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Base polyhedra and the linking property

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Abstract

An integer polyhedron $P \subseteq \mathbb{R}^n$ has the linking property if for any $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$ with $f \leq g$, P has an integer point between f and g if and only if it has both an integer point above f and an integer point below g. We prove that an integer polyhedron in the hyperplane $\sum_{j=1}^{n} x_j = 0$ is a base polyhedron if and only if it has the linking property. The result implies that an integer polyhedron has the strong linking property, as defined in [A. Frank, T. Király, A survey on covering supermodular functions, 2009], if and only if it is a generalized polymatroid.

1 Introduction

The linking property is a powerful and elegant property that appears in several fundamental combinatorial optimization problems. Loosely speaking, it states that if there is a solution satisfying a given lower bound f and there is one satisfying a given upper bound g (where $f \leq g$), then there is a solution satisfying both bounds at the same time. A well-known example is the Mendelsohn–Dulmage theorem [6], which states that if a bipartite graph $G = (S, T; E)$ has a matching covering a given node set $S' \subseteq S$ and also one covering a given $T' \subseteq T$, then it has a matching covering $S' \cup T'$. (To interpret this as upper and lower bounds, we may orient the graph from S to T, so the conditions correspond to upper and lower bounds on the balance.) Another example is the linking property of graph orientations: if a graph has an orientation satisfying some given lower bounds on in-degrees, and it also has one satisfying given upper bounds, then there is an orientation whose in-degrees satisfy both bounds simultaneously. More examples are presented in the book Connections in Combinatorial Optimization by Frank [2].

The linking property of combinatorial problems is typically explained by an underlying polyhedral structure. In fact, the property can be defined explicitly for integer polyhedra. An *integer box* is a polyhedron of the form $B(f, g) = \{x \in \mathbb{R}^n : f \leq x \leq g\}$ for some $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$. The notation $B(f, g)$ will also be used in the case when f and g may have infinite values. An integer polyhedron P has the *linking property* if for every $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$ such that $f \leq g$ and both $P \cap B(f, \infty)$ and $P \cap B(-\infty, g)$ have integer points, the polyhedron $P \cap B(f, g)$ also has an integer point.

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A notable class of polyhedra having the linking property is the class of *generalized polymatroids*, or *g-polymatroids* for short, see e.g. [2, Section 14.3.2]. This class of polyhedra was introduced in [1], and it has many equivalent characterizations; some recent characterization and recognition results are presented in [4]. A *base polyhedron* is a g-polymatroid that lies in the affine hyperplane $\sum_{j=1}^n x_j = \beta$ for some β. Equivalently, a base polyhedron is a polyhedron defined as $\{x \in \mathbb{R}^n : \sum_{j\in Z} x_j \leq b(Z)\ \forall Z \subseteq [n],\ \sum_{j=1}^n x_j = b([n])\}$, where $b : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a submodular set function.

Not all polyhedra with the linking property are g-polymatroids. For example, the triangle $(0,0),(2,0),(0,1)$ has the linking property but it is not a g-polymatroid. More generally, it is easy to see that any integer polyhedron of the form $\{x \in \mathbb{R}^n : Ax \leq b,\ x \geq \mathbf{0}\}$, where A is a nonnegative matrix, has the linking property. In contrast to this, we prove that among the integer polyhedra on the affine hyperplane $\sum_{j=1}^n x_j = \beta$ for some constant β, the linking property holds only for base polyhedra. A useful consequence is that by proving the linking property for a particular combinatorial structure, we automatically get all the other nice properties of base polyhedra, like the greedy algorithm for optimization and the intersection theorem. The result also implies that integer g-polymatroids are characterized by the strong linking property, which is presented in the last section of the paper.

Our proof uses a characterization of base polyhedra in terms of their tangent cones, due to Tomizawa [7]; see also [5, Theorem 17.1] for a proof and discussion. Let e_i denote the ith unit vector. The *tangent cone* of a polyhedron P at point $x \in P$ is the cone $\{\lambda z : \lambda \geq 0,\ x + z \in P\}$.

Theorem 1.1 ([7])**.** *A polyhedron $P \subseteq \mathbb{R}^n$ is a base polyhedron if and only if for each $x \in P$, the tangent cone of P at x has a generating set which is a subset of $\{e_i - e_j : i, j \in [n]\}$.*

We denote the segment with endpoints u and v by $[u, v]$. If P is a bounded polyhedron and v is a vertex of P, then the tangent cone of P at v is generated by the edge vectors $\{u-v : [u, v]$ is an edge of $P\}$. If $x \in P$ is not a vertex, then the tangent cone at x is generated by the tangent cones at the vertices of the smallest face containing x. Thus tangent cones of bounded polyhedra are always generated by edge vectors; it follows from Theorem 1.1 that a bounded polyhedron P is a base polyhedron if and only if all of its edge vectors are of the form $\mu(e_i - e_j)$ for some $i, j \in [n]$ and $\mu \in \mathbb{R}$.

The situation for unbounded but pointed polyhedra is similar, but in addition to the edge vectors, the extreme directions must also be of the form $e_i - e_j$. If the polyhedron is not pointed, then the characterization in Theorem 1.1 is somewhat less intuitive. Our first result is a lemma that offers a useful alternative characterization in case of integer polyhedra. For a rational polyhedron P, let P_I denote the convex hull of $P \cap \mathbb{Z}^n$. The *size of the support* of a vector is the number of non-zero components. Given two vectors u and v in $\mathbb{Z}^n$, let $\min(u, v)$ $(\max(u, v))$ denote the componentwise minimum (maximum) of the two vectors.

Lemma 1.2. *If P is an integer polyhedron in the hyperplane $\sum_{j=1}^n x_j = \beta$ that is not a base polyhedron, then there exist integer points $u, v \in P \cap \mathbb{Z}^n$ such that (i) the size of the support of $u-v$ is at least 3, and (ii) $[u, v]$ is an edge of $(P \cap B(\min(u, v), \max(u, v)))_I$.*

Proof. First observe that if u and v are distinct integer points in P and the size of the support of $u - v$ is at most 2, then $u - v = \mu(e_i - e_j)$ for some i, j and μ, because P lies in the hyperplane $\sum_{j=1}^n x_j = \beta$.

If P is bounded, then, by Theorem 1.1 and the above argument, it has an edge $[u, v]$ such that the support of $u - v$ has size at least 3. The segment $[u, v]$ is clearly also an edge of $(P \cap B(\min(u, v), \max(u, v)))_I$, so u and v satisfy the conditions. If P is pointed and no such edge exists, then there is a vertex u and an extreme ray $u + \lambda z$ such that the support of z has size at least 3. There is a value λ for which $v := u + \lambda z$ is integer; u and v satisfy conditions (i) and (ii).

Assume now that P is not pointed, so it has a characteristic subspace H of dimension $d \geq 1$, and every minimal face is a translate of H. If a point x is on a minimal face of P, then the tangent cone at x is generated by $\{u - x : u \text{ is on a } (d+1)\text{-dimensional face containing } x\}$. If $x \in P$ is on a face F that is not minimal, then the tangent cone at x is generated by the tangent cones of the minimal faces of F. By Theorem 1.1, there is a point u on a minimal face F_1 of P and a $(d+1)$-dimensional face F_2 containing u such that the linear hull of F_2 is not generated by vectors having support of size at most 2. Since F_2 is an integer polyhedron, it has an integer point w not in F_1. Let H_1 be the translate of H containing w; note that $H_1 \subseteq F_2$ and $H_1 \cap F_1 = \emptyset$. First, we will show that there is an integer point $v \in H_1$ such that $(F_2 \cap B(\min(u, v), \max(u, v)))_I$ is $(d+1)$-dimensional – note that here we do not require $[u, v]$ to be an edge of this polyhedron.

Let $J = \{j \in [n] : \exists\, x \in H,\ x_j \neq 0\}$. Since H is a subspace, it has an integer point z such that $z_j \neq 0$ for every $j \in J$. Let $\{b^1, \ldots, b^d\}$ be an arbitrary integer basis of H. Choose $\mu \in \mathbb{Z}_+$ such that $\mu|z_j| \geq |w_j - u_j| + \sum_{i=1}^d |b^i_j|$ for every $j \in J$. Let $v = w + 2\mu z$. We claim that the box $B(\min(u, v), \max(u, v))$ contains $w + \mu z$ and $w + \mu z + b^i$ for every i. Indeed, if $z_j > 0$, then $u_j \leq w_j + \mu z_j \leq w_j + 2\mu z_j$ and $u_j \leq w_j + \mu z_j + b^i_j \leq w_j + 2\mu z_j$ by the choice of μ. Analogously, if $z_j < 0$, then $u_j \geq w_j + \mu z_j \geq w_j + 2\mu z_j$ and $u_j \geq w_j + \mu z_j + b^i_j \geq w_j + 2\mu z_j$ for every i. Obviously, u is also in the box, so $Q := (F_2 \cap B(\min(u, v), \max(u, v)))_I$ is $(d+1)$-dimensional.

The edges of Q generate the linear hull of F_2, so Q must have an edge $[u', v']$ such that $u' - v'$ has support of size at least 3. We claim that u' and v' satisfy condition (ii) of the lemma. Indeed, $[u', v']$ is an edge of Q, so it is also an edge of $(F_2 \cap B(\min(u', v'), \max(u', v')))_I$. Since F_2 is a face of P, $[u', v']$ is also an edge of $(P \cap B(\min(u', v'), \max(u', v')))_I$. □

2 Proof of the main result

As mentioned before, integer g-polymatroids, and hence base polyhedra, satisfy the linking property. We prove the converse for base polyhedra.

Theorem 2.1. *If $P \subseteq \{x \in \mathbb{R}^n : \sum_{j=1}^n x_j = \beta\}$ is an integer polyhedron having the linking property, then P is a base polyhedron.*

Proof. Let $P \subseteq \{x \in \mathbb{R}^n : \sum_{j=1}^n x_j = \beta\}$ be an integer polyhedron that is not a base polyhedron. By Lemma 1.2, there are integer vectors $u, v \in P \cap \mathbb{Z}^n$ such that the size of the support of $u-v$ is at least 3, and $[u,v]$ is an edge of $(P\cap B(\min(u,v),\max(u,v)))_I$. Choose a pair u,v with these properties such that $\|u-v\|_1$ is minimal, and let $P' = P\cap B(\min(u,v),\max(u,v)))_I$. Let d be the dimension of $B(\min(u,v),\max(u,v))$, i.e. the number of components in which u and v differ. In order to make the notation simpler, we consider P' as a polyhedron in $\mathbb{R}^d$.

By the minimality of $\|u-v\|_1$, all edge vectors of P', except for $u-v$, are of the form $\mu(e_i-e_j)$. As $[u,v]$ is an edge of P', there is a vector $c\in\mathbb{R}^d$ such that $c^Tu=c^Tv$, and $c^Tx<c^Tv$ for every $x\in P'\setminus[u,v]$. Let k be an index for which $|c_k|$ is maximal. Our first aim is to show the existence of an index $\ell\neq k$ such that $u_\ell-v_\ell$ has the same sign as u_k-v_k.

The equations $\sum_{j=1}^d u_j=\sum_{j=1}^d v_j$ and $c^Tu=c^Tv$ together imply $\sum_{j\neq k}(c_k-c_j)(u_j-v_j)=0$. If no index ℓ exists such that $u_\ell-v_\ell$ has the same sign as u_k-v_k, then u_j-v_j has the same sign for every $j\neq k$, thus $c_k=c_j$ for every j by the maximality of $|c_k|$. As $d\geq 3$, we can modify the choice of k in this case in such a way that an index ℓ with the desired property exists. We can also assume w.l.o.g. that c_k and u_k-v_k are both positive, because otherwise we can exchange the role of u and v and/or replace P by $-P$ (these operations do not invalidate our choice of k and ℓ).

Let $[v,w]$ be an edge of P', where $w\neq u$. The vector $w-v$ is of the form $\mu(e_i-e_j)$ for some i,j, and $\mu\in\mathbb{Z}_+$. On one hand, $j\neq k$ because P' is a subset of $B(\min(u,v),\max(u,v))$. On the other hand, $i=k$ would imply $c^Tw\geq c^Tv$ by the maximality of $|c_k|$, but this contradicts the choice of c, so $i\neq k$. This means that all edges of P' incident to v, apart from $[u,v]$, are in the hyperplane $x_k=v_k$. In particular, the face $\{x\in P' : x_\ell=v_\ell\}$ is in the hyperplane $\{x_k=v_k\}$, so $\{x\in P' : x_\ell=v_\ell,\ x_k=u_k\}=\emptyset$.

Let us define vectors f and g by $f_k=g_k=u_k$, $f_\ell=g_\ell=v_\ell$, and $f_j=\min\{u_j,v_j\}$, $g_j=\max\{u_j,v_j\}$ if $j\neq k,\ell$. By the assumptions $u_k>v_k$ and $u_\ell>v_\ell$, we have $u\in P\cap B(f,\infty)$ and $v\in P\cap B(-\infty,g)$, but $P\cap B(f,g)$ does not contain an integer point, because $P\cap B(f,g)\cap\mathbb{Z}^n\subseteq\{x\in P' : x_k=u_k,\ x_\ell=v_\ell\}=\emptyset$ (here we slightly abuse notation because we consider P' again as a polyhedron in $\mathbb{R}^n$). This means that P does not satisfy the linking property. □

3 Strong linking property

The characterization of base polyhedra in Theorem 2.1 leads to a characterization of generalized polymatroids in terms of the *strong linking property*, as defined by Frank [2, 3]. An integer polyhedron P has the strong linking property if the polyhedron $P\cap\{x\in\mathbb{R}^n : f\leq x\leq g,\ \alpha\leq\sum_{j=1}^n x_j\leq\beta\}$ has an integer point for all $f\in\mathbb{Z}^n$, $g\in\mathbb{Z}^n$, $\alpha\in\mathbb{Z}$, and $\beta\in\mathbb{Z}$ that satisfy

(i) $f\leq g$

(ii) $\alpha\leq\beta$,

(iii) $P\cap\{x\in\mathbb{Z}^n : x\geq f,\ \sum_{j=1}^n x_j\leq\beta\}\neq\emptyset$,

(iv) $P\cap\{x\in\mathbb{Z}^n : x\leq g,\ \sum_{j=1}^n x_j\geq\alpha\}\neq\emptyset$.

Corollary 3.1. *An integer polyhedron has the strong linking property if and only if it is a g-polymatroid.*

Proof. Let us associate an $(n+1)$-dimensional polyhedron P^+ to an n-dimensional integer polyhedron P by the definition

$$P^+ = \left\{ \left(x, -\sum_{j=1}^{n} x_j \right) : x \in P \right\}.$$

It is clear that P^+ is an integer polyhedron in the hyperplane $\sum_{j=1}^{n} x_j = 0$. It is also well-known that P is a g-polymatroid if and only if P^+ is a base polyhedron. The characterization thus follows from Theorem 2.1 by the observation that the strong linking property for P is equivalent to the linking property for P^+. □

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