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Covering complete partite hypergraphs by monochromatic components

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András Gyárfás* and Zoltán Király**

Abstract

A well-known special case of a conjecture attributed to Ryser (actually appeared in the thesis of Henderson [7]) states that k-partite intersecting hypergraphs have transversals of at most $k-1$ vertices. An equivalent form of the conjecture in terms of coloring of complete graphs is formulated in [1]: if the edges of a complete graph K are colored with k colors then the vertex set of K can be covered by at most $k-1$ sets, each connected in some color. It turned out that the analogue of the conjecture for hypergraphs can be answered: Z. Király proved [8] that in every k-coloring of the edges of the r-uniform complete hypergraph K^r ($r \geq 3$), the vertex set of K^r can be covered by at most $\lceil k/r \rceil$ sets, each connected in some color.

Here we investigate the analogue problem for complete r-uniform r-partite hypergraphs. An edge coloring of a hypergraph is called **spanning** if every vertex is incident to edges of any color used in the coloring. We propose the following analogue of Ryser conjecture.

In every spanning $(r+t)$-coloring of the edges of a complete r-uniform r-partite hypergraph, the vertex set can be covered by at most $t+1$ sets, each connected in some color.

We show that the conjecture (if true) is best possible. Our main result is that the conjecture is true for $1 \leq t \leq r-1$. We also prove a slightly weaker result for $t \geq r$, namely that $t+2$ sets, each connected in some color, are enough to cover the vertex set.

To build a bridge between complete r-uniform and complete r-uniform r-partite hypergraphs, we introduce a new notion. A hypergraph is complete r-uniform (r,ℓ)-partite if it has all r-sets that intersect each partite class in at most ℓ vertices (where $1 \leq \ell \leq r$).

Extending our results achieved for $\ell = 1$, we prove that for any $r \geq 3$, $2 \leq \ell \leq r$, $k \geq 1 + r - \ell$, in every spanning k-coloring of the edges of a complete r-uniform (r,ℓ)-partite hypergraph, the vertex set can be covered by at most $1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ sets, each connected in some color.

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1 Introduction

For an edge-colored hypergraph H let H_i denote its subhypergraph consisting of edges colored by i. The connected components of H_i are called monochromatic components of color i, and a **monochromatic component** refers to a monochromatic component of color i for some i. Here connectivity is understood in its weakest sense, a hypergraph is connected if either it has only one vertex or any two distinct vertices can be connected by a sequence of edges each intersecting the next. Every hypergraph can be uniquely partitioned into connected components. Components with a single vertex are called *trivial*.

Given an edge-colored hypergraph H, let $c(H)$ denote the minimum integer m such that $V = V(H)$, the vertex set of H, can be covered by m monochromatic components of H. An edge coloring of a hypergraph is called **spanning** if every vertex is incident to edges of any color used in the coloring. Note that in spanning colorings every monochromatic component is non-trivial. The importance of this definition is shown in Theorem 1.1.

A conjecture attributed to Ryser which actually appeared in [7] is that k-partite intersecting hypergraphs have transversals of at most $k-1$ vertices. An equivalent form is formulated in [1] as follows: if K is a complete graph with a k coloring on its edges, then $c(K) \leq k-1$. The conjecture is true for $k \leq 5$ and seems very difficult in general (further information can be found in [3], [6]). A particular feature of the conjecture is that $c(K) \leq k$ is obvious since the monochromatic stars at any vertex form monochromatic components. Note that the conjecture is obvious for colorings that are not spanning.

Surprisingly, the problem for hypergraphs is easier, Z. Király in [8] showed that if the edges of the complete r-uniform hypergraph K ($r \geq 3$) are colored with k colors, then $c(K) \leq \lceil k/r \rceil$ and this is best possible (the $k = r$ case were already in [1] extending the well-known remark of Erdős and Rado stating that a graph or its complement is connected).

The problem naturally extends for sparser host graphs (or hypergraphs). Gyárfás and Lehel conjectured that for k-colored complete bipartite graphs G, $c(G) \leq 2k-2$ (see [2]), here again $c(G) \leq 2k-1$ is obvious. For the hypergraph case [4, 5] initiated the study of $c(H)$ when H has bounded independence number.

The main subject of the present paper is the case when the target hypergraph K is a complete r-uniform r-partite hypergraph, i.e., when $V = V(K)$ is partitioned into nonempty classes $V_1 \cup \ldots \cup V_r$ and the edges of K are the sets containing one vertex from each class. Let $\mathrm{cov}(r,k)$ denote the maximum of $c(K)$ when K ranges over spanning k-colorings of complete r-uniform r-partite hypergraphs, and $\mathrm{COV}(r,k)$ denote the maximum of $c(K)$ when K ranges over (not necessarily spanning) k-colorings of complete r-uniform r-partite hypergraphs.

Throughout the paper we **always assume** $r \geq 3$. Our introductory theorem shows that only the spanning colorings are the interesting ones. For any positive integer k we use the standard notation $[k] = \{1, 2, \ldots, k\}$.

Theorem 1.1. *If $r \geq 3$, then $\mathrm{COV}(r,k) = k$.*

Proof. Let K be a k-edge-colored r-uniform r-partite complete hypergraph. Take an edge e of K. Let $C_1, \dots, C_\ell$ be the monochromatic components with $|C_i \cap e| \geq r-1$. As $r > 2$, clearly no two of them have the same color, so $\ell \leq k$. For every vertex $v \in V$ there is an edge $f \ni v$ with $|f \cap e| = r-1$, so v is covered by one of these components.
For the sharpness let $V_1 = [k]$ and color each edge e by color $e \cap V_1$. □

We remark that if a coloring of the r-uniform r-partite complete hypergraph is spanning, then *all monochromatic components meet every class.* An edge of color i in a k-colored r-uniform hypergraph K is called **essential** if it is not contained in monochromatic components of any color different from i. When $\text{cov}(r, k)$ is studied we may restrict ourselves to colorings having at least one essential edge in every used color, since otherwise a color can be eliminated by recoloring all edges of that color to some other color and the resulting hypergraph would still have a spanning coloring and the same set of (maximal) monochromatic components. This concept is established in [8] and works well in the proof of our initial result.

Theorem 1.2. $\text{cov}(r, k) = 1$ *for every* $1 \leq k \leq r \geq 3$.

Proof. Let $e = \{v_1, \dots, v_r\}$ be an essential edge of color 1 in a complete r-uniform r-partite hypergraph with vertex set $V = \cup_{i=1}^r V_i$ where $v_i \in V_i$. Let $R_i = e - \{v_i\}$ and denote by $\text{Col}(R_i) \subseteq [k]$ the set of colors appearing on any edge of the form $R_i \cup \{v_i'\}$ (where $v_i' \in V_i$). As $\text{Col}(R_i) \cap \text{Col}(R_j) = \{1\}$ for $i \neq j$, by the pigeonhole principle there exists j such that $\text{Col}(R_j) = \{1\}$. Now V_j is covered by the monochromatic component containing e (of color 1), and, as the coloring is spanning, it necessarily covers the whole V. □

By Theorem 1.2 from this point we may assume that $k = r + t$ with some integer $t \geq 1$.

Conjecture 1.3. $\text{cov}(r, r+t) = t + 1$ *for every* $r \geq 3,\ t \geq 1$.

It is worth formulating this conjecture in dual form. Assume K is a complete r-uniform r-partite hypergraph with a spanning k-coloring. Consider a new hypergraph H with vertex set $V(K)$ whose edges are the vertex sets of the monochromatic components in the coloring. The dual F of this new hypergraph H is a k-uniform k-partite complete hypergraph whose edges are partitioned into r classes with the property that any r edges from different partite classes have nonempty intersection. As the coloring of K was spanning, monochromatic components have at least r vertices. In this setting Conjecture 1.3 can be stated in terms of the transversal number $\tau(F)$, the minimum number of vertices intersecting all edges of F.

Conjecture 1.4. *Assume that the edges of a k-uniform k-partite hypergraph F with minimum degree at least $r \geq 3$ are partitioned into r classes so that any r edges from different classes have nonempty intersection. Then $\tau(F) \leq k - r + 1$.*

In Section 2 we show that Conjecture 1.3 (if true) is best possible, and it is "almost" true, i.e., $\text{cov}(r, r+t) \leq t + 2$ for every $t \geq 1$ (Theorem 2.6). We also prove that the conjecture is true for $1 \le t \le r-2$ (Theorem 2.5). Our most difficult result makes one further step, proving Conjecture 1.3 for $t = r-1$ (Theorem 2.7).

In Section 3 we investigate $c(H)$ for hypergraphs "between" complete and complete partite, in order to build a bridge between the results proved in Section 2 and the results of [8]. We call a hypergraph *(r,ℓ)-partite* if its vertex set is partitioned into r nonempty classes, such that the intersection of any edge and any class has at most ℓ vertices. We call a hypergraph *complete r-uniform (r,ℓ)-partite* if it contains all r-element sets as edges which meet every partition class in at most ℓ vertices. Let $\mathrm{cov}(r,\ell,k)$ denote the minimum number of monochromatic components needed to cover the vertex set of any complete r-uniform (r,ℓ)-partite hypergraph in any spanning k-coloring. For $2 \le \ell \le r$ we determine exactly the values of $\mathrm{cov}(r,\ell,k)$. We conclude our paper by summarizing the results achieved. Our main result is Theorem 3.6, stating that

$$\mathrm{cov}(r,\ell,k) = 1 + \left\lfloor \frac{k-r+\ell-1}{\ell} \right\rfloor$$

for every $r \ge 3,\ k \ge 1+r-\ell,\ 1 \le \ell \le r$, *except* for the cases ($\ell = 1$ and $k \ge 2r$), where only we could prove a slightly weaker upper bound.

2 Results for complete r-uniform r-partite hypergraphs

2.1 Lower bound

Construction 1. For $t \ge 1,\ r \ge 3,\ k = r+t$, we define a complete r-uniform r-partite hypergraph $K(r,t)$ with a k-coloring of its edges as follows. The vertex set V of $K(r,t)$ is partitioned into r classes, $V_1,\dots,V_r$. The first class V_1 has $\binom{k}{t}$ vertices associated to the t-element subsets of $[k]$. For $2 \le j \le r$ set $V_j = A_j^1 \cup \dots \cup A_j^k$, where the A_j^i-s are disjoint and have $\binom{k-1}{t-1}$ vertices. Fix an arbitrary linear order on every A_j^i.

First we define special edges of color i for any $i \in [k]$. Consider the set W_i of $\binom{k-1}{t-1}$ vertices of V_1 associated to t-sets of $[k]$ containing i.

- *Special edges of color i* are the $\binom{k-1}{t-1}$ edges whose vertex from W_i is the ℓ-th in lexicographic order, and for all $2 \le j \le r$ whose vertex from V_j is the ℓ-th in the fixed linear order of A_j^i for $\ell = 1,\dots,\binom{k-1}{t-1}$. Thus special edges of color i form a matching for all i, $i = 1,\dots,k$.

- *Non-special edges* with vertices $v_1 \in V_1,\dots,v_r \in V_r$ get their color as the smallest $c \in [k]$ such that c is not in the set associated to v_1 and $v_j \notin A_j^c$ for all $2 \le j \le r$.

Note that every non-special r-tuple $v_1,\dots,v_r$ gets a color because the conditions forbid at most $t+r-1$ colors. Observe also that a special edge of color i is always disjoint from any other edge of color i. Consequently a special edge of color i forms a monochromatic component of color i having r vertices, we call them small monochromatic components.

We claim that the coloring given is spanning. Suppose first that $v \in V_1$ representing wlog the set $[t] \subset [k]$. For any $1 \le i \le t$, v is in a special edge of color i. On the other hand, for any $t < i \leq r + t$ we can select vertices $v_2 \in A_2^{j_2} \ldots, v_r \in A_r^{j_r}$ so that the upper indices j_t take all values except i from $t + 1, \ldots, t + r$. Then the non-special edge $v, v_2, \ldots, v_r$ is colored by i.

On the other hand, let $v \in A_j^i$ for some $1 < j \leq r$, $1 \leq i \leq k$. Clearly v is in a special edge of color i. For any $c \neq i$ such that $1 \leq c \leq k$ we can take any vertex $w \in V_1$ associated to a t-set A of $[k]$ such that $c, i \notin A$. Set $B = [k] \setminus (A \cup \{i\} \cup \{c\})$. Then from the $(r-2)$ V_t-s where $t \notin \{1, j\}$ we can pick a set of $r - 2$ vertices with distinct superscripts in B. These vertices together with v, w define an edge that must be colored with c. Thus the coloring of $K(r, t)$ is spanning.

Theorem 2.1. $\operatorname{cov}(r, r + t) \geq t + 1$ *for every* $r \geq 3$, $t \geq 1$.

Proof. Consider the hypergraph $K(r, t)$. Note that the union of at most t large monochromatic components do not cover V_1. Let their colors be $c_1, \ldots, c_s$ with $s \leq t$, and take any t-set that contains $\{c_1, \ldots, c_s\}$; the vertex in V_1 associated to this set is not covered.

The uncovered vertices of V_1 must be covered by small monochromatic components, and every such component can contain just one vertex of V_1. Therefore we need $\binom{k-s}{t-s} > t - s$ small monochromatic components to cover them, thus altogether we need more than $s + (t - s) = t$ monochromatic components to cover the vertices of $K(r, t)$. □

2.2 Upper bounds

We need some additional notation. We assign vectors of length k to every element of the base set $V = V_1 \cup \ldots \cup V_r$. For $v \in V$ the ith coordinate $\mathbf{v}(i)$ of the associated vector $\mathbf{v}$ is the serial number of the monochromatic component of color i containing v. The Hamming distance of two vertices $\delta(v, w) = \delta(\mathbf{v}, \mathbf{w})$ is the number of places the two associated vectors differ.

Statement 2.2. *For* $i = 1, \ldots, r$ *let* $v_i \in V_i$. *Then there exists* $c \in [k]$ *and an integer* s, *such that* $\mathbf{v}_i(c) = s$ *for all* $i \leq r$.

Proof. The edge $e = \{v_1, v_2, \ldots, v_r\}$ is colored by a color, say, by color c. Then the vertices of e belong to the same monochromatic component of color c. □

Lemma 2.3. *Either* $\operatorname{cov}(r, r + t) = 1$, *or for any two vertices* v, w *from different classes,* $\delta(v, w) \leq t + 1$.

Proof. Wlog $v \in V_1$, $w \in V_2$ and $\mathbf{v} = 1 \ldots 1$ and $\mathbf{w} = 1 \ldots 12 \ldots 2$, where the number of ones is at most $r - 2$. As the coloring is spanning and no monochromatic component covers V, we can choose $v_3, \ldots, v_r$, such that $v_i \in V_i$ and $\mathbf{v}_i(i - 2) > 1$. However, this contradicts to Statement 2.2. Thus the number of twos in $\mathbf{w}$ is at most $t + 1$, so $\delta(v, w) \leq t + 1$. □

Lemma 2.4. *If* $\operatorname{cov}(r, r + t) > 1$ *and* $w_1, \ldots, w_\ell$ *are vertices from different classes, then for* $J = \{j \in [k] \mid \mathbf{w}_1(j) = \mathbf{w}_2(j) = \ldots = \mathbf{w}_\ell(j)\}$ *we have* $|J| \geq r + 1 - \ell$.

Proof. If $\ell = r$, then this statement coincides with Statement 2.2. Otherwise suppose $|J| \leq r-\ell$ and $J = \{j_1, \ldots, j_{|J|}\}$. We may choose at most $r-\ell$ vertices $u_1, \ldots, u_{|J|}$ from the classes not having a w_i with $\mathbf{u}_i(j_i) \neq \mathbf{w}_1(j_i)$, contradicting to Statement 2.2. $\square$

Theorem 2.5. $\mathrm{cov}(r, r+t) \leq t+1$ *for every* $1 \leq t \leq r-2$ *and* $r \geq 3$.

Proof. Suppose the statement does not hold. First we claim that for any $i \neq j$ and for any $a \in V_i$, $b \in V_j$ we have $\delta(a, b) \leq t$. Suppose not, wlog $a \in V_1$, $b \in V_2$, such that $\mathbf{a} = 1 \ldots 1$ and $\mathbf{b} = 1 \ldots 12 \ldots 2$, where $\mathbf{b}$ ends with q 2-values, and $q \geq t+1$, consequently $q = t+1$ by Lemma 2.3.

As two monochromatic components do not cover V, there exists $d \in V_3$, such that $\mathbf{d}(t+r) > 2$. By the assumption we have a vertex $c \in V_i$ for some i with $\mathbf{c}(j) \neq 1$ for $j = 1, \ldots, t+1$.

If $i > 1$, then by Lemma 2.3 $\delta(a, c) \leq t+1$ and so $\mathbf{c}(j) = 1$ for $j = t+2, \ldots, t+r$. As $t+1 \leq r-1$, $\delta(b, c) \geq 2t+2$, so $i = 2$. Now $\delta(d, b) \leq t+1$ and $\delta(d, c) \leq t+1$ but $\delta(b, c) = 2t+2$, so $\mathbf{d}$ has to agree with either $\mathbf{b}$ or $\mathbf{c}$ in every coordinate where $\mathbf{b}$ and $\mathbf{c}$ differ. However, this is not the case for the $(t+r)$-th coordinate.

If $i = 1$, then by Lemma 2.3 $\delta(b, c) \leq t+1$ and so $\mathbf{c}(j) = \mathbf{b}(j)$ for $j = t+2, \ldots, t+r$, as $t+1 \leq r-1$. Now $\delta(d, a) \leq t+1$ and $\delta(d, c) \leq t+1$ but $\delta(a, c) = 2t+2$, so $\mathbf{d}$ has to agree with either $\mathbf{a}$ or $\mathbf{c}$ in every coordinate where $\mathbf{a}$ and $\mathbf{c}$ differ. However, this is not the case for the $(t+r)$-th coordinate and the claim is proved.

Let a, b, c as before, now $q \leq t$, so we have $\delta(a, c) \geq t+1$ and $\delta(b, c) \geq t+1$ but this contradicts to the claim because either $c \notin V_1$ or $c \notin V_2$. $\square$

Theorem 2.6. $\mathrm{cov}(r, r+t) \leq t+2$ *for every* $2 \leq r-1 \leq t$.

Proof. Suppose the statement does not hold. First we claim that there exist $i \neq j$, $a \in V_i$, $b \in V_j$, such that $\delta(a, b) \geq r-2$. Wlog $a_\emptyset \in V_1$ with $\mathbf{a}_\emptyset = 11 \ldots 1$. For each $J \subseteq [k]$, $|J| = t+2$, there exists a vertex a_J with $\mathbf{a}_J(j) \neq 1$ for each $j \in J$. (Note, that for $J \neq J'$, $a_J = a_{J'}$ is possible.) These vertices have Hamming distance $\delta(a_J, a_\emptyset) > t+1$ from $a_\emptyset$, consequently, by Lemma 2.3, they all are in V_1. Take any vertex $b \in V_2$, we claim that $\delta(b, a_J) \geq r-2$ for either $J = \emptyset$ or for a $|J| = t+2$. Let $I = \{i \mid \mathbf{b}(i) = 1\}$. If $|I| < t+2$, then $\delta(b, a_\emptyset) \geq r-2$ and we are done, otherwise take any $J \subseteq I$ with $|J| = t+2$. Now obviously $\delta(b, a_J) \geq t+2 \geq r+1$, proving the claim.

By the claim we have wlog $b_1 \in V_1$, $b_2 \in V_2$ where $\mathbf{b}_1 = 11 \ldots 1$ and $\mathbf{b}_2 = 22 \ldots 211 \ldots 1$, and $\mathbf{b}_2$ starts with $r-2+q$ twos ($q \geq 0$). If two monochromatic components cover V, then we are done, otherwise we have $b_i \in V_i$ for $i = 3, 4, \ldots, r$, such that $\mathbf{b}_i(i-2) > 2$. Take also a vertex $d \in V$, where $\mathbf{d}(j) \neq \mathbf{b}_r(j)$ for $r-2 < j \leq r-2+q$ and $\mathbf{d}(j) \neq 1$ for $r-2+q < j \leq r+t$; this involves $(r+t)-(r-1)+1 = t+2$ coordinates, by our assumption such a vertex must exist.

Thus $d \in V_i$ for some $1 \leq i \leq r$. Take the edge $e = \{b_j \mid j \neq i\} \cup \{d\}$, it is colored by some color c. Observe that $\mathbf{d}$ differs form both $\mathbf{b}_1$ an $\mathbf{b}_2$ in the last $t+2-q$ coordinates, so $c \leq r-2+q$. If $c \leq r-2$, then we have $i_1, i_2 \in [r]-\{i\}$ such that $\mathbf{b}_{i_1}(c) \neq \mathbf{b}_{i_2}(c)$, so $r-2 < c \leq r-2+q$. However, if $i < r$, then $\mathbf{b}_r(c) \neq \mathbf{d}(c)$, otherwise $\mathbf{b}_1(c) \neq \mathbf{b}_2(c)$. Thus c does not exist, contradiction. $\square$

2.3 The case $t = r - 1$

Theorem 2.7. $\mathrm{cov}(r, 2r - 1) = r$ *if* $r \geq 3$.

Suppose the statement does not hold, let $k = 2r - 1$ and fix a k-colored r-uniform r-partite hypergraph K where $c(K) \geq r + 1$ (and the coloring is spanning).

Claim 2.8. *For any* $i \neq j$ *and* $a \in V_i, b \in V_j$ *we have*

$$r - 1 \leq \delta(a, b) \leq r.$$

Proof. The upper bound comes from Lemma 2.3. To prove the lower bound, wlog assume that we have $a \in V_1,\ b \in V_2$ with vectors

$$\mathbf{a} = 11\dots111,\ \mathbf{b} = 2\dots21\dots1$$

where $\mathbf{b}$ begins with $q \leq r - 2$ twos. Suppose first that $q > 0$.

As two monochromatic components do not cover V, we can choose vertices $d_3, \dots, d_r$ with $d_i \in V_i$ and $\mathbf{d}_i(i - 2) = 3$. We claim that $\mathbf{d}_i(j) = 1$ for all $3 \leq i \leq r$ and $r-1 \leq j \leq 2r-1$. Otherwise the index set $J = \{j \mid \mathbf{d}_3(j) = \mathbf{d}_4(j) = \dots = \mathbf{d}_r(j) = 1\}$ has size at most r, so there is a set I such that $J \subseteq I \subseteq \{r - 1, \dots, 2r - 1\}$ and $|I| = r$. There is a vertex c_I with the property $\mathbf{c}_I(j) \neq 1$ for all $j \in I$. As either $\delta(c_I, a) > r$ or $\delta(c_I, b) > r$, $c_I \in V_1 \cup V_2$. If $c_I \in V_1$, then $c_I, b, d_3, \dots, d_r$, otherwise $a, c_I, d_3, \dots, d_r$ contradicts to Lemma 2.3, so $\mathbf{d}_3(j) = 1$ for all $r-1 \leq j \leq 2r-1$. Now for $I = \{r, \dots, 2r - 1\}$ we also have $c_I \in V_1 \cup V_2$, and $\mathbf{c}_I(1) = \mathbf{a}(1) = 1$ (if $c_I \in V_2$) or $\mathbf{c}_I(1) = \mathbf{b}(1) = 2$ (if $c_I \in V_1$) also follows, so $\delta(c_I, d_3) \geq 1 + |I| = 1 + r$ contradicting to Lemma 2.3.

We conclude that $q = 0$, thus $\mathbf{a} = \mathbf{b}$ are both the all-1 vectors. Then for all $I \subset [2r - 1]$, $|I| = r$ there exist vertices c_I such that $\mathbf{c}_I(j) \neq 1$ for all $j \in I$. As either $\delta(c_I, a) \leq r$ or $\delta(c_I, b) \leq r$ must hold, $\mathbf{c}_I(j) = 1$ for all $j \in [2r - 1] - I$. Suppose that for $I_1, I_2 \subset [2r-1]$ the complementary sets $\overline{I_1}, \overline{I_2}$ are disjoint. Then the corresponding vertices c_{I_1} and c_{I_2} must be in the same vertex class, otherwise the Hamming distance of their vectors would be at least $2(r - 1) \geq r + 1$. Since the Kneser graph defined by disjoint $(r-1)$-element subsets of a $(2r-1)$-element ground set is a connected graph, all the c_I-s are in the same class, call it the full class; by symmetry we may assume that it is not V_1. Select vertices $d_2 \in V_2$ and $d_3 \in V_3$ such that $\mathbf{d}_i(i) = 3$. Observe that $\mathbf{d}_i$ has at most r ones because $1 \leq \delta(a, d_i) \leq r - 2$ cannot happen. If the full class is V_2, let I contain the positions where $\mathbf{d}_3$ is 1, then a, c_I, d_3 violate Statement 2.2. If the full class is V_j for $j \geq 3$, let I contain the positions where $\mathbf{d}_2$ is 1, now a, d_2, c_I violate Statement 2.2. □

Claim 2.9. *If* a, b *are two vertices from different partite classes such that* $\delta(a, b) = r - 1$, *then some of these classes contain two vertices with Hamming distance* $2r - 1$.

Proof. Assume wlog that there are $a \in V_1,\ b \in V_2$ such that $\mathbf{a} = 11\dots111,\ \mathbf{b} = 2\dots21\dots1$ where $\mathbf{b}$ ends with exactly r ones. There exists a vertex $c \in V$ with $\mathbf{c}(j) \neq 1$ if $r \leq j \leq 2r - 1$. By Statement 2.2, $c \in V_1 \cup V_2$. If $c \in V_1$, then (as $\delta(c, b) \leq r$) its vector starts with $r-1$ twos, so a, c is a pair required. If $c \in V_2$, then its vector starts with $r-1$ ones, so b, c is a pair required. □

Claim 2.10. *For any two vertices v, w from the same partite class,*

$$\delta(\mathbf{v}, \mathbf{w}) < 2r - 1.$$

Proof. Assume indirectly that we have two vertices wlog $v, w \in V_1$,

$$\mathbf{v} = 11\dots11, \ \mathbf{w} = 22\dots22.$$

For any $2 \le i \le r$, $1 \le j \le 2r-1$ there exist vertices $v_i^j \in V_i$ such that $\mathbf{v}_i^j(j) = 3$ by our assumption. They are all distinct because their vectors must contain exactly $r-1$ ones and exactly $r-1$ twos since their distance from both v, w must be at most $r-1$.

Statement 2.11. *Let $\mathbf{v}$ and $\mathbf{w}$ be 1-2 vectors of the same length. If the number of ones in $\mathbf{v}$ and $\mathbf{w}$ have the same parity, then $\delta(\mathbf{v}, \mathbf{w})$ is even, otherwise it is odd.* □

Statement 2.12. $\delta(v_i^j, v_i^{j'}) \le 2r-2$ *for any $2 \le i \le r$, $1 \le j < j' \le 2r-1$.*

Proof. Suppose wlog $\delta(v_2^1, v_2^2) = 2r - 1$ and

$$\mathbf{v}_2^1 = 31\dots12\dots2, \ \mathbf{v}_2^2 = 232\dots21\dots1.$$

At this point the parity of r comes into play. If r is even, then $\delta(v_3^3, v_2^1) \le r$ and $\delta(v_3^3, v_2^2) \le r$ by Lemma 2.3, thus for each $\ell \ne 3$ either $\mathbf{v}_3^3(\ell) = \mathbf{v}_2^1(\ell)$ or $\mathbf{v}_3^3(\ell) = \mathbf{v}_2^2(\ell)$ and $\delta(v_3^3, v_2^1) = \delta(v_3^3, v_2^2) = r$. Accordingly $\mathbf{v}_3^3(1) = 2$, $\mathbf{v}_3^3(2) = 1$ and in the positions $\ell > 3$ $\mathbf{v}_3^3$ has $r-2$ ones but $\mathbf{v}_2^1$ has $r-3$ ones and $\mathbf{v}_2^2$ has $r-1$ ones, leading to a contradiction by Statement 2.11. If r is odd, then we consider v_3^{2r-1} instead of v_3^3, here for each $\ell \ne 2r-1$ either $\mathbf{v}_3^3(\ell) = \mathbf{v}_2^1(\ell)$ or $\mathbf{v}_3^3(\ell) = \mathbf{v}_2^2(\ell)$, so $\mathbf{v}_3^3(1) = 2$, $\mathbf{v}_3^3(2) = 1$. Now we focus to positions $\ell = 3, \dots, 2r-2$ where $\mathbf{v}_2^1, \mathbf{v}_2^2, \mathbf{v}_3^{2r-1}$ have $r-2$ ones. By Statement 2.11 $\delta(v_3^{2r-1}, v_2^1)$ and $\delta(v_3^{2r-1}, v_2^2)$ is even, however, they should be exactly r which is odd. □

Statement 2.13. *Suppose $2 \le i < i' \le r$. Then*

$$\delta(v_i^j, v_{i'}^j) = \begin{cases} r & \text{if } r \text{ is even} \\ r-1 & \text{if } r \text{ is odd} \end{cases}$$

Also, $\delta(v_i^j, v_{i'}^{j'}) = r$ for any $j \ne j'$. Moreover

$$\mathbf{v}_i^j(j') = \begin{cases} \mathbf{v}_{i'}^{j'}(j) & \text{if } r \text{ is even} \\ 3 - \mathbf{v}_{i'}^{j'}(j) & \text{if } r \text{ is odd} \end{cases}$$

Proof. The first part is a consequence of Claim 2.8 and Statement 2.11. If $\delta(v_i^j, v_{i'}^{j'}) = r-1$ then by Claim 2.9 we get a vertex a wlog in V_i such that $\delta(a, v_i^j) = 2r-1$, moreover $\mathbf{a}$ agrees with $\mathbf{v}_{i'}^{j'}$ in all positions where $\mathbf{v}_{i'}^{j'}$ and $\mathbf{v}_i^j$ differ. Thus $\mathbf{a}(j') = 3$, so we may call it $v_i^{j'}$ for getting a contradiction by Statement 2.12. Having this, the last part is a consequence of Statement 2.11. □

We associate matrices to the selected vertices as follows. For $i = 2 \dots r$ let $A_i(j, j') = 0$ if $j = j'$, $A_i(j, j') = 1$ if $\mathbf{v}_i^j(j') = 1$, and $A_i(j, j') = -1$ if $\mathbf{v}_i^j(j') = 2$. These are $(2r-1) \times (2r-1)$ matrices, we introduce the following operation for them.

$$A_i^* := (-1)^r \cdot A_i^{\mathrm{T}}.$$

Now $A_{i'} = A_i^*$ for any $i \neq i'$ by the last part of Statement 2.13. For the case $r \geq 4$ it is easy to complete the proof of Claim 2.10. Indeed, as $A_3 = A_4 = A_2^*$, we have e.g., $\mathbf{v}_3^1 = \mathbf{v}_4^1$ contradicting to Statement 2.13. For the case $r = 3$ we need some extra work. As $2r - 1 = 5$ now, we have 5×5 matrices, and every row contains two 1 and two -1 entries. We define an auxiliary graph G on vertex set $[5]$. Let ij is an edge iff $A_2(i, j) = A_2(j, i) = 1$. We claim that in this graph all five vertices have degree 1, leading to a contradiction. If $d_G(i) = 0$, then the ith row of A_2 is the negative of the ith column. As $A_3 = A_2^*$ we have $\mathbf{v}_2^i = \mathbf{v}_3^i$ contradicting to Statement 2.13. If $d_G(i) \geq 2$, then the ith row of A_2 equals to the ith column. As $A_3 = A_2^*$ we have $\delta(\mathbf{v}_2^i, \mathbf{v}_3^i) = 2r - 2 = 4$, contradicting again to Statement 2.13. Thus Claim 2.10 is proved. □

Combining the claims we conclude with the following corollary.

Corollary 2.14. *For any two vertices v, w from different classes, $\delta(v, w) = r$, and for any two vertices u, v from the same class, $\delta(u, v) \leq 2r - 2$.*

Now we are ready for finishing the proof of Theorem 2.7. Select two vertices $v \in V_1$, $w \in V_2$, wlog $\mathbf{v} = 11 \dots 11$, $\mathbf{w} = 22 \dots 2211 \dots 1$ where $\mathbf{w}$ starts with exactly r twos. Accordingly, for a vector of length $2r-1$ we call its first part the first r coordinates, and its last part the last $r-1$ coordinates. There exists a $y \in V_3$ with $\mathbf{y}(1) = 3$, and let $I = \{i \mid \mathbf{y}(i) = 1\}$ and $\alpha = |I \cap [r]|$ (i.e., the number of ones in its first part). Since $\delta(y, v) = r$, we have $|I| = r - 1$, and since $\delta(y, v) = \delta(y, w)$, $\mathbf{y}$ has exactly α twos in its first part, so $\alpha \leq \frac{r-1}{2}$. There exists a $J \subset [2r-1]$, $J \supset I$, $|J| = r$, $|J \cap [r]| \leq \frac{r-1}{2}$, and by the assumption (r monochromatic components do not cover V) there exist a vertex v_J with the property $\mathbf{v}_J(j) \neq 1$ for all $j \in J$.

If $v_J \notin V_1 \cup V_2$ then each of it coordinates is 1 outside J, as $\delta(v_J, v) = r$. By the definition of J, it means that $\mathbf{v}_J$ has $\beta \geq \frac{r+1}{2}$ ones in the first part and β non-ones in the second part, so $\delta(v_J, w) \geq \beta + \beta \geq r + 1$, a contradiction.

Therefore $v_J \in V_1 \cup V_2$, then $\delta(y, v_J) = r$ implies that $\mathbf{y}$ and $\mathbf{v}_J$ are equal outside I, with possibly one exception. However, $\mathbf{y}(j) \neq 1$ for any $j \in [2r-1] - J$, consequently $\mathbf{v}_J$ can have at most one coordinate that is 1. Thus $v_J \in V_1$, and $\delta(v_J, v) \geq 2r - 2$, consequently by Corollary 2.14 it equals to $2r - 2$ and $\mathbf{v}_J$ has exactly one coordinate that is 1.

Let $I' = \{i \mid \mathbf{v}_J(i) = \mathbf{w}(i) = \mathbf{y}(i)\}$, by Lemma 2.4 we have $|I'| \geq r - 2$. However, $I' \subset \{j \leq r \mid \mathbf{y}(j) = 2\}$, and this latter set has cardinality $\alpha \leq \frac{r-1}{2}$, so $r - 2 \leq \frac{r-1}{2}$, i.e., $r \leq 3$, which leads to a contradiction, except for the case $r = 3$, $\alpha = 1$.

Now $\mathbf{v}_J(i) = 1$ for an $i \leq 3$, let $J' = \{i, 4, 5\}$ and define $v_{J'}$ as $\mathbf{v}_{J'}(j) \neq 1$ if $j \in J'$. Now $v_{J'} \in V_1 \cup V_2$ because otherwise $\delta(v_{J'}, v) = \delta(v_{J'}, w) = 3$ would lead to a contradiction. If $v_{J'} \in V_2$, then $\mathbf{v}_{J'}(j) = 1$ for $j \notin J'$, now we choose $z \in V_3$ with

$\mathbf{z}(i) = 1$. As $\delta(z, v) = 3$, $\mathbf{z}$ has exactly one other 1 but if its position is in $\{1, 2, 3\}$, then $\delta(z, w) \geq 4$, and if in $\{4, 5\}$, then $\delta(z, v_{J'}) \geq 4$.

So $v_{J'} \in V_1$, consequently, by $\delta(v_J, w) = \delta(v_{J'}, w) = 3$, both v_J and $v_{J'}$ have 2 twos in the first part, let $\ell \in [3]$ the position of a common 2 and we now choose $t \in V_3$ with $\mathbf{t}(\ell) = 3$. Since $\delta(t, v) = \delta(t, w) = 3$, t has one 1 and one 2 in the first part, and one 2 in the second part, contradicting to $\delta(v_J, t) = \delta(v_{J'}, t) = 3$. □

Corollary 2.15. *If* $r \geq 3$, *then* $\text{cov}(r, k) = 1$ *for every* $1 \leq k \leq r$,
$\text{cov}(r, k) = k - r + 1$ *for every* $r \leq k \leq 2r - 1$,
and for any $k \geq 2r$ *we have* $k - r + 1 \leq \text{cov}(r, k) \leq k - r + 2$.

3 Generalized complete uniform hypergraphs

Definition 3.1. A hypergraph is called (r, ℓ)-**partite** if the ground set V is partitioned into nonempty classes $V_1 \cup \ldots \cup V_r$, and no edge intersects any V_i in more than ℓ vertices. A hypergraph is complete r-uniform (r, ℓ)-partite if its edge set consists of all r-tuples intersecting each class in at most ℓ vertices. An edge of an (r, ℓ)-partite hypergraph is called *friendly* if it intersects at most one class in exactly ℓ vertices; otherwise we call it *unfriendly*. An r-uniform (r, ℓ)-partite hypergraph is called **semi-complete** if its edge set consists of all r-tuples intersecting at most one class in exactly ℓ vertices (that is, it consists of the friendly edges of the complete r-uniform (r, ℓ)-partite hypergraph). An r-uniform (r, ℓ)-partite hypergraph is called **rich** if it contains all edges of the semicomplete hypergraph.

Among r-uniform hypergraphs the complete $(r, 1)$-partite hypergraphs are the complete r-partite ones and complete (r, r)-partite hypergraphs are the complete ones. The complete $(r, r-1)$-partite hypergraphs are also interesting, containing all r-tuples of V except those that are contained in some V_i. The purpose of this section is to build a bridge between the two known extreme cases ($\ell = r$ was solved in [8], $\ell = 1$ was handled in the previous section).

For $1 \leq \ell \leq r$, let $\text{cov}(r, \ell, k)$ denote the minimum number of monochromatic components needed to cover the vertex set of any complete r-uniform (r, ℓ)-partite hypergraph in any spanning k-coloring.

Conjecture 3.2.

$$\text{cov}(r, \ell, k) = 1 + \left\lfloor \frac{k - r + \ell - 1}{\ell} \right\rfloor$$

for every $r \geq 3,\ k \geq 1 + r - \ell,\ 1 \leq \ell \leq r$.

We start with giving the lower bound.

Construction 2. This construction is a straightforward generalization of Construction 1. We have r, k, ℓ fixed with $k \geq r + 1 \geq 4$ and $1 \leq \ell \leq r$, and let $q = \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ and $k' = q \cdot \ell + r - \ell + 1 \leq k$. First we fix the sizes and labels of the classes. $|V_1| = \binom{k'}{q}$ and elements V_1 are labeled with the q-element subsets of $[k']$. For $2 \leq j \leq r$ set V_j is a disjoint union of $V_j = A_j^1 \cup \cdots \cup A_j^{k'}$ where $|A_j^i| = \binom{k'-1}{q-1}$, all elements of A_j^i are labeled with set $\{i\}$ and have an arbitrary fixed linear order. Now take an arbitrary rich r-uniform (r,ℓ)-partite hypergraph H_{rich} on $V = V_1 \cup \ldots \cup V_r$, we are going to define a spanning k'-coloring of its edges.

First we define special edges of color i for any $i \in [k']$. Consider the set W_i of $\binom{k'-1}{q-1}$ vertices of V_1 associated to q-sets of $[k']$ containing i.

Special edges of color i are the $\binom{k'-1}{q-1}$ edges whose vertex from W_i is the ℓ-th in lexicographic order, and for all $2 \le j \le r$ whose vertex from V_j is the ℓ-th in the fixed linear order of A^i_j for $\ell = 1, \ldots, \binom{k'-1}{q-1}$. Thus special edges of color i form a matching for all i.

Non-special edges with vertices $v_1, \ldots, v_r$ get their color as the smallest $c \in [k']$ such that c is not in the union of sets associated to $v_1, \ldots, v_r$.

Note that every non-special r-tuple $v_1, \ldots, v_r$ gets a color because the conditions forbid at most $\ell \cdot q + (r - \ell) < k'$ colors. Observe also that a special edge of color i is always disjoint from any other edge of color i. Consequently a special edge of color i forms a monochromatic component of color i having r vertices, we call them small monochromatic components.

We claim that the coloring is spanning. Suppose first that $v \in V_1$ representing the set $Q_1 \subset [k']$. For any $i \in Q_1$, v is in a special edge of color i. On the other hand, for any $i \notin Q_1$ we can select vertices $v_2, \ldots, v_\ell \in V_1$ with associated q-sets $Q_2, \ldots, Q_\ell \subseteq [k'] - \{i\}$, such that for every $j \neq j'$ sets Q_j and $Q_{j'}$ are disjoint. Then we may select $v_{\ell+1}, \ldots, v_r$ from $V_2, \ldots, V_{r-\ell+1}$, such that the associated one-element subsets are distinct, and are subsets of $[k'] - \{i\} - \cup Q_j$. Now the union of the associated sets of our selected r-tuple is $[k'] - \{i\}$, thus it was colored by i.

On the other hand, let $v_r \in A^i_j$ for some $2 \le j \le r$, $1 \le i \le k'$. Clearly v_r is in a special edge of color i. For any $1 \le c \le k'$ if $c \neq i$, then we can take vertices $v_1, \ldots, v_\ell \in V_1$ with associated q-sets $Q_1, \ldots, Q_\ell \subseteq [k'] - \{i\} - \{c\}$, such that for every $j \neq j'$ sets Q_j and $Q_{j'}$ are disjoint. Then we may select $v_{\ell+1}, \ldots, v_{r-1}$ from $V_2 \cup \ldots \cup V_r - V_j$, such that the associated one-element subsets are distinct, and are subsets of $[k'] - \{i\} - \{c\} - \cup Q_j$. Now the union of the associated sets of our selected r-tuple is $[k'] - \{c\}$, thus it was colored by c.

Theorem 3.3. $\mathrm{cov}(r,\ell,k) \ge 1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ *for every* $r \ge 3,\ k \ge 1 + r - \ell,\ 1 \le \ell \le r.$

Proof. The statement is obvious if $k \le r$. Consider Construction 2. Note that the union of at most $q = \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ large monochromatic components do not cover V_1. Let their colors are $c_1, \ldots, c_s$ with $s \le q$, and take any q-set that contains $\{c_1, \ldots, c_s\}$; the vertex in V_1 associated to this set is not covered.

The uncovered vertices of V_1 must be covered by small monochromatic components, and every such component can contain just one vertex of V_1. Therefore we need $\binom{k'-s}{q-s} > q - s$ small monochromatic components to cover them, thus altogether we need more than $s + (q - s) = q$ monochromatic components to cover all vertices. □

Remark 3.4. The basic idea of the above construction is from [8] where the constructed coloring for complete r-uniform hypergraphs is not spanning (this was not an issue of that paper). Here, when $\ell = r$, we gave another construction for complete r-uniform hypergraphs where the coloring is spanning.

Theorem 3.5. $\mathrm{cov}(r,\ell,k) \le 1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ *for every* $r \ge 3,\ k \ge 1+r-\ell,\ 2 \le \ell \le r$.

Proof. The proof goes similarly as in the proof of Theorem 1.2. Fix the nonempty classes $V_1,\dots,V_r$ and take any rich r-uniform (r,ℓ)-partite hypergraph H_{rich} with a spanning k-coloring of its edges. We are going to show by induction on k that $c(H_{\text{rich}}) \le 1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$. The cases $k \le r$ are obvious.

Let $e = \{u_1,\dots,u_r\}$ be an essential edge of H_{rich} colored by 1, if no such edge exists, then recolor edges having color 1 and use induction. Until there exists an essential friendly edge colored by 1, we choose that edge for e. If all essential edges colored by 1 are unfriendly, then simply delete them from H_{rich} getting a $(k-1)$-colored rich hypergraph, where the coloring is still spanning, so we are done by induction.

So e is a friendly essential edge, wlog $\ell \ge |e \cap V_1| \ge |e \cap V_j|$ for all j. As e is friendly, we also have $|e \cap V_j| < \ell$ for $j > 1$. Take $R_{u_1},\dots,R_{u_r}$, where $R_{u_j} = e - \{u_j\}$, for any $i \ne j$ we have $\mathrm{Col}(R_{u_i}) \cap \mathrm{Col}(R_{u_j}) = \{1\}$, so there is a j with $|\mathrm{Col}(R_{u_j})| \le 1 + \lfloor \frac{k-1}{r} \rfloor$.

First consider the case $|R_{u_j} \cap V_1| < \ell$ (note that this is always true for $\ell = r$). We also emphasize here that for this case we do not need the coloring to be spanning. For any vertex $v \in V$ the set $R_{u_j} \cup \{v\}$ is a friendly edge of H_{rich}, consequently the monochromatic components of colors in $\mathrm{Col}(R_{u_j})$ containing R_{u_j} cover the whole V. We need to prove $\lfloor \frac{k-1}{r} \rfloor \le \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$. For $k-1 < r$ both are zero, otherwise $(r-\ell)(k-1) \ge (r-\ell)r$, so $\frac{k-1}{r} \le \frac{k-r+\ell-1}{\ell}$.

So we are left with the case $|R_{u_j} \cap V_1| = \ell$. There are two possibilities. Either one of $\mathrm{Col}(R_{u_i}) = \{1\}$ for an $i > 1$, in this case the monochromatic component containing u_i and colored by 1 covers V because it covers $V - V_1$, as for all $v \in V - V_1$ the set $e - \{u_i\} \cup \{v\}$ is an edge of H_{rich}, and (using that the coloring is spanning), every $w \in V_1$ is incident to an edge colored by 1 and this edge meets $V - V_1$.

Otherwise $|\mathrm{Col}(R_{u_i})| \ge 2$ for all $i > 1$, so by the pigeonhole principle there is a $2 \le i \le \ell$ with $|\mathrm{Col}(R_{u_i})| \le 1 + \lfloor \frac{k-1-(r-\ell)}{\ell} \rfloor$, and the monochromatic components of colors in $\mathrm{Col}(R_{u_i})$ containing $e - \{u_i\}$ cover the whole V because $e - \{u_i\} \cup \{v\}$ is an edge of H_{rich} for every $v \in V - e$. □

Summarizing the results of this section and Corollary 2.15, we proved Conjecture 3.2 for almost all cases. We also proved that Conjecture 3.2 is equivalent to Conjecture 1.3.

Theorem 3.6 (Main theorem)**.**

$$\mathrm{cov}(r,\ell,k) = 1 + \left\lfloor \frac{k-r+\ell-1}{\ell} \right\rfloor$$

for every $r \ge 3,\ k \ge 1+r-\ell,\ 1 \le \ell \le r$, except *when* $\ell = 1$ *and* $k \ge 2r$, *where only* $1 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor \le \mathrm{cov}(r,\ell,k) \le 2 + \lfloor \frac{k-r+\ell-1}{\ell} \rfloor$ *was proved.*

4 Open problems

Besides the missing case ($k \ge 2r$) of Conjecture 1.3 and the above mentioned conjecture of Gyárfás and Lehel (stating that $\mathrm{COV}(2,k) = 2k-2$), we list some more open problems.

In [2] it is shown that $2k-2 \leq \mathrm{COV}(2,k) \leq 2k-1$. Much less is known about $\mathrm{cov}(2,k)$. The best known upper bound is still $2k-1$ but no reasonable lower bound is known. The second author conjectures that $2k-4\sqrt{k} \leq \mathrm{cov}(2,k)$.

For $r \geq 3$ we did not study $\mathrm{COV}(r,\ell,k)$ (that is similar to $\mathrm{cov}(r,\ell,k)$ but the coloring need not to be a spanning one), it was only determined for $\ell = 1$ (see Theorem 1.1) and for $\ell = r$ (either in [8] or in the proof of Theorem 3.5).

We can naturally generalize further.

Definition 4.1. For $1 \leq \ell \leq r \leq R\ell$, let $\mathrm{cov}(r,R,\ell,k)$ denote the minimum number of monochromatic components needed to cover the vertex set of any complete r-uniform (R,ℓ)-partite hypergraph in any spanning k-coloring.

Determining $\mathrm{cov}(r,R,\ell,k)$ for all possible ranges seems to be very challenging. At the moment we do not have a conjecture about the value of $\mathrm{cov}(2,3,1,k)$.

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