Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2015-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C,
H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

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July 2015

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Abstract

In this paper, we investigate reachability questions in the chip-firing game. For strongly connected digraphs we show that if a position is recurrent (i.e. reachable from itself), then the question whether it is reachable from another given position can be decided in polynomial time. We also prove a similar result for weakly connected digraphs. We prove some basic properties of recurrent positions on digraphs, and give a characterization of recurrent positions on Eulerian digraphs.

Keywords: chip-firing game; computational complexity

1 Introduction

Chip-firing is a solitary game on a directed graph, defined by Björner, Lovász and Shor [4]. Each vertex contains a pile of chips. A legal move is to choose a vertex with at least as many chips as its outdegree and let it send a chip along each outgoing edge. We analyze the complexity of the following reachability question: given two chip-distributions x and y, decide whether y can be reached from x by playing a legal game. This question was first considered by Björner and Lovász, who gave an algorithm that decides this question and runs in polynomial time for simple Eulerian digraphs [3]. For the general case, however, the complexity of the reachability question remains open.

Our main result is the following: If G is a strongly connected directed graph and the chip-distribution y is recurrent, i.e., it is reachable from itself by a legal game,

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then the set of distributions from which y is reachable can be characterized in a simple way. This characterization enables one to decide in polynomial time whether a recurrent distribution y is reachable by a legal game from a given distribution x. For weakly connected directed graphs, we generalize the characterization theorem for distributions y that are recurrent restricted to each strongly connected component. This theorem also gives rise to a polynomial algorithm.

Recurrent chip-distributions play an important role in the dynamics of non-terminating chip-firing games. Their study originates in the work of Jeffs and Seager [6]. Prior to our paper, investigations were focusing on undirected graphs. In Section 4 we state some basic properties of recurrent chip-distributions on directed graphs and examine the algorithmic decidability of recurrence. For the case of Eulerian directed graphs we also give a combinatorial characterization of recurrent states, which is a generalization of the former results of [6, 5] in the undirected case.

2 Preliminaries

Throughout this paper, *digraph* means a (weakly) connected directed graph that can have multiple edges but no loops. A digraph is usually denoted by G. The vertex set and edge set of a digraph G are denoted by $V(G)$ and $E(G)$ (or simply V and E), respectively. For a vertex v, the indegree and the outdegree of v are denoted by $d^-(v)$ and $d^+(v)$, respectively. We denote a directed edge leading from vertex u to vertex v by $\overrightarrow{uv}$. The multiplicity of a directed edge $\overrightarrow{uv}$ is denoted by $\overrightarrow{d}(u,v)$.

A digraph is *simple*, if $\overrightarrow{d}(u,v) \leq 1$ and $\overrightarrow{d}(v,u) \leq 1$ for each pair of vertices $u, v \in V$. A digraph is *Eulerian*, if $d^+(v) = d^-(v)$ for each $v \in V$. A digraph is *strongly connected*, if for each pair of vertices u, v, there is a directed path from u to v, and also from v to u. A connected Eulerian digraph is always strongly connected. Each digraph has a unique decomposition to strongly connected components. A component is called a *sink component*, if there is no edge leaving the component.

Many objects in this paper are integer vectors indexed by the vertices of a digraph G. We denote the set of such vectors by $\mathbb{Z}^V$, whereas $\mathbb{Z}^V_+$ denotes the set of vectors with non-negative coordinates. We denote by $\mathbf{0}_G$ ($\mathbf{1}_G$) the vector in $\mathbb{Z}^V$ with each coordinate equal to 0 (1). For a vertex $v \in V$, the characteristic vector of v is denoted by $\mathbf{1}_v$.

The *Laplacian* of a digraph G is the following matrix $L \in \mathbb{Z}^{V \times V}$:

$$L(u,v) = \begin{cases} -d^+(v) & \text{if } u = v, \\ \overrightarrow{d}(v,u) & \text{if } u \neq v. \end{cases}$$

A non-negative vector $p \in \mathbb{Z}^V_+$ is called a *period vector* for G if $Lp = \mathbf{0}_G$. A non-zero period vector is called *primitive* if its entries have no non-trivial common divisor. The following proposition follows from [3, 3.1 and 4.1].

Proposition 2.1. *For a strongly connected digraph G there exists a unique primitive period vector p_G, moreover, it is strictly positive. If G is connected Eulerian, then $p_G = \mathbf{1}_G$. For a general digraph G, if $G_1, \ldots, G_k$ are the sink components of G and a* *vector* $z \in \mathbb{Z}^V$ *satisfies* $Lz = \mathbf{0}_G$ *then* $z = \sum_{i=1}^k \lambda_i p_i$*, where for* $i \in \{1, \dots, k\}$*,* $\lambda_i \in \mathbb{Z}$ *and* p_i *is the primitive period vector of* G_i *restricted to* $V(G_i)$ *and zero otherwise.*

For a strongly connected digraph G, let us denote the sum of the coordinates of p_G by $\mathrm{per}(G)$. For a general digraph G let $\mathrm{per}(G) = \sum_{i=1}^l \mathrm{per}(G_i)$ where $G_1, \dots, G_l$ are the strongly connected components of G.

In a chip-firing game we consider a digraph G with a pile of chips on each of its nodes. A position of the game, called a *chip-distribution* (or just distribution) is described by a vector $x \in \mathbb{Z}_+^V$, where $x(v)$ denotes the number of chips on vertex $v \in V$. We denote the set of all chip-distributions on G by $\mathrm{Chip}(G)$.

The basic move of the game is *firing* a vertex. It means that this vertex passes a chip to its neighbors along each outgoing edge, and so its number of chips decreases by its outdegree. In other words, firing a vertex v means taking the new chip-distribution $x + L\mathbf{1}_v$ instead of x.

The firing of a vertex $v \in V(G)$ is *legal*, if v still has a non-negative amount of chips after the firing (i.e. $x(v) \geq d^+(v)$). A *legal game* is a sequence of distributions in which every distribution is obtained from the previous one by a legal firing. For a legal game, let us call the vector $z \in \mathbb{Z}^V$, where $z(v)$ equals the number of times v has been fired, the *firing vector* of the game. A game terminates if no firing is legal with respect to the last distribution. By a result of Björner, Lovász and Shor, whether a chip-firing game terminates after finitely many steps or it can be continued indefinitely depends only on the initial chip-distribution [4, Remark 2.4]. Based on this fact, we call a distribution x *terminating* if a legal game (hence, all legal games) started from x terminates, and we call x *non-terminating* otherwise.

We define the chip-firing game on an undirected graph as the game on the corresponding bidirected graph.

Let us introduce an equivalence-relation on chip-distributions, motivated by the theory of graph divisors [2].

Definition 2.2. For $x, y \in \mathbb{Z}^V$, let $x \sim y$ if there exists $z \in \mathbb{Z}^V$ such that $x = y + Lz$.

One can easily check that $\sim$ defines an equivalence relation on $\mathbb{Z}^V$.

3 Reachability of chip-distributions

An interesting question about the chip-firing game is the following: given two chip-distributions $x, y \in \mathrm{Chip}(G)$, is it possible to reach y from x by playing a legal game? Let us denote by $x \rightsquigarrow y$ if such a legal game exists.

To the best of our knowledge, the only previous result in the reachability question is an algorithm from [3], that decides for given $x, y \in \mathrm{Chip}(G)$ whether $x \rightsquigarrow y$ holds. This algorithm runs in $O(|V|^2 D^2 \mathrm{per}(G) \log(|V| D N \mathrm{per}(G)))$ time, where $D = \max\{d^+(v) : v \in V\}$ and N is the number of chips in x. Thus, for simple Eulerian digraphs (which includes simple undirected graphs as a special case), reachability can be decided in polynomial time. However, as $\mathrm{per}(G)$ can be exponentially large, for general digraphs, this algorithm is not polynomial. Also, if we encode multigraphs so that we only write down edge-multiplicities, then D can also be exponentially large in the size of the input. Hence the complexity of the reachability question remains open for undirected multigraphs, too.

Problem 3.1. Is there a characterization for chip-distributions $x, y \in \mathrm{Chip}(G)$ such that $x \rightsquigarrow y$?
What is the complexity of deciding whether $x \rightsquigarrow y$ for given chip-distributions $x, y \in \mathrm{Chip}(G)$?

In contrast with this, checking whether $x \sim y$ holds can be done in polynomial time using Gaussian elimination.

The condition $x \sim y$ is clearly necessary for $x \rightsquigarrow y$, as a legal game defines a vector $z \in \mathbb{Z}^V$ such that $y = x + Lz$. The following theorem, which is the main result of our paper, shows a case where this necessary condition is also sufficient. Our theorem uses the notion of recurrent chip-distributions. Here we only give the definition, while Section 4.1 is dedicated to a detailed study of recurrent chip-distributions.

Definition 3.2. We call a chip-distribution $x \in \mathrm{Chip}(G)$ *recurrent* if there exists a nonempty sequence of legal firings that transforms x to itself.

Theorem 3.3. *Let G be strongly a connected digraph and $x, y \in \mathrm{Chip}(G)$. If y is recurrent and $x \sim y$, then $x \rightsquigarrow y$.*

Proof. First we claim that if $x \sim y$ then there exists $z \in \mathbb{Z}_+^V$ such that $x = y + Lz$. Indeed, $x \sim y$ implies the existence of $w \in \mathbb{Z}^V$ with $x = y + Lw$. From Proposition 2.1, for a sufficiently large $k \in \mathbb{Z}_+$, the vector $z = w + kp_G$ is non-negative, while $x = y + Lw = y + Lw + kLp_G = y + Lz$ holds.

Fix such a z. We proceed by induction on $\sum_{v\in V(G)} z(v)$. If $\sum_{v\in V(G)} z(v) = 0$, then $x = y$, thus $x \rightsquigarrow y$. Now suppose $\sum_{v\in V(G)} z(v) > 0$. As y is recurrent, there exists a sequence $v_1, v_2, \dots, v_k$ of vertices (a vertex may occur multiple times) such that from initial distribution y, firing them in this order is a legal game that leads back to y. Fix such a sequence. We claim that in this sequence, each vertex occurs at least once. Indeed, for the firing vector w of the game, $y = y + Lw$ thus w is a multiple of the primitive period vector, and the primitive period vector of a strongly connected digraph is strictly positive, as claimed in Proposition 2.1.

Let i be the smallest index such that $z(v_i) > 0$. Such an index exists because each vertex is listed at least once in $v_1, v_2, \dots, v_k$. Starting from y, fire the vertices $v_1, \dots, v_{i-1}$. This is a legal game by definition. Let the resulting distribution be y'. We claim that the sequence of firings $v_1, \dots, v_{i-1}$ is also legal starting from x. To prove this, it is enough to show that $x(v_j) \geq y(v_j)$ for all $1 \leq j \leq i-1$. This is true, because $x(v_j) = y(v_j) + (Lz)(v_j)$, where $(Lz)(v_j) \geq 0$, since the only negative element in the row corresponding to v_j is $L(v_j, v_j)$, but $z(v_j) = 0$. Hence the firing of the vertices $v_1, \dots, v_{i-1}$ from distribution x is legal. Let the distribution obtained by this game be x'. Thus $x \rightsquigarrow x'$.

For x' and y', we also have $x' = y' + Lz$. At position y', firing v_i is legal, by definition of the sequence $v_1, \dots, v_k$. Denote by y'' the distribution we get by firing v_i at y'. The distribution y'' is recurrent, since firing $v_{i+1} \dots, v_k, v_1, \dots, v_i$ is a legal game that leads back to y''. Now for x' and y'' we have $x' = y'' + Lz'$, where $z'(v_i) = z(v_i) - 1$, and for each other vertex $v \neq v_i$, $z'(v) = z(v)$. This way $\sum_{v \in V(G)} z'(v) = \sum_{v \in V(G)} z(v) - 1$, hence by the induction hypothesis, $x' \rightsquigarrow y''$.

We claim that $y'' \rightsquigarrow y$. Indeed, firing $v_{i+1}, \ldots, v_k$ starting from y'' is a legal game that leads to y. We also have $x \rightsquigarrow x'$. Summarizing, we have $x \rightsquigarrow y$. □

Our aim is now to generalize Theorem 3.3 for weakly connected digraphs. In Section 4, after proving some basic properties of recurrent chip-distributions, we give an example (Example 4.5) which shows that Theorem 3.3 does not remain true in its original form for general digraphs. With a somewhat stronger condition, however, we can generalize Theorem 3.3 to weakly connected digraphs.

Theorem 3.4. *Let G be a weakly connected digraph, and $x, y \in \mathrm{Chip}(G)$ be two chip-distributions such that there exists $z \in \mathbb{Z}_+^V$ with $y = x + Lz$. Suppose that for each strongly connected component $G' = (V', E')$ of G, $z|_{V'} = \mathbf{0}_{G'}$ or $y|_{V'} \in \mathrm{Chip}(G')$ is recurrent. Then $x \rightsquigarrow y$.*

Note that the existence of a non-negative z such that $y = x + Lz$ is a necessary condition for $x \rightsquigarrow y$. Indeed, the firing vector of a legal game realizing $x \rightsquigarrow y$ is such a vector. For strongly connected digraphs, $x \sim y$ also implies the existence of a non-negative $z \in \mathbb{Z}^V$ such that $y = x + Lz$. However, for general digraphs this is not the case.

Proof. Fix a non-negative vector $z \in \mathbb{Z}_+^V$ with $y = x + Lz$. Let $V_1, V_2, \ldots, V_k$ be a topological ordering of the strongly connected components of G, i.e., $V = V_1 \cup \cdots \cup V_k$, for each i the digraph $G_i = (V_i, E|_{V_i \times V_i})$ is strongly connected, and there is no directed edge from $v_i \in V_i$ to $v_j \in V_j$ if $i > j$.

Let x' be the chip-distribution obtained from x by passing $z(u) \cdot \overrightarrow{d}(u, v)$ chips from u to v for each pair of vertices $u, v \in V$ where u and v are in different strongly connected components. Note that $x \not\rightsquigarrow x'$ is possible. The proof of the theorem is based on the following lemma.

Lemma 3.5. *For each i, $x'|_{V_i} \sim y|_{V_i}$ on the digraph G_i. Moreover, if $y|_{V_i}$ is recurrent on G_i, then there exists a legal game on G_i with firing vector $z|_{V_i}$ that transforms $x'|_{V_i}$ to $y|_{V_i}$.*

Proof. Let L_i be the Laplacian matrix of G_i. We first prove that $x'|_{V_i} \sim y|_{V_i}$ (as chip-distributions on G_i) by showing that $x'|_{V_i} + L_i z|_{V_i} = y|_{V_i}$. For this, let $v \in V_i$. Then

$$
\begin{aligned}
&x'(v) + (L_i z|_{V_i})(v) = \\
x(v) + \sum_{v' \in V \setminus V_i} &\left(\overrightarrow{d}(v', v) \cdot z(v') - \overrightarrow{d}(v, v') \cdot z(v) \right) + (L_i z|_{V_i})(v) = \\
&x(v) + (Lz)(v) = y(v).
\end{aligned}
$$

Now, if $y|_{V_i}$ is recurrent, by Theorem 3.3, $x'|_{V_i} \rightsquigarrow y|_{V_i}$ in G_i. Let $w_i \in \mathbb{Z}^{V_i}$ be the firing vector of a legal game transforming $x'|_{V_i}$ to $y|_{V_i}$. Then $L_i(z|_{V_i} - w_i) = 0$, hence by Proposition 2.1, $w_i - z|_{V_i} = c \cdot p_{G_i}$ with $c \in \mathbb{Z}$. If $c = 0$ then $z|_{V_i}$ is the firing vector of a legal game, proving the lemma. In the followings, we treat separately the case $c < 0$ and $c > 0$. For both cases, the following lemma of Björner and Lovász is useful.

Lemma 3.6 ([3, 4.3])**.** *Let $w \in \mathbb{Z}_+^{V_i}$ be a firing vector of a legal game from some initial distribution. If $w \geq p_{G_i}$, then $w - p_{G_i}$ is also the firing vector of a legal game from the same initial distribution.*

Suppose that $c < 0$. Since $y|_{V_i}$ is recurrent, there is a legal game on G_i that transforms $y|_{V_i}$ back to itself. For the firing vector w of this game, $L_i w = 0$, hence $w = \lambda \cdot p_{G_i}$ with $\lambda \in \mathbb{Z}$, $\lambda > 0$. From Lemma 3.6, we can suppose that $\lambda = 1$. Now starting from distribution $x'|_{V_i}$ on G_i, after playing the legal game with firing vector w_i, we get to the distribution $y|_{V_i}$. Then iterate $-c$ times the legal game with firing vector p_{G_i}. This gives us a legal game with firing vector $z|_{V_i}$, finishing the proof for the $c < 0$ case.

Now suppose that $c > 0$. Then Lemma 3.6 guarantees that there is a legal game from $x'|_{V_i}$ with firing vector $w_i - c \cdot p_{G_i} = z|_{V_i}$. This finishes the proof of the lemma. □

For each $1 \leq i \leq k$ let z_i be the vector with $z_i(v) = z(v)$ if $v \in V_i$, and $z_i(v) = 0$ otherwise. Let $s_i = \sum_{j \leq i} z_j$, i.e., $s_i(v) = z(v)$ if $v \in \bigcup_{j \leq i} V_j$, and $s_i(v) = 0$ otherwise. Let $x_i = x + Ls_i$ and $x_0 = x$. We show that for $i = 1, \ldots, k$, starting from the distribution x_{i-1}, there is a legal game on G with firing vector z_i. Since $x_{i-1} + Lz_i = x_i$, and $x_k = y$, this is enough to finish the proof of the theorem.

So let i be fixed. It is easy to see that for each $v \in V_i$

$$x'(v) = x_{i-1}(v) - z(v) \cdot \sum_{v' \in V \setminus V_i} \overrightarrow{d}(v, v'). \tag{1}$$

If $z|_{V_i} = \mathbf{0}_{G_i}$, then $z_i = \mathbf{0}_G$, hence we have nothing to prove. If this is not the case, then $y|_{V_i}$ is recurrent by the assumptions. Using the lemma, from initial distribution $x'|_{V_i}$ there exists a legal game on G_i with firing vector $z|_{V_i}$. We claim that the same sequence of firings on G, with initial distribution x_{i-1} remains a legal game. Indeed, we can see from (1) that by playing the game on G from initial distribution x_{i-1}, at any moment we have a distribution that is greater or equal on V_i than the distribution we get by playing the game on G_i with initial distribution $x'|_{V_i}$. Hence there exists a legal game on G with initial distribution x_{i-1} and firing vector z_i. This finishes the proof of the theorem. □

Note that for distributions y such that y is recurrent restricted to each strongly connected component, Theorem 3.4 gives a necessary and sufficient condition for the reachability of y. The condition of the theorem, i.e. whether there exists $z \in \mathbb{Z}_+^V$ with $y = x + Lz$, can also be decided in polynomial time: One can compute a vector w with $y = x + Lw$ using Gaussian elimination. By Proposition 2.1, there exists $z \in \mathbb{Z}_+^V$ with $y = x + Lz$ if and only if w is integer, and non-negative on every non-sink component.

4 Recurrent chip-distributions

Recurrent chip-distributions were first investigated by Jeffs and Seager [6] in the special case of undirected cycle graphs. They gave a characterization of recurrent distributions on undirected graphs using the notion of *diffuse* configurations. Connections between recurrent distributions and acyclic orientations of undirected graphs are demonstrated in [5, Section 14.11]

While former results considered undirected graphs, in this section we study recurrent distributions on digraphs. In the special case of Eulerian digraphs we also give a characterization of recurrent distributions, which generalizes the characterization for undirected graphs given in [6] and [5].

Recurrent distributions were also defined and studied in the Abelian sandpile model (for more about this model, see e.g. [1]). Let us briefly describe the connection between the two notions. If a chip-distribution $x \in \mathrm{Chip}(G)$ is recurrent, then for any choice of sink $v_0 \in V$, the stabilization of $x|_{V-v_0}$ is recurrent in the Abelian sandpile model. Conversely, if a configuration $y \in \mathbb{Z}^{V-v_0}$ is recurrent in the Abelian sandpile model, then if a chip-configuration is non-terminating and agrees with y on $V - v_0$, then it is recurrent.

4.1 Properties of recurrent chip-distributions

Claim 4.1. *If x is a recurrent chip-distribution and $y \geq x$ (coordinatewise), then y is also recurrent.*

Proof. By definition, there exists a nonempty legal game starting from x, that leads back to x. This game remains legal if it is started from y, as y is coordinatewise greater than or equal to x. Moreover, started from y, it leads back to y. □

Proposition 4.2. *A chip-distribution $x \in \mathrm{Chip}(G)$ on a digraph G is recurrent if and only if there is a sink-component G_i of G such that $x|_{V(G_i)}$ is recurrent on G_i.*

Proof. First we show the "if" direction. Suppose that there is a sink-component G_i of G such that $x|_{V(G_i)}$ is recurrent on G_i. Then x is recurrent, since we can perform on G the sequence of firings that transforms $x|_{V(G_i)}$ back to itself on G_i. These firings have the same effect when they are performed on G, since G_i is a sink component. In particular, the game remains legal on G, and the distribution is not modified outside $V(G_i)$. Hence the game leads us back to x.

Now we show the "only if" direction. If x is recurrent, there is a legal game that transforms it back to itself. For the firing vector $z \neq \mathbf{0}_G$ of this game, $x = x + Lz$, hence $Lz = \mathbf{0}_G$. By Proposition 2.1, z is of the form $z = \sum_{i=1}^{k} \lambda_i p_i$, where $\lambda_i \in \mathbb{Z}$ for $i = 1, \ldots, k$, $G_1, \ldots G_k$ are the sink components of G, and p_i is the primitive period vector of G_i restricted to $V(G_i)$ and zero otherwise. In particular, z is zero outside the sink-components, and there is at least one sink-component G_i such that $\lambda_i > 0$. Hence the game consists of some firings in come sink components. A firing in a sink component does not affect the chip-distribution on other sink components, thus if we restrict the game to G_i (i.e., we only perform the firings where the vertex is in $V(G_i)$), then we also get a legal game. Moreover, since in the original game, no vertex is fired outside sink components, the effect of this game for the chip-distribution on G_i is the same as the effect of the original game. Hence the resulting distribution on $V(G_i)$ is $x|_{V(G_i)}$. Also, since G_i is a sink component, it does not make a difference whether we play this game on G or on G_i. Hence we have a legal game on G_i that transforms $x|_{V(G_i)}$ back to itself. □

Proposition 4.3. *A recurrent chip-distribution is non-terminating. Moreover, a legal game started from a recurrent chip-distribution takes only recurrent positions.*

Proof. A recurrent chip-distribution x is non-terminating, since we can repeat the nonempty legal game transforming x to itself infinitely.

Now we show that a legal game from a recurrent distribution takes only recurrent positions. Let x be a recurrent distribution. It is enough to show that any legal firing from x leads to a recurrent distribution. Let $v_1, \dots v_k$ be a sequence of vertices (one vertex can occur more than once) such that firing them in this order from initial distribution x is a legal game that leads back to x (such a sequence exists because x is recurrent). Suppose that $v \in V$ is a vertex such that starting from x the firing of v is legal. Fire v and let the obtained distribution be x'. We need to show that x' is recurrent. If v does not occur in the sequence $v_1, v_2, \dots, v_k$, then on any vertex in $\{v_1, v_2, \dots, v_k\}$, there are at least as many chips in x' as in x, thus $v_1, v_2, \dots, v_k$ is still a legal game from x', and it leads back to x'. This shows that in this case x' is indeed recurrent.

On the other hand, if v occurs in the sequence $v_1, v_2, \dots, v_k$, let v_i be its first occurrence. Then $v_i, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ is a legal game from x, as the firing of v_i in x is legal, the vertices $v_1, v_2, \dots, v_{i-1}$ have at least as many chips when they are fired as in the original sequence, and the vertices $v_{i+1}, \dots, v_k$ have the same number of chips. But this means that $v_1, \dots, v_{i-1}, v_{i+1}, \dots v_k, v_i$ is a legal game from x' that leads back to x', showing that x' is recurrent also in this second case. This finishes the proof. □

Proposition 4.4. *Every non-terminating chip-firing game takes a recurrent position after a finite number of firing moves.*

Proof. As there are only finitely many chip-distributions with a given number of chips, after finitely many steps, a distribution appears for the second time, which must therefore be recurrent. □

Now we are able to give an example showing that Theorem 3.3 does not remain true for general digraphs, i.e. for general digraphs, $x \sim y$ and y being recurrent is not sufficient for $x \rightsquigarrow y$.

Example 4.5. Let G_1 and G_2 be strongly connected digraphs, both of them with at least two vertex-disjoint directed cycles. Let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Let G be the following graph: $V(G) = \{v\} \cup V(G_1) \cup V(G_2)$, $E(G) = \{\overrightarrow{vv_1}, \overrightarrow{vv_2}\} \cup E(G_1) \cup E(G_2)$. Then the strongly connected components of G are $\{v\}$, G_1 and G_2, and the sink components are G_1 and G_2.

For $i = 1, 2$, let $x_i \in \mathrm{Chip}(G_i)$ be a non-recurrent chip-distribution and $y_i \sim x_i$ a recurrent chip-distribution. Such distributions exist: By the proof of [3, Theorem 2.2], a recurrent chip-distribution has at least one chip on each directed cycle. Also, by [3], if a distribution has at least $|E(G)| - |V(G)| + 1$ chips, then it is non-terminating. Hence, if x_i has $|E(G_i)|$ chips on one vertex of G_i and zero elsewhere, then x_i is non-recurrent and non-terminating. By Proposition 4.4, after playing the chip-firing game from x_i on G_i for finitely many steps, we arrive at a recurrent distribution $y_i \sim x_i$.

Let $x \in \mathrm{Chip}(G)$ be the chip-distribution that is zero on v, agrees with x_1 on $V(G_1)$ and agrees with y_2 on $V(G_2)$. Let $y \in \mathrm{Chip}(G)$ be the chip-distribution that is zero on v, agrees with y_1 on $V(G_1)$ and agrees with x_2 on $V(G_2)$. Then clearly $x \sim y$. Moreover both x and y are recurrent, since both of them are recurrent restricted to a sink component.

Since $x(v) = y(v) = 0$, if y were reachable from x on G, then x_1 were reachable from y_1 on G_1. But this is impossible, since y_1 is recurrent, while x_1 is not, and by Proposition 4.3, a legal game started from a recurrent distribution never leads to a non-recurrent distribution.

4.2 Characterization of recurrence

The results of Section 3 suggest the problem of deciding whether a given chip-distribution is recurrent. For Eulerian digraphs (this case includes undirected graphs as a special case), it follows from results of Björner and Lovász, that a chip-distribution x is recurrent if and only if one can play a legal game starting from x such that each vertex fires exactly once. This can be decided in polynomial time. For general digraphs, the complexity of deciding whether a given chip-distribution is recurrent is unknown.

Problem 4.6. Let G be a digraph. What is the complexity of deciding whether a chip-distribution $x \in \mathrm{Chip}(G)$ is recurrent?

In this section, we give a combinatorial characterization of recurrent chip-distributions on Eulerian digraphs, that generalizes a result of [6, 5] for the case of undirected graphs. For general graphs, we only obtain a necessary condition. Our characterization is based on ideas of [8, 7].

Let us first recall the result concerning undirected graphs. For this, we need the notion of *diffuse* distributions.

Definition 4.7 ([6])**.** Let G be an undirected graph. A chip-distribution $x \in \mathrm{Chip}(G)$ is called *diffuse* if for any nonempty subset U of V there is a vertex $v \in U$ such that $x(v) \geq d_{G[U]}(v)$, where $G[U]$ denotes the subgraph of G induced by U.

For undirected graphs, the equivalence of recurrent and diffuse distributions was proved by Jeffs and Seager [6]. The following theorem in this form can be found in [5].

Theorem 4.8. *Let G be an undirected graph and $x \in \mathrm{Chip}(G)$ be a chip-distribution. Then the following three statements are equivalent:*

(1) x is recurrent,

(2) there exists an acyclic orientation of G such that for each vertex v, $x(v)$ is at least the in-degree of v in the orientation,

(3) x is diffuse.

The notion of diffuse chip-distribution can be naturally generalized to directed graphs:

Definition 4.9. Let G be a digraph. A chip-distribution $x \in \mathrm{Chip}(G)$ is called *diffuse* if for any nonempty subset $U \subseteq V$ there is a vertex $v \in U$ such that $x(v) \geq d^-_{G[U]}(v)$, where $G[U]$ denotes the subgraph of G induced by U.

In the directed case, acyclic orientations correspond to feedback arc sets.

Definition 4.10. A *feedback arc set* of a digraph D is a set of edges $F \subseteq E(D)$ such that the digraph $D' = (V(D), E(D) \setminus F)$ is acyclic.

Definition 4.11. For a digraph G, let us say that a chip-distribution $x \in \mathrm{Chip}(G)$ is *above a feedback arc set*, if there exists a feedback arc set of G such that for all $v \in V(G)$, $x(v)$ is at least the indegree of v restricted to the feedback arc set.

Theorem 4.12. *Let G be a digraph and $x \in \mathrm{Chip}(G)$ be a chip-distribution. Consider the following three statements:*

(1) x is recurrent,

(2) x is above a feedback arc set,

(3) x is diffuse.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). *If G is Eulerian, then* (2) $\Rightarrow$ (1) *also holds. Hence for Eulerian digraphs, all three statements are equivalent.*

For the case (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) in Eulerian digraphs, our proof is a slight modification of ideas in [8, 7].

Proof. (1) $\Rightarrow$ (2). Suppose that $x \in \mathrm{Chip}(G)$ is recurrent. Take a non-empty legal game that transforms it back to itself. Let z be the firing vector of this game. Then $Lz = \mathbf{0}_G$, hence by Proposition 2.1, $z = c \cdot p_G$ for some constant c. Thus, each vertex fired at least once. Take the vertices in the order of their last firing. Then

$$F = \{\overrightarrow{uv} : \text{the last firing of } v \text{ precedes the last firing of } u\}$$

is a feedback arc set. Moreover, for the final distribution (which is also x by our assumption), for each $v \in V$, $x(v) \geq d^-_F(v)$, since after its last firing, v had a non-negative number of chips, and after that time, it received a chip through all of its incoming F-arcs. Hence x is indeed above a feedback arc set.

(2) ⇒ (3). Suppose that x is above a feedback arc set F of G. By the definition of feedback arc set, $G' = (V, E \setminus F)$ is an acyclic digraph, i.e. it has a topological ordering $\mathcal{O}$. Let U be any nonempty subset of V. Let v be the vertex of U which occurs first in $\mathcal{O}$. Then the indegree of v in $G'[U]$ is zero, so $d^-_{G[U]}(v) = d^-_F(v) \le x(v)$, proving that x is diffuse.

(3) ⇒ (2). Now suppose that x is diffuse. Then one can recursively find an ordering $(v_1, v_2, \ldots, v_n)$ of V such that $x(v_i) \ge d^-_{G[V_i]}$, where $V_i = \{v_i, v_{i+1}, \ldots, v_n\}$. Then $F = \{\overrightarrow{v_i v_j} \in E(G) : i > j\}$ is a feedback arc set with $d^-_F(v) = d^-_{G[V_i]} \le x(v)$ for all $v \in V$.

Finally, we show that (2) ⇒ (1) if G is an Eulerian digraph. Suppose that $x \in \mathrm{Chip}(G)$ is above a feedback arc set F of the Eulerian digraph G. By the definition of feedback arc set, $G' = (V, E \setminus F)$ is an acyclic digraph. Take a topological ordering of the vertices of G'. It is easy to check that firing the vertices in this order is a legal game on G started from x. In this game each vertex is fired once, hence each vertex v loses $d^+(v)$ chips, and receives $d^-(v)$ chips. Since G is Eulerian, this means that the resulting chip-distribution is again x, showing that x is recurrent. □

Example 4.13. We show that being above a feedback arc set is not sufficient for being recurrent in general digraphs. Let $G = (V, E)$ be the following strongly connected digraph: $V = \{1, 2, 3\}$ and $E = \{(2, 1), (1, 2), (2, 3), (3, 1)\}$. It is easy to see that $F = \{(1, 2)\} \subset E$ is a feedback arc set, hence the distribution $x \in \mathrm{Chip}(G)$ with $x(1) = x(3) = 0$ and $x(2) = 1$ is above a feedback arc set. But no vertex can fire, thus x cannot be recurrent.

Acknowledgements

We would like to thank András Frank and Ágoston Weisz for calling our attention to reachability questions and the definition of recurrent distributions. Investigations of Ágoston in some special cases [9] were also very helpful for us.

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