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Old and new results on packing arborescences

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Abstract

We propose a further development in the theory of packing arborescences. First we review some of the existing results on packing arborescences and then we provide common generalizations of them. We introduce and solve the problem of reachability-based packing of matroid-rooted hyperarborescences and we also solve the minimum cost version of this problem. Furthermore, we introduce and solve the problem of matroid-based packing of matroid-rooted mixed hyperarborescences.

1 Introduction

We study packings of arborescences in this paper. An ***r*-arborescence** is a directed tree on a vertex-set containing the **root** vertex r in which each vertex has in-degree 1 except r. (For other definitions, see the next section.) The starting point of the research on arborescence-packings is the following famous result of Edmonds [6] on packing spanning arborescences.

Theorem 1.1 ([6]). *There exists a packing of k spanning r-arborescences in a digraph $\vec{G} = (V, A)$ if and only if*

$$\varrho_A(X) \geq k \tag{1}$$

holds for all $\emptyset \neq X \subseteq V \setminus r$ where $\varrho_A(X)$ denotes the in-degree of X. ■

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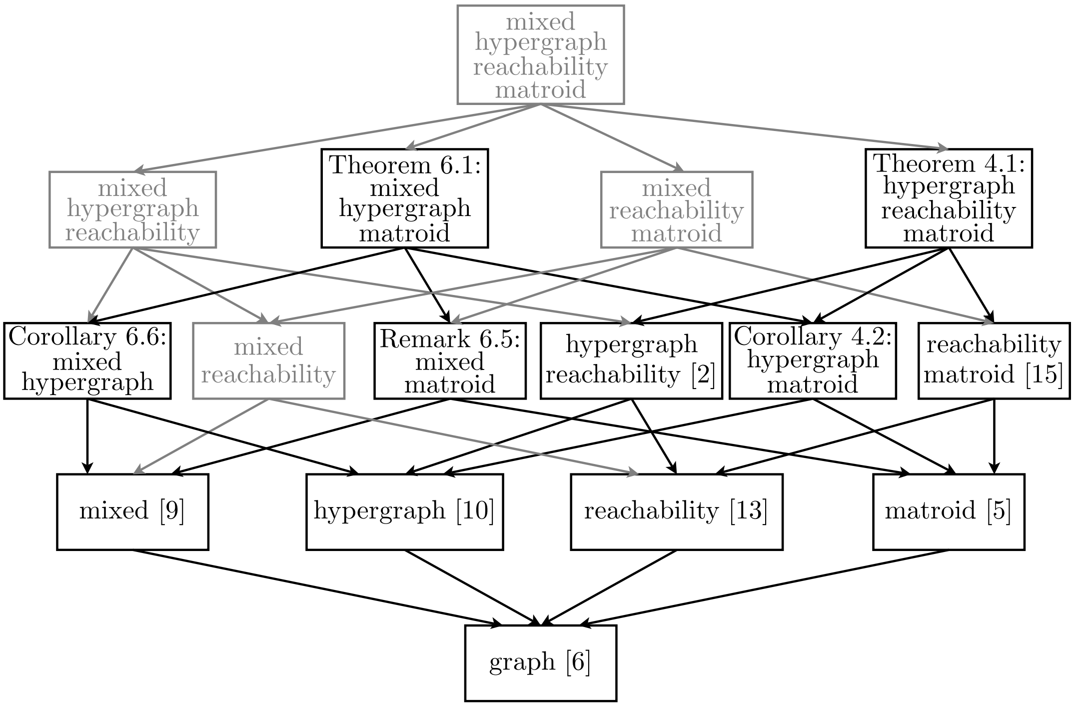

Figure 1: All possible common generalizations of the 4 problems mentioned in the introduction.

This result has extensions in many directions. For our purposes let us mention four of them: the result of Kamiyama, Katoh, Takizawa [13] on packing *reachability* arborescences (Theorem 3.3 in this paper), Theorem 3.4 on packing matroid-rooted arborescences with *matroid* constraint by Durand de Gevigney, Nguyen, Szigeti [5], Theorem 3.2 on packing spanning *hyper*arborescences (Frank, T. Király, Z. Király [10]) and Theorem 3.6 on packing spanning *mixed* arborescences (Frank [9]). Figure 1 shows all the possible common generalizations of these extensions. The results without citations corresponding to black boxes of the diagram are presented in this paper, the ones in gray are yet to be proved.

The main contribution of this work is to show how the existing hypergraphical results can be derived directly from their graphical counterparts. We note that the original proofs of these results were different. Both of Frank, Z. Király and T. Király [10] and Bérczi and Frank [2] showed that a directed hypergraph satisfying their condition for the packing problem can be reduced – by an operation called *trimming* – to a digraph satisfying the condition of the graphical counterpart of their problem. Our method looks a bit similar to this however we also add some extra vertices to the digraph to ensure that the condition of the graphical result holds automatically for the digraph if the hypergraphical condition holds for the directed hypergraph. We also note that this method allows us to find a minimum cost solution of these problems for any cost function on the set of dyperedges.

We introduce and solve with the same method the problem of reachability-based packing of matroid-rooted hyperarborescences, that is, a common generalization of three of the above four extensions, not the mixed one. We also consider a generalization of other three of the above four extensions, not the reachability one this time, namely the problem of matroid-based packing of matroid-rooted mixed hyperarborescences. Using a new orientation result (Theorem 6.2) on hypergraphs covering intersecting supermodular functions, we reduce this problem to its directed version, the problem of matroid-based packing of matroid-rooted hyperarborescences, which in turn is a special case of the problem of reachability-based packing of matroid-rooted hyperarborescences. The proof of Theorem 6.2 (that is presented in the Appendix) imitates the proof (see in [8, 9]) of its special case in graphs.

2 Definitions

In this paper, $\boldsymbol{\mathcal{H}} = (V, \mathcal{E})$ will be a hypergraph. We suppose that all the hyperedges in $\mathcal{E}$ are of size at least 2. When all the hyperedges are of size 2, that is, when the hypergraph is a graph, we will denote it by $\boldsymbol{G} = (V, E)$. For a vertex set X, $\boldsymbol{i_{\mathcal{E}}(X)}$ denotes the number of hyperedges in $\mathcal{E}$ that are contained in X. For a partition $\boldsymbol{\mathcal{P}}$ $= \{V_0, V_1, \dots, V_\ell\}$ of V where only $\boldsymbol{V_0}$ can be empty, we denote by $\boldsymbol{e_{\mathcal{E}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ intersecting at least two members of $\mathcal{P}$. Since every hyperedge is either completely contained in some V_i or intersects at least two V_i's, the following formula holds.

$$e_{\mathcal{E}}(\mathcal{P}) + \sum_{0}^{\ell} i_{\mathcal{E}}(V_i) \quad = \quad |\mathcal{E}|. \tag{2}$$

Let $\vec{\boldsymbol{\mathcal{H}}} = (V, \mathcal{A})$ be a directed hypergraph (shortly **dypergraph**) where $\boldsymbol{V}$ denotes the set of vertices and $\boldsymbol{\mathcal{A}}$ denotes the set of dyperedges of $\vec{\mathcal{H}}$. By a **dyperedge** we mean a pair (Z, z) such that $z \in Z \subseteq V$, where z is the **head** of the dyperedge (Z, z) and the elements of $Z \setminus z$ are the **tail**s of the dyperedge (Z, z). We suppose that each dyperedge has one head and at least one tail. When a dypergraph is a digraph, we will denote it by $\vec{\boldsymbol{G}} = (V, A)$. Let $X \subseteq V$. We say that the dyperedge (Z, z) **enters** X if the head of (Z, z) is in X and at least one tail of (Z, z) is not in X. We define the **in-degree** $\boldsymbol{\varrho_{\mathcal{A}}(X)}$ of X as the number of dyperedges in $\mathcal{A}$ entering X.

For a set function h on V, we say that the dypergraph $\vec{\mathcal{H}}$ **covers** h if

$$\varrho_{\mathcal{A}}(X) \geq h(X) \text{ for all } X \subseteq V. \tag{3}$$

By **trimming** the dypergraph $\vec{\mathcal{H}}$ we mean replacing each dyperedge (Z, z) of $\vec{\mathcal{H}}$ by an arc tz where t is one of the tails of the dyperedge (Z, z).

By an **orientation** of $\mathcal{H}$, we mean a dypergraph $\vec{\mathcal{H}}$ obtained from $\mathcal{H}$ by choosing, for every $Z \in \mathcal{E}$, an orientation of Z, that is by choosing a head z for Z.

For a function $m : V \to \mathbb{Z}$ and a set $X \subseteq V$, we define $\boldsymbol{\widetilde{m}(X)}$ as in [8], that is

$$\widetilde{m}(X) = \sum_{v \in X} m(v). \tag{4}$$

Let p be a set function on V. We call p **supermodular** if for every $X, Y \subseteq V$,

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{5}$$

We say that p is **intersecting supermodular** if (5) is satisfied for every $X, Y \subseteq V$ when $X \cap Y \neq \emptyset$. A set function b is called **submodular** if $-b$ is supermodular. It is well known that $i_{\mathcal{E}}$ is supermodular and that $\varrho_{\mathcal{A}}$ is submodular (see e.g. in [8]).

In a dypergraph $\vec{\mathcal{H}} = (V, \mathcal{A})$, we say that a vertex w **can be reached** from a vertex u if there exists an alternating sequence $v_1 = u, Z_1, v_2, \ldots, v_i, Z_i, v_{i+1}, \ldots, v_j = w$ of vertices and dyperedges such that v_i is a tail of Z_i and v_{i+1} is the head of Z_i. For a set $X \subseteq V$, we denote by $\boldsymbol{P_{\mathcal{A}}(X)}$ the set of vertices from which X can be reached in $\vec{\mathcal{H}}$ and by $\boldsymbol{Q_{\mathcal{A}}(X)}$ the set of vertices that can be reached from X in $\vec{\mathcal{H}}$. Let $\boldsymbol{\mathcal{R}}$ $= \{R_1, \ldots, R_t\}$ be a list of t not necessarily distinct sets of vertices of $\vec{\mathcal{H}}$. We call the pair $(\vec{\mathcal{H}}, \mathcal{R})$ a **rooted dypergraph**. For $X \subseteq V$, we define $\boldsymbol{p^{\mathcal{R}}(X)}$ as the number of members of $\mathcal{R}$ disjoint from X and $\boldsymbol{q^{\mathcal{R}}_{\mathcal{A}}(X)}$ as the number of R_i's which do not intersect X but from which X is reachable in $\vec{\mathcal{H}}$, in other words: $q^{\mathcal{R}}_{\mathcal{A}}(X) = |\{i : R_i \cap X = \emptyset, Q_{\mathcal{A}}(R_i) \cap X \neq \emptyset\}|$. When each R_i consists of a single vertex r_i, we denote $\mathcal{R}$ by R.

For a non-empty set $R \subseteq U$, the subdigraph $\vec{T} = (U, A')$ of $\vec{G} = (V, A)$ is said to be an R-**branching** if it consists of $|R|$ vertex-disjoint arborescences whose roots are in R. Let $\vec{\mathcal{T}} = (U, \mathcal{A}')$ be a subdypergraph of $\vec{\mathcal{H}} = (V, \mathcal{A})$ such that U is the vertex set induced by $\mathcal{A}'$ and $R \subseteq U$. Let U' be the set of vertices in U whose in-degree in $\vec{\mathcal{T}}$ is not 0. We say that $\vec{\mathcal{T}}$ is an R-**hyperbranching** if it can be trimmed to an R-branching with vertex-set $U' \cup R$. (It is easy to see that this is equivalent to the following: $R \subseteq U$, $\varrho_{\mathcal{A}'}(r) = 0$ for all $r \in R$, $\varrho_{\mathcal{A}'}(u) = 1$ for all $u \in U'$, $\varrho_{\mathcal{A}'}(X) \geq 1$ for all $X \subseteq V \setminus R, X \cap U' \neq \emptyset$.) When $R = \{r\}$, an R-hyperbranching is also called an r-**hyperarborescence**.

Remark 2.1. Observe that the definitions of R-hyperbranchings and R-branchings coincide for digraphs. This statement also holds for the subsequent definitions, that is, our definitions for hypergraphs are straightforward generalizations of the original definitions for graphs. Therefore, we will define everything only for the general hypergraphical case.

We call $\vec{\mathcal{T}}$ a **reachability** R-**hyperbranching** in $\vec{\mathcal{H}}$ if $U' \cup R$ contains the set $Q_{\mathcal{A}}(R)$, in other words, if $Q_{\mathcal{A}'}(R) = Q_{\mathcal{A}}(R)$. If all the vertices can be reached from R in $\vec{\mathcal{H}}$, then a reachability R-hyperbranching is called **spanning**. In a rooted dypergraph $(\vec{\mathcal{H}}, \mathcal{R} = \{R_1, \ldots, R_k\})$, a set of arc-disjoint spanning (reachability, resp.) hyperbranchings is called a **packing of spanning** (**reachability**, resp.) $\mathcal{R}$-**hyperbranchings**. Examples for a spanning hyperarborescence and for a reachability hyperarborescence can be found in Figure 2.

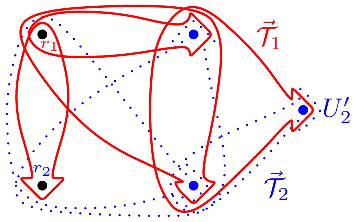

Figure 2: $\vec{\mathcal{T}}_1$ is a spanning r_1-hyperarborescence while $\vec{\mathcal{T}}_2$ is a reachability r_2-hyperarborescence of the dypergraph.

We also need some basic notions from matroid theory (for more details we refer to [8, Chapter 5]). Let $\boldsymbol{\mathcal{M}}$ be a matroid on S with rank function $\boldsymbol{r_{\mathcal{M}}}$. It is well known that $r_{\mathcal{M}}$ is non-negative, monotone, subcardinal and submodular. We define $\mathbf{Span}_{\boldsymbol{\mathcal{M}}}\mathbf{(Q)} := \{\mathsf{s} \in \mathsf{S} : r_{\mathcal{M}}(\mathsf{Q} \cup \{\mathsf{s}\}) = r_{\mathcal{M}}(\mathsf{Q})\}$.

A **matroid-rooted dypergraph** is a quadruple $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ where $\vec{\mathcal{H}}$ is a dypergraph, $\mathcal{M}$ is a matroid on the set $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_{|\mathsf{S}|}\}$ with rank function $r_{\mathcal{M}}$ and π is a map from S to V. In general, π is not injective; different elements of S may be mapped to the same vertex of V. The elements $\{\mathsf{s}_1, \dots, \mathsf{s}_{|\mathsf{S}|}\}$ mapped to the vertices of V are called the **matroid-roots**. For $X \subseteq V$, we denote by $\boldsymbol{\mathsf{S}_X}$ the set of matroid-roots mapped to X by π. We say that π is **$\mathcal{M}$-independent** if S_v is independent in $\mathcal{M}$ for all $v \in V$.

A **matroid-rooted hyperarborescence** is a triple $(\vec{\mathcal{T}}, r, \mathsf{s})$ where $\vec{\mathcal{T}}$ is an r-hyperarborescence and s is an element of S mapped to r. We say that s is the **matroid-root** of the matroid-rooted hyperarborescence $(\vec{\mathcal{T}}, r, \mathsf{s})$. A **matroid-based packing of matroid-rooted hyperarborescences** in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise dyperedge-disjoint matroid-rooted hyperarborescences such that for each $v \in V$, the set $\mathbf{B}_{\boldsymbol{v}}$ of matroid-roots of the matroid-rooted hyperarborescences in which the vertex v can be reached from the roots forms a base of the matroid $\mathcal{M}$, that is $\mathsf{B}_v = \{\mathsf{s}_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ is a base of S. A **reachability-based packing of matroid-rooted hyperarborescences** in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise dyperedge-disjoint matroid-rooted hyperarborescences such that for each $v \in V$, the set B_v is a base of $\mathsf{S}_{P_{\mathcal{A}}(v)}$.

Remark 2.2. Let $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{R} = \{R_1, ..., R_k\})$ be a rooted dypergraph. Let $\mathsf{S}_{\mathcal{R}} := \bigcup^* \mathcal{R}$ (as a multiset), let π map each occurrence of r in $\mathsf{S}_{\mathcal{R}}$ to the vertex $r \in V$, and let $\boldsymbol{\mathcal{M}_{\mathcal{R}}}$ be the partition matroid on $\mathsf{S}_{\mathcal{R}}$ given by $\mathcal{R}$ where a set $\mathsf{P} \subseteq \mathsf{S}_{\mathcal{R}}$ is independent if and only if $|\mathsf{P} \cap R_i| \leq 1$ for $i = 1, ..., k$. Then the problem of matroid-based (reachability-based, resp.) packing of matroid-rooted hyperarborescences in $(\mathcal{H}, \mathcal{M}_{\mathcal{R}}, \mathsf{S}, \pi)$ and of packing spanning (reachability, resp.) $\mathcal{R}$-hyperbranchings coincide.

Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph where $\mathcal{E}$ is the set of hyperedges and $\mathcal{A}$ is the set of dyperedges of $\mathcal{F}$. The definitions of a **rooted mixed hypergraph**

$(\mathcal{F}, \mathcal{R})$ and a **matroid-rooted mixed hypergraph** $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ are similar to the previous definitions of a rooted and a matroid-rooted dypergraph, respectively. Under a **mixed r-hyperarborescence** (**mixed R-hyperbranching**, respectively) in a mixed hypergraph, we mean a mixed subhypergraph which, after a proper orientation of its hyperedges, can become an r-hyperarborescence (R-hyperbranching, respectively). A **matroid-rooted mixed hyperarborescence** is a triple $(\mathcal{T}, r, \mathsf{s})$ where $\mathcal{T}$ is a mixed r-hyperarborescence and s is an element of S mapped to r. We define a **matroid-based packing of matroid-rooted mixed hyperarborescences** in $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ as a set $\{(\mathcal{T}_1, r_1, \mathsf{s}_1), \dots, (\mathcal{T}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ of pairwise (hyper-and-dyperedge)-disjoint matroid-rooted mixed hyperarborescences in $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ such that, by a proper orientation of the hyperedges of each $(\mathcal{T}_i, r_i, \mathsf{s}_i)$, one can get a matroid-based packing of matroid-rooted hyperarborescences $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ with the same roots. When a rooted (matroid-rooted, respectively) mixed hypergraph has no dyperedges, it is a **rooted** (**matroid-rooted**, respectively) **hypergraph**. We call a mixed hyperarborescence without dyperedges a **hypertree**.

3 Preliminaries

First we mention the strong form of Theorem 1.1 that considers a more general problem when we want to find a packing of spanning $\mathcal{R}$-branchings in $\vec{G}$.

Theorem 3.1 ([6]). *In a rooted digraph $(\vec{G} = (V, A), \mathcal{R})$, there exists a packing of spanning $\mathcal{R}$-branchings if and only if*

$$\varrho_A(X) \geq p^{\mathcal{R}}(X) \tag{6}$$

holds for all $\emptyset \neq X \subseteq V$. ■

This result was generalized for rooted dypergraphs by Frank, T. Király and Z. Király [10] by observing that a dypergraph satisfying condition (7) of the following theorem can be trimmed to a digraph satisfying (6). We should also cite here the paper of Frank, T. Király and Kriesell [11].

Theorem 3.2 ([10]). *In a rooted dypergraph $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{R})$, there exists a packing of spanning $\mathcal{R}$-hyperbranchings if and only if*

$$\varrho_{\mathcal{A}}(X) \geq p^{\mathcal{R}}(X) \tag{7}$$

holds for all $\emptyset \neq X \subseteq V$. ■

A generalization of Theorem 3.1 for reachability branchings was given by Kamiyama, Katoh and Takizawa [13], as follows.

Theorem 3.3 ([13]). *There exists a packing of reachability $\mathcal{R}$-branchings in a rooted digraph $(\vec{G} = (V, A), \mathcal{R})$ if and only if*

$$\varrho_A(X) \geq q_A^{\mathcal{R}}(X) \tag{8}$$

holds for all $\emptyset \neq X \subseteq V$. ■

Observe that, (6) holds if and only if (8) holds and each vertex $v \in V$ is reachable from each set $R_i \in \mathcal{R}$. Bérczi and Frank [2] noted that Theorem 3.3 extends to dypergraphs.

Recently, Durand de Gevigney, Nguyen and Szigeti [5] and Cs. Király [15] extended Theorems 3.1 and 3.3 for matroid-rooted digraphs, as follows.

Theorem 3.4 ([5]). *Let $(\vec{G} = (V, A), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted digraph. There exists a matroid-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{9}$$

holds for all $\emptyset \neq X \subseteq V$. ■

Theorem 3.5 ([15]). *Let $(\vec{G} = (V, A), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted digraph. There exists a reachability-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P_A(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{10}$$

holds for all $\emptyset \neq X \subseteq V$. ■

In Section 4, we extend Theorems 3.4 and 3.5 for dypergraphs.

An extension of Theorem 1.1 for mixed graphs was given by Frank [9] as an application of an orientation result. We provide a common generalization of this result and Theorem 3.1 later.

Theorem 3.6 ([9]). *There exist a packing of k spanning mixed r-arborescences in a mixed graph $F = (V, E \cup A)$ if and only if*

$$e_E(\mathcal{P}) \geq \sum_1^{\ell}(k - \varrho_A(V_i)) \tag{11}$$

holds for every partition $\mathcal{P} = \{r \in V_0, V_1, \ldots, V_\ell\}$ of V. ■

4 Reachability-based packing of matroid-rooted hyperarborescences

The following theorem which is the main contribution of the present paper provides a common generalization of Theorems 3.2 and 3.5.

Theorem 4.1. *Let $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph. There exists a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{12}$$

holds for all $X \subseteq V$.

Proof. To prove the necessity, let $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \ldots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ be a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$. For any $v \in V$, since $\mathsf{S}_v \subseteq \mathsf{B}_v$ and B_v is independent in $\mathcal{M}$, so is S_v, and hence π is $\mathcal{M}$-independent. Let now $X \subseteq V$ and $\mathsf{B} = \bigcup_{v \in X} \mathsf{B}_v$. Since $\mathrm{Span}_{\mathcal{M}}$ is monotone, B_v is a base of $\mathsf{S}_{P_{\mathcal{A}}(v)}$ and by definition of $P_{\mathcal{A}}(X)$, we have $\mathrm{Span}_{\mathcal{M}}(\mathsf{B}) \supseteq \bigcup_{v \in X} \mathrm{Span}_{\mathcal{M}}(\mathsf{B}_v) \supseteq \bigcup_{v \in X} \mathsf{S}_{P_{\mathcal{A}}(v)} = \mathsf{S}_{P_{\mathcal{A}}(X)}$. Then, since $r_{\mathcal{M}}$ is monotone, $(\star)$ $r_{\mathcal{M}}(\mathsf{B}) \geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)})$.

For each matroid-root $\mathsf{s}_i \in \mathsf{B} \setminus \mathsf{S}_X$, there exists a vertex $v \in X$ such that $\mathsf{s}_i \in \mathsf{B}_v$ and then since $\vec{\mathcal{T}}_i$ is an r_i-hyperarborescence and $v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i) \cap X$, there exists a dyperedge of $\vec{\mathcal{T}}_i$ that enters X. Since these matroid-rooted hyperarborescences are dyperedge-disjoint, $r_{\mathcal{M}}$ is subcardinal, submodular, and monotone, and by $(\star)$, we have $\varrho_{\mathcal{A}}(X) \geq |\mathsf{B} \setminus \mathsf{S}_X| \geq r_{\mathcal{M}}(\mathsf{B} \setminus \mathsf{S}_X) \geq r_{\mathcal{M}}(\mathsf{B} \cup \mathsf{S}_X) - r_{\mathcal{M}}(\mathsf{S}_X) \geq r_{\mathcal{M}}(\mathsf{B}) - r_{\mathcal{M}}(\mathsf{S}_X) \geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$ that is, (12) is satisfied.

To prove the sufficiency, let $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph such that π is $\mathcal{M}$-independent and (12) holds. First we define a matroid-rooted digraph $(\vec{G} = (V', A), \mathcal{M}, \mathsf{S}, \pi)$ for which the conditions of Theorem 3.5 hold. We define $V' := V \cup \mathcal{A}$ hence π is still well defined and the matroid-rooted digraph is still $\mathcal{M}$-independent. Let $A_1 := \{(Z, z)z : (Z, z) \in \mathcal{A}\}$ and $A_2 := \{t(Z, z) : (Z, z) \in \mathcal{A}, t \in Z \setminus z\}$. Let $A := A_1 \cup (r_{\mathcal{M}}(\mathsf{S}) \cdot A_2)$ where $r_{\mathcal{M}}(\mathsf{S}) \cdot A_2$ denotes the multiset consisting of the union of $r_{\mathcal{M}}(\mathsf{S})$ copies of A_2. For the construction see Figure 3.

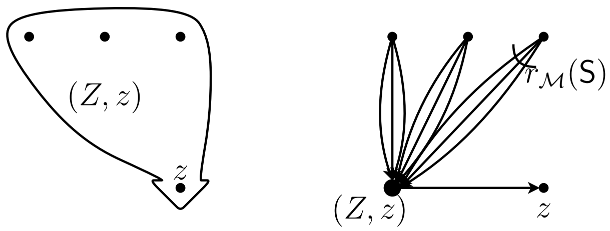

Figure 3: The construction.

Observe that $\varrho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P_A(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$ for a subset $X \subseteq V'$ whenever there exist a dyperedge (Z, z) and a tail vertex $t \in Z \setminus z$ in $\vec{\mathcal{H}}$ such that, in $\vec{G}$, $(Z, z) \in X$ and $t \notin X$ since then the $r_{\mathcal{M}}(\mathsf{S})$ copies of the arc $t(Z, z)$ enter X in $\vec{G}$ and hence $\varrho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}) \geq r_{\mathcal{M}}(\mathsf{S}_{P_A(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$. Moreover, if there is no such dyperedge, then $\varrho_A(X) = \varrho_{\mathcal{A}}(X \cap V), r_{\mathcal{M}}(\mathsf{S}_{P_A(X)}) = r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X \cap V)}), r_{\mathcal{M}}(\mathsf{S}_X) = r_{\mathcal{M}}(\mathsf{S}_{X \cap V})$ and hence (10) follows from (12).

Therefore, there exists a reachability-based packing of matroid-rooted arborescences $\{(\vec{T}_1, r_1, \mathsf{s}_1), \ldots, (\vec{T}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ by Theorem 3.5. We define $\vec{\mathcal{T}}_i$ $(i = 1, \ldots, |\mathsf{S}|)$ to be the subdypergraph of $\vec{\mathcal{H}}$ induced by dyperedges $(Z, z) \in \mathcal{A}$ such that the vertex (Z, z) has out-degree 1 in $\vec{T}_i$. It is easy to check that $\vec{\mathcal{T}}_i$ is an r_i-hyperarborescence with matroid-root s_i and the set of vertices of $\vec{\mathcal{T}}_i$ with in-degree 1 is the same as the set of vertices in V of in-degree 1 in $\vec{T}_i$. Moreover, the hyperarborescences $\vec{\mathcal{T}}_1, \ldots, \vec{\mathcal{T}}_{|\mathsf{S}|}$ are dyperedge-disjoint since each vertex $(Z, z) \in \mathcal{A}$ has out-degree 1 in $\vec{G}$. Hence, as the reachability of the vertices in V from r_i coincides in $\vec{T}_i$ and

$\vec{\mathcal{T}}_i$ $(i = 1, \ldots, |\mathsf{S}|)$, $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \ldots, (\vec{\mathcal{T}}_{|\mathsf{S}|}, r_{|\mathsf{S}|}, \mathsf{s}_{|\mathsf{S}|})\}$ is a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$. ■

As a corollary of Theorem 4.1 (or from Theorem 3.4 with a proof similar to the previous one), one can get the following result on matroid-based packing of matroid-rooted hyperarborescences.

Corollary 4.2. *Let $(\vec{\mathcal{H}} = (V, \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph. There exists a matroid-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\varrho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \tag{13}$$

holds for all $\emptyset \neq X \subseteq V$. ■

Similarly, one can get Theorem 3.2 and the result of Bérczi and Frank [2], that is, the extensions of Theorems 3.1 and 3.3 for dypergraphs.

5 Algorithmic aspects

Bérczi and Frank [1] gave a TDI polyhedral description of the – so called – arborescence packable subgraphs. Using this result it can be shown that there is a polynomial algorithm to find a minimum cost packing of spanning (reachability, resp.) $\mathcal{R}$-branchings for any cost function on the arc-set of a rooted digraph $(\vec{G}, \mathcal{R})$. [5] provided also an algorithm for the problem of minimum cost matroid-based packing of matroid-rooted arborescences and recently Bérczi, T. Király and Kobayashi [3, 4] solved the problem of minimum cost reachability-based packing of matroid-rooted arborescences.

As noted before, Frank, T. Király and Z. Király [10] showed that it is possible to trim a dypergraph satisfying (7) to a digraph satisfying (6). However, this method fails to work for the generalization of the problem when we are seeking minimum cost dyperedge-disjoint spanning hyperbranchings as Figure 4 shows. Note that the minimum cost spanning hyperarborescence of solid, red dyperedges of the dypergraph on the left-hand side of the figure has cost 0 while in the trimmed digraph on the right-hand side there is only one spanning arborescence and it has cost 1.

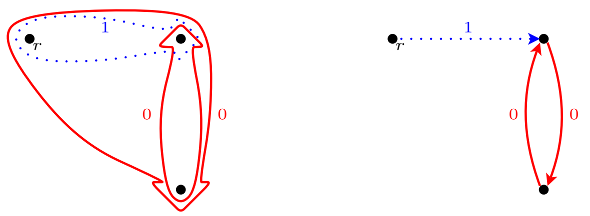

Figure 4: The trimming operation does not preserve minimum cost arborescence packings.

Let us now assume that, in a matroid-rooted dypergraph $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$, a cost function c is given on the dyperedges. Recall the proof of Theorem 4.1. Observe that if we take a cost function c' on the arc-set of the defined digraph to be 0 on the arcs with a head in $\mathcal{A}$ and $c((Z,z))$ on the arc with a tail (Z,z) for every $(Z,z) \in \mathcal{A}$, then a minimum cost reachability-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ gives rise to a minimum cost reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$ with the same cost. By using the above algorithms and similar deductions, we obtain the following result.

Theorem 5.1. *There exist polynomial algorithms that, for an input consisting of a dypergraph $\vec{\mathcal{H}} = (V, \mathcal{A})$, a cost function c on $\mathcal{A}$, and a family $\mathcal{R}$ of some non-empty subsets of V or a matroid $\mathcal{M}$ on S along with a map $\pi : \mathsf{S} \to V$, output the following:*
(a) a minimum cost packing of spanning $\mathcal{R}$-hyperbranchings in $(\vec{\mathcal{H}}, \mathcal{R})$,
(b) a minimum cost packing of reachability $\mathcal{R}$-hyperbranchings in $(\vec{\mathcal{H}}, \mathcal{R})$,
(c) a minimum cost matroid-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$,
(d) a minimum cost reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{H}}, \mathcal{M}, \mathsf{S}, \pi)$. ■

6 Packing mixed hyperarborescences

A common generalization of Theorem 3.6 and Corollary 4.2 can be formulated as follows.

Theorem 6.1. *There exists a matroid-based packing of matroid-rooted mixed hyperarborescences in a matroid-rooted mixed hypergraph $(\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell} (r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_{V_i}) - \varrho_{\mathcal{A}}(V_i)) \tag{14}$$

holds for every partition $\mathcal{P} = \{V_0, V_1, \ldots, V_\ell\}$ of V.

We prove this theorem using the method of Frank [9]. To this end, we need the following general orientation result on hypergraphs. The proof of [8, Theorem 15.4.13] (the corresponding result for graphs) – with the necessary straightforward modifications – can be extended for hypergraphs. For the sake of completeness we provide a proof in the Appendix. We mention that this result can also be obtained by using the techniques from [10].

Theorem 6.2. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and h an integer-valued, intersecting supermodular function (with possible negative values) such that $h(V) = 0$. There exists an orientation of $\mathcal{H}$ that covers h if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell} h(V_i) \tag{15}$$

holds for every partition $\mathcal{P} = \{V_0, V_1, \ldots, V_\ell\}$ of V.

Please note that in (14) and in (15) the index i starts at 1 (and not at 0). This means that we consider here all the subpartitions of V.

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $(\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A}), \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted mixed hypergraph. Let us introduce the following function $h^\star$, which is integer-valued, intersecting supermodular and satisfies $h^\star(V) = 0$.

$$h^\star(X) = \begin{cases} r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) - \varrho_{\mathcal{A}}(X) & \text{if } \emptyset \neq X \subseteq V, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 6.2, applied for $(V, \mathcal{E})$ (the undirected part of the mixed hypergraph $\mathcal{F}$) and $h^\star$, provides the following result.

Lemma 6.3. *There exists an orientation of a matroid-rooted mixed hypergraph* $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ *satisfying* (13) *if and only if* (14) *is satisfied.* ■

We get Theorem 6.1 by Corollary 4.2 and Lemma 6.3. ■■

Note that Theorem 6.1 reduces to the following result when $\mathcal{A} = \emptyset$. This result is a generalization of a result of Katoh and Tanigawa [14] for hypergraphs. Recall that under a **matroid-based packing of matroid-rooted hypertrees** we mean that the hypertrees can be *oriented* such that we get a matroid-based packing of matroid-rooted hyperarborescences with the same roots.

Corollary 6.4. *Let* $(\mathcal{H}, \mathcal{M}, \mathsf{S}, \pi)$ *be a matroid-rooted hypergraph. There exists a matroid-based packing of matroid-rooted hypertrees in* $(\mathcal{H}, \mathcal{M}, \mathsf{S}, \pi)$ *if and only if* π *is* $\mathcal{M}$*-independent and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)) \tag{16}$$

holds for every partition $\mathcal{P}$ *of* V. ■

Remark 6.5. We note that Theorem 6.1 in the case where $\mathcal{F}$ is a mixed graph is a common generalization of the above mentioned result of Katoh and Tanigawa [14] and Theorem 3.4.

By Remark 2.2, we get the following corollary of Theorem 6.1 that, in the case where $\mathcal{F}$ is a mixed graph, generalizes Theorem 3.6 for packing of mixed branchings.

Corollary 6.6. *In a rooted mixed hypergraph* $(\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A}), \mathcal{R})$*, there exists a packing of spanning mixed* $\mathcal{R}$*-hyperbranchings if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell} (p^{\mathcal{R}}(V_i) - \varrho_{\mathcal{A}}(V_i)) \tag{17}$$

holds for every partition $\mathcal{P} = \{V_0, V_1, \ldots, V_\ell\}$ *of* V. ■

7 Concluding remarks

We finish this paper by mentioning some remarks on other possible generalizations.

7.1 Packing reachability mixed arborescences

The first problem is about packing reachability mixed arborescences. We just mention the orientation version of the problem. Let $(F = (V, E \cup A), \mathcal{R})$ be a rooted mixed graph. For a set $X \subseteq V$, we denote by $\boldsymbol{Q_{E\cup A}(X)}$ the set of vertices that can be reached from X in F and by $\boldsymbol{q^{\mathcal{R}}_{E\cup A}(X)}$ the number of indices i such that $R_i \cap X = \emptyset$ and $Q_{E\cup A}(R_i) \cap X \neq \emptyset$. When does there exist an orientation $\vec{E}$ of E such that $(V, \vec{E} \cup A)$ covers $q^{\mathcal{R}}_{E\cup A}$? Let us consider the following two conditions that are clearly necessary: for every partition $\mathcal{P} = \{V_0, V_1, \dots, V_\ell\}$ of V,

$$e_E(\mathcal{P}) \geq \sum_1^\ell (q^{\mathcal{R}}_{E\cup A}(V_i) - \varrho_A(V_i)), \tag{18}$$

$$e_E(\mathcal{P}) \geq \sum_1^\ell (q^{\mathcal{R}}_{E\cup A}(V \setminus V_i) - \varrho_A(V \setminus V_i)). \tag{19}$$

The following example shows that conditions (18) and (19) are not sufficient. Let $F = (V, E\cup A)$ and $\mathcal{R} = \{\{r_1\}, \{r_2\}\}$ be defined as follows. $V = \{a, b, c, d\}$, $E = \{ab\}$, $A = \{ca, cb, ad, bd\}$, $r_1 = a$ and $r_2 = b$. It is easy to check that (18) and (19) are satisfied. However, the required orientation does not exist since the edge ab should be oriented in both directions.

7.2 Infinite dypergraphs

In this paper, we considered finite dypergraphs however some results can also be proved for infinite dypergraphs. In a recent paper, Joó [12] showed that Theorem 3.1 is also true in infinite digraphs that contain no forward-infinite paths. Hence using the proof technique of Theorem 4.1 to this result one can extend Theorem 3.2 for infinite dypergraphs that contain no forward-infinite paths.

7.3 Covering intersecting bi-set families under matroid constraints in dypergraphs

Finally, we mention that Bérczi, T. Király, Kobayashi [3] have provided an abstract result on covering intersecting bi-set families under matroid constraints that generalizes Theorem 3.5 and another result of Bérczi and Frank [2]. We do not want to go into details, we just mention that their proof also works for dypergraphs.

Acknowledgements

We note some results of this paper were published before as a manuscript by some of the authors of the present work. It was shown in a research project report of Léonard [16] – that was written under the supervision of Szigeti – that the trimming method can be applied to extend Theorem 3.4. Fortier, Léonard, Szigeti and Talon

[7] proved Theorem 4.1 using trimming however the proof was quite long and did not extend for the minimum cost version of the problem.

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8 Appendix

For completeness we give the proof of Theorem 6.2 here.

Proof of Theorem 6.2. If such an orientation exists then, for every partition $\mathcal{P} = \{V_0, V_1, \ldots, V_\ell\}$, we have by (3), $e_\mathcal{E}(\mathcal{P}) \geq \sum_1^\ell \varrho_\mathcal{A}(V_i) \geq \sum_1^\ell h(V_i)$, so (15) is satisfied.

To prove the sufficiency, let us suppose that (15) is satisfied. Let us introduce the following two integer-valued set functions.

$$b(X) = |\mathcal{E}| - i_\mathcal{E}(V \setminus X) \tag{20}$$
$$p(X) = h(X) + i_\mathcal{E}(X). \tag{21}$$

Since $i_\mathcal{E}$ is supermodular and h is intersecting supermodular, it follows that b is submodular and p is intersecting supermodular. Note that, by (20), (21) and $h(V) = 0$,

$$b(V) = |\mathcal{E}| - i_\mathcal{E}(\emptyset) = |\mathcal{E}| \tag{22}$$
$$p(V) = h(V) + i_\mathcal{E}(V) = |\mathcal{E}|. \tag{23}$$

For every $Z \subseteq V$ and every partition $\{Z_1, \ldots, Z_\ell\}$ of Z and $Z_0 = V \setminus Z$, we have, by (21), (15), (2) and (20), $\sum_1^\ell p(Z_i) = \sum_1^\ell h(Z_i) + \sum_1^\ell i_\mathcal{E}(Z_i) \leq e_\mathcal{E}(\{Z_0, Z_1, \ldots, Z_\ell\}) + \sum_1^\ell i_\mathcal{E}(Z_i) = |\mathcal{E}| - i_\mathcal{E}(V - Z) = b(Z)$.

Then, by Theorem 12.2.2 in [8], there exists an integral vector m on V such that

$$p(X) \leq \widetilde{m}(X) \leq b(X) \qquad \text{for every } X \subseteq V. \tag{24}$$

Inequalities (22), (23) and (24) provide

$$\widetilde{m}(V) = |\mathcal{E}|. \tag{25}$$

By (25), (4), (20) and (24), we have

$$|\mathcal{E}| = \widetilde{m}(V) = \widetilde{m}(X) + \widetilde{m}(V \setminus X) \text{ for every } X \subseteq V, \tag{26}$$
$$|\mathcal{E}| - i_\mathcal{E}(X) = b(V \setminus X) \geq \widetilde{m}(V \setminus X) \text{ for every } X \subseteq V, \tag{27}$$

and hence, by (27) and (27),

$$(0 \leq)\ i_{\mathcal{E}}(X) \leq \widetilde{m}(X) \qquad \text{for every } X \subseteq V. \tag{28}$$

Then, by (25), (28) and Theorem 9.4.2 in [8], there exists an orientation $\vec{\mathcal{H}} = (V, \mathcal{A})$ of $\mathcal{H}$ such that

$$\varrho_{\mathcal{A}}(v) = m(v). \tag{29}$$

Then, for every $X \subseteq V$, we have by (29), (24) and (21), $\varrho_{\mathcal{A}}(X) = \sum_{v \in X} \varrho_{\mathcal{A}}(v) - i_{\mathcal{A}}(X) = \sum_{v \in X} m(v) - i_{\mathcal{E}}(X) = \widetilde{m}(X) - i_{\mathcal{E}}(X) \geq p(X) - i_{\mathcal{E}}(X) = h(X)$ that is $\vec{\mathcal{H}}$ covers h. ■