.2 (Brualdi, [3])**.** Let $G^+ = (S, T; E^+)$ be a bigraph with a matroid $M_S = (S, r_S)$ on S and with a matroid $M_T = (T, r_T)$ on T for which $r_S(S) = r_T(T) = \ell$. There is a matching of G^+ covering bases of M_S and M_T if and only if

$$\begin{cases} r_S(X) + r_T(Z) \geq \ell \\ \text{holds whenever } X \cup Z \text{ hits every edge of } G^+ \quad (X \subseteq S, Z \subseteq T). \end{cases} \tag{14}$$

We need the following equivalent version of Theorem 3.2.

Lemma 3.3. We are given a bigraph $G^+ = (S, T; E^+)$, a matroid M_S on S with rank function r_S and a matroid M_T on T with a co-rank function p_T for which $r_S(S) = p_T(T) = \ell$. There is a matching of G^+ covering bases of M_S and M_T if and only if

$$r_S(\Gamma_{G^+}(Y)) \geq p_T(Y) \text{ for every } Y \subseteq T. \tag{15}$$

Proof. The necessity is straightforward. The sufficiency follows from Theorem 3.2 once we show that (14) holds. Suppose, indirectly, that there are X and Z for which $r_S(X)+r_T(Z) < \ell$, that is, $r_S(X) < \ell - r_T(Z) = p_T(T-Z)$. Since $X \cup Z$ hits every edge of G^+, for $Y := T - Z$ we have $\Gamma_{G^+}(Y) \subseteq X$. Therefore $r_S(\Gamma_{G^+}(Y)) \leq r_S(X) < p_T(Y)$, contradicting (15). •

THEOREM 3.4. We are given a simple bigraph $H_0 = (S, T; F_0)$, matroids $M_S = (S, r_S)$ and matroid $M_T = (T, r_T)$ with $r_S(S) = r_T(T) = \ell$, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and includes a matching covering a basis of M_S and a basis of M_T if and only if *(7)* holds and

$$\begin{cases} \widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y') \leq \gamma \\ \text{for } X \subseteq X' \subseteq S,\ Y \subseteq Y' \subseteq T,\ \ X' \cup Y' \text{ hits all the edges of } H_0, \end{cases} \tag{16}$$

where G_0 is the bipartite complement of H_0 and $d_{G_0}(X, Y)$ is the number of edges in G_0 connecting X and Y.

Proof. Necessity. Suppose that the requested bigraph G and its ℓ-element matching M exist. The number of edges of G with at least one end-node in $X \cup Y$ is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y)$. The number of edges in M with at least one end-node in $X' \cup Y'$ is at most $r_S(X') + r_T(Y')$. Therefore M has at least $\ell - r_S(X') - r_T(Y')$ elements connecting $S - X'$ and $T - Y'$. But these elements must be in E since $X' \cup Y'$ hits all edges of H_0. Therefore the total number of edges of G is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_S(X') - r_T(Y')$, and (16) follows.

Sufficiency. Let p_T denote the co-rank function of M_T, that is, $p_T(Z) = \ell - r_T(Z)$ for $Z \subseteq T$. This is fully supermodular and monotone non-decreasing and the inequality in Condition (9) transforms to

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - r_T(Y \cup \Gamma_{H_0}(S - X)) - r_S(X) \leq \gamma. \tag{17}$$

No sets $X \subseteq S$, $Y \subseteq T$ can violate this inequality since then by letting $Y' := Y \cup \Gamma_{H_0}(S - X)$ and $X' := X$, the quadruple $(X,\ Y,\ X',\ Y')$ would violate (16).

By the second part of Theorem 2.16, there is bigraph G fitting m_V for which $G^+ = G + H_0$ is simple and M_S-covers p_T. The last property, by definition, means that (15) holds, and therefore Lemma 3.3 implies that G^+ has a requested matching. •

Remarks It follows that requiring inequality (17) for every pair of sets $X \subseteq S$ and $Y \subseteq T$ is also a necessary and sufficient condition for the existence of the bigraph G described in the theorem. This condition has the advantage that it is more compact than (16) in the sense that X' and Y' play no role in it. On the other hand, Condition (16) has the advantage that it is symmetric in S and T.

When $m_V \equiv 0$ and $\gamma = 0$, it suffices to require (16) only for $X = Y = \emptyset$ in which case it transforms to

$$\begin{cases} \ell - r_S(X') - r_T(Y') \leq 0 \\ \text{for } X' \subseteq S,\ Y' \subseteq T,\ \ X' \cup Y' \text{ hits all the edges of } H_0, \end{cases} \tag{18}$$

which is the same as (14). In other words, Theorem 3.4 may be considered as a generalization of Brualdi's theorem.

By specializing Theorem 3.4 to the case when $F_0 = \emptyset$, one obtains the following.

Corollary 3.5. Let S and T be two disjoint sets and $m_V = (m_S, m_T)$ a degree-specification on $S \cup T$ for which $m_S(S) = m_T(T) = \gamma$ and $\mathcal{G}(m_S, m_T)$ is non-empty, that is, ***(12)*** holds. Let $M_S = (S, r_S)$ and matroid $M_T = (T, r_T)$ be matroids for which $r_S(S) = r_T(T) = \ell$. There is a simple graph fitting m that includes a matching covering bases of M_S and M_T if and only if

$$\widetilde{m}_S(X) + \widetilde{m}_T(Y) - |X||Y| + \ell - r_S(X) - r_T(Y) \leq \gamma \tag{19}$$

holds for every $X \subseteq S$ and $Y \subseteq T$. •

By specializing Theorem 3.4 to the case when M_S and M_T are ℓ-uniform matroids on S and T, respectively, one obtains the following.

Corollary 3.6. We are given a simple bigraph $H_0 = (S, T; F_0)$, an integer ℓ, and a degree-specification $m_V = (m_S, m_T)$ for which $\widetilde{m}_S(S) = \widetilde{m}_T(T) = \gamma$. There is a bigraph $G = (S, T; E)$ fitting m_V for which $G^+ = G + H_0$ is simple and includes an ℓ-element matching if and only if

$$\begin{cases} \widetilde{m}_S(X) + \widetilde{m}_T(Y) - d_{G_0}(X, Y) + \ell - |X'| - |Y'| \leq \gamma \\ \text{for } X \subseteq X' \subseteq S,\ Y \subseteq Y' \subseteq T,\ \ X' \cup Y' \text{ hits all the edges of } H_0, \end{cases} \tag{20}$$

where G_0 denotes the bipartite complement of H_0, and $d_{G_0}(X, Y)$ is the number of edges in G_0 connecting X and Y. •

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