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Base polyhedra and the linking property

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Abstract

An integer polyhedron $P \subseteq \mathbb{R}^n$ has the linking property if for any $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$ with $f \leq g$, P has an integer point between f and g if and only if it has both an integer point above f and an integer point below g. We prove that an integer polyhedron in the hyperplane $\sum_{j=1}^n x_j = \beta$ is a base polyhedron if and only if it has the linking property. The result implies that an integer polyhedron has the strong linking property, as defined in [A. Frank, T. Király, A survey on covering supermodular functions, 2009], if and only if it is a generalized polymatroid.

1 Introduction

The linking property, a notion introduced by Ford and Fulkerson [1], is a powerful and somewhat surprising feature of several fundamental combinatorial optimization problems. Loosely speaking, it means that if there is a solution satisfying a given lower bound f and there is one satisfying a given upper bound g (where $f \leq g$), then there is a solution satisfying both bounds at the same time. A well-known example is the Mendelsohn–Dulmage theorem [10], which states that if a bipartite graph $G = (S, T; E)$ has a matching covering a given node set $S' \subseteq S$ and also one covering a given $T' \subseteq T$, then it has a matching covering $S' \cup T'$. (To interpret this as upper and lower bounds, we may orient the graph from S to T, so the conditions correspond to upper and lower bounds on the balance at each node.) Another example is the linking property of graph orientations: if a graph has an orientation satisfying some given lower bounds on in-degrees, and it also has one satisfying given upper bounds, then there is an orientation whose in-degrees satisfy both bounds simultaneously. The same is true for strong orientations, and, more generally, for k-arc-connected orientations [3]. However, the linking property does not hold for strong orientations of mixed graphs [4]. Further examples are presented in the book *Connections in Combinatorial Optimization* by Frank [5].

The linking property of combinatorial problems is usually a consequence of an underlying polyhedral structure. In fact, the property can be defined explicitly for integer polyhedra. An *integer box* is a polyhedron of the form $B(f, g) = \{x \in \mathbb{R}^n :$

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$f \le x \le g\}$ for some $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$. The notation $B(f,g)$ will also be used in the case when f and g may have infinite values. An integer polyhedron P has the *linking property* if for every $f \in \mathbb{Z}^n$ and $g \in \mathbb{Z}^n$ such that $f \le g$ and both $P \cap B(f,\infty)$ and $P \cap B(-\infty,g)$ have integer points, the polyhedron $P \cap B(f,g)$ also has an integer point.

A notable class of polyhedra having the linking property is the class of *generalized polymatroids*, or *g-polymatroids* for short, see e.g. [5, Section 14.3.2]. This class of polyhedra was introduced in [2], and it has many equivalent characterizations; some recent characterization and recognition results are presented in [7]. A *base polyhedron* is a g-polymatroid that lies in the affine hyperplane $\sum_{j=1}^n x_j = \beta$ for some β. Equivalently, a base polyhedron is a polyhedron defined as $\{x \in \mathbb{R}^n : \sum_{j\in Z} x_j \le b(Z)\ \forall Z \subseteq [n],\ \sum_{j=1}^n x_j = b([n])\}$, where $b : \{0,1\}^n \to \mathbb{R} \cup \{\infty\}$ is a submodular set function, and $[n]$ denotes the set $\{1,2,\dots,n\}$.

Not every polyhedron with the linking property is a g-polymatroid. For example, the triangle $\mathrm{conv}((0,0),(2,0),(0,1))$ has the linking property but it is not a g-polymatroid. More generally, it is easy to see that any integer polyhedron of the form $\{x \in \mathbb{R}^n : Ax \le b,\ x \ge \mathbf{0}\}$, where A is a nonnegative matrix, has the linking property. In contrast to this, we prove that among the integer polyhedra on the affine hyperplane $\sum_{j=1}^n x_j = \beta$, the linking property holds only for base polyhedra. A practical consequence of this result is that a proof of the linking property for a particular combinatorial structure automatically implies all the other useful features of base polyhedra, like the greedy algorithm for optimization and the intersection theorem. Our result also has the corollary that integer g-polymatroids are characterized by the so-called strong linking property (see Section 2.2).

Our proof uses a characterization of base polyhedra in terms of their tangent cones, due to Tomizawa [11]; see also [8, Theorem 17.1] for a proof and discussion. Let e_i denote the ith unit vector. The *tangent cone* of a polyhedron P at point $x \in P$ is defined as $\mathrm{cone}_P(x) = \{\lambda z : \lambda \ge 0,\ x+z \in P\}$.

Theorem 1.1 ([11]). *A polyhedron $P \subseteq \mathbb{R}^n$ is a base polyhedron if and only if for each $x \in P$, the tangent cone of P at x has a generating set which is a subset of $\{e_i - e_j : i,j \in [n]\}$.*

We denote the segment with endpoints u and v by $[u,v]$. If P is a bounded polyhedron and v is a vertex of P, then the extreme directions of the tangent cone of P at v are the edge vectors $\{u-v : [u,v]$ is an edge of $P\}$. If $x \in P$ is not a vertex, then the tangent cone at x is generated by the tangent cones at the vertices of the smallest face containing x. Therefore, tangent cones of bounded polyhedra are always generated by edge vectors, and it follows from Theorem 1.1 that a bounded polyhedron P is a base polyhedron if and only if all of its edge vectors are of the form $\mu(e_i - e_j)$ for some $i,j \in [n]$ and $\mu \in \mathbb{R}$.

For unbounded polyhedra, the meaning of the characterization in Theorem 1.1 is less intuitive; some explanation will be given in Section 3. Here we state a consequence concerning integer polyhedra that is essential for the proof of the main result. For a rational polyhedron P, let P_I denote the convex hull of $P \cap \mathbb{Z}^n$. The *size of the support* of a vector is the number of non-zero components. Given two vectors u and v

in $\mathbb{Z}^n$, let $\min(u, v)$ and $\max(u, v)$ denote the componentwise minimum and maximum of the two vectors, respectively.

Lemma 1.2. *If P is an integer polyhedron in the hyperplane $\sum_{j=1}^{n} x_j = \beta$ that is not a base polyhedron, then there exist integer points $u, v \in P \cap \mathbb{Z}^n$ such that (i) the size of the support of $u - v$ is at least 3, and (ii) $[u, v]$ is an edge of the polyhedron $(P \cap B(\min(u, v), \max(u, v)))_I$.*

The proof of this lemma, along with some other observations about Theorem 1.1, is presented in Section 3.

2 Main results

2.1 Sufficiency of the linking property

As mentioned in the introduction, integer g-polymatroids, and hence base polyhedra, satisfy the linking property. Our main result is the following reverse implication for base polyhedra.

Theorem 2.1. *If $P \subseteq \{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j = \beta\}$ is an integer polyhedron having the linking property, then P is a base polyhedron.*

Proof. Let $P \subseteq \{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_j = \beta\}$ be an integer polyhedron that is not a base polyhedron. By Lemma 1.2, there are integer vectors $u, v \in P \cap \mathbb{Z}^n$ such that the size of the support of $u - v$ is at least 3 and $[u, v]$ is an edge of $(P \cap B(\min(u, v), \max(u, v)))_I$. Choose a pair u, v with these properties such that $\|u - v\|_1$ is minimal, and let $P' = P \cap B(\min(u, v), \max(u, v)))_I$. Let d be the dimension of $B(\min(u, v), \max(u, v))$, i.e. the number of components in which u and v differ. In order to simplify the notation, we consider P' as a polyhedron in $\mathbb{R}^d$.

By the minimality of $\|u - v\|_1$, all edge vectors of P', except for $u - v$, are of the form $\mu(e_i - e_j)$. As $[u, v]$ is an edge of P', there is a vector $c \in \mathbb{R}^d$ such that $c^T u = c^T v$, and $c^T x < c^T v$ for every $x \in P' \setminus [u, v]$. Let k be an index for which $|c_k|$ is maximal. Our first aim is to show the existence of an index $\ell \neq k$ such that $u_\ell - v_\ell$ has the same sign as $u_k - v_k$.

The equalities $\sum_{j=1}^{d} u_j = \sum_{j=1}^{d} v_j$ and $c^T u = c^T v$ together imply that $\sum_{j \neq k} (c_k - c_j)(u_j - v_j) = 0$. If no index ℓ exists such that $u_\ell - v_\ell$ has the same sign as $u_k - v_k$, then $u_j - v_j$ has the same sign for every $j \neq k$, thus $c_k = c_j$ for every j by the maximality of $|c_k|$. As $d \geq 3$, we can modify the choice of k in this case in such a way that an index ℓ with the desired property exists. We can also assume w.l.o.g. that c_k and $u_k - v_k$ are both positive, because otherwise we can exchange the role of u and v and/or replace P by $-P$ (these operations do not invalidate our choice of k and ℓ).

Let $[v, w]$ be an edge of P', where $w \neq u$. The vector $w - v$ is of the form $\mu(e_i - e_j)$ for some i, j, and $\mu \in \mathbb{Z}_+$. On one hand, $j \neq k$ because P' is a subset of the box $B(\min(u, v), \max(u, v))$. On the other hand, $i = k$ would imply $c^T w \geq c^T v$ by the maximality of $|c_k|$, but this contradicts the choice of c, so $i \neq k$. This means that all edges of P' incident to v, apart from $[u, v]$, are in the hyperplane $x_k = v_k$. In particular, the face $\{x \in P' : x_\ell = v_\ell\}$ is in the hyperplane $\{x_k = v_k\}$, so $\{x \in P' : x_\ell = v_\ell,\ x_k = u_k\} = \emptyset$.

Let us define vectors f and g by $f_k = g_k = u_k$, $f_\ell = g_\ell = v_\ell$, and $f_j = \min\{u_j, v_j\}$, $g_j = \max\{u_j, v_j\}$ if $j \neq k, \ell$. By the assumptions $u_k > v_k$ and $u_\ell > v_\ell$, we have $u \in P \cap B(f, \infty)$ and $v \in P \cap B(-\infty, g)$, but $P \cap B(f, g)$ does not contain an integer point, because $P \cap B(f, g) \cap \mathbb{Z}^n \subseteq \{x \in P' : x_k = u_k,\ x_\ell = v_\ell\} = \emptyset$ (here we slightly abuse notation because we consider P' again as a polyhedron in $\mathbb{R}^n$). This means that P does not satisfy the linking property. □ □

2.2 Strong linking property

The characterization of base polyhedra in Theorem 2.1 leads to a characterization of generalized polymatroids in terms of the *strong linking property*, as defined by Frank [5, 6]. An integer polyhedron P has the strong linking property if the polyhedron $P \cap \{x \in \mathbb{R}^n : f \leq x \leq g,\ \alpha \leq \sum_{j=1}^n x_j \leq \beta\}$ has an integer point for all $f \in \mathbb{Z}^n$, $g \in \mathbb{Z}^n$, $\alpha \in \mathbb{Z}$, and $\beta \in \mathbb{Z}$ that satisfy

(i) $f \leq g$

(ii) $\alpha \leq \beta$,

(iii) $P \cap \{x \in \mathbb{Z}^n : x \geq f,\ \sum_{j=1}^n x_j \leq \beta\} \neq \emptyset$,

(iv) $P \cap \{x \in \mathbb{Z}^n : x \leq g,\ \sum_{j=1}^n x_j \geq \alpha\} \neq \emptyset$.

To illustrate the difference between this and the linking property, consider again the triangle conv$((0,0), (2,0), (0,1))$, which has the linking property. Let $f = (0,1)$, $g = (2,1)$, and $\alpha = \beta = 2$. Then $(0,1) \in P \cap \{x \in \mathbb{Z}^n : x \geq f,\ \sum_{j=1}^n x_j \leq \beta\}$ and $(2,0) \in P \cap \{x \in \mathbb{Z}^n : x \leq g,\ \sum_{j=1}^n x_j \geq \alpha\}$, but $P \cap \{x \in \mathbb{R}^n : f \leq x \leq g,\ \alpha \leq \sum_{j=1}^n x_j \leq \beta\} = \emptyset$.

Corollary 2.2. *An integer polyhedron has the strong linking property if and only if it is a g-polymatroid.*

Proof. Let us associate an $(n+1)$-dimensional polyhedron P^+ to an n-dimensional integer polyhedron P by the definition

$$P^+ = \left\{ \left(x, -\sum_{j=1}^n x_j \right) : x \in P \right\}.$$

It is clear that P^+ is an integer polyhedron in the hyperplane $\sum_{j=1}^n x_j = 0$. It is also well-known [9] that P is a g-polymatroid if and only if P^+ is a base polyhedron. The characterization thus follows from Theorem 2.1 by the observation that the strong linking property for P is equivalent to the linking property for P^+. □ □

3 Additional proofs

To interpret Tomizawa's Theorem 1.1 for unbounded polyhedra, we rely on the following property of tangent cones. A proof is included for completeness.

Lemma 3.1. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron, and let u be a point in the relative interior of a non-minimal face F. Then $\mathrm{cone}_P(u)$ is generated by*

$$\cup\{\mathrm{cone}_P(y) \cup -\mathrm{cone}_F(y) : \ y \textit{ is on a face of } F\}.$$

Proof. It is easy to see that $\mathrm{cone}_P(y)$ and $-\mathrm{cone}_F(y)$ are both contained in $\mathrm{cone}_P(u)$. For a set $X \subseteq \mathbb{R}^n$, let $\mathrm{lin}(X)$ denote the linear hull of X. Clearly, $\mathrm{lin}(F)$ is generated by $\cup\{\mathrm{cone}_F(y) \cup -\mathrm{cone}_F(y) : \ y$ is on a face of $F\}$, since F is not a minimal face. It remains to show that $\mathrm{lin}(F) \cup \{\mathrm{cone}_P(y) : \ y$ is on a face of $F\}$ generates $\mathrm{cone}_P(u)$. Let $Ax \leq b$ be a linear description of P, and let $A^=$ be the submatrix formed by the active rows for F. Let $z \in \mathrm{cone}_P(u)$, i.e. $A^= z \leq \mathbf{0}$. Let a be a row of A that determines a facet F' of F, let A' be the matrix obtained by appending the row a to $A^=$, and let y be a point in the relative interior of F'. There exists $q \in \mathrm{lin}(F)$ such that $A'(z+q) \leq \mathbf{0}$, and therefore $z + q \in \mathrm{cone}_P(y)$, which proves the claim. □ □

Corollary 3.2. *If $P \subseteq \mathbb{R}^n$ is not a base polyhedron, then it has a point x on a minimal face such that $\mathrm{cone}_P(x)$ is not generated by vectors of the form $\{e_i - e_j : i, j \in [n]\}$.*

Proof. Immediate from Theorem 1.1 and Lemma 3.1. □ □

We now use the above Corollary to prove Lemma 1.2. For convenience, we repeat the lemma here.

Lemma 3.3. *If P is an integer polyhedron in the hyperplane $\sum_{j=1}^n x_j = \beta$ that is not a base polyhedron, then there exist integer points $u, v \in P \cap \mathbb{Z}^n$ such that (i) the size of the support of $u - v$ is at least 3, and (ii) $[u, v]$ is an edge of the polyhedron $(P \cap B(\min(u, v), \max(u, v)))_I$.*

Proof. First, observe that if u and v are distinct integer points in P and the size of the support of $u - v$ is at most 2, then $u - v = \mu(e_i - e_j)$ for some i, j and μ, because P lies in the hyperplane $\sum_{j=1}^n x_j = \beta$.

If P is pointed, then, by Corollary 3.2, there exists a vertex u such that $\mathrm{cone}_P(u)$ is not generated by vectors of the form $\{e_i - e_j : i, j \in [n]\}$. This implies that there is an edge or extreme ray incident to u, and an integer point v on the edge or extreme ray, such that the size of the support of $u - v$ is at least 3. Clearly, $[u, v]$ is an edge of $(P \cap B(\min(u, v), \max(u, v)))_I$, because it is an edge of $P \cap B(\min(u, v), \max(u, v))$.

Assume now that P has a characteristic subspace H of dimension $d \geq 1$, so every minimal face is a translate of H. First, we consider the case when H is not generated by vectors having support of size at most 2. Let $J = \{j \in [n] : \exists\ x \in H,\ x_j \neq 0\}$. Since H is a subspace, it has an integer point z such that $z_j \neq 0$ for every $j \in J$. Let $\{b^1, \ldots, b^d\}$ be an arbitrary integer basis of H, and choose $\mu \in \mathbb{Z}_+$ such that $\mu|z_j| \geq \sum_{i=1}^d |b^i_j|$ for every $j \in J$. Let u be an integer point on a minimal face F of P, and let $v = u + 2\mu z$. We claim that the box $B(\min(u, v), \max(u, v))$ contains

$u + \mu z$ and $u + \mu z + b^i$ for every i. Indeed, if $z_j > 0$, then $u_j \leq u_j + \mu z_j \leq u_j + 2\mu z_j$ and $u_j \leq u_j + \mu z_j + b^i_j \leq u_j + 2\mu z_j$ by the choice of μ. Analogously, if $z_j < 0$, then $u_j \geq u_j + \mu z_j \geq u_j + 2\mu z_j$ and $u_j \geq u_j + \mu z_j + b^i_j \geq u_j + 2\mu z_j$ for every i. As a consequence, $Q := (F \cap B(\min(u,v), \max(u,v)))_I$ is d-dimensional, and the edges of Q generate H. By our assumption, Q has an edge $[u', v']$ such that $u' - v'$ has support of size at least 3. Now u' and v' satisfy condition (ii) of the lemma, because $[u', v']$ is an edge of $(F \cap B(\min(u', v'), \max(u', v')))_I$, and hence also an edge of $(P \cap B(\min(u', v'), \max(u', v')))_I$.

The remaining case is when H is generated by vectors having support of size at most 2. By Corollary 3.2, there is a point u on a minimal face F_1 whose tangent cone is not generated by vectors having support of size at most 2. By the integrality of P, we can assume that u is integer. As $\text{cone}_P(u)$ is generated by the vectors $\{y - u : y \text{ is on a } (d+1)\text{-dimensional face containing } u\}$, there must be a $(d+1)$-dimensional face F_2 containing u such that $\text{lin}(F_2)$ is not generated by vectors having support of size at most 2 (here we use the assumption that H itself can be generated by such vectors).

Since F_2 is an integer polyhedron, it has an integer point w not in F_1. Let H_1 be the translate of H containing w; note that $H_1 \subseteq F_2$ and $H_1 \cap F_1 = \emptyset$. As in the previous case, let $J = \{j \in [n] : \exists\ x \in H,\ x_j \neq 0\}$, let z be an integer point such that $z_j \neq 0$ for every $j \in J$, and let $\{b^1, \dots, b^d\}$ be an integer basis of H. Choose $\mu \in \mathbb{Z}_+$ such that $\mu |z_j| \geq |w_j - u_j| + \sum_{i=1}^d |b^i_j|$ for every $j \in J$. Let $v = w + 2\mu z$. We claim that the box $B(\min(u,v), \max(u,v))$ contains $w + \mu z$ and $w + \mu z + b^i$ for every i. If $z_j > 0$, then $u_j \leq w_j + \mu z_j \leq w_j + 2\mu z_j$ and $u_j \leq w_j + \mu z_j + b^i_j \leq w_j + 2\mu z_j$ by the choice of μ. Analogously, if $z_j < 0$, then $u_j \geq w_j + \mu z_j \geq w_j + 2\mu z_j$ and $u_j \geq w_j + \mu z_j + b^i_j \geq w_j + 2\mu z_j$ for every i. Obviously, u is also in the box, so $Q := (F_2 \cap B(\min(u,v), \max(u,v)))_I$ is $(d+1)$-dimensional.

The edges of Q generate $\text{lin}(F_2)$, so Q must have an edge $[u', v']$ such that $u' - v'$ has support of size at least 3. We claim that u' and v' satisfy condition (ii) of the lemma. Indeed, $[u', v']$ is an edge of Q, so it is also an edge of the polyhedron $(F_2 \cap B(\min(u', v'), \max(u', v')))_I$. Since F_2 is a face of P, $[u', v']$ is also an edge of $(P \cap B(\min(u', v'), \max(u', v')))_I$. □ □

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