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Abstract

In this paper, we study the complexity of the chip-firing reachability problem. We show that for Eulerian digraphs, the reachability problem can be decided in polynomial time, even if the digraph has multiple edges. We also show a special case when the reachability problem can be decided in polynomial time for general digraphs: if the target distribution is recurrent restricted to each strongly connected component. Both of these algorithms are strongly polynomial. As a further positive result, we show that the chip-firing reachability problem is in co-NP for general digraphs. We also show that the chip-firing halting problem is in co-NP for Eulerian digraphs.

Keywords: chip-firing game; computational complexity

1   Introduction

Chip-firing is a solitary game on a directed graph, defined by Björner, Lovász and Shor [3]. Each vertex contains a pile of chips. A legal move is to choose a vertex with at least as many chips as its out-degree and let it send a chip along each outgoing edge. We analyze the complexity of the following reachability question: given two chip-distributions x and y, decide whether y can be reached from x by playing a legal game. This question is a special case of the reachability problem for integral vector addition systems [2]. It was first considered by Björner and Lovász, who gave an algorithm that decides the reachability question and runs in weakly polynomial time for simple Eulerian digraphs [2]. The complexity of the reachability problem was left

open for Eulerian digraphs with multiple edges, and for non-Eulerian digraphs. The question whether the reachability problem is in NP or in co-NP was also left open.

In this paper, we show that the chip-firing reachability problem can be decided in polynomial time for Eulerian digraphs with multiple edges. Our algorithm is strongly polynomial. The main ingredient of the algorithm is a lemma ensuring that if one chip-distribution is reachable from another, then it can be reached by firing an "ascending chain of sets of vertices".

For general digraphs, we show that the chip-firing reachability problem is in co-NP. Also, we show a special case when the chip-firing reachability problem is polynomial time solvable even on non-Eulerian digraphs. If G is a strongly connected digraph and the chip-distribution y is recurrent, i.e., it is reachable from itself by a non-empty legal game, then we characterize the set of chip-distributions from which y is reachable. This characterization enables one to decide in strongly polynomial time whether a recurrent chip-distribution y is reachable by a legal game from a given chip-distribution x. For weakly connected directed graphs, we generalize the characterization theorem for chip-distributions y that are recurrent restricted to each strongly connected component. This theorem also gives rise to a strongly polynomial algorithm.

Finally, in Section 5, we collect some open problems related to the reachability problem. In this last section, we show that the chip-firing halting problem (which is a problem similar to the chip-firing reachability problem) is in NP∩co-NP for Eulerian digraphs, which makes it a good candidate for the search of a polynomial algorithm.

2 Preliminaries

2.1 Digraphs

Throughout this paper, *digraph* means a (weakly) connected directed graph that can have multiple edges but no loops. A digraph is usually denoted by G. The vertex set and edge set of a digraph G are denoted by $V(G)$ and $E(G)$ (or simply V and E), respectively. For a vertex v, the in-degree and the out-degree of v are denoted by $d^-(v)$ and $d^+(v)$, respectively. We denote a directed edge leading from vertex u to vertex v by $\overrightarrow{uv}$. The multiplicity of a directed edge $\overrightarrow{uv}$ is denoted by $\overrightarrow{d}(u,v)$.

A digraph is *simple*, if $\overrightarrow{d}(u,v) \leq 1$ and $\overrightarrow{d}(v,u) \leq 1$ for each pair of vertices $u, v \in V$. A digraph is *Eulerian*, if $d^+(v) = d^-(v)$ for each $v \in V$. A digraph is *strongly connected*, if for each pair of vertices u, v, there is a directed path from u to v, and also from v to u. A connected Eulerian digraph is always strongly connected. Each digraph has a unique decomposition to strongly connected components. A component is called a *sink component*, if there is no edge leaving the component.

Throughout this paper, we identify undirected graphs with the digraph obtained by replacing each edge with a pair of oppositely directed edges. This way, undirected graphs become special Eulerian digraphs.

If we give a digraph as an input to an algorithm, we always encode it by its adjacency matrix. Hence the size of the input is not increased by the values of the edge multiplicities, just the logarithms of it.

We denote by $\mathbb{Z}^V$ the set of integer vectors indexed by the vertices of a digraph G. $\mathbb{Z}_+^V$ denotes the set of vectors with non-negative integer coordinates. For $S \subseteq V$, we denote the characteristic vector of S by $\mathbf{1}_S$. If $S = \{v\}$, we use the notation $\mathbf{1}_v$. We denote the vector with each coordinate equal to zero by $\mathbf{0}$.

The *Laplacian* of a digraph G is the following matrix $L \in \mathbb{Z}^{V \times V}$:

$$L(u,v) = \begin{cases} -d^+(v) & \text{if } u = v, \\ \overrightarrow{d}(v,u) & \text{if } u \neq v. \end{cases}$$

2.2 Chip-firing

In a chip-firing game we consider a digraph G with a pile of chips on each of its nodes. A position of the game, called a *chip-distribution* (or just distribution) is described by a vector $x \in \mathbb{Z}_+^V$, where $x(v)$ is interpreted as the number of chips on vertex $v \in V$. We denote the set of all chip-distributions on G by $\text{Chip}(G)$.

The basic move of the game is *firing* a vertex. It means that this vertex passes a chip to its neighbors along each outgoing edge, and so its number of chips decreases by its out-degree. In other words, firing a vertex v means taking the new chip-distribution $x + L\mathbf{1}_v$ instead of x.

The firing of a vertex $v \in V$ is *legal* with respect to a distribution x, if v has a non-negative amount of chips after the firing (i.e. $x(v) \geq d^+(v)$). A *legal game* is a sequence of distributions in which every distribution is obtained from the previous one by a legal firing. For a legal game, let us call the vector $f \in \mathbb{Z}_+^V$, where $f(v)$ equals the number of times v has been fired, the *firing vector* of the game. A game terminates if no firing is legal with respect to the last distribution. The following theorem of Björner, Lovász and Shor describes a very important "Abelian" property of the chip-firing game.

Theorem 2.1. *[3, Remark 2.4] From a given initial chip-distribution, either every legal game can be continued indefinitely, or every legal game terminates after finitely many steps. The firing vector of every maximal legal game is the same.*

Based on this fact, we call a distribution x *terminating* if a legal game (hence, all legal games) started from x terminates, and we call x *non-terminating* otherwise.

For a given vector $b \in \mathbb{Z}_+^V$, let us call the following game *chip-firing game with upper bound b*: We are only allowed to make legal firings, and each vertex v can be fired at most $b(v)$ times during the whole game. Björner and Lovász show the "Abelian" property for the bounded chip-firing game as well.

Lemma 2.2. *[2, Lemma 1.4] For a given bound $b \in \mathbb{Z}_+^V$ and initial distribution x, each maximal bounded game with upper bound b and initial distribution x has the same firing vector.*

A non-negative vector $p \in \mathbb{Z}_+^V$ is called a *period vector* for G if $Lp = \mathbf{0}$. A non-zero period vector is called *primitive* if its entries have no non-trivial common divisor. The following proposition follows from [2, 3.1 and 4.1].

Proposition 2.3. *For a strongly connected digraph G there exists a unique primitive period vector p_G, moreover, it is strictly positive. If G is connected Eulerian, then $p_G = \mathbf{1}_V$. For a general digraph G, if $G_1, \ldots, G_k$ are the sink components of G and a vector $z \in \mathbb{Z}^V$ satisfies $Lz = \mathbf{0}$ then $z = \sum_{i=1}^k \lambda_i p_i$, where for $i \in \{1, \ldots, k\}$, $\lambda_i \in \mathbb{Z}$ and p_i is the primitive period vector of G_i restricted to $V(G_i)$ and zero elsewhere.*

For a strongly connected digraph G, let us denote the sum of the coordinates of p_G by $\mathrm{per}(G)$. For a general digraph G let $\mathrm{per}(G) = \sum_{i=1}^{\ell} \mathrm{per}(G_i)$ where $G_1, \ldots, G_\ell$ are the strongly connected components of G.

2.2.1   Reachability

A basic question about the chip-firing game is the so-called reachability question: Given two chip-distributions $x, y \in \mathrm{Chip}(G)$, is it possible to reach y from x by playing a legal game? Let us denote by $x \rightsquigarrow y$ if such a legal game exists.

Our main goal in this paper is to investigate the computational complexity of the reachability question. Let us sum up the previous results about the problem. To do so, we state an important lemma of Björner and Lovász.

Lemma 2.4. *[2, Lemma 4.3] Let p be a period vector of a digraph G, and suppose that $\alpha = (v_1, v_2, \ldots, v_s)$ is a legal sequence of firings on G from some initial distribution. Let α' be the sequence obtained from α by deleting the first $p(v)$ occurrence of each vertex v (if v occurs less than $p(v)$ times in α, then we delete all of its occurrences). Then α' is also a legal sequence of firings from the same initial distribution.*

A vector $f \in \mathbb{Z}_+^V$ is called *reduced* if $f \not\geq p$ for every non-zero period vector p. The following phenomenon is a direct consequence of the previous lemma:

Lemma 2.5. *[2, Lemma 5.2] If $x \rightsquigarrow y$, then there exists a legal game transforming x to y with a reduced firing vector.*

Note that if $x \rightsquigarrow y$ then for the firing vector f of a legal game transforming x to y, $y = x + Lf$. Among the vectors $g \in \mathbb{Z}_+^V$ satisfying $y = x + Lg$, there is a unique one that is reduced.

The previous lemmas imply, that the reachability question can be decided "greedily": For given $x, y \in \mathrm{Chip}(G)$ one can decide if there exists a reduced vector f with $y = x + Lf$. If no such vector exists then $x \not\rightsquigarrow y$. If such a vector f exists, it can be computed. By Lemma 2.5, $x \rightsquigarrow y$ if and only if there is a legal game from x to y with firing vector f. By Lemma 2.2, we can find greedily a maximal chip-firing game from x with upper bound f. We have $x \rightsquigarrow y$ if and only if this maximal bounded chip-firing game has firing vector f.

This reasoning gives an algorithm for deciding the reachability problem, however, this algorithm is in general not polynomial, as the firing vector f may have exponentially large elements.

Björner and Lovász improve this naive algorithm, and obtain the following:

Theorem 2.6 ([2])**.** *There is an algorithm that for given $x, y \in \mathrm{Chip}(G)$ on a digraph G decides whether $x \rightsquigarrow y$ holds, and runs in*

$$O(|V|^2 D^2 \mathrm{per}(G) \log(|V| D N \mathrm{per}(G)))$$

time, where $D = \max\{d^+(v) : v \in V\}$ and N is the number of chips in x.

This algorithm is not polynomial in general, as $\mathrm{per}(G)$ and D may be exponentially large. However, as for simple Eulerian digraphs, $\mathrm{per}(G) = |V|$ and $D \leq |V|$, the algorithm is weakly polynomial for simple Eulerian digraphs.

In this paper, we show that the reachability problem can be decided in polynomial time also for general Eulerian digraphs (i.e. also for Eulerian digraphs with multiple edges). In addition, our algorithm is strongly polynomial. For general digraphs, we show that the reachability problem is in co-NP. We also show that in the special case if y is recurrent restricted to each strongly connected component, whether $x \rightsquigarrow y$ holds can be decided in polynomial time for general digraphs.

Before presenting our results, let us describe two simple, polynomial-time computable necessary conditions for the reachability problem:

The first necessary condition is the linear equivalence of x and y:

Definition 2.7 (Linear equivalence [1])**.** For $x, y \in \mathbb{Z}^V$, let $x \sim y$ if there exists $z \in \mathbb{Z}^V$ such that $x = y + Lz$. In this case we say that x and y are linearly equivalent.

One can easily check that $\sim$ defines an equivalence relation on $\mathbb{Z}^V$. This equivalence relation has been defined by Baker and Norine [1], and it plays an important role in the theory of graph divisors.

It is easy to see, that $x \sim y$ is a necessary condition for $x \rightsquigarrow y$. Indeed, if there is a legal game leading from x to y, let its firing vector be f. Then $y = x + Lf$. Note that checking whether $x \sim y$ holds can be done in strongly polynomial time using Gaussian elimination.

A slightly stronger necessary condition for $x \rightsquigarrow y$ is the existence of $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$. (This condition is stronger than the linear equivalence only in that we require f to be non-negative.) As the firing vector of a legal game from x to y gives a non-negative f such that $y = x + Lf$, this condition is also necessary. For strongly connected digraphs, the two conditions are equivalent, since a strong digraph has a period vector that is positive on every coordinate. In that case it is more convenient to work with the notion of linear equivalence, as that defines an equivalence relation, while the later condition does not. Note, however, that the existence of $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$ can also be decided in strongly polynomial time. Indeed, we can decide using Gaussian elimination whether $x \sim y$, and if so, compute a $g \in \mathbb{Z}^V$ such that $y = x + Lg$. By Proposition 2.3, there exists $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$ if and only if g is non-negative on every non-sink strongly connected component.

3 A polynomial algorithm for Eulerian digraphs

In this section, we describe our algorithm deciding the reachability problem on Eulerian digraphs, which is polynomial even for graphs with multiple edges. The following

lemma is the heart of our algorithm. Informally, it says, that if one chip-distribution is reachable from another, then it can be reached so that we fire an ascending chain of subsets of vertices.

Lemma 3.1. *Let G be an Eulerian digraph. Suppose that we have two chip-distributions x and y such that $x \rightsquigarrow y$. Then there exists a sequence of legal firings $(v_1, v_2, \ldots, v_s)$ that transforms x to y, and there exist indices $i_0 = 0, i_1, i_2, \ldots i_t = s$ such that for each $j = 1, \ldots, t$, no vertex appears twice in the sequence $v_{i_{j-1}+1}, \ldots, v_{i_j}$, and by setting $S_j = \{v_{i_{j-1}+1}, \ldots, v_{i_j}\}$, we have $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t \subsetneq V$.*

Proof. Lemma 2.4 plays a key role in this proof. Note that for Eulerian digraphs, Lemma 2.4 says, that if for a legal sequence of firings, we leave out the first occurrence of each vertex that occurs in the sequence, we still get a legal game.

Since $x \rightsquigarrow y$, there exists a sequence of firings $\alpha = (w_1, \ldots w_s)$ that is legal, and transforms x to y. Let $f_\alpha \in \mathbb{Z}_+^V$ be the firing vector of α, and S_α be the set of vertices that occur at least once in α. Notice that we can suppose that $S_\alpha \neq V$. Indeed, if $S_\alpha = V$ that means that each vertex occurs in α. Then by Lemma 2.4, leaving out the first occurrence of each vertex, we still get a legal game α' with firing vector $f_\alpha - \mathbf{1}_V$. As $Lf_\alpha = L(f_\alpha - \mathbf{1}_V)$, this game still transforms x to y. We can continue this until there is a vertex that does not occur in our legal sequence of firings, hence we can suppose $S_\alpha \neq V$.

We use induction for the largest coordinate of f_α. If the largest coordinate of f_α is one, then we have $t = 1$ and $S_1 = S_\alpha \neq V$, and we are ready.

Suppose that the largest coordinate of f_α is larger than one. We prove that from initial distribution x, there exists a legal sequence of firings α' with firing vector $f_{\alpha'} = f_\alpha - \mathbf{1}_{S_\alpha}$ that can be extended legally by a sequence β of firings with firing vector $\mathbf{1}_{S_\alpha}$. Indeed, by Lemma 2.4, the sequence of firings α' that we get from α by deleting the first occurrence of each vertex is still legal. The firing vector of this sequence is $f_\alpha - \mathbf{1}_{S_\alpha}$. Play the bounded chip-firing game with upper bound f_α from initial distribution x. Then α' is a valid beginning. As α is a legal chip-firing game with upper bound f_α, and its firing vector is f_α, by Lemma 2.2, each maximal bounded chip-firing game with upper bound f_α has firing vector f_α. Hence α' can be extended to a legal game with firing vector f_α. Let us call the sequence of the last $|S_\alpha|$ firings β.

The largest coordinate of $f_{\alpha'}$ is strictly smaller than the largest coordinate of f_α, hence by the induction hypothesis, there is a legal sequence $\gamma = (v_1, \ldots, v_{s'})$ of firings with firing vector $f_{\alpha'}$ such that there exist indices $i_0 = 0, i_1, i_2, \ldots i_t = s'$ such that for each $j = 1, \ldots, t$, no vertex appears twice in the sequence $v_{i_{j-1}+1}, \ldots v_{i_j}$, and after setting $S_j = \{v_{i_{j-1}+1}, \ldots v_{i_j}\}$, we have $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t$. As α' can be legally extended by β, γ can also be legally extended by β, since the chip-distribution after a sequence of firings only depends on the firing vector, which is the same for γ and for α'. Now let $i_{t+1} = s$, thus $S_{t+1} = S_\alpha$. As $S_t \subseteq S_{\alpha'} \subseteq S_\alpha \neq V$, we proved the statement for α. $\square$

Remark 3.2. 'Ascending chains' also play a role in the related field of graph divisor theory, see for example [6, Lemma 1.3.] or the notion of 'level sets' in [7].

Theorem 3.3. *There is a polynomial algorithm that decides whether $x \rightsquigarrow y$ for two chip-distributions x and y on an Eulerian digraph G.*

Proof. The idea of the proof is to use Lemma 3.1: $x \rightsquigarrow y$ if and only if y can be reached from x by firing an "ascending chain of vertex sets". Since the chain is ascending, the number of distinct sets is at most $|V|$ (but one may occur in the sequence exponentially many times). By Gaussian elimination, we can compute whether $x \sim y$, and if so, we can compute what these ever growing sets should be. The main idea is that it is enough to check for each set, whether it can be fired at its last occurrence.

Let us write this formally. The algorithm is the following:

Using Gaussian elimination, we first decide whether $x \sim y$, that is, whether there exists an integer vector $g \in \mathbb{Z}^V$ such that $y = x + Lg$. If no such vector exists then $x \not\sim y$, hence $x \not\rightsquigarrow y$. If such a g exists, then let $k = \min_{v \in V} g(v)$ and $f = g - k \cdot \mathbf{1}_V$. Since $L\mathbf{1}_V = \mathbf{0}$, $y = x + Lf$. Moreover, the coordinates of f are non-negative, and it has a coordinate that is zero. Let $t = \max_{v \in V} f(v)$ and for $1 \leq j \leq t$, let $S_j = \{v \in V : f(v) \geq t - j + 1\}$. It is easy to see that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t \subsetneq V$. Let k be the number of distinct S_j's, and for $1 \leq i \leq k$ let a_i be the index of the first occurrence of the i'th smallest set among the S_j's. Also, set $a_{k+1} = t + 1$. With these notations, $S_j = S_{a_i}$ if $a_i \leq j < a_{i+1}$.

Now let $x_1 = x$ and define $x_j = x + \sum_{\ell=1}^{j-1} L\mathbf{1}_{S_\ell}$ for $j = 1, \ldots, t+1$. We do not compute all of these chip-distributions (as there can be exponentially many), but note that for a fixed j, x_j can be computed in polynomial time: If $a_i \leq j < a_{i+1}$ then

$$x_j = x_1 + L\left((j - a_i)\mathbf{1}_{S_{a_i}} + \sum_{\ell=1}^{i-1} (a_{\ell+1} - a_\ell)\mathbf{1}_{S_{a_\ell}} \right).$$

Now the algorithm proceeds as follows: For each $1 \leq i \leq k$ compute $x_{a_{i+1}-1}$ and $x_{a_{i+1}}$ and check whether $x_{a_{i+1}-1} \rightsquigarrow x_{a_{i+1}}$. This can also be done in polynomial time, since by Lemma 2.2 and Lemma 2.5 for each i we only need to check greedily whether the firing vector $\mathbf{1}_{S_{a_i}}$ can be fired from initial distribution $x_{a_{i+1}-1}$. If $x_{a_{i+1}-1} \rightsquigarrow x_{a_{i+1}}$ for each $1 \leq i \leq k$, then the algorithm returns $x \rightsquigarrow y$, otherwise the algorithm returns $x \not\rightsquigarrow y$.

Now we prove the correctness of the algorithm. First we prove that if the algorithm returns $x \rightsquigarrow y$ then $x \rightsquigarrow y$.

Note that $f = \sum_{j=1}^t \mathbf{1}_{S_j}$, hence $x_{t+1} = x + \sum_{j=1}^t L\mathbf{1}_{S_j} = x + Lf = y$. Thus for proving $x \rightsquigarrow y$, it is enough to prove for each $1 \leq j \leq t$ that $x_j \rightsquigarrow x_{j+1}$. So let $1 \leq j \leq t$, then for some $i \leq k$, $a_i \leq j < a_{i+1}$. Hence $x_{j+1} = x_j + L\mathbf{1}_{S_{a_i}}$.

Since the algorithm returned $x \rightsquigarrow y$, we have $x_{a_{i+1}-1} \rightsquigarrow x_{a_{i+1}}$. Let β be a legal game witnessing this. By Lemma 2.4, we can suppose that β has firing vector $\mathbf{1}_{S_{a_i}}$. We prove that β is also a legal game starting from the distribution x_j. For this, it is enough to show that $x_j(v) \geq x_{a_{i+1}-1}(v)$ for each $v \in S_j = S_{a_i}$. But this is true, since G is Eulerian, and $x_{a_{i+1}-1} = x_j + L \cdot (a_{i+1} - 1 - j)\mathbf{1}_{S_j}$. Hence each vertex $v \in S_j$ fires $(a_{i+1} - 1 - j) \cdot d^+(v)$ chips and gains

$$(a_{i+1} - 1 - j) \cdot \sum_{u \in S_j} \overrightarrow{d}(u, v) \leq (a_{i+1} - 1 - j) \cdot d^-(v) = (a_{i+1} - 1 - j) \cdot d^+(v)$$

chips as we change from x_j to $x_{a_{i+1}-1}$. Thus $x_j \rightsquigarrow x_{j+1}$ for every $j \leq t$, hence $x \rightsquigarrow y$.

Now it remains to show that if $x \rightsquigarrow y$ then the algorithm returns $x \rightsquigarrow y$. Let f' be the firing vector of the legal game $(v_1, v_2, \ldots, v_s)$ from x to y provided by Lemma 3.1. One can easily see that $f' = f$, hence the sets $S_1, \ldots, S_t$ coincide with those given by the lemma. Consequently, after firing $(v_1, \ldots v_{i_{j-1}})$ from initial distribution x, we arrive at x_j. If we continue the firing of the legal sequence and fire $(v_{i_{j-1}+1}, \ldots v_{i_j})$, then we arrive at x_{j+1}. Hence $(v_{i_{j-1}+1}, \ldots v_{i_j})$ is a legal game from initial distribution x_j, showing $x_j \rightsquigarrow x_{j+1}$. In particular, $x_{a_{i+1}-1} \rightsquigarrow x_{a_{i+1}}$ for each $1 \leq i \leq k$, hence the algorithm returns $x \rightsquigarrow y$.

The algorithm needs $O(|V|^3)$ basic arithmetical operations:

1. Preprocessing: computing the (diagonal elements d_i^+ of the) Laplacian: $O(|V|^2)$.

2. The firing vector $g \in \mathbb{Z}^V$ can be determined by Gaussian elimination in $O(|V|^3)$ time.

3. Having g, numbers a_i and characteristic vectors of the sets S_{a_i} can be determined in $O(|V|^2)$. (Note that $k \leq |V|$ as $\emptyset \subsetneq S_{a_1} \subsetneq S_{a_2} \subsetneq \cdots \subsetneq S_{a_k} \subsetneq V$ is a strictly ascending chain.)

4. Any x_j can be computed in $O(|V|^2)$. We need the vector x_j for at most $2|V|$ instances of j.

5. Checking whether $x_{a_{i+1}-1} \rightsquigarrow x_{a_{i+1}}$ needs at most $|S_{a_j}| \leq |V|$ firings. Checking whether a vertex can fire can be done in 1 step, since the out-degree of every vertex is already computed. Hence we can find in $|V|$ steps a vertex that can legally fire (if there is any). The effect of a firing can be computed in $O(|V|)$ time.

$\square$

4 General digraphs

Unfortunately, the algorithm of Section 3 is not valid for non-Eulerian digraphs. It is conjectured by Björner and Lovász in [2] that the reachability problem is NP-hard for general digraphs. In this section, we give two positive results: We show that the reachability problem is in co-NP, and we show a special case when it is decidable in polynomial time for general digraphs.

Theorem 4.1. *Let G be a digraph (with possibly multiple edges) and $x, y \in \mathrm{Chip}(G)$. Then deciding whether $x \rightsquigarrow y$ is in co-NP.*

Proof. As we noted in Section 2, the existence of $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$ is a necessary condition for $x \rightsquigarrow y$, that can be checked in strongly polynomial time. Hence in case there exists no $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$, our certificate for $x \not\rightsquigarrow y$ is simply the statement that there exists no $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$.

In case there exists $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$, our certificate is a pair of vectors $f, g \in \mathbb{Z}^V$ satisfying the following properties.

1. $x + Lf = y$, $f \geq \mathbf{0}$ and $f \not\geq p$ for any non-zero period vector p of G;

2. $\mathbf{0} \leq g \leq f$, and there exists $v \in V$ such that $g(v) < f(v)$;

3. For any $v \in V$, $g(v) = f(v)$ or $x_g(v) < d^+(v)$, where $x_g = x + Lg$.

All three conditions can be checked in polynomial time.

We claim that if $x \not\rightsquigarrow y$ and there exists $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$ then such f and g exist. Firstly, by subtracting an appropriate period vector of G from the vector $f \in \mathbb{Z}_+^V$ with $y = x + Lf$, we can ensure that $f \geq \mathbf{0}$ and $f \not\geq p$ for any non-zero period vector p of G.

Let g be the firing vector of the maximal bounded chip-firing game from initial distribution x with upper bound f. By Lemma 2.2, g is well defined. By the definition of the bounded game, $\mathbf{0} \leq g \leq f$. If $g = f$ then $x \rightsquigarrow y$, hence if $x \not\rightsquigarrow y$, then necessarily there exists $v \in V$ such that $g(v) < f(v)$. The third condition follows because g is the firing vector of a maximal game with upper bound f.

Now we prove that if such an f and g exist then $x \not\rightsquigarrow y$. Suppose for a contradiction that $x \rightsquigarrow y$. By Lemma 2.5, there exists a legal sequence $(v_1, v_2, \ldots, v_t)$ of firings with firing vector f that leads from x to y. Let j be the largest index such that $\sum_{i=1}^{j} \mathbf{1}_{v_i} \leq g$. Let h be the firing vector of the sequence $(v_1, v_2, \ldots, v_j)$ and let $x_h = x + Lh$. By the choice of j, $g \geq h$ and $g(v_{j+1}) = h(v_{j+1}) < f(v_{j+1})$, hence

$$x_g(v_{j+1}) - x_h(v_{j+1}) \geq L(g - h)(v_{j+1}) \geq 0.$$

Since $(v_1, v_2, \ldots, v_j, v_{j+1})$ is a legal sequence of firings, we get

$$d^+(v_{j+1}) \leq x_h(v_{j+1}) \leq x_g(v_{j+1}),$$

contradicting Condition 3.   □

4.1   Reachability of recurrent distributions

In this section, we show a case when the reachability problem can be decided in polynomial time also for general digraphs. More exactly, we give a case where the necessary condition of $x \sim y$ is also sufficient for $x \rightsquigarrow y$. Our theorem uses the notion of recurrent chip-distributions.

Definition 4.2. We call a chip-distribution $x \in \mathrm{Chip}(G)$ *recurrent* if there exists a non-empty sequence of legal firings that transforms x to itself.

Theorem 4.3. *Let G be strongly a connected digraph and $x, y \in \mathrm{Chip}(G)$. If y is recurrent and $x \sim y$, then $x \rightsquigarrow y$.*

Proof. First we claim that if $x \sim y$ then there exists $f \in \mathbb{Z}_+^V$ such that $x = y + Lf$. Indeed, $x \sim y$ implies the existence of $g \in \mathbb{Z}^V$ with $x = y + Lg$. From Proposition 2.3, for a sufficiently large $k \in \mathbb{Z}_+$, the vector $f = g + kp_G$ is non-negative, while $x = y + Lg = y + Lg + kLp_G = y + Lf$ holds.

Fix such an f. We proceed by induction on $\sum_{v \in V(G)} f(v)$. If $\sum_{v \in V(G)} f(v) = 0$, then $x = y$, thus $x \rightsquigarrow y$. Now suppose $\sum_{v \in V(G)} f(v) > 0$. As y is recurrent, there exists a sequence $(v_1, v_2, \ldots, v_k)$ of legal firings from initial distribution y (a vertex may occur multiple times), that leads back to y. Fix such a sequence. We claim that in this sequence, each vertex occurs at least once. Indeed, for the firing vector g of the game, $y = y + Lg$ thus g is a multiple of the primitive period vector, and the primitive period vector of a strongly connected digraph is strictly positive, as claimed in Proposition 2.3.

Let i be the smallest index such that $f(v_i) > 0$. Such an index exists because each vertex is listed at least once in $v_1, v_2, \ldots, v_k$. Starting from y, fire the vertices $v_1, \ldots, v_{i-1}$. This is a legal game by definition. Let the resulting distribution be y'. We claim that the sequence of firings $v_1, \ldots, v_{i-1}$ is also legal starting from x. To prove this, it is enough to show that $x(v_j) \geq y(v_j)$ for all $1 \leq j \leq i - 1$. This is true, because $x(v_j) = y(v_j) + (Lf)(v_j)$, where $(Lf)(v_j) \geq 0$, since the only negative element in the row corresponding to v_j is $L(v_j, v_j)$, but $f(v_j) = 0$. Hence the firing of the vertices $v_1, \ldots, v_{i-1}$ from distribution x is legal. Let the distribution obtained by this game be x'. Thus $x \rightsquigarrow x'$.

For x' and y', we also have $x' = y' + Lf$. At position y', firing v_i is legal, by definition of the sequence $v_1, \ldots, v_k$. Denote by y'' the distribution we get by firing v_i at y'. The distribution y'' is recurrent, since firing $v_{i+1} \ldots, v_k, v_1, \ldots, v_i$ is a legal game that leads back to y''. Now for x' and y'' we have $x' = y'' + Lf'$, where $f' = f - \mathbf{1}_{v_i}$. This way $\sum_{v \in V(G)} f'(v) = \sum_{v \in V(G)} f(v) - 1$, hence by the induction hypothesis, $x' \rightsquigarrow y''$.

We claim that $y'' \rightsquigarrow y$. Indeed, firing $v_{i+1}, \ldots, v_k$ starting from y'' is a legal game that leads to y. We also have $x \rightsquigarrow x'$. By transitivity, we have $x \rightsquigarrow y$.   □

This theorem raises the question of the complexity of deciding whether a given chip-distribution is recurrent. By results of Björner and Lovász (Lemmas 2.2 and 2.4), a chip-distribution x is recurrent if and only if there exists a nonzero primitive period vector p, such that started from x, the maximal chip-firing game with upper bound p has firing vector p. For Eulerian digraphs, this can be checked in polynomial time (even if the digraph has multiple edges). However, for general digraphs, the complexity of deciding recurrence is open.

Our aim is now to generalize Theorem 4.3 for weakly connected digraphs. Here, we need to use the stronger necessary condition of the existence of a non-negative f such that $y = x + Lf$. However, the condition of y being recurrent is not enough to make this condition sufficient on general digraphs. We show this by an example at the end of this section (Example 4.7). With a somewhat stronger condition, however, we can generalize Theorem 4.3 to weakly connected digraphs.

Theorem 4.4. *Let G be a weakly connected digraph, and $x, y \in \mathrm{Chip}(G)$ be two chip-distributions such that there exists $f \in \mathbb{Z}_+^V$ with $y = x + Lf$. Suppose that for each strongly connected component $G' = (V', E')$ of G, $f|_{V'} = \mathbf{0}$ or $y|_{V'} \in \mathrm{Chip}(G')$ is recurrent. Then $x \rightsquigarrow y$.*

Proof. Fix a non-negative vector $f \in \mathbb{Z}_+^V$ with $y = x + Lf$. Let $V_1, V_2, \ldots, V_k$ be a topological ordering of the strongly connected components of G, i.e., $V = V_1 \cup \cdots \cup V_k$,

for each i the digraph $G_i = (V_i, E|_{V_i \times V_i})$ is strongly connected, and there is no directed edge from $v_i \in V_i$ to $v_j \in V_j$ if $i > j$.

Let x' be the chip-distribution obtained from x by passing $f(u) \cdot \overrightarrow{d}(u, v)$ chips from u to v for each pair of vertices $u, v \in V$ where u and v are in different strongly connected components. Note that $x \not\sim x'$ is possible. The proof of the theorem is based on the following lemma.

Lemma 4.5. *For each i, $x'|_{V_i} \sim y|_{V_i}$ on the digraph G_i. Moreover, if $y|_{V_i}$ is recurrent on G_i, then there exists a legal game on G_i with firing vector $f|_{V_i}$ that transforms $x'|_{V_i}$ to $y|_{V_i}$.*

Proof. Let L_i be the Laplacian matrix of G_i. We first prove that $x'|_{V_i} \sim y|_{V_i}$ (as chip-distributions on G_i) by showing that $x'|_{V_i} + L_i f|_{V_i} = y|_{V_i}$. For this, let $v \in V_i$. Then

$$x'(v) + (L_i f|_{V_i})(v) =$$

$$x(v) + \sum_{v' \in V \setminus V_i} \left(\overrightarrow{d}(v', v) \cdot f(v') - \overrightarrow{d}(v, v') \cdot f(v) \right) + (L_i f|_{V_i})(v) =$$

$$x(v) + (Lf)(v) = y(v).$$

Now, if $y|_{V_i}$ is recurrent, by Theorem 4.3, $x'|_{V_i} \rightsquigarrow y|_{V_i}$ in G_i. Let $g_i \in \mathbb{Z}^{V_i}$ be the firing vector of a legal game transforming $x'|_{V_i}$ to $y|_{V_i}$. Then $L_i(f|_{V_i} - g_i) = 0$, hence by Proposition 2.3, $g_i - f|_{V_i} = c \cdot p_{G_i}$ with $c \in \mathbb{Z}$. If $c = 0$ then $f|_{V_i}$ is the firing vector of a legal game, proving the lemma. In the followings, we treat separately the cases $c < 0$ and $c > 0$.

Suppose that $c < 0$. Since $y|_{V_i}$ is recurrent, there is a legal game on G_i that transforms $y|_{V_i}$ back to itself. For the firing vector g of this game, $L_i g = 0$, hence $g = \lambda \cdot p_{G_i}$ with $\lambda \in \mathbb{Z}$, $\lambda > 0$. By Lemma 2.4, we can suppose that $\lambda = 1$. Now starting from distribution $x'|_{V_i}$ on G_i, after playing the legal game with firing vector g_i, we get to the distribution $y|_{V_i}$. Then iterate $-c$ times the legal game with firing vector p_{G_i}. This gives us a legal game with firing vector $f|_{V_i}$, finishing the proof for the $c < 0$ case.

Now suppose that $c > 0$. Then Lemma 2.4 guarantees that there is a legal game from $x'|_{V_i}$ with firing vector $g_i - c \cdot p_{G_i} = f|_{V_i}$. This finishes the proof of the lemma. $\square$

For each $1 \leq i \leq k$ let f_i be the vector with $f_i(v) = f(v)$ if $v \in V_i$, and $f_i(v) = 0$ otherwise. Let $s_i = \sum_{j \leq i} f_j$, i.e., $s_i(v) = f(v)$ if $v \in \bigcup_{j \leq i} V_j$, and $s_i(v) = 0$ otherwise. Let $x_i = x + L s_i$ and $x_0 = x$. We show that for $i = 1, \ldots, k$, starting from the distribution x_{i-1}, there is a legal game on G with firing vector f_i. Since $x_{i-1} + L f_i = x_i$, and $x_k = y$, this is enough to finish the proof of the theorem.

So let i be fixed. It is easy to see that for each $\in V_i$

$$x'(v) = x_{i-1}(v) - f(v) \cdot \sum_{v' \in V \setminus V_i} \overrightarrow{d}(v, v'). \tag{1}$$

If $f|_{V_i} = \mathbf{0}_{V_i}$, then $f_i = \mathbf{0}$, hence we have nothing to prove. If this is not the case, then $y|_{V_i}$ is recurrent by the assumptions. Using the lemma, from initial distribution

$x'|_{V_i}$ there exists a legal game on G_i with firing vector $f|_{V_i}$. We claim that the same sequence of firings on G, with initial distribution x_{i-1} remains a legal game. Indeed, we can see from (1) that by playing the game on G from initial distribution x_{i-1}, at any moment we have a distribution that is greater or equal on V_i than the distribution we get by playing the game on G_i with initial distribution $x'|_{V_i}$. Hence there exists a legal game on G with initial distribution x_{i-1} and firing vector f_i. This finishes the proof of the theorem. □

Note that for distributions y such that y is recurrent restricted to each strongly connected component, Theorem 4.4 gives a necessary and sufficient condition for the reachability of y. The condition of the theorem, i.e. whether there exists $f \in \mathbb{Z}_+^V$ with $y = x + Lf$, can also be decided in strongly polynomial time, as discussed in Section 2.

Now we give an example showing that Theorem 4.3 does not remain true for general digraphs, i.e. for general digraphs, the existence of $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$ and y being recurrent is not sufficient for $x \rightsquigarrow y$. First let us state some basic properties of recurrent chip-distributions.

Proposition 4.6. *Let G be a digraph.*

    (a) *If G is strongly connected and $x \in Chip(G)$ is recurrent, then x has at least one chip on some vertex of each directed cycle.*

    (b) *A chip-distribution $x \in \mathrm{Chip}(G)$ is recurrent if and only if there is a sink-component G_i of G such that $x|_{V(G_i)}$ is recurrent on G_i.*

    (c) *A recurrent chip-distribution is non-terminating.*

    (d) *A chip-firing game started from a recurrent chip-distribution takes only recurrent positions.*

    (e) *A chip-firing game started from a non-terminating chip-distribution reaches a recurrent chip-distribution after finitely many steps.*

Proof. Part (a) follows from the proof of [2, Theorem 2.2]. The other four statements can be proved by straightforward application of the definition of recurrence and of Proposition 2.3. □

Example 4.7. Let G_1 and G_2 be strongly connected digraphs, both of them with at least two vertex-disjoint directed cycles. Let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Let G be the following graph: $V(G) = \{v\} \cup V(G_1) \cup V(G_2)$, $E(G) = \{\overrightarrow{vv_1}, \overrightarrow{vv_2}\} \cup E(G_1) \cup E(G_2)$. Then the strongly connected components of G are $\{v\}$, G_1 and G_2, and the sink components are G_1 and G_2.

For $i = 1, 2$, let $x_i \in \mathrm{Chip}(G_i)$ be a non-recurrent chip-distribution and $y_i \sim x_i$ a recurrent chip-distribution. Such distributions exist: By part (a) of Proposition 4.6, a recurrent chip-distribution on G_i has at least one chip on each directed cycle. Also, by [2], if a distribution has at least $|E(G)| - |V(G)| + 1$ chips, then it is non-terminating. Hence, if x_i has $|E(G_i)|$ chips on one vertex of G_i and zero elsewhere,

then x_i is non-recurrent and non-terminating. By part (e) of Proposition 4.6, after playing the chip-firing game from x_i on G_i for finitely many steps, we arrive at a recurrent distribution $y_i \sim x_i$.

Let $x \in \text{Chip}(G)$ be the chip-distribution that is zero on v, agrees with x_1 on $V(G_1)$ and agrees with y_2 on $V(G_2)$. Let $y \in \text{Chip}(G)$ be the chip-distribution that is zero on v, agrees with y_1 on $V(G_1)$ and agrees with x_2 on $V(G_2)$. Then clearly there exists $f \in \mathbb{Z}_+^V$ such that $y = x + Lf$. Moreover both x and y are recurrent, since both of them are recurrent restricted to a sink component.

Since $x(v) = y(v) = 0$, if y were reachable from x on G, then x_1 were reachable from y_1 on G_1. But this is impossible, since y_1 is recurrent, while x_1 is not, and by part (d) of Proposition 4.6, a legal game started from a recurrent distribution never leads to a non-recurrent distribution.

5 Open questions and related problems

The most intriguing open question in the area is the complexity of the reachability problem on general digraphs. An interesting special case of this problem is deciding whether a chip-distribution on a general digraph is recurrent.

Problem 5.1. Let G be a digraph and $x, y \in \text{Chip}(G)$. What is the complexity of deciding whether $x \rightsquigarrow y$?

Problem 5.2. Let G be a digraph. What is the complexity of deciding whether a chip-distribution $x \in \text{Chip}(G)$ is recurrent?

We conjecture that both of these questions are co-NP-hard.

A related problem to the chip-firing reachability problem is the so-called chip-firing halting problem.

   Chip-firing halting problem Given a digraph G and a chip-distribution $x \in \text{Chip}(G)$, decide if x is terminating.

Informally, the halting problem and the reachability problem are both about determining the firing vector of a maximal game, only this game is a chip-firing game for the halting problem, and a bounded chip-firing game for the reachability problem.

The halting problem is known to be in **P** for simple Eulerian digraphs [2], and it is known to be NP-complete for general digraphs [5]. The complexity of the problem is open both for simple digraphs, and for Eulerian digraphs. We point out the following:

Proposition 5.3. *The chip-firing halting problem is in co-NP for Eulerian digraphs.*

Proof. Our certificate for "x is non-terminating" is a recurrent chip-distribution y such that $x \sim y$. Since the graph is Eulerian, both the fact that y is recurrent, and that $x \sim y$, can be checked in polynomial time.

We show that x is non-terminating if and only if such a y exists. If x is non-terminating, then by part (e) of Proposition 4.6, starting a legal chip-firing game

from x, we arrive at a recurrent chip-distribution y in finitely many steps. Moreover, then $y \sim x$, since $y = x + Lf$ for the firing vector of the game.

For the other direction, we use a lemma of Bond and Levine [4, Lemma 4.3.]: if for two chip-distributions x and y on a strongly connected digraph G, $x \sim y$, then x is terminating if and only if y is terminating. Note that now G is strongly connected since it is connected and Eulerian. If a recurrent y exists such that $x \sim y$, then y is non-terminating by part (c) of Proposition 4.6, therefore x is also non-terminating. $\square$

This means, that for Eulerian digraphs, the chip-firing halting problem is in NP $\cap$ co-NP.

Problem 5.4. Is there a polynomial time algorithm that decides the chip-firing halting problem for Eulerian digraphs (with multiple edges possible)?

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