Chip-firing based methods in the Riemann–Roch theory of directed graphs

Bálint Hujter and Lilla Tóthmérész

Chip-firing based methods in the Riemann–Roch theory of directed graphs

Bálint Hujter[*] and Lilla Tóthmérész[**]

Abstract

Baker and Norine proved a Riemann–Roch theorem for divisors on undirected graphs. The notions of graph divisor theory are in duality with the notions of the chip-firing game of Björner, Lovász and Shor. We use this connection to prove Riemann–Roch-type results on directed graphs. We give a simple proof for a Riemann–Roch inequality on Eulerian directed graphs, improving a result of Amini and Manjunath. We also give a graph-theoretic version of the abstract Riemann–Roch criterion of Baker and Norine, and explore the natural Riemann–Roch property introduced by Asadi and Backman. Our proofs are based on connections between graph divisor theory and some well-known concepts of combinatorial optimization such as graph orientations and feedback arc sets.

1 Introduction

In 2007, Baker and Norine proved a graph-theoretic analogue of the classical Riemann–Roch theorem for algebraic curves [4]. This result inspired much research about Riemann–Roch theorems on tropical curves, lattices and directed graphs [1, 2, 11].

As pointed out already by Baker and Norine, Riemann–Roch theory on graphs is in close relationship with the chip-firing game of Björner, Lovász and Shor [4, Section 5.5]. There is a duality between these theories, that enables one to translate the notions of one theory to the language of the other. This connection has already turned out to be fruitful in both directions. NP-hardness results in both chip-firing (halting problem of chip-firing games on directed multigraphs [9]) and Riemann–Roch theory (the computation of the rank of a divisor [12]) have recently been proved by translating the problem to the dual language. We note that the graph divisor theory of Baker and Norine is sometimes also referred to as chip-firing, but in this paper by chip-firing we always mean the game of Björner, Lovász and Shor.

Our aim in this paper is to show, that the chip-firing framework can be used very effectively to prove Riemann–Roch type results both for graphs and digraphs. First, we give a proof of the Riemann–Roch theorem on undirected graphs using the chip-firing game. Though this proof has been found independently by the authors, one could also get it by translating the proof of Cori and le Borgne [8] to the chip-firing language. Then we show that in the chip-firing language this proof can be naturally generalized to Eulerian digraphs to give a weak Riemann–Roch theorem which improves on a previous result of Amini and Manjunath [1, Section 6.2]. These proofs come from a deeper understanding of the connections between Riemann-Roch theory and some well-known concepts of combinatorial optimization such as graph orientations and feedback arc sets.

In the second half of the paper, we investigate the Riemann–Roch property for general (strongly connected) digraphs. Using the language of chip-firing, we give a necessary and sufficient condition for the Riemann–Roch theorem to hold on a strongly connected directed graph. At first sight, this setting seems to be somewhat restricted, but it follows from a result of Perkinson, Perlman and Wilmes [13, Theorem 4.11], that divisor theory on strongly connected digraphs is equivalent to divisor theory on lattices (in the sense of Amini and Manjunath [1]), and to the setting of the abstract Riemann–Roch theorem of Baker and Norine [4, Section 2]. Hence our graphical Riemann–Roch condition has the same power as the abstract Riemann–Roch condition of Baker and Norine, and the condition of Amini and Manjunath. These three theorems are essentially equivalent.

We also investigate the natural Riemann–Roch property defined by Asadi and Backman [2, Definition 3.12], proving that an Eulerian digraph has the natural Riemann–Roch property if and only if it is bidirected i.e. corresponds to an undirected graph. On the other hand, in Section 8, we show that there exist non-Eulerian digraphs with the natural Riemann–Roch property. We also give examples for Eulerian digraphs with non-natural Riemann–Roch property, and for digraphs without the Riemann–Roch property.

2 Preliminaries

2.1 Basic notations

Throughout this paper, *digraph* means a strongly connected directed graph that can have multiple edges but no loops. A digraph is usually denoted by G.

The vertex set and edge set of a digraph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex v, the indegree and the outdegree of v are denoted by $d^-(v)$ and $d^+(v)$, respectively. A digraph G is *Eulerian* if $d^+(v) = d^-(v)$ for each vertex $v \in V(G)$. We denote a directed edge leading from vertex u to vertex v by $\overrightarrow{uv}$. The multiplicity of the directed edge $\overrightarrow{uv}$ is denoted by $\overrightarrow{d}(u,v)$.

By *graph*, we mean a connected undirected graph. Throughout this paper, we think of undirected graphs as special digraphs, replacing each undirected edge uv by a pair of oppositely directed edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$. We call digraphs with $\overrightarrow{d}(u,v) = \overrightarrow{d}(v,u)$

for each $u, v \in V$ *bidirected.* (These are exactly the digraphs corresponding to an undirected graph). If we are talking specially about undirected graphs, we use the notation $d(v)$ for the degree of a vertex v.

For a graph G, an *orientation* of G is a directed graph $\overrightarrow{G}$ obtained from G by giving an orientation to each edge. We identify the vertices of $\overrightarrow{G}$ with the corresponding vertices of G. We denote the indegree and the outdegree of a vertex $v \in V(G)$ in the orientation $\overrightarrow{G}$ by $d^-_{\overrightarrow{G}}(v)$ and $d^+_{\overrightarrow{G}}(v)$, respectively.

2.2   Integer vectors and linear equivalence

This paper focuses on two related fields, graph divisor theory, and the chip-firing game. The basic objects in both fields are integer vectors indexed by the vertices of a (di)graph G. We denote the set of such vectors by $\mathbb{Z}^{V(G)}$.

We think of the elements of $\mathbb{Z}^{V(G)}$ in three ways simultaneously:

as vectors $f \in \mathbb{Z}^{|V(G)|}$ with coordinates indexed by the vertices of G;

as functions $f : V(G) \to \mathbb{Z}$;

as elements of the free Abelian group on the set of vertices of G.

For any $f \in \mathbb{Z}^{V(G)}$, the *degree* of f is the sum of its coordinates; we denote it by $\deg(f)$, i.e. $\deg(f) = \sum_{v \in V(G)} f(v)$. We denote by $f \geq g$ if $f(v) \geq g(v) \; \forall v \in V(G)$.

We denote by $\mathbf{0}_G$ ($\mathbf{1}_G$) the vector in $\mathbb{Z}^{V(G)}$ with each coordinate equal to 0 (1). For a vertex $v \in V(G)$, the characteristic vector of v is denoted by $\mathbf{1}_v$, i.e., $\mathbf{1}_v(v) = 1$ and $\mathbf{1}_v(u) = 0$ for $u \neq v$. We use the notation $\mathbb{Z}^{V(G)}_+ = \{x \in \mathbb{Z}^{V(G)} : \; x \geq \mathbf{0}_G\}$.

For an undirected graph G, $\mathbf{d}_G \in \mathbb{Z}^{V(G)}$ is the vector with $\mathbf{d}_G(v) = d(v)$ for all $v \in V(G)$. For a directed graph G, $\mathbf{d}^+_G \in \mathbb{Z}^{V(G)}$ ($\mathbf{d}^-_G \in \mathbb{Z}^{V(G)}$) is the vector with $\mathbf{d}^+_G(v) = d^+(v)$ ($\mathbf{d}^-_G(v) = d^-(v)$) for all $v \in V(G)$. If there is no danger of confusion, we omit the subscripts.

The *Laplacian matrix* of a digraph G means the following matrix $L \in \mathbb{Z}^{V(G) \times V(G)}$:

$$L(u, v) = \begin{cases} -d^+(v) & \text{if } u = v \\ \overrightarrow{d}(v, u) & \text{if } u \neq v. \end{cases}$$

The Laplacian matrix of an undirected graph is the Laplacian matrix of the bidirected graph corresponding to it.

The Laplacian of a digraph G defines an equivalence relation on $\mathbb{Z}^{V(G)}$, that plays a key role in graph divisor theory.

Definition 2.1 (Linear equivalence on $\mathbb{Z}^{V(G)}$)**.** For $x, y \in \mathbb{Z}^{V(G)}$, $x \sim y$ if there exists a $z \in \mathbb{Z}^{V(G)}$ such that $x = y + Lz$.

Note that this is indeed an equivalence relation.

2.3 Riemann–Roch theory on graphs

In this section we introduce those notions of the Riemann–Roch theory that we need in this paper. Let G be a strongly connected digraph. $\operatorname{Div}(G)$ denotes the free Abelian group on the set of vertices of G. We identify $\operatorname{Div}(G)$ with $\mathbb{Z}^{V(G)}$. Elements of $\operatorname{Div}(G)$ are called *divisors*. A divisor $f \in \operatorname{Div}(G)$ is called *effective*, if $f \geq \mathbf{0}$. A divisor is called *equi-effective*, if it is linearly equivalent to an effective divisor.

An important quantity associated to a divisor is its rank.

Definition 2.2 (The rank of a divisor, [4])**.** Let $f \in \operatorname{Div}(G)$.

$$\operatorname{rank}(f) = \min\{\deg(g) - 1 : g \in \operatorname{Div}(G), g \text{ is effective}, f - g \text{ is not equi-effective }\}.$$

In [4], for undirected graphs, Baker and Norine proved the following analogue of the classical Riemann–Roch theorem.

Theorem 2.3 (Riemann–Roch theorem for graphs [4, Theorem 1.12])**.** *Let G be an undirected graph and let f be a divisor on G. Then*

$$\operatorname{rank}(f) - \operatorname{rank}(K_G - f) = \deg(f) - \mathfrak{g} + 1$$

where $\mathfrak{g} = |E(G)| - |V(G)| + 1$ and $K_G(v) = d(v) - 2$ for each $v \in V(G)$.

A key notion in the proof of Baker and Norine is the following:

Definition 2.4 (Non-special divisor [4, Section 2])**.** A divisor $f \in \operatorname{Div}(G)$ on an undirected graph G is *non-special* if $\deg(f) = \mathfrak{g} - 1$ and f is not equi-effective.

It is a natural question whether Riemann–Roch type theorems can be true for directed graphs. For previous research in the area, see [2], [1, Section 6]. In this paper, we approach this question by translating the concepts of graph divisor theory to the language of chip-firing games, which allows a natural interpretation of non-special divisors and gives ways to generalizations to directed graphs.

2.4 Chip-firing

Chip-firing is a solitary game on a digraph, introduced by Björner, Lovász and Shor [6, 5]. In this game we consider a digraph G with a pile of chips on each of its nodes. A position of the game, called a *chip-distribution* (or just distribution) is described by a vector $x \in \mathbb{Z}^{V(G)}$, where $x(v)$ denotes the number of chips on vertex $v \in V$. Note that though originally chip-firing was defined only for non-negative chip-distributions, in this paper, we allow vertices to have a negative number of chips. We denote the set of all chip-distributions on G by $\operatorname{Chip}(G)$, which we again identify with $\mathbb{Z}^{V(G)}$.

The basic move of the game is *firing* a vertex. It means that this vertex passes a chip to its neighbors along each outgoing edge, and so its number of chips decreases by its outdegree. In other words, firing a vertex v means taking the new chip-distribution $x + L\mathbf{1}_v$ instead of x. Note that $\deg(x + L\mathbf{1}_v) = \deg(x)$.

A vertex $v \in V(G)$ is *active* with respect to a chip-distribution x, if $x(v) \geq d^+(v)$. The firing of a vertex $v \in V(G)$ is *legal*, if v was active before the firing. A *legal game*

is a sequence of distributions in which every distribution is obtained from the previous one by a legal firing. A game terminates if there is no active vertex with respect to the last distribution. Chip-firing on an undirected graph is defined as chip-firing on the corresponding bidirected graph.

The following theorem was proved by Björner and Lovász. They state their theorem only for chip-distributions $x \in \mathrm{Chip}(G)$ with $x \geq \mathbf{0}_G$, but it is easy to check that the proof also works for chip-distributions with negative entries.

Theorem 2.5 ([5, Theorem 1.1]). *Let G be a digraph and let $x \in \mathrm{Chip}(G)$ be a chip-distribution. Then starting from x, either every legal game can be continued indefinitely, or every legal game terminates after the same number of moves with the same final distribution. Moreover, the number of times a given node is fired is the same in every maximal legal game.*

Let us call a chip-distribution $x \in \mathrm{Chip}(G)$ *terminating*, if every legal chip-firing game played starting from x terminates, and call it *non-terminating*, if every legal chip-firing game played starting from x can be continued indefinitely. According to Theorem 2.5, a chip-distribution is either terminating or non-terminating.

One can easily check by the pigeonhole principle that if for a digraph G, a distribution $x \in \mathrm{Chip}(G)$ has $\deg(x) > |E(G)| - |V(G)|$ then x is non-terminating (see [5]). From this it follows that the following quantity, which measures how far a given distribution is from being non-terminating, is well defined.

Definition 2.6. For a distribution $x \in \mathrm{Chip}(G)$, the *distance* of x from non-terminating distributions is

$$\mathrm{dist}(x) = \min\{\deg(y) : y \in \mathrm{Chip}(G),\ y \geq \mathbf{0}_G,\ x + y \text{ is non-terminating}\}.$$

The following is a useful technical lemma from [5]. Since in [5] it is only proved for non-negative distributions, but we need it for integer valued distributions, we give a proof.

Proposition 2.7. *On a strongly connected digraph G, in any infinite legal game every vertex is fired infinitely often.*

Proof. In a chip-firing game a vertex can only loose chips if it is fired, but if it is fired, it is not allowed to go negative. Hence in a game with initial distribution x, on any vertex v, the number of chips is always at least $\min\{0, x(v)\}$. Hence the number of chips on any vertex is at most $\sum_{v \in V} \max\{x(v), 0\}$ at any time.

If a legal game is infinitely long, then there is a vertex that fires infinitely often. If a vertex is fired infinitely often, then it passes infinitely many chips to its out-neighbors, hence the out-neighbors also need to be fired infinitely often, otherwise they would have more chips than possible. By induction, every vertex reachable on directed path from an infinitely often fired vertex is also fired infinitely often. As the digraph is strongly connected, each vertex is reachable on directed path from each vertex, thus every vertex fires infinitely often. □

From Proposition 2.7 it follows that for a strongly connected digraph G if $x \in$ Chip(G) is non-terminating, then playing a legal game, after finitely many steps we arrive at a distribution which is nowhere negative. Hence there exists a non-negative chip-distribution among the non-terminating distributions of minimum degree. From this, it follows that dist$(\mathbf{0}_G)$ equals to the minimum degree of a non-terminating distribution on the digraph G.

3 Linear equivalence of chip distributions and duality with graph divisors

In this section, we describe the duality between graph divisor theory and the chip-firing game. This duality was first discovered by Baker and Norine [4]. First we need to generalize the idea of linear equivalence to the chip-firing game. The next lemma, which appears first in [7, Lemma 4.3.], plays a key role in this. To be self-contained, we give a proof.

Lemma 3.1. *[7] Let G be a strongly connected digraph and $x, y \in$ Chip(G). If $x \sim y$, then x is terminating if and only if y is terminating.*

Proof. By symmetry, it is enough to prove that if x is terminating, then y is also terminating.

Let x be a terminating chip-distribution. Play the chip-firing game starting from x until it terminates. Let the final configuration be x^*. Clearly, $x^* \sim x \sim y$. Let $z \in \mathbb{Z}^{V(G)}$ be the vector with $x^* = y + Lz$. We can suppose that $z \in \mathbb{Z}_+^{V(G)}$, since the Laplacian of a strongly connected digraph has a strictly positive eigenvector with eigenvalue zero [5, Lemma 4.1]. Start a game from y in the following way: If there is an active vertex v that has been fired less than $z(v)$ times, then one such vertex is fired. If there is no such vertex, the game ends. Clearly, after at most $\sum_{v \in V(G)} z(v)$ steps, this modified game ends. We claim that for the final distribution $y' = y + Lz'$ (where $z' \leq z$), $y'(v) < d^+(v)$ for each vertex v. Indeed, as the game stopped, for any vertex v with $y'(v) \geq d^+(v)$, $z'(v) = z(v)$. As x^* is stable, $x^*(v) < d^+(v)$. But then from $x^* = y' + L(z - z')$ and $z(v) = z'(v)$, we get $d^+(v) > x^*(v) \geq y'(v)$, which is a contradiction. $\square$

Corollary 3.2. *If $x \sim y$, then* dist$(x) =$ dist(y).

Using Lemma 3.1, we can easily prove the following proposition establishing a duality between divisors and chip-distributions. This proposition is the generalization of a result of Baker and Norine [4, Corollary 5.4] to the case of directed graphs.

Proposition 3.3. *$x \in$ Chip(G) is terminating if and only if $\mathbf{d}^+ - \mathbf{1} - x$ is equi-effective, i.e. there exists $\mathbf{0} \leq y \in$ Div(G) such that $y \sim (\mathbf{d}^+ - \mathbf{1} - x)$.*

Proof. On the one hand, if x is terminating then we play the game until it terminates at some chip distribution $x^* \leq \mathbf{d}^+ - \mathbf{1}$. Now $\mathbf{0} \leq (\mathbf{d}^+ - \mathbf{1} - x^*) \sim (\mathbf{d}^+ - \mathbf{1} - x)$.

On the other hand, if for $y \geq \mathbf{0}$, $y \sim (\mathbf{d}^+ - \mathbf{1} - x)$, then the chip distribution $x' = \mathbf{d}^+ - \mathbf{1} - y$ has no active vertex. As $y \sim (\mathbf{d}^+ - \mathbf{1} - x)$, we have $x \sim x'$, and then x is terminating by Lemma 3.1. $\qquad \square$

The following is a straightforward consequence of Proposition 3.3.

Corollary 3.4. *For any $f \in \mathrm{Div}(G)$, the following holds:*

$$\mathrm{rank}(f) = \mathrm{dist}(\mathbf{d}^+ - \mathbf{1} - f) - 1$$

Corollary 3.4 enables us to state equivalent forms of Riemann–Roch-type theorems for graphs using the notion of dist, see Sections 5 and 6.

4 Non-terminating chip-distributions and turnback arc sets

In this section, for Eulerian digraphs, we explain the connection between non-terminating chip-distributions and turnback arc sets, which are a special type of feedback arc sets. This connection will be crucial in our proofs of Riemann-Roch-type theorems. The connection of non-terminating distributions and feedback arc sets has been known, but in this paper, we need the connections to turnback arc sets. The proofs included here are slight modifications of proofs in [14, 5, 12].

Definition 4.1. A *feedback arc set* of a digraph G is a set of edges $F \subseteq E(G)$ such that the digraph $G' = (V(G), E(G) \setminus F)$ is acyclic. We denote

$$\mathrm{minfas}(G) = \min\{|F| : F \subseteq E(G) \text{ is a feedback arc set}\}.$$

Definition 4.2. A *turnback arc set* of a digraph G is a set of edges $T \subseteq E(G)$ such that the digraph G' we get by reversing the edges in T is acyclic.

We need the following theorem of Gallai [10].

Proposition 4.3. *[10] Let G be a digraph. A minimal feedback arc set of G is also a turnback arc set of G.*

As [10] is not an easily accessible source, we include a short proof that we learned from Darij Grinberg.

Proof. Let $F \subseteq E(G)$ be any minimal feedback arc set. Let G_1 denote the digraph that we get form G by deleting the edges of F and let G_2 denote the digraph that we get from G by reversing the edges in F. We know that G_1 is acyclic and we need to show that G_2 is acyclic, too.

As G_1 is acyclic, there is a topological ordering $v_1, v_2, \ldots, v_{|V(G)|}$ of $|V(G)|$ such that for each edge $\overrightarrow{v_i v_j} \in E(G) \setminus F$, $i < j$ holds. For any edge $e = \overrightarrow{v_i v_j} \in F$ with $i < j$, $F \setminus \{e\}$ would be a feedback arc set of G, contradicting the minimality of F. Therefore $i > j$ holds for any edge $\overrightarrow{v_i v_j} \in F$, hence all the edges of G_2 are directed according to to our topological ordering, so G_2 is acyclic. $\qquad \square$

Corollary 4.4. *The cardinality of a minimum cardinality turnback arc set of G equals to* $\mathrm{minfas}(G)$.

Proposition 4.5. *Let G be a strongly connected Eulerian digraph. A chip-distribution $x \in \mathrm{Chip}(G)$ is non-terminating if and only if there exists a turnback arc set T of G, and a chip-distribution $a \in \mathrm{Chip}(G)$ with $a \geq 0$, such that if ϱ is the indegree distribution of T, then $x \sim \varrho + a$.*

For the proof of this proposition we need a lemma, which is a variant of [14, Lemma 2.4.].

Lemma 4.6. *Let G be an Eulerian digraph, and $T \subseteq E(G)$ be a turnback arc set. Denote by $d_T^+(v)$ and $d_T^-(v)$ the outdegree and indegree of a vertex v in the digraph $G_T = (V(G), T)$. Then a distribution $x \in \mathrm{Chip}(G)$ satisfying*

$$x(v) \geq d_T^-(v) \ \text{for every } v \in V(G) \tag{1}$$

is non-terminating.

Proof. Let G' be the digraph we get by reversing the edges of T. From the definition of turnback arc set, G' is an acyclic digraph. Therefore, it has a source v_0.

Hence no out-edge of v_0 in G is in T, but all the in-edges of v_0 in G are in T. From (1), the choice of v_0 and the fact that G is Eulerian, we have that $x(v_0) \geq d_T^-(v_0) = d^-(v_0) = d^+(v_0)$, therefore v_0 is active with respect to x. Fire v_0. Let x' be the resulting distribution. Let T' be the set of arcs obtained from T by removing the in-edges of v_0 and adding the out-edges of v_0. Then the digraph G'' that we get by reversing the edges in T' is acyclic, since compared to G', we only transformed a source to be a sink. Hence T' is a turnback arc set. It is straightforward to check that $x'(v) \geq d_{T'}^-(v)$ for every $v \in V(G)$. Thus, we are again in the starting situation, which shows that x is indeed non-terminating. $\square$

Proof of Proposition 4.5. Suppose that there is a chip-distribution $x \in \mathrm{Chip}(G)$ such that $x \sim \varrho + a$ where $\varrho \in \mathrm{Chip}(G)$ is the indegree sequence of a turnback arc set T of G, and $a \in \mathrm{Chip}(G)$ with $a \geq 0$. Then from the lemma, $\varrho + a$ is non-terminating. From Lemma 3.1, $x \sim \varrho + a$ is non-terminating.

For the other direction, take a non-terminating distribution x. Let us play a chip-firing game with initial distribution x. Proposition 2.7 says that after finitely many steps, every vertex has fired. Play until such a moment, and let the distribution at that moment be x'. Then $x \sim x'$.

Let A be the following set of edges:

$$A = \{\overrightarrow{uv} \in E(G) : \text{ the last firing of } u \text{ precedes the last firing of } v\}.$$

As every vertex has fired, A is well defined. Let $v_1, v_2, \ldots v_{|V(G)|}$ be the ordering of the vertices by the time of their last firing.

Then $v_1, v_2, \ldots v_{|V(G)|}$ is a topological ordering of the digraph G' that we get by reversing the edges in $E(G) \setminus A$, hence $T = E(G) \setminus A$ is a turnback arc set. We show that $x'(v) \geq d_T^-(v)$ for every $v \in V(G)$. For $1 \leq i \leq |V(G)|$ the vertex v_i has

$d_T^-(v_i) = \sum_{j>i} \overrightarrow{d}(v_j, v_i)$. After its last firing, v_i had a nonnegative number of chips. Since then, it kept all chips it received. And as $v_{i+1}, \ldots, v_{|V(G)|}$ all fired since the last firing of v_i, it received at least $\sum_{j>i} \overrightarrow{d}(v_j, v_i) = d_T^-(v_i)$ chips. So indeed, we have $x'(v_i) \geq d_T^-(v_i)$.

Therefore $x' = \varrho + a$, where $\varrho(v) = d_T^-(v_i)$, and $a \geq 0$. Since $x \sim x'$, we are ready. $\qquad\square$

Let us state the above proposition for the special case of undirected, i.e. bidirected graphs. In a bidirected graph, a turnback arc set must include exactly one version of each bidirected edge, i.e for each turnback arc set T, for each $uv \in E$, either $\overrightarrow{uv} \in T$ or $\overrightarrow{vu} \in T$. Hence a turnback arc set corresponds to an orientation of G. This orientation also needs to be acyclic, therefore turnback arc sets in undirected graphs correspond to acyclic orientations. Hence we can deduce the following corollary (whose statement is equivalent to [6, Theorem 2.3])

Proposition 4.7. *Let G be an undirected graph. A chip-distribution $x \in \mathrm{Chip}(G)$ is non-terminating if and only if there exists an acyclic orientation $\overrightarrow{G}$ of G, and a chip-distribution $a \in \mathrm{Chip}(G)$ with $a \geq \mathbf{0}_G$, such that if ϱ is the indegree distribution of $\overrightarrow{G}$, then $x \sim \varrho + a$.*

This proposition together with Proposition 3.3 shows, that for an undirected graph G, $f \in \mathrm{Div}(G)$ is non-special if and only if $x = \mathbf{d}_G - \mathbf{1}_G - f$ is linearly equivalent to the in-degree sequence of an acyclic orientation.

5 A weak Riemann–Roch theorem for Eulerian digraphs

Using the connection between dist and rank (Corollary 3.4), we can state the following equivalent form of Theorem 2.3.

Theorem 5.1 (Riemann–Roch)**.** *Let G be an undirected graph. Then for any $x \in \mathrm{Chip}(G)$:*

$$\mathrm{dist}(x) - \mathrm{dist}(\mathbf{d}_G - x) = |E(G)| - \deg(x).$$

The equivalence of the two forms can be seen by choosing $x = \mathbf{d}_G - \mathbf{1}_G - f$. We give a short proof for this version of theorem. Note that by translating this proof from the language of chip-firing to the language of graph divisor theory, one gets back the proof of Cori and le Borgne for Theorem 43 in the Annex of [8].

Proof. Let $|E(G)| = m$. First, we prove that $\mathrm{dist}(\mathbf{d} - x) \leq \mathrm{dist}(x) - m + \deg(x)$.

From the definition of dist, there exists a chip-distribution $a \geq \mathbf{0}$, with $\deg(a) = \mathrm{dist}(x)$ such that $x + a$ is non-terminating.

Then, from Proposition 4.7, there exists an acyclic orientation $\overrightarrow{G}$ of G with indegree distribution ϱ, and a chip-distribution $b \geq \mathbf{0}$, such that $\varrho + b \sim x + a$.

Let $x' = \varrho + b - a$. Clearly, $x' \sim x$, and $\mathbf{d} - x' \sim \mathbf{d} - x$. Therefore, by Corollary 3.2, it is enough to show that $\mathrm{dist}(\mathbf{d} - x') \leq \mathrm{dist}(x) - m + \deg(x') = \mathrm{dist}(x) - m + \deg(x)$.

As $x' + a = \varrho + b$,

$$(\mathbf{d} - x') + b = (\mathbf{d} - \varrho) + a.$$

Note that $\mathbf{d} - \varrho$ is the indegree vector of the reverse of $\overrightarrow{G}$, which is also an acyclic orientation. Therefore, again by Proposition 4.7, $(\mathbf{d} - x') + b$ is non-terminating, showing that

$$\mathrm{dist}(\mathbf{d} - x') \leq \deg(b) = \deg(x') + \deg(a) - \deg(\varrho) = \deg(x) + \mathrm{dist}(x) - m.$$

From this, we have $\mathrm{dist}(x) - \mathrm{dist}(\mathbf{d} - x) \geq m - \deg(x)$.

Now let $y = \mathbf{d} - x$. Then $x = \mathbf{d} - y$. From the above argument, we have

$$\mathrm{dist}(x) = \mathrm{dist}(\mathbf{d} - y) \leq \mathrm{dist}(y) - m - \deg(y) = \mathrm{dist}(\mathbf{d} - x) - m - (2m - \deg(x)),$$

giving $\mathrm{dist}(x) - \mathrm{dist}(\mathbf{d} - x) \leq m - \deg(x)$. $\qquad\square$

Remark 5.2. Several proofs of the Riemann–Roch theorem for graphs have been published (see for example [4, 1, 3, 15]). The key idea of each proof is to understand the relationship of the non-special divisors to each other and to other divisors. From the chip-firing point of view, dual pairs of non-special divisors are exactly the chip-distributions linearly equivalent to the in-degree sequence of an acyclic orientation. In this setting, the pairing of non-special divisors (property (RR2) of Baker and Norine) depends on the fact that the reversal of an acyclic orientation is also acyclic. On the other hand, the relationship of non-special divisors to other divisors is established by Proposition 4.7. The fact that Proposition 4.7 has a version for Eulerian digraphs (Proposition 4.5) enables us to generalize the proof for Eulerian digraphs.

Theorem 5.3. *Let G be an Eulerian digraph. For each $x \in \mathrm{Chip}(G)$,*

$$\mathrm{minfas}(G) - \deg(x) \leq \mathrm{dist}(x) - \mathrm{dist}(\mathbf{d}_G^- - x) \leq |E(G)| - \mathrm{minfas}(G) - \deg(x).$$

This theorem is proved by Amini and Manjunath in [1, Section 6.2.] for Eulerian digraphs where each edge has multiplicity at least one by using a limiting argument. For the general case they only obtain a lower bound of $\mathrm{minfas}(G) - \deg(x) - 2$ and an upper bound of $|E(G)| - \mathrm{minfas}(G) - \deg(x) + 2$. Here we give a proof of the tighter bound in the general case.

Proof. First, we prove that $\mathrm{dist}(\mathbf{d}^- - x) \leq \mathrm{dist}(x) - \mathrm{minfas}(G) + \deg(x)$.

From the definition of $\mathrm{dist}(x)$, there exists a chip-distribution $a \geq 0$ with $\deg(a) = \mathrm{dist}(x)$ such that $x + a$ is non-terminating.

Then, from Proposition 4.5, there exists a turnback arc set T of G with indegree vector ϱ, and a chip-distribution $b \geq 0$ such that $x + a \sim \varrho + b$. As ϱ is the indegree vector of a turnback arc set of G, using Corollary 4.4 we get $\deg(\varrho) \geq \mathrm{minfas}(G)$.

Let $x' = \varrho + b - a$. Clearly, $x' \sim x$, and $\mathbf{d}^- - x' \sim \mathbf{d}^- - x$. Therefore, by Corollary 3.2, it is enough to show that $\mathrm{dist}(\mathbf{d}^- - x') \leq \mathrm{dist}(x') - \mathrm{minfas}(G) + \deg(x') = \mathrm{dist}(x) - \mathrm{minfas}(G) + \deg(x)$.

As $x' + a = \varrho + b$,

$$(\mathbf{d}^- - x') + b = (\mathbf{d}^- - \varrho) + a.$$

Note that $\mathbf{d}^- - \varrho$ is the indegree vector of the complement edge-set of T, which is also a turnback arc set. Moreover, $a \geq 0$. Therefore, again by Proposition 4.5, $(\mathbf{d}^- - x') + b$ is non-terminating showing that

$$\text{dist}(\mathbf{d}^- - x') \leq \deg(b) = \deg(x') + \deg(a) - \deg(\varrho) \leq \deg(x) + \text{dist}(x) - \text{minfas}(G).$$

From this, we have $\text{dist}(x) - \text{dist}(\mathbf{d}^- - x) \geq \text{minfas}(G) - \deg(x)$.

Now let $y = \mathbf{d}^- - x$. Then $x = \mathbf{d}^- - y$. From the above argument, we have $\text{dist}(x) = \text{dist}(\mathbf{d}^- - y) \leq \text{dist}(y) - \text{minfas}(G) + \deg(y) = \text{dist}(\mathbf{d}^- - x) - \text{minfas}(G) + (|E(G)| - \deg(x)) = \text{dist}(\mathbf{d}^- - x) + |E(G)| - \text{minfas}(G) - \deg(x)$, giving $\text{dist}(x) - \text{dist}(\mathbf{d}^- - x) \leq |E(G)| - \text{minfas}(G) - \deg(x)$. $\qquad\square$

Remark 5.4. The theorem is sharp in the following sense: for any digraph G, taking x to be the indegree-distribution of a minimum cardinality turnback arc set, $\text{dist}(x) = \text{dist}(\mathbf{d}^- - x) = 0$, hence $\text{dist}(x) - \text{dist}(\mathbf{d}^- - x) = \text{minfas}(G) - \deg(x)$. On the other hand, for $y = \mathbf{d}^- - x$, $\text{dist}(y) - \text{dist}(\mathbf{d}^- - y) = |E(G)| - \text{minfas}(G) - \deg(y)$.

6 Riemann–Roch for general digraphs

In this section, we investigate the Riemann–Roch property for strongly connected digraphs. We note that divisor theory on strongly connected digraphs is equivalent to two previous settings in which the Riemann–Roch property was investigated; the setting of the abstract Riemann–Roch condition of Baker and Norine [4], and the setting of Amini and Manjunath [1]. In the setting of Baker and Norine, $\text{Div}(X)$ is a free Abelian group on a finite set X, which is equipped with an equivalence relation satisfying two given properties, (E1) and (E2). These properties are equivalent to the fact that the differences of equivalent divisors form a lattice $\Gamma \subset \mathbb{Z}_0^n$. Amini and Manjunath consider this latter situation, i.e., for them, divisors are elements of $\mathbb{Z}^n$ and for a fixed lattice $\Gamma \subset \mathbb{Z}_0^n$ they call two divisors equivalent if their difference is in Γ. The case of divisor theory on strongly connected digraphs corresponds to the case if the lattice Γ is generated by the Laplacian matrix of a strongly connected digraph. Since by a theorem of Perkinson, Perlman and Wilmes [13, Theorem 4.11], each lattice $\Gamma \subset \mathbb{Z}_0^n$ can be generated by the Laplacian matrix of a strongly connected digraph, the graphical case is indeed equivalent to the other two.

Baker–Norine and Amini–Manjunath both obtain a necessary and sufficient condition in their setting for the existence of a Riemann–Roch formula ([4, Theorem 2.2] and [1, Theorem 1.4]). In this section, we deduce the graphical equivalent of these necessary and sufficient conditions.

We say that a strongly connected digraph G *has the Riemann–Roch property* if there exists some $K \in \text{Chip}(G)$ and integer t, such that for each $x \in \text{Chip}(G)$,

$$\text{dist}(x) - \text{dist}(K - x) = t - \deg(x).$$

In this case we say that K is a *canonical distribution* for G. Examples of Section 8 show that such K and t does not always exist for a digraph.

From the divisor-theoretic point of view, the existence of such K and t implies, that for the divisor $\tilde{K} = 2 \cdot \mathbf{d}_G^+ - 2 \cdot \mathbf{1}_G - K$, each divisor $f \in \mathrm{Div}(G)$ has

$$\mathrm{rank}(f) - \mathrm{rank}(\tilde{K} - f) = t - |E(G)| + |V(G)| + \deg(f).$$

Proposition 6.1. *If for a digraph G, the Riemann–Roch formula*

$$\mathrm{dist}(x) - \mathrm{dist}(K - x) = t - \deg(x)$$

holds for all $x \in \mathrm{Chip}(G)$ for some $K \in \mathrm{Chip}(G)$ and value t, then

(i) $\deg(K) = 2t$,

(ii) $t = \mathrm{dist}(\mathbf{0}_G)$.

Proof. *(i)* Take an arbitrary $x \in \mathrm{Chip}(G)$, and write up the Riemann–Roch formula for x and for $K - x$. Note that $K - (K - x) = x$.

$$\begin{aligned}
\mathrm{dist}(x) - \mathrm{dist}(K - x) &= t - \deg(x) \\
\mathrm{dist}(K - x) - \mathrm{dist}(x) &= t - \deg(K - x)
\end{aligned}$$

Summing these two equalities, we get $2t = \deg(K - x) + \deg(x) = \deg(K)$.

(ii) Let $\mathrm{dist}(K) = k$. The Riemann–Roch formula for $x = \mathbf{0}_G$ says that $\mathrm{dist}(\mathbf{0}_G) = \mathrm{dist}(K) + t - 0 = k + t$. By definition k is non-negative, hence it is enough to show that it is non-positive (i.e. K is non-terminating). By the definition of $\mathrm{dist}(\mathbf{0}_G)$, there exists a non-terminating chip-distribution x_0 with $\deg(x_0) = \mathrm{dist}(\mathbf{0}_G) = k + t$. The Riemann–Roch formula for $x = x_0$ says that

$$\mathrm{dist}(x_0) - \mathrm{dist}(K - x_0) = t - \deg(x_0) = -k.$$

Since x_0 is non-terminating, $\mathrm{dist}(x_0) = 0$. Hence we have $\mathrm{dist}(K - x_0) = k$. Using part *(i)*, we have

$$\deg(K - x_0) = \deg(K) - \deg(x_0) = 2t - (k + t) = \mathrm{dist}(\mathbf{0}_G) - 2k.$$

As a non-terminating distribution has degree at least $\mathrm{dist}(\mathbf{0}_G)$, we necessarily have $\mathrm{dist}(K - x_0) \geq 2k$, which means $k \geq 2k$. This implies $k = 0$. $\square$

Now we give a necessary, and a necessary and sufficient condition for a strongly connected digraph to have the Riemann–Roch property.

Definition 6.2. Let us call a chip-distribution $x \in \mathrm{Chip}(G)$ *minimally non-terminating*, if it is non-terminating, but for each $v \in V(G)$, $x - \mathbf{1}_v$ is terminating.

Proposition 6.3. *If the Riemann–Roch formula holds for a digraph G, then all minimally non-terminating distributions have degree $\mathrm{dist}(\mathbf{0}_G)$.*

Proof. Suppose that the Riemann–Roch formula holds for the digraph G, $x \in \mathrm{Chip}(G)$ is non-terminating, and $\deg(x) = \mathrm{dist}(\mathbf{0}_G) + k$ with $k > 0$. We show that x is not minimally non-terminating.

From the Riemann–Roch formula, $\mathrm{dist}(x) - \mathrm{dist}(K - x) = \mathrm{dist}(\mathbf{0}_G) - \deg(x)$, thus $\mathrm{dist}(K - x) = k$. Since $\deg(K - x) = \mathrm{dist}(\mathbf{0}_G) - k$, this means that there exists a chip-distribution $a \in \mathrm{Chip}(G)$, $a \geq 0$, $\deg(a) = k$, such that $K - x + a$ is non-terminating, and is of degree $\mathrm{dist}(\mathbf{0}_G)$.

By the Riemann–Roch formula, $\mathrm{dist}(K - x + a) - \mathrm{dist}(x - a) = \mathrm{dist}(\mathbf{0}_G) - \deg(K - x + a)$. As $\mathrm{dist}(K - x + a) = 0$ and $\deg(K - x + a) = \mathrm{dist}(\mathbf{0}_G)$, we have $\mathrm{dist}(x - a) = 0$, hence $x - a$ is non-terminating. Since $a \geq 0$, we conclude that x is not minimally non-terminating. □

Theorem 6.4. *Let G be a digraph. G has the Riemann–Roch property if and only if each minimally non-terminating distribution has degree $\mathrm{dist}(\mathbf{0}_G)$, and there exists a distribution $K \in \mathrm{Chip}(G)$ such that for any minimally non-terminating distribution $x \in \mathrm{Chip}(G)$, $K - x$ is also minimally non-terminating. If the above condition holds, then the Riemann–Roch formula holds for G with K as canonical distribution.*

Proof. First we show the "only if" direction. Suppose that G has the Riemann–Roch property with canonical distribution K. Then by Proposition 6.3, each minimally non-terminating distribution has degree $\mathrm{dist}(\mathbf{0}_G)$. Suppose that x is minimally non-terminating. Then $\mathrm{dist}(x) = 0$ and $\deg(x) = \mathrm{dist}(\mathbf{0}_G)$. Since $\mathrm{dist}(x) - \mathrm{dist}(K - x) = \mathrm{dist}(\mathbf{0}_G) - \deg(x) = 0$, we have $\mathrm{dist}(K - x) = 0$, thus $K - x$ is non-terminating. Also, since $0 = \mathrm{dist}(K - x) - \mathrm{dist}(x) = \mathrm{dist}(\mathbf{0}_G) - \deg(K - x)$, we have $\deg(K - x) = \mathrm{dist}(\mathbf{0}_G)$, thus $K - x$ is minimally non-terminating.

Now, we show the "if" direction.

First note that $\deg(K) = 2 \cdot \mathrm{dist}(\mathbf{0}_G)$ is a direct consequence of our conditions. It is sufficient to show that $\mathrm{dist}(x) - \mathrm{dist}(K - x) \geq \mathrm{dist}(\mathbf{0}_G) - \deg(x)$ holds for any $x \in \mathrm{Chip}(G)$. Indeed, by plugging $K - x$ into x, and using that $\deg(K - x) = \deg(K) - \deg(x) = 2 \cdot \mathrm{dist}(\mathbf{0}_G) - \deg(x)$, we get

$$\mathrm{dist}(K - x) - \mathrm{dist}(x) \geq \mathrm{dist}(\mathbf{0}_G) - \deg(K - x) = \deg(x) - \mathrm{dist}(\mathbf{0}_G)$$

which implies the equality.

x is either terminating or non-terminating.

First suppose that x is terminating. Then $\mathrm{dist}(x) = k > 0$. This means that there exists $a \in \mathrm{Chip}(G)$, $a \geq 0$, $\deg(a) = k$ such that $x + a$ is non-terminating.

There exists at least one minimally non-terminating distribution y such that $x + a = y + b$ where $b \geq 0$. (We can take off chips until our distribution gets minimally non-terminating.)

Then by our assumption, $K - y$ is also a minimally non-terminating distribution. We have $K - (x + a) = K - (y + b)$, thus $(K - x) + b = (K - y) + a$. $(K - y) + a$ is non-terminating, as $K - y$ is non-terminating and $a \geq 0$. Thus $\mathrm{dist}(K - x) \leq \deg(b) = \deg(x) + \deg(a) - \deg(y) = \deg(x) + \mathrm{dist}(x) - \mathrm{dist}(\mathbf{0}_G)$.

Hence

$$\mathrm{dist}(x) - \mathrm{dist}(K - x) \geq \mathrm{dist}(x) - (\deg(x) + \mathrm{dist}(x) - \mathrm{dist}(\mathbf{0}_G)) = \mathrm{dist}(\mathbf{0}_G) - \deg(x).$$

Now suppose that x is non-terminating. Then there exists a minimally non-terminating distribution $y \in \mathrm{Chip}(G)$ such that $x = y + b$ where $b \geq 0$. By our assumptions, $\deg(y) = \mathrm{dist}(\mathbf{0}_G)$ and $K - y$ is also minimally non-terminating.

$K - x + b = K - y$. As $K - y$ is non-terminating, $\mathrm{dist}(K - x) \leq \deg(b)$. On the other hand, $\deg(x) = \deg(y) + \deg(b) = \mathrm{dist}(\mathbf{0}_G) + \deg(b)$. Thus

$$\mathrm{dist}(x) - \mathrm{dist}(K - x) \geq 0 - \deg(b) = 0 - (\deg(x) - \mathrm{dist}(\mathbf{0}_G)) = \mathrm{dist}(\mathbf{0}_G) - \deg(x).$$

$\square$

7 The natural Riemann–Roch property in Eulerian digraphs

Asadi and Backman introduced the following variant of the Riemann–Roch property [2, Definition 3.12]: A digraph G has the *natural Riemann–Roch property*, if it satisfies a Riemann–Roch formula with canonical divisor $K(v) = d^+(v) - 2$ for each $v \in V$. This definition translates to the language of chip-firing in the following way:

Definition 7.1. A digraph G has the natural Riemann–Roch property, if for each $x \in \mathrm{Chip}(G)$

$$\mathrm{dist}(x) - \mathrm{dist}(\mathbf{d}_G^+ - x) = \frac{1}{2}|E(G)| - \deg(x)$$

From the Riemann–Roch theorem for undirected graphs, it follows that each undirected (that is, bidirected) graph has the natural Riemann–Roch property. However it is left open in [2] whether there are any other such graphs.

In Section 8.3 we show an example that a non-bidirected graph can also have the natural Riemann–Roch property. However, the following theorem shows that this is not possible if the digraph is Eulerian.

Theorem 7.2. *Let G be an Eulerian digraph. Then G has the natural Riemann–Roch property if and only if it is a bidirected graph corresponding to an undirected graph (i.e. $\overrightarrow{d}(u, v) = \overrightarrow{d}(v, u)$ for any pair of vertices u, v).*

Proof. Suppose that G has the natural Riemann–Roch property. Then we have $K(v) = d^+(v) \ \forall v \in V(G)$, thus, $\deg(K) = |E(G)|$. Proposition 6.1 implies that $\mathrm{dist}(\mathbf{0}_G) = \frac{1}{2}\deg(K) = \frac{1}{2}|E(G)|$. Proposition 4.5 says that for Eulerian digraphs, $\mathrm{dist}(\mathbf{0}_G) = \mathrm{minfas}(G)$. As a consequence, we have $\mathrm{minfas}(G) = \frac{1}{2}|E(G)|$.

For an ordering $v_1, v_2, \ldots, v_n$ of $V(G)$, we call an arc $\overrightarrow{v_i v_j}$ with $i < j$ a *forward arc*, and an arc $\overrightarrow{v_i v_j}$ with $j < i$ a *backward arc*.

For any ordering, the set of forward arcs forms a feedback arc set. The same is true for the set of backward arcs. As $E(G)$ is the disjoint union of these two sets and $\mathrm{minfas}(G) = \frac{1}{2}|E(G)|$, it follows that for any ordering of vertices there are exactly $\frac{1}{2}|E(G)|$ forward arcs and exactly $\frac{1}{2}|E(G)|$ backward arcs.

Suppose that there is a pair of vertices (u, v) with $\overrightarrow{d}(u, v) \neq \overrightarrow{d}(v, u)$. Then consider orderings $u, v, v_3, \ldots, v_n$ and $v, u, v_3, \ldots, v_n$. The set of forward arcs has different cardinality for these two orderings, therefore they cannot both have size $\frac{1}{2}|E(G)|$, which is a contradiction. $\square$

8 Examples

In this section we provide examples showing that a digraph may not have the Riemann–Roch property, but for certain digraphs such a theorem can still hold.

8.1 A digraph without Riemann–Roch property

Consider the following digraph G_1 (see also Figure 1):

$$
\begin{aligned}
V(G_1) &= \{v_1, v_2, v_3, v_4, v_5, v_6\}, \\
E(G_1) &= \{\overrightarrow{v_1 v_2}; \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_4}, \overrightarrow{v_4 v_5}, \overrightarrow{v_5 v_6}, \overrightarrow{v_6 v_1}, \overrightarrow{v_1 v_5}, \overrightarrow{v_2 v_6}, \overrightarrow{v_3 v_1}, \overrightarrow{v_4 v_2}, \overrightarrow{v_5 v_3}, \overrightarrow{v_6 v_4}\}.
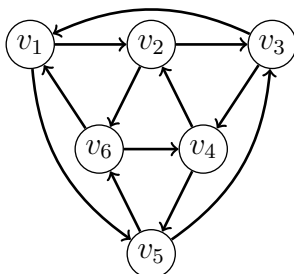\end{aligned}
$$

Figure 1: G_1, a graph without the Riemann-Roch property

It is easy to check that for G_1, $x_1 = (1, 0, 0, 1, 0, 2)$ and $x_2 = (2, 1, 1, 1, 0, 0)$ are both minimally non-terminating. Since $\deg(x_1) \neq \deg(x_2)$, Proposition 6.3 tells us that the Riemann–Roch formula does not hold for G_1.

8.2 Eulerian digraphs with non-natural Riemann–Roch property

A very simple example of an Eulerian digraph with the Riemann–Roch property is a directed cycle. It is straightforward that for a directed cycle, any chip-distribution of degree at least one is non-terminating. On the other hand, since minfas=1, any chip-distribution of degree less than one is terminating. Thus, the minimally non-terminating distributions are exactly the distributions of degree 1. Let K be any distribution of degree two. It is straightforward that the conditions of Theorem 6.4 hold, thus a directed cycle has the Riemann–Roch property.

Another example, where there are more than one equivalence classes with the same number of chips is the following graph G_2 (see also Figure 2):

$$
V(G_2) = \{v_1, v_2, v_3, v_4\}; \quad E(G_2) = \{\overrightarrow{v_1 v_2}, \overrightarrow{v_1 v_4}, \overrightarrow{v_2 v_1}, \overrightarrow{v_2 v_3}, \overrightarrow{v_3 v_1}, \overrightarrow{v_4 v_2}\}.
$$

For this graph, $\mathrm{dist}(\mathbf{0}_{G_2}) = \mathrm{minfas}(G_2) = 2$. It is well-known that for an Eulerian digraph, the number of equivalence classes of chip-distributions of a fixed degree equals the number of spanning in-arborescences rooted at an arbitrary vertex v. Thus this
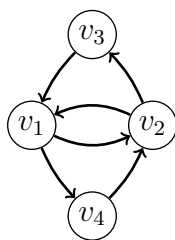
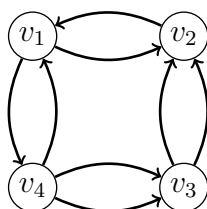Figure 2: G_2, a graph with non-natural Riemann–Roch property

graph has 2 equivalence classes of degree two. It easy to check that the distribution $(2, 0, 0, 0)$ is non-terminating, while the distribution $(1, 1, 0, 0)$ is terminating. So for $K = (4, 0, 0, 0)$ the conditions of Theorem 6.4 hold.

8.3 A non-Eulerian digraph with natural Riemann–Roch property

We have seen in Section 7 that an Eulerian digraph has the natural Riemann–Roch property if and only if it is bidirected. Here we show that there exist also non-Eulerian digraphs with the natural Riemann–Roch property.

Let G_3 be the following digraph (see also Figure 3).

$$
\begin{aligned}
V(G_3) &= \{v_1, v_2, v_3, v_4\}; \\
E(G_3) &= \{\overrightarrow{v_1v_2}, \overrightarrow{v_1v_4}, \overrightarrow{v_2v_1}, \overrightarrow{v_3v_2}, \overrightarrow{v_3v_2}, \overrightarrow{v_4v_3}, \overrightarrow{v_4v_3}, \overrightarrow{v_4v_1}\}.
\end{aligned}
$$

Figure 3: G_3, a non-Eulerian digraph with the natural Riemann–Roch property

We claim that in this digraph, there is only one minimally non-terminating equivalence class, which is the equivalence class of $(1, 0, 0, 3)$.

First, we show that a minimally non-terminating distribution needs to have degree 4. Note that G_3 is strongly connected. By Proposition 2.7, in a non-terminating game on a strongly connected digraph, each vertex is fired infinitely often. Hence in a non-terminating game on G_3, there must be a time when each vertex has already fired, and v_4 can be fired at the moment. Thus, each non-terminating equivalence class on G_3 contains an element with at least 0 chips on each vertex, and at least 3 chips on v_4. Hence a non-terminating chip-distribution needs to have at least 3 chips. Moreover,

a non-terminating degree-3 equivalence class could only be the class of $(0, 0, 0, 3)$, but it is easy to check that this distribution is terminating.

In a non-terminating equivalence class of degree 4, there is also an element x with $x \geq (0, 0, 0, 3)$. We have four choices for this: $(1, 0, 0, 3), (0, 1, 0, 3), (0, 0, 1, 3)$ and $(0, 0, 0, 4)$. Among these, $(1, 0, 0, 3) \sim (0, 1, 0, 3) \sim (0, 0, 0, 4)$ and these are non-terminating, and $(0, 0, 1, 3)$ is terminating.

In the case if the degree of a chip-distribution x is 5, and $x \geq (0, 0, 0, 3)$: If $x - (0, 0, 0, 3)$ has at least one chip on v_1 or on v_2 or on v_4, then $x \geq (1, 0, 0, 3)$ or $x \geq (0, 1, 0, 3)$ or $x \geq (0, 0, 0, 4)$, hence it cannot be minimally non-terminating. The only remaining chip-distribution x with $\deg(x) = 5$ and $x \geq (0, 0, 0, 3)$ is $(0, 0, 2, 3)$. However, it is easy to check that $(0, 0, 2, 2)$ is non-terminating, hence $(0, 0, 2, 3)$ is also not minimally non-terminating.

We have $|E(G_3)| - |V(G_3)| + 1 = 5$, hence by [5], each chip-distribution on G_3 with degree at least 5 is non-terminating. Hence a chip-distribution with degree at least 6 cannot be minimally non-terminating.

Since we only have one minimally non-terminating equivalence class, the conditions of Theorem 6.4 trivially hold with $K = (2, 0, 0, 6) \sim (2, 1, 2, 3) = \mathbf{d}_{\mathbf{G_3}}^{+}$. Thus, G_3 has the natural Riemann–Roch property.

Acknowledgement

References

[1] O. Amini and M. Manjunath. Riemann-Roch for sub-lattices of the root lattice A_n. *Electron. J. Combin.*, 17(1):R124, 2010.

[2] A. Asadi and S. Backman. Chip-firing and Riemann-Roch theory for directed graphs. *Preprint*, http://arxiv.org/abs/1012.0287, 2011.

[3] S. Backman. Riemann–Roch theory for graph orientations. *Adv. Math.*, 309:655–691, 2017.

[4] M. Baker and S. Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[5] A. Björner and L. Lovász. Chip-firing games on directed graphs. *J. Algebraic Combin.*, 1(4):305–328, 1992.

[6] A. Björner, L. Lovász, and P. W. Shor. Chip-firing games on graphs. *European J. Combin.*, 12(4):283–291, 1991.

[7] B. Bond and L. Levine. Abelian networks I. Foundations and examples. *SIAM J. Discrete Math.*, 30(2):856–874., 2016.

[8] R. Cori and Y. le Borgne. On computation of Baker and Norine's rank on complete graphs. *Electron. J. Combin.*, 23(1):P1.31, 2016.

[9] M. Farrell and L. Levine. Coeulerian graphs. *Proc. Amer. Math. Soc.*, 144:2847–2860, 2016.

[10] T. Gallai. On directed paths and circuits. In *Theory of Graphs (Proc. Coll. held at Tihany, Hungary, September, 1966)*, pages 115–118. Akadémiai Kiadó, Budapest, 1968.

[11] A. Gathmann and M. Kerber. A Riemann-Roch theorem in tropical geometry. *Math. Zeitschrift*, 259:217–230, 2008.

[12] V. Kiss and L. Tóthmérész. Chip-firing games on Eulerian digraphs and **NP**-hardness of computing the rank of a divisor on a graph. *Discrete Appl. Math.*, 193:48–56, 2015.

[13] D. Perkinson, J. Perlman, and J. Wilmes. Primer for the algebraic geometry of sandpiles. *Contemp. Math.*, 605, 2014.

[14] K. Perrot and T. Van Pham. Feedback Arc Set Problem and NP-Hardness of Minimum Recurrent Configuration Problem of Chip-Firing Game on Directed Graphs. *Ann. Comb.*, 19(1):1–24, 2015.

[15] J. S. Wilmes. Algebraic invariants of sandpile graphs. Bachelor's thesis, The Division of Mathematics and Natural Sciences, Reed College, 2010.