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Hardness results for stable exchange problems

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Abstract

In this paper we study variants of the stable exchange problem which can be viewed as a model for kidney exchange. The b-way stable l-way exchange problem is a generalization of the stable roommates problem. For $b = l = 3$ Biró and McDermid proved that the problem is NP-complete and asked whether a polynomial time algorithm exists for $b = 2,\ l = 3$. We prove that the problem is NP-complete and it is W[1]-hard with the number of 3-cycles in the exchange as a parameter. We answer a question of Biró by proving that it is NP-hard to maximize the number of covered nodes in a stable exchange. We also prove some related results.

1 Introduction

Given a simple digraph $D = (V, A)$, a set of disjoint directed cycles is called an **exchange**. In an instance of a stable exchange problem, every $v \in V$ has a strictly ordered preference list containing the nodes to which there is an arc from v. We say that u gets v in the exchange if uv is an arc of one of the directed cycles in the exchange. We say that $v \in V$ is **covered** by the exchange E if v belongs to a cycle in E. An exchange is called **stable** if there is no directed cycle C such that for each arc $e = uv$ of C, u is not covered by the exchange or u prefers v over what he got in the exchange. An exchange is called **strongly stable** if there is no directed cycle C not in the exchange such that for each arc $e = uv$ of C, u is not covered by the exchange or e is in the exchange or u prefers v over what he got in the exchange. In both cases the node set of a violating cycle C is called a **blocking coalition**.

An important motivation of this model is kidney exchange. (This was first described in [14].) Currently the best known treatment for kidney failure is transplantation. Since there are a large number of people on the deceased donor waiting list, the more efficient solution is living donation. However, a kidney of a willing living donor is often not suitable for the patient for immunological reasons. Therefore incompatible patient-donor pairs might want to exchange kidneys with other pairs in the same situation. Kidney exchanges have been organized in several countries, for an overview of the different approaches see [14, 4]. In the model described above, the nodes of the

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digraph correspond to the incompatible patient-donor pairs and $uv \in A$ if and only if the kidney of the donor corresponding to v is suitable for the patient corresponding to u. Each patient has a strict preference order over the kidneys suitable for him. In an exchange the patient-donor pairs exchange kidneys backwards along the cycles.

Shapley and Scarf [16] showed that the stable exchange problem (SE) is always solvable, and a stable exchange can be found by the Top Trading Cycles (TTC) algorithm proposed by Gale.

In case of kidney exchanges the cycles in the exchange should be short, since all operations along a cycle have to be carried out at the same time (to avoid someone backing out). If all the cycles in the exchange have length at most l, we call it an **l-way exchange**. An exchange is called **b-way stable** if there is no blocking coalition of size at most b. The definition is analogous for strong stability. Biró and McDermid [1] proved that the decision problem of finding a 3-way stable 3-way exchange is NP-complete, and asked whether a polynomial time algorithm exists for the decision problem of finding a 2-way stable 3-way exchange. In section 2 we prove that the problem is NP-complete and it is W[1]-hard with the number of 3-cycles in the exchange as a parameter, even in complete digraphs. We also prove that the decision problem of finding a b-way strongly stable l-way exchange is NP-complete for any $b \geq 2,\ l \geq 3$ and the same result holds for b-way stable l-way exchanges and stable l-way exchanges.

An instance might admit more than one stable exchanges, therefore it is a natural goal to maximize the number of covered nodes in the exchange. The complexity of this problem was mentioned as an open problem in [2] as well as the same question for 2-way stable exchanges. An exchange is called **complete** if it covers every node. In section 3 we show that deciding if an instance admits a complete stable exchange is NP-complete and the same holds for b-way stable exchanges for any $b \geq 2$. Roth and Postlewaite [15] proved that the exchange found by the TTC algorithm is strongly stable and it is the only strongly stable solution. However, there might be more than one b-way strongly stable exchanges. We prove that deciding if an instance admits a complete b-way strongly stable exchange is NP-complete for any $b \geq 2$. We show that if the digraph is symmetric, then TTC is a $\frac{1}{2}$-approximation algorithm, while the stable partition algorithm is a $\frac{2}{3}$-approximation algorithm for maximizing the number of covered nodes in a 2-way (strongly) stable exchange. All the NP-hardness reductions are from the k-clique in k-partite graph problem, which is specified as follows:
Instance: An integer k, and a k-partite graph $G = (V_1 \cup V_2 \cup ... \cup V_k, E)$.
Question: Is there a clique of size k in G?
The NP-completeness of this problem was implicitly proved in [9], and the W[1]-completeness of the problem was proved in [7].

1.1 Related work

An instance of the stable marriage problem (SM) consists of n men and n women. Each person has a strictly ordered preference list containing all members of the opposite sex. The problem is to find a matching which is stable in a sense that there is no blocking pair, i.e. a man and a woman who prefer each other over their partners in the matching. The Gale-Shapley algorithm [8] always finds a stable matching in an instance of SM.

In the stable roommates problem (SR) there are $2n$ persons, each of whom ranks all the others in strict order of preference. The goal is to find a complete stable matching. Gale and Shapley [8] gave an instance of SR for which no stable matching is possible. Irving [10] proposed an $O(n^2)$ time algorithm which finds a complete stable matching if there is one, or reports that none exists.

The stable roommates with incomplete lists problem (SRI) is a generalization of SR, where each person's preference list only contains his acceptable partners. The problem can be represented by a graph, where there is an edge between two persons if and only if they are acceptable to each other. Here the number of people is not necessarily even, and the stable matching does not need to be complete. However, the same persons are matched in every stable matching and Irving's algorithm can be extended to SRI [11].

The stable exchange problem and the definition of b-way stable l-way exchanges have already been described above. We may assume that if $uv \in A$ in an instance of the 2-way stable 2-way exchange problem, then $vu \in A$, since otherwise uv does not belong to any 2-cycle or blocking coalition. A digraph satisfying this property is called a **symmetric digraph**. We call two arcs in opposite directions between the same two nodes a **bidirected edge**. The 2-way stable 2-way exchange problem is equivalent to SRI hence solvable in polynomial time. (We can replace the bidirected edges with edges and vice versa.) Irving [12] proved that it is NP-complete to decide if an instance admits a stable 2-way exchange, and the same holds for 3-way stable 2-way exchanges.

Cechlárová et al. [5] defined another version of the stable exchange problem, where ties are allowed in the preference lists and a node prefers an exchange E to another exchange E' if he prefers what he got in E over what he got in E' or if he is indifferent between them, but he belongs to a shorter cycle in E than in E'. The notion of stable exchange can be defined analogously. Cechlárová and Lacko [6] proved that it is NP-complete to decide if an instance admits a complete stable exchange in this sense. It was proved in [3] that the problem of finding a stable exchange that covers the maximum number of nodes in this model is not approximable within $n^{1-\epsilon}$ for any $\epsilon > 0$ unless P = NP.

Another generalization of SM is the so-called 3-dimensional stable marriage problem (3DSM). Here there are three sets: men, women and dogs. The sets have cardinality n. Each man has a strict preference order over all the woman-dog pairs. The preference lists of the women and dogs are defined analogously. A matching is a set of n disjoint families, that is triples of the form (man, woman, dog). A matching is stable if there is no blocking family, i.e. a family such that all of its members prefer this family over their current family in the matching. Ng and Hirschberg [13] proved that the problem of deciding whether a stable matching exists is NP-complete. They mentioned the cyclic 3DSM as an open problem, where men only care about women, woman only care about dogs and dogs only care about men. In case of strong stability the cyclic 3DSM problem is NP-complete [1]. If the preference lists may be incomplete we refer to the problem as cyclic 3DSMI. Here the cardinality of the sets are not necessarily equal, and the matching does not need to cover everyone. Biró and McDermid [1] showed that it is NP-complete to decide if an instance of cyclic 3DSMI admits a stable matching. Cyclic 3DSMI is equivalent to the 3-way stable 3-way exchange problem in tripartite graphs, therefore the NP-completeness result applies to this problem as well.

2 Stable exchanges with restrictions

First we state a few straightforward observations that can be found in [2].

- If an exchange is strongly stable, then it is also stable.
- An l-way exchange is also an $(l+1)$-way exchange.
- A b-way (strongly) stable (l-way) exchange is also a $(b-1)$-way (strongly) stable (l-way) exchange.

We will prove that the b-way stable l-way exchange problem is NP-complete for any $b \geq 2$, $l \geq 3$. (For $b = l = 3$, this was proved in [1].) For sake of simplicity first we prove the special case where $b = 2$ and $l = 3$.

Theorem 1. *The decision problem of finding a 2-way stable 3-way exchange is NP-complete.*

Proof. We reduce from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable 3-way exchange problem. By adding isolated nodes we may assume that $|X_i| = n_i$ is odd and $n_i \geq 5$ for $i = 1, ..., k$.

First we define an undirected graph, which then we transform into a digraph by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a circuit $U_i = (u_{i,1}, u_{i,2}, ..., u_{i,n_i})$, and for each $u_{i,j}$ we add a new edge $u_{i,j}v_{i,j}$. Let $(v_{i,j}, w_{i,j}, z_{i,j})$ be a circuit with two new nodes, $w_{i,j}$ and $z_{i,j}$. For every $x \in X_i$ there is a distinct corresponding node in $V_i = \{v_{i,1}, ..., v_{i,n_i}\}$. For $x \in X_i$, $y \in X_j$, $i \neq j$, there is an edge between the corresponding nodes if and only if $xy \notin E$.

The preference lists are shown in the following table. Let the neighbours of $v_{i,j}$ in $\cup_{l \neq i} V_l$ be denoted by $N_{i,j}$. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list			
$u_{i,j},\ i \in [k],\ j \in [n_i]$	$u_{i,j+1}$	$v_{i,j}$	$u_{i,j-1}$	
$v_{i,j},\ i \in [k],\ j \in [n_i]$	$w_{i,j}$	$N_{i,j}$	$u_{i,j}$	$z_{i,j}$
$w_{i,j},\ i \in [k],\ j \in [n_i]$	$z_{i,j}$	$v_{i,j}$		
$z_{i,j},\ i \in [k],\ j \in [n_i]$	$v_{i,j}$	$w_{i,j}$		

See Figure 1. We will prove that the constructed instance admits a 2-way stable

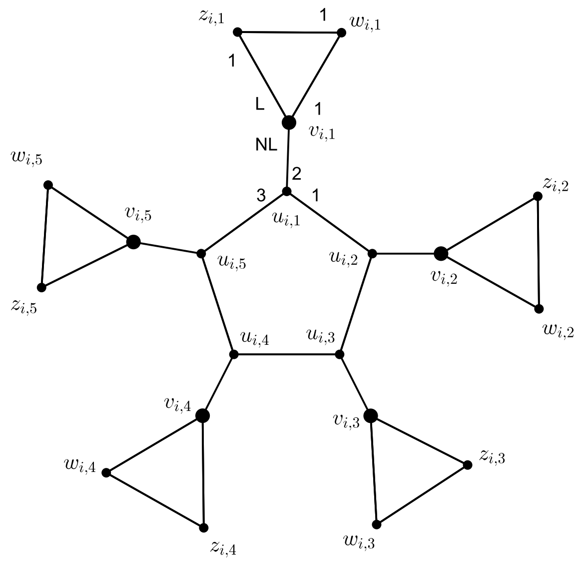

Figure 1: All the edges of the graph are bidirected edges. L:=last, NL:=next to last.

3-way exchange if and only if there is a k-clique in G. First suppose that the constructed instance admits a 2-way stable 3-way exchange.

Lemma 1. *For every i, if the arcs $u_{i,j-1}u_{i,j}$ and $u_{i,j}u_{i,j+1}$ are not in the exchange, then $u_{i,j}v_{i,j}$ must be in the exchange for every $j = 1, ..., n_i$ (subscripts modulo n_i).*

Proof. Suppose that $u_{i,j-1}u_{i,j}$ and $u_{i,j}u_{i,j+1}$ are not in the exchange. Notice that $u_{i,j}u_{i,j-1}$ cannot be in the exchange either, because the arcs of U_i are not in any cycle of length 3. $u_{i,j}$ is first on $u_{i,j-1}$'s preference list and $u_{i,j}$ does not get the first item on his list, therefore if $u_{i,j}v_{i,j}$ would not be in the exchange, that is $u_{i,j}$ would not get the second item on his list either, then $\{u_{i,j-1}, u_{i,j}\}$ would be a blocking pair. □

$|U_i| = n_i$ is odd, and it is not possible to have two consecutive arcs of U_i in the exchange (because the arcs of U_i are not in any cycle of length 3). Therefore it follows from the above lemma, that for every i there is a node u_{i,j_i} in U_i such that $u_{i,j_i}v_{i,j_i}$ is in the exchange. Since this arc is not in any cycle of length 3, $v_{i,j_i}u_{i,j_i}$ has to be in the exchange as well.

Now we prove that the nodes corresponding to v_{i,j_i} for $i = 1, ..., k$ form a k-clique. Suppose there are two which are not connected. Then there is a bidirected edge between the nodes corresponding to them. But then these two nodes form a blocking pair, because they prefer each other to their current partner.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, v_{2,1}, ..., v_{k,1}$.

Lemma 2. *The 3-way exchange defined by the 2-cycles* (u_{i,j_i}, u_{i,j_i+1}), $j_i = 2, 4, ..., n_i - 1$, $(u_{i,1}, v_{i,1})$, $(w_{i,1}, z_{i,1})$ *and the 3-cycles:* $(v_{i,j_i}, w_{i,j_i}, z_{i,j_i})$, $j_i = 2, 3, .., n_i$ *is stable (thus 2-way stable).*

Proof. In the 3-cycles everyone gets his first choice, so none of them can belong to a blocking coalition. The same applies to the nodes $u_{i,2}, u_{i,4}, ..., u_{i,n_i-1}$ and $w_{i,1}$. $u_{i,1}$ gets the second item on his list, therefore he would get his first item $u_{i,2}$ in a blocking coalition, but $u_{i,2}$ cannot belong to a blocking coalition. $u_{i,3}, u_{i,5}, ..., u_{i,n_i}$ get their third choice, so in a blocking coalition they would get the first or second item on their list, but we have seen that these nodes cannot belong to a blocking coalition, therefore they cannot either. $z_{i,1}$ gets his second choice, therefore he would get his first choice $v_{i,1}$ in a blocking coalition. However $\{z_{i,1}, v_{i,1}\}$ is not a blocking pair, and in a bigger coalition only $w_{i,1}$ could get $z_{i,1}$, but he does not belong to a blocking coalition. Therefore the nodes that could belong to a blocking coalition are only $v_{i,1}$ for $i = 1, ..., k$, but these correspond to the nodes of a k-clique in G, thus they are not even connected. □

□

We used in the proof that the exchange we are looking for only contains cycles of length at most three when we showed that certain arcs cannot belong to a cycle of length at least three. If we wish to extend this proof to l-way exchanges, where $l > 3$, we need to modify the construction.

Theorem 2. *The decision problem of finding a stable l-way exchange is NP-complete for any $l \geq 3$. The same holds for b-way stable l-way exchanges for any $b \geq 2$.*

Proof. For the first part of the theorem, we must show that the problem is in NP. Suppose we are given an l-way exchange. To check that it is stable, first we delete every arc uv such that u does not prefer v over what he got in the exchange. These arcs cannot belong to a blocking coalition. (Note that we deleted all the arcs of the exchange). The exchange was stable if and only if the remaining digraph does not contain a directed cycle.

For the NP-hardness proof, we modify the construction of the previous theorem. Let $t = l - 2$ if l is odd, $t = l - 1$ if l is even. We replace the bidirected edges $u_{i,j}u_{i,j+1}$ for $i = 1, ..., k,\ j = 1, ..., n_i$ subscripts modulo n_i with a bidirected $u_{i,j}u_{i,j+1}$ path of length t. The new nodes prefer the node succeeding them in the cycle over the one preceding them.

Lemma 3. *If there are two consecutive arcs of the cycle* $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ *which are not in the exchange, then there is a subscript j for which* $(u_{i,j}, v_{i,j})$ *is a 2-cycle in the exchange.*

Proof. The proof is very similar to the proof of Lemma 1. We suppose there are two consecutive arcs not in the exchange. The endpoints of the first arc form a blocking pair, unless the common endpoint of the arcs gets his second choice. Since the length of the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ is odd and the nodes of the cycle are not in any cycle of length at most l and at least 3, there must be two consecutive arcs of the cycle not in the exchange, hence $u_{i,j}v_{i,j}$ is in the exchange for some j. This arc does not belong to a cycle of length at most l and at least 3, therefore $(u_{i,j}, v_{i,j})$ is a 2-cycle in the exchange. □

From the above lemma, there is a node v_{i,j_i} for every i, who gets his next to last item in the exchange. The nodes corresponding to these form a k-clique in G, from the same proof as in Theorem 1.

Now we prove that if there is a k-clique in G, the instance of SE admits a stable 3-way (and thus l-way) exchange. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, v_{2,1}, ..., v_{k,1}$. We modify the 3-way exchange defined in Lemma 2, such that in the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ every second arc from $u_{i,2}$ should belong to a 2-cycle. This 3-way exchange is still stable. (The proof is very similar to the proof of Lemma 2.) □

Remark 1. This construction does not work for b-way strongly stable l-way exchanges, because the cycles $(v_{i,1}, w_{i,1}, z_{i,1})$ are blocking in the strongly stable sense.

Theorem 3. *The decision problem of finding a b-way strongly stable l-way exchange is NP-complete for any $b \geq 2$, $l \geq 3$.*

Proof. Given an instance of the k-clique in k-partite graph problem, we create an instance of the b-way strongly stable l-way exchange problem by modifying the construction of Theorem 1. First we make the modifications we made in the proof of Theorem 2 but with t being the smallest odd number at least $\max\{b-2, l-2\}$. Let $b' = b-1$ if b is odd, and $b' = b$ if b is even. We add new bidirected paths of length $b'+1$: $w_{i,j}y^1_{i,j}...,y^{b'}_{i,j}z_{i,j}$ for $i = 1,...,k,\ j = 1,...,n_i$. The new nodes prefer the node succeeding them over the one preceding them. We modify $w_{i,j}$ and $z_{i,j}$'s preference list:

$$w_{i,j}:\ y^1_{i,j},\ z_{i,j},\ v_{i,j}$$

$$z_{i,j}:\ v_{i,j},\ y^{b'}_{i,j},\ w_{i,j}$$

We prove that this instance of SE admits a b-way strongly stable l-way exchange if and only if there is a k-clique in G. Suppose the instance admits a b-way strongly stable l-way exchange. Then it is also a b-way stable l-way exchange, and Lemma 3 clearly holds in this case too. Just like before, it follows from the lemma that there is a k-clique in G.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, v_{2,1}, ..., v_{k,1}$. We prove that the 3-way (and thus l-way) exchange defined by the following 2-cycles and 3-cycles is b-way strongly stable.

- $(u_{i,1}, v_{i,1})$
- the 2-cycles defined by every second arc of the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ starting from $u_{i,2}$
- $(w_{i,1}, y^1_{i,1}),\ (y^2_{i,1}, y^3_{i,1})\ (y^4_{i,1}, y^5_{i,1}), ..., (y^{b'-2}_{i,1}, y^{b'-1}_{i,1}),\ (y^{b'}_{i,1}, z_{i,1})$

- $(y_{i,j}^1, y_{i,j}^2), (y_{i,j}^3, y_{i,j}^4), ..., (y_{i,j}^{b'-1}, y_{i,j}^{b'})$
- $(v_{i,j}, w_{i,j}, z_{i,j})$ $i = 1, ..., k,\ j = 2, ..., n_i$

Those nodes of the new paths added to the construction who got their first choice in the exchange (namely $y_{i,j}^m$ for $i = 1, ..., k,\ j = 2, ..., n_i, m = 1, 3, ..., b' - 1$ and $y_{i,1}^m$ for $i = 1, ..., k,\ m = 2, 4, ..., b'$), cannot belong to a blocking coalition of size at most b, because they are in a 2-cycle with their first choice in the exchange and they do not belong to any longer cycle of length at most b. For $j \geq 2,\ z_{i,j}$ gets his first choice, $v_{i,j}$, therefore he must get it in every blocking coalition he might belong to. $(z_{i,j}, v_{i,j})$ and $(z_{i,j}, v_{i,j}, w_{i,j})$ are the only cycles of length at most b which contain the arc $z_{i,j}v_{i,j}$, but these are not blocking, therefore $z_{i,j}$ cannot belong to a blocking coalition of size at most b. Those nodes of the new paths added to the construction who got their second choice in the exchange (namely $y_{i,j}^m$ for $i = 1, ..., k,\ j = 2, ..., n_i, m = 2, 4, ..., b'$ and $y_{i,1}^m$ for $i = 1, ..., k,\ m = 1, 3, ..., b' - 1$), cannot belong to a blocking coalition of size at most b. This is because they are in a 2-cycle with their second choice in the exchange, we have seen that their first choice cannot belong to a blocking coalition, and they do not belong to any longer cycle of length at most b. For $j \geq 2,\ w_{i,j}$ gets his second choice. He cannot belong to a blocking coalition of size at most b, since his first choice $y_{i,j}^1$, and his second choice $z_{i,j}$ cannot either. For $j \geq 2,\ v_{i,j}$ gets his first choice, $w_{i,j}$ in the exchange, who cannot belong to a blocking coalition of size at most b, therefore $v_{i,j}$ cannot either. Now we show that the nodes of the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ cannot belong to a blocking coalition of size at most b. These nodes are in 2-cycles in the exchange, and they do not belong to any longer cycle of length at most b. It follows from this, that the nodes of the cycle that get their first choice cannot belong to a blocking coalition. The ones who get their second choice cannot belong to a blocking coalition either, because they would get their first choice in it, but we have just seen that these nodes cannot belong to a blocking coalition. $w_{i,1}$ is in a 2-cycle with his first choice, and this arc does not belong to any longer cycle of length at most b, which means that $w_{i,1}$ cannot belong to a blocking coalition of size at most b. This also holds for $z_{i,1}$, because he is in a 2-cycle with his second choice thus he would get his first choice $v_{i,1}$ in any blocking coalition he might be in, but the only cycles of length at most b which contain this arc are $(z_{i,1}, v_{i,1})$ and $(z_{i,1}, v_{i,1}, w_{i,1})$, and these are not blocking. We have shown for every node that it does not belong to a blocking coalition of size at most b except for $v_{i,1},\ i = 1, ..., k$. But these nodes correspond to the nodes of a k-clique thus they are not connected, therefore they cannot belong to a blocking coalition either. □

Now we return to 2-way stable 3-way exchanges, and prove that the problem remains NP-complete even in complete digraphs. If we added the missing arcs to the end of the preference lists in our construction in Theorem 1, this would not work for proving this, because we used that certain arcs do not belong to a cycle of length 3, which would not be true in the modified version. Therefore we need a new construction.

Theorem 4. *The decision problem of finding a 2-way stable 3-way exchange in a complete digraph is NP-complete.*

Proof. The reduction is from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable 3-way exchange problem. By adding isolated nodes we may assume that $|X_i| = n_i$ is odd for $i = 1, ..., k$, and $n_i \geq 5$. For every i, we create a set of nodes $C_i = \{v_{i,1}, v_{i,2}, ..., v_{i,n_i}\}$. The digraph of the 2-way stable 3-way exchange problem is the complete digraph defined on these nodes. For every node in X_i, there is a distinct corresponding node in C_i. For $v_{i,j}$ we denote the set of nodes in $\cup_{t \neq i} C_t$ for which the corresponding node in $\cup_{t \neq i} X_t$ and the node corresponding to $v_{i,j}$ are not connected with $N_{i,j}$. Let $T_{i,j} = (\cup_{t \neq i} C_t) - N_{i,j}$. (These are the nodes for which the corresponding node in G is connected with the node corresponding to $v_{i,j}$ in G.)

Now we describe the preference lists. A set in the preference list means the nodes of the set in arbitrary order. For $i \in [k]$, $j \in [n_i]$ the preference list of $v_{i,j}$ is shown in the following table. (subscripts modulo n_i).

node					preference list
$v_{i,j}$	$v_{i,j+1}$	$v_{i,j-1}$	$N_{i,j}$	$C_i - v_{i,j}$	$T_{i,j}$

We prove that there is a k-clique in G if and only if the constructed instance admits a 2-way stable 3-way exchange.

First suppose that there is a k-clique in G. Without loss of generality we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,3}, v_{2,3}, ..., v_{k,3}$.

Lemma 4. *The 3-way exchange defined by the directed 3-cycles* $(v_{i,1}, v_{i,2}, v_{i,3})$ *and 2-cycles* $(v_{i,j}, v_{i,j+1})$ *for* $i = 1, 2, ..., k$; $j = 4, 6, ..., n_i - 1$ *is 2-way stable.*

Proof. The nodes $v_{i,j}$ for $j = 1, 2, 4, 6, ..., n_i - 1$ get the first item on their preference list, therefore they cannot belong to a blocking pair. For $5 \leq 2t + 1 \leq n_i$, $v_{i,2t+1}$ cannot belong to a blocking pair either, because he gets the second item on his list, so the only way he could belong to a blocking pair is if $\{v_{i,2t+1}, v_{i,2t+2}\}$ would be blocking, but v_{2t+2} cannot belong to a blocking pair. Finally a pair $\{v_{i,3}, v_{j,3}\}$ cannot be blocking because $v_{i,3}$ and $v_{j,3}$ correspond to two nodes of a k-clique in G so they prefer what they got in the exchange over each other. □

Now suppose that the constructed instance admits a 2-way stable 3-way exchange.

Lemma 5. *Suppose we have an odd bidirected cycle* $C = (c_1, c_2, ..., c_t)$ *in the digraph, such that each node of the cycle has the successive node in the cycle as his first choice and the preceding node as his second choice. If there is a 2-way stable exchange, then either there are two consecutive arcs of* C *in that exchange, or the whole backwards cycle is in the exchange.*

Proof. Suppose that there are two consecutive arcs of C: $c_i c_{i+1}$ and $c_{i+1} c_{i+2}$ so that neither of them is in the exchange. If $c_{i+1} c_i$ is not in the exchange, then $\{c_i, c_{i+1}\}$ is a blocking pair, because c_{i+1} is first on c_i's preference list, c_i is second on c_{i+1}'s preference list, and $c_{i+1} c_{i+2}$ and $c_{i+1} c_i$ are not in the exchange, which means that c_{i+1} cannot get the first or the second item on his list in the exchange. If $c_{i+1} c_i$ is in the exchange, then $c_{i-1} c_i$ cannot be in the exchange, because there is only one arc entering c_i in the exchange. Therefore if c_ic_{i-1} is not in the exchange, then it follows from the same argument, that $\{c_{i-1}, c_i\}$ is a blocking pair, so c_ic_{i-1} has to belong to the exchange. Repeating the same argument yields that the whole backwards cycle has to be in the exchange.

If there are no two consecutive arcs of C such that neither of them is in the exchange, there have to be two consecutive arcs of C which are in the exchange, since C is odd. $\square$

The cycles $C_i = (v_{i,1}, v_{i,2}, ..., v_{i,n_i})$ meet the conditions of the above lemma for every $i \in [k]$, $j \in [n_i]$, and since $n_i \geq 5$, the backwards cycles cannot belong to the 3-way exchange. Therefore there exist two consecutive arcs $v_{i,j_i}v_{i,j_i+1}$ and $v_{i,j_i+1}v_{i,j_i+2}$ in each C_i, which are in the exchange. Thus the cycles $(v_{i,j_i}, v_{i,j_i+1}, v_{i,j_i+2})$ are in the exchange.

Now we prove that the nodes corresponding to v_{i,j_i+2} for $i = 1, ..., k$ form a k-clique. Suppose there are two, which are not connected in G. Then there exist i, l such that $v_{i,j_i+2} \in N_{l,j_l+2}$ and $v_{l,j_l+2} \in N_{i,j_i+2}$. For $t = i, l$, v_{t,j_t+2} gets v_{t,j_t} in the exchange over which he prefers the nodes of N_{t,j_t+2}, therefore v_{i,j_i+2} and v_{l,j_l+2} prefer each other, which means that $\{v_{i,j_i+2}, v_{l,j_l+2}\}$ is a blocking pair, thus we have reached a contradiction. $\square$

Remark 2. It follows from the same proof, that the decision problem of finding a (b-way) stable 3-way exchange in a complete digraph is NP-complete for any $b \geq 2$.

Remark 3. It can be proved in a similar way that the decision problem of finding a (b-way) stable 4-way exchange in a complete digraph is NP-complete for any $b \geq 2$. The same proof does not work for l-way exchanges if $l \geq 5$ (because we cannot guarantee that the cycle containing the two consecutive arcs from C_i does not contain nodes from other C_j's).

Theorem 5. *The 2-way stable 3-way exchange problem with the number of 3-cycles in the exchange as a parameter is $W[1]$-hard even in complete digraphs.*

Proof. The reduction described in the NP-hardness proof the 2-way stable 3-way exchange problem (Theorem 4) is a parameterized reduction from the W[1]-complete k-clique in k-partite graph problem since the number of 3-cycles in the constructed 2-way stable 3-way exchange is k. $\square$

Remark 4. We can decide in polynomial time whether an instance admits a 2-way stable 3-way exchange, where the number of 3-cycles in the exchange is at most a given constant. (We can reduce the problem of deciding whether an exchange with given 3-cycles can be extended to a 2-way stable 3-way exchange to SRI).

3 Maximizing the number of covered nodes

3.1 NP-hardness

An instance might admit more than one stable exchanges, therefore it is a natural goal to maximize the number of covered nodes in the exchange. It follows from the theorem below that this problem is NP-hard.

Theorem 6. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete stable exchange.*

Proof. We reduce from the k-clique in k-partite graph problem. For every k-partite graph $G = (V_1, ..., V_k, E)$, we define an instance of the stable exchange problem. First we describe an undirected graph, which we then transform into a digraph, by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a node w_i, and n_i circuits: $(w_i, x_{i,j}, y_{i,j}, z_{i,j})$ $j = 1, ..., n_i$. Let $x_{i,j}z_{i,j}$ be an edge for every $i \in [k]$, $j \in [n_i]$. Let $X_i = \{x_{i,1}, ..., x_{i,n_i}\}$. For every $v \in V_i$ there is a distinct corresponding node in X_i. For $v \in V_i, v' \in V_j$ there is an edge between the corresponding nodes if and only if $vv' \notin E$. Let the neighbours of $x_{i,j}$ in $\cup_{l \neq i} X_l$ be denoted by $N_{i,j}$. The preference lists are shown in the following table. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list			
$z_{i,j},\ i \in [k],\ j \in [n_i]$	$y_{i,j}$	w_i		
$y_{i,j},\ i \in [k],\ j \in [n_i]$	$x_{i,j}$	$z_{i,j}$		
$x_{i,j},\ i \in [k],\ j \in [n_i]$	$z_{i,j}$	$N_{i,j}$	w_i	$y_{i,j}$
$w_i,\ i \in [k]$	arbitrary	X_i		

See Figure 2.

Lemma 6. *In every complete exchange there is a node in every X_i who gets one of the last two items on his list.*

Proof. If there is a complete stable exchange, then someone gets w_i in the exchange, for every i. If someone from X_i gets w_i, then this person gets the next to last item on his list. Otherwise someone from Z_i must get w_i, suppose $z_{i,j}$ does. Then $z_{i,j}$ cannot get $y_{i,j}$ and $y_{i,j}$ must be covered, therefore $x_{i,j}$ must get him, which means that $x_{i,j}$ gets the last item on his list. □

Take a node form every X_i who gets his last or next to last choice. These nodes prefer each other to what they get in the exchange, therefore no two of them are connected in the digraph, therefore the corresponding nodes form a k-clique in G.

Suppose there is a k-clique in G. We may assume that the nodes corresponding to the nodes of the k-clique are $x_{1,1}, x_{2,1}, ..., x_{k,1}$. We prove that the complete exchange defined by the following directed cycles is stable:

$$(x_{i,1}, w_i, z_{i,1}, y_{i,1}), \quad (x_{i,j}, z_{i,j}, y_{i,j}), \quad i = 1, ..., k, \quad j = 2, ..., n_i.$$

The nodes of the cycles $(x_{i,j}, z_{i,j}, y_{i,j})$, $i = 1, ..., k$, $j = 2, ..., n_i$ get their first choice thus cannot belong to a blocking coalition. The same holds for $z_{i,1}$ and $y_{i,1}$ for $i = 1, ..., k$. We have seen that all the nodes other then $x_{i,1}$ connected to w_i cannot belong to a blocking coalition, therefore w_i cannot either. This leaves us only with $x_{1,1}, ..., x_{k,1}$. But these correspond to the nodes of a k-clique, meaning that they are not connected in the digraph, thus no subset of them forms a blocking coalition. □

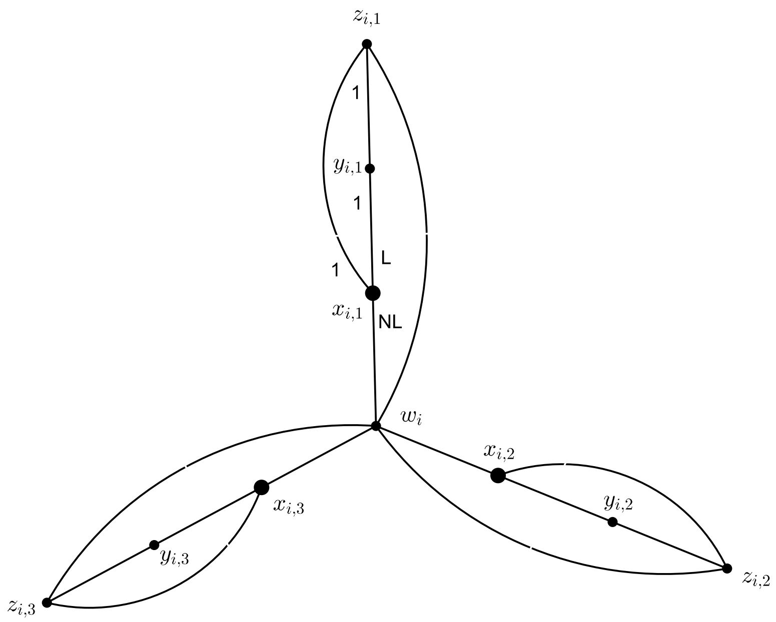

Figure 2: All the edges of the graph are bidirected edges. L:=last, NL:=next to last.

For proving that there is a k-clique in G, we only used that the instance admits a 2-way stable exchange. Therefore the same proof applies to the following theorem.

Theorem 7. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete b-way stable exchange for any $b \geq 2$.*

Roth and Postlewaite [15] proved that the exchange found by the TTC algorithm is the only strongly stable solution. However, there might be more then one b-way stable exchanges.

Theorem 8. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete b-way strongly stable exchange for any $b \geq 2$.*

Proof. We modify our previous construction. Let us replace the bidirected $x_{i,j}z_{i,j}$ path of length 2 with a bidirected path of length t, where $t = b$ if b is even, and $t = b + 1$ otherwise. The new nodes prefer the neighbour which is closer to $x_{i,j}$. Now we show that Lemma 6 remains true for this construction too. If there is a complete stable exchange, then someone gets w_i in the exchange. If it is $x_{i,j}$ for some $j \in [n_i]$, then he gets his next to last choice, so we are done. Otherwise $z_{i,j}$ gets w_i for some $j \in [n_i]$. This $z_{i,j}$ does not get the next node on the path from $z_{i,j}$ to $x_{i,j}$, so its other neighbour has to get it. If this arc is in a cycle of length at least 3, then, since $z_{i,j}$ gets w_i, it could only be in the cycle $(w_i, x_{i,j}, ..., z_{i,j})$. Therefore in that case $x_{i,j}$ gets his last choice. So we only need to cover the case when the first and second nodes after $z_{i,j}$ on the path from $z_{i,j}$ to $x_{i,j}$ are switched in the exchange. In that case, the nodes of the $z_{i,j}x_{i,j}$ path cannot belong to a cycle of length at least 3 and every one of them is covered in the exchange, therefore every second arc of the $z_{i,j}x_{i,j}$ path belongs to a 2-cycle. Since the length of the path is even $x_{i,j}$ is paired with the previous node of the $z_{i,j}x_{i,j}$ path, thus gets his last choice. Just like in Theorem 6, the lemma implies that there is a k-clique in G.

If there is a k-clique in G, without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $x_{1,1}, x_{2,1}, ..., x_{k,1}$. We prove that the complete exchange defined by the following directed cycles is strongly stable:

$$(w_i, z_{i,1}, ..., x_{i,1}), \quad (z_{i,j}, ..., x_{i,j}), \quad i = 1, ..., k, \quad j = 2, ..., n_i.$$

The nodes of the following cycles of length at least b get their first choice thus cannot belong to a blocking coalition: $(z_{i,j}, ..., x_{i,j})$, $i = 1, ..., k$, $j = 2, ..., n_i$. The nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ except for the $x_{i,1}$'s and w_i's get their first choice in the exchange, therefore they get the same item in any blocking coalition which they belong to. The only cycles of length at most b which contain one of these nodes are 2-cycles, but the nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ prefer the item succeeding them to the one preceding them, therefore these pairs are not blocking. This implies that the nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ except for the $x_{i,1}$'s and w_i's cannot belong to a blocking pair. We have seen that all the nodes other than $x_{i,1}$ connected to w_i cannot belong to a blocking coalition, and $\{w_i, x_{i,1}\}$ is not a blocking pair, therefore w_i cannot belong to a blocking coalition either. This leaves us with only $x_{1,1}, ..., x_{k,1}$. But these correspond to nodes of a k-clique, meaning that they are not connected in the digraph, and thus no subset of them forms a blocking coalition. □

Remark 5. All the NP-completeness theorems in this subsection apply for the special case when the digraph is symmetric.

3.2 Approximation algorithms

We have seen that it is NP-hard to maximize the number of covered nodes in a 2-way (strongly) stable exchange. Now we will check how well the known algorithms for finding a 2-way (strongly) stable exchange approximate the problem. First we do not assume that the digraph is symmetric.

Claim 1. *There does not exist an $\alpha < 1$ for which the Top Trading Cycles algorithm is an α-approximation algorithm for finding a 2-way (strongly) stable exchange which covers the maximum number of nodes.*

Proof. Suppose there is such an α. Take an integer k, such that $\frac{2}{k} < \alpha$. Take two directed cycles of length k: $(u_1, u_2, ..., u_k)$ and $(v_1, v_2, ..., v_k)$ and two additional arcs u_2v_1 and v_2u_1. u_2 prefers v_1 over u_3 and v_2 prefers u_1 over v_3, all the other nodes have only one acceptable node on their lists. TTC takes the cycle (v_1, v_2, u_1, u_2) in the exchange in the first step and then terminates. Therefore it finds a 2-way (strongly) stable exchange which covers 4 nodes. However, the 2-way (strongly) stable exchange defined by the cycles $(u_1, u_2, ..., u_k)$ and $(v_1, v_2, ..., v_k)$ covers all the $2k$ nodes. Since $\alpha > \frac{2}{k}$, $4 < \alpha(2k)$ thus TTC is not an α-approximation. □

However, if we do assume that the digraph is symmetric, the following theorem holds.

Theorem 9. *If the digraph is symmetric, TTC is a $\frac{1}{2}$-approximation algorithm, and the approximation ratio is sharp.*

Proof. Let U be the set of nodes that are not covered in a 2-way (strongly) stable exchange E_{TTC} given by TTC. The nodes of U are independent in the digraph, because if any two were connected the pair would block E_{TTC}. Let E_{OPT} be the optimal 2-way (strongly) stable exchange. Suppose that some nodes from U are covered by E_{OPT}. Since U is an independent set, these nodes are given to some nodes outside of U in E_{OPT} and each of them is given to a different node. Therefore E_{OPT} covers at most twice as many nodes as E_{TTC} which means that TTC is a $\frac{1}{2}$-approximation algorithm.

Now we present a sharp example. Take a bidirected circuit $(u_1, ..., u_k)$ and additional bidirected edges u_iv_i for $i = 1, ..., k$. The first item on u_i's preference list is u_{i+1}, the second is v_i and the third is u_{i-1} for $i = 1, ..., k$ subscripts modulo k. TTC takes the cycle $(u_1, u_2, ..., u_k)$ in the exchange in the first step, and then terminates, so the exchange covers half of the nodes. However, the 2-way (strongly) stable exchange defined by the 2-cycles (u_i, v_i) $i = 1, ..., k$ covers all the nodes. □

We still assume that the digraph is symmetric. In this case, another way of finding a 2-way (strongly) stable exchange is by finding a stable partition. In an instance of SRI, we are given a graph $G = (V, E)$, and each node has strict preferences over its neighbours. Now we are interested in finding a complete stable matching. We have already mentioned is Section 1.1 that a modified version of Irving's algorithm finds a (not necessarily complete) stable matching if there is one, or reports that none exists. The covered nodes are the same in any stable matching, therefore this algorithm can also decide whether an instance of SRI admits a complete stable matching. However, if the instance does not admit a complete stable matching, Irving's algorithm does not provide a simple evidence for why not. Tan gave a necessary and sufficient condition for the existence of a complete stable matching which we will describe later on. He defined a new structure called **stable partition**, and proved that an instance of the stable roommates problem always admits one.

Definition 1. *For $A \subseteq V$ a cyclic permutation $\pi(A) = \langle a_1, a_2, ...a_k \rangle$ of the nodes in A is called a* ***semi-party permutation*** *if*

$|A| = 1$, *or*

$|A| = 2$ *and* $a_1a_2 \in E$, *or*

$|A| \geq 3$ *and* $a_ia_{i+1} \in E$ *and* a_i *prefers* a_{i+1} *over* a_{i-1} *for* $i = 1, ..., k$, *subscripts modulo* k.

Definition 2. *A stable partition π consists of a partition of V, and a specified semi-party permutation for each set in the partition, such that the following stability condition holds:*

Let A and B be two (not necessarily distinct) sets of the partition with specified semi-party permutations $\pi(A) = \langle a_1, ...\rangle$ and $\pi(B) = \langle b_1, ...\rangle$. Let $a_i \in A,\ b_j \in B$. If $|A| = 1$ or a_i prefers b_j over a_{i-1}, then, (unless $A = B$ and $|A| = 1$), $|B| \neq 1$ and b_j prefers b_{j-1} over a_i (if $|B| \geq 3$, $b_{j-1} = a_i$ is also possible).

*A set A of the partition is called a **party** in π, and the associated semi-party permutation $\pi(A)$ is called a **party permutation** for A.*

Definition 3. *A subset A of V is called a **party**, if there exists a stable partition π, such that A is a party in π. A party with odd cardinality is called an **odd party**.*

Another modified version of Irving's algorithm proposed by Tan always finds a stable partition in an instance of SRI. Moreover, Tan proved that the odd parties and the corresponding party permutations are the same in any two stable partitions. This provides a necessary and sufficient condition for the existence of a complete stable matching, namely the nonexistence of odd parties. These results can be found in [17].

Let π be a stable partition in G, with party permutations $\pi(A_i) = \langle a_1, ...a_{k_i}\rangle$. If we replace the edges of G with bidirected edges, the exchange defined by the directed cycles $(a_1, ..., a_{k_i})$ is 2-way stable in the arising digraph. (This is straightforward from the definitions.) Therefore the above mentioned algorithm for finding a stable partition also gives an algorithm for finding a 2-way (strongly) stable exchange, which will be referred to as stable partition algorithm.

Theorem 10. *The stable partition algorithm is a $\frac{2}{3}$-approximation for finding a 2-way (strongly) stable exchange and the approximation ratio is sharp.*

Proof. We denote the optimal 2-way (strongly) stable exchange by E_{OPT}. Let U be the set of nodes that are not covered in the 2-way stable exchange given by the stable partition algorithm. The nodes of U are the parties of size 1 in the stable partition, therefore they are not connected. Suppose the nodes in $U' \subseteq U$ are covered by E_{OPT}. Just like in Theorem 9, the set of nodes X that get a node from U' in E_{OPT} are such that $X \cap U = \emptyset$ and each node from U' is given to a different node from X. (Thus $|X| = |U'|$.) We will show that the set of nodes Y that get a node from X in the stable partition algorithm are such that $Y \cap X = \emptyset$ (clearly $Y \cap U = \emptyset$ and $|X| = |Y|$). Suppose $x' \in Y \cap X$ and x' gets $x \in X$ in the stable partition algorithm. In E_{OPT}, x gets u and x' gets u', $u, u' \in U$. u and u' are parties of size one in the stable partition, therefore x prefers x' over u and x' prefers x over u'. But this means that $\{x, x'\}$ blocks E_{OPT} which is a contradiction. We proved that $Y \cap X = \emptyset$, from which it follows that the optimal solution covers at most $\frac{3}{2}$ times the number of nodes covered by the 2-way (strongly) stable exchange found by the stable partition algorithm.

Now we present a sharp example. The digraph consists of $2k$ distinct bidirected circuits of length three: $(v_i, u_i, w_{i+1}),\ (v_{i+1}, u_{i+1}, w_i),\ i = 1, 3, ..., 2k-1$, and bidirected edges: $v_iw_i,\ i = 1, ..., 2k$. The preference lists are shown in the table below.

node	preference list		
$u_i,\ i \in [2k],\ i$ is odd	w_{i+1}	v_i	
$u_i,\ i \in [2k],\ i$ is even	w_{i-1}	v_i	
$v_i,\ i \in [2k],\ i$ is odd	w_i	u_i	w_{i+1}
$v_i,\ i \in [2k],\ i$ is even	w_i	u_i	w_{i-1}
$w_i,\ i \in [2k],\ i$ is odd	v_{i+1}	v_i	u_{i+1}
$w_i,\ i \in [2k],\ i$ is even	v_{i-1}	w_i	u_{i-1}

The stable partition algorithm finds the 2-way (strongly) stable exchange defined by the 2-cycles (w_i, v_i), $i = 1, ..., 2k$, which covers $4k$ nodes. However, the optimal solution covers all the $6k$ nodes: the exchange defined by the cycles (v_i, u_i, w_{i+1}), (v_{i+1}, u_{i+1}, w_i), $i = 1, 3..., 2k - 1$ is 2-way (strongly) stable. $\square$

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