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Abstract

Given a digraph $D = (V, A)$ and a positive integer k, an arc set $F \subseteq A$ is called a k**-arborescence** if it is the disjoint union of k spanning arborescences. The problem of finding a minimum cost k-arborescence is known to be polynomial-time solvable using matroid intersection. In this paper we study the following problem: find a minimum cardinality subset of arcs that contains at least one arc from every minimum cost k-arborescence. For $k = 1$, the problem was solved in [A. Bernáth, G. Pap , Blocking optimal arborescences, IPCO 2013]. In this paper we give an algorithm for general k that has polynomial running time if k is fixed.

1   Introduction

The *cuts* of a matroid are the minimal transversals of the family of bases; in other words, a subset of the elements is a cut if it is an inclusionwise minimal subset that contains at least one element from each base. The problem of finding minimum cuts in matroids has been studied in several different contexts (note the distinction between *minimal* and *minimum*: minimal is shorthand for *inclusionwise minimal*, while minimum means *minimum size*). Perhaps the best known special case is the minimum cut problem in graphs, which can be solved using network flows, and faster algorithms have also been developed (e.g. the Nagamochi-Ibaraki algorithm [11]). More generally, the minimum cut of kM, where M is a graphic matroid (or even a hypergraphic matroid, see [9]), can be found in polynomial time. A notable open question is the complexity of finding a minimum cut in a rigidity matroid.

The minimum cut of a transversal matroid can also be found in polynomial time; however, the problem of finding a minimum *circuit* of a transversal matroid is NP-complete [10], which implies that the minimum cut problem is NP-complete for gammoids. Another line of research considers the problem for binary matroids. NP-completeness was proved by Vardy [14]; Geelen, Gerards, and Whittle [7] conjecture

that the problem is in P for any minor-closed proper subclass of binary matroids. Partial results in this direction have been achieved by Geelen and Kapadia [8].

If we consider *minimum cost bases* (or *optimal bases* for brevity) of a matroid M, then these form the bases of another matroid which can be obtained by taking the direct sum of certain minors of M. Thus we can find a minimum transversal of the family of optimal bases of M by solving minimum cut problems in some minors of M. In particular, if the minimum cut problem is solvable in polynomial time in a minor-closed class of matroids, then a minimum transversal of optimal bases can also be found in polynomial time in this class. For example, since the class of graphic matroids is minor-closed and the minimum cut problem can be solved efficiently, we can also efficiently find a minimum transversal of optimal spanning trees in a graph with edge costs.

Our paper belongs to a line of research that considers directed versions of this problem. Let $D = (V, A)$ be a digraph with node set V and arc set A. A **spanning arborescence** is an arc set $F \subseteq A$ that is a spanning tree in the undirected sense and every node has in-degree at most one. Thus there is exactly one node, the **root node**, with in-degree zero. If the node set is clear from the context, spanning arborescences will be called **arborescences** for brevity. Arborescences can be considered as common bases of two matroids, so the problem of finding a minimum transversal of the family of arborescences is a special case of the minimum transversal problem for common bases of two matroids. This problem is NP-hard in general (as mentioned above, it is NP-hard even when the two matroids coincide). However, the special case for arborescences can be formulated as the minimization of the sum of the in-degrees of two disjoint node sets of the digraph, which can be solved efficiently using network flows. The problem of finding a minimum transversal of the family of *minimum cost arborescences* is considerably more difficult. It can still be solved in polynomial time as shown in [1], but the solution requires more sophisticated tools than network flows.

The arc-disjoint union of k spanning arborescences is called a k-**arborescence**. If $F \subseteq A$ is a k-arborescence in a digraph $D = (V, A)$, then its **root vector** is the vector $q \in \mathbb{Z}_+^V$ for which $q(v)$ counts the number of arborescences in F that are rooted at $v \in V$. Note that the root vector is determined by the in-degrees, as $q(v) = k - \varrho_F(v)$ for every $v \in V$, so it does not depend on the way a k-arborescence is decomposed into arborescences. If every arborescence has the same root node s, then F is called an s-**rooted k-arborescence**. Given $D = (V, A)$, k and a cost function $c : A \to \mathbb{R}_+$, a minimum cost k-arborescence or a minimum cost s-rooted k-arborescence can be found efficiently using the matroid intersection algorithm; see [12, Chapter 53.8] for a reference, where several related problems are considered. The existence of an s-rooted k-arborescence is characterized by Edmonds' disjoint arborescence theorem, while the existence of a k-arborescence is characterized by a theorem of Frank [4]. Frank also gave a linear programming description of the convex hull of k-arborescences, generalizing Edmonds' linear programming description of the convex hull of s-rooted k-arborescences.

In this paper we consider the following two problems.

Problem 1 (Blocking optimal k-arborescences). *Given a digraph $D = (V, A)$,*

a positive integer k, and a cost function $c : A \to \mathbb{R}_+$, find a minimum cardinality transversal of the family of minimum cost k-arborescences.

Problem   2 (Blocking optimal s-rooted k-arborescences). *Given a digraph $D = (V, A)$, a node $s \in V$, a positive integer k, and a cost function $c : A \to \mathbb{R}_+$, find a minimum cardinality transversal of the family of minimum cost s-rooted k-arborescences.*

In Section 2 we show that the two problems are polynomial-time equivalent. For $k = 1$, these problems have been solved in [1]. Moreover, Problem 1 is solved in [2] in the special case when $c \equiv 1$ (note that Problem 2 is a minimum cut problem when $c \equiv 1$). The papers [1, 2] also consider more general weighted versions of these problems.

The main result of the present paper is an algorithm for Problems 1 and 2 that has polynomial running time when k is constant. It remains open whether there is a polynomial-time algorithm when k is not fixed, or indeed whether there is an FPT algorithm where k is the parameter. Along the way we obtain the following result of independent interest: the convex hull of root vectors of minimum cost k-arborescences is a base polyhedron. This generalizes the result of Frank [4] stating that the root vectors of k-arborescences form a base polyhedron.

The paper is organized as follows. After a brief section on notation, the relationship between different versions of the problem is discussed in Section 2, including a dual characterization of optimal k-arborescences. The next section describes the **matroid-restricted k-arborescence** problem, a generalization of k-arborescences introduced by Frank [5] that is essential to the proof of the main result. In Section 4, we describe the connection between matroid-restricted k-arborescences and the dual characterization of optimal k-arborescences. A corollary of this connection is that the convex hull of the root vectors of optimal k-arborescences is a base polyhedron (Theorem 21).

The structure of minimal transversals is analyzed in Section 5. In the case when the size of the minimum transversal is at least k, we derive that there is a minimum transversal with a special structure (Theorem 31). This leads to the main result of the paper, an algorithm that finds a minimum transversal of optimal k-arborescences in polynomial time if k is constant.

1.1   Notation

Let us overview some of the notation and definitions used in the paper. Given a digraph $D = (V, A)$ and a node set $Z \subseteq V$, let $D[Z]$ be the subdigraph induced by Z. If $E \subseteq A$ is a subset of the arc set, then we will identify E and the subgraph (V, E). Thus $E[Z]$ is obtained from (V, E) by deleting the nodes of $V - Z$. The arc set of the digraph D will also be denoted by $A(D)$. The set of arcs of D entering a node set Z is denoted $\delta_D^{in}(Z)$, and $\varrho_D(Z) = |\delta_D^{in}(Z)|$. For an undirected or directed graph $G = (V, E)$ and a subset $X \subseteq V$, $i_G(X)$ denotes the number of edges with both endpoints in X.

A **subpartition** of a subset X of V is a collection of pairwise disjoint non-empty subsets of X. Note that $\emptyset$ cannot be a member of a subpartition, but $\emptyset$ is a valid subpartition, having no members at all. A set family $\mathcal{L} \subseteq 2^V$ is said to be **laminar** if any two members of $\mathcal{L}$ are either disjoint, or one contains the other. For a vector $x : A \to \mathbb{R}$ and subset $Z \subseteq A$ we use the notation $x(Z) = \sum_{a \in Z} x_a$.

In the paper we will use the $-$ (minus) operator in many roles beyond subtraction of numbers: for example we will use it for set-theoretical difference instead of $\backslash$. Furthermore, for a digraph $D = (V, A)$ and $E \subseteq A$ we will use the notation $D - E$ to mean the digraph $(V, A - E)$. A one-element set $\{e\}$ will be denoted without braces by e in some contexts; for example, $E - e$ means $E - \{e\}$, and this is used even if $e \notin E$, in which case $E - e = E$. Similarly, for a subpartition $\mathcal{X}$ and for a member $X \in \mathcal{X}$, we write $\mathcal{X} - X$ instead of $\mathcal{X} - \{X\}$.

For general background on **matroids** and **base polyhedra** we refer the reader to [6]. Given a matroid $M = (S, r)$ (where S is the ground set and r is the rank function) and a positive integer k, the k-**shortening** of M is the matroid (S, r') where $r'(E) = \min\{r(E), k\}$.

Given a function $p : 2^S \to \mathbb{R}$, a subset $X \subseteq S$ is called **separable** if there exists a partition $X_1, X_2, \ldots, X_t$ of X such that $p(X) \leq \sum_i p(X_i)$. The function p is called **near supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ holds for every intersecting pair $X, Y \subseteq V$ of non-separable sets. The **(upper) truncation** of a set function $p : 2^S \to \mathbb{R}$ (satisfying $p(\emptyset) = 0$) is a set function $p^\wedge : 2^S \to \mathbb{R}$ defined by

$$p^\wedge(X) = \max\{\sum\{p(Z) : Z \in \mathcal{Z}\} : \mathcal{Z} \text{ is a partition of } X\}.$$

Theorem 1. *[6, Theorems 15.1.1 and 15.1.3] The truncation of a near supermodular function is fully supermodular. The truncation of a nonnegative function is monotone increasing. If p is near supermodular and the polyhedron $B(p) = \{x \in \mathbb{R}^S : x(S) = p(S), x(Z) \geq p(Z) \ \forall Z \subseteq S\}$ is non-empty, then $B(p)$ is a base polyhedron and $B(p) = B(p^\wedge)$.*

Given a digraph $D = (V, A)$ and a positive integer α, we will often use an extended digraph $D^+ = (V + s, A^+)$, called the α-**extension** of D, that has a new node $s \notin V$ and α parallel arcs from s to every node in V. If a cost function $c : A \to \mathbb{R}$ is also given, then we extend c to a function $c^+ : A^+ \to \mathbb{R}$ so that $c^+(uv) = c(uv)$ for any $uv \in A$ and $c^+(sv) = \beta$ for any new arc $sv \in A^+ - A$, where β is some nonnegative real number. The weighted digraph (D^+, c^+) is then called the (α, β)-**extension of** (D, c).

2 Relationship between different versions of the problem

Theorem 2. *Problem 1 (Blocking optimal k-arborescences) and Problem 2 (Blocking optimal s-rooted k-arborescences) are polynomial-time equivalent.*

Proof. Problem 2 reduces to Probem 1 by deleting all arcs entering node s from the input digraph. For the other direction, consider an instance D, k, c of Problem 1, and let $\alpha = |A| + k$, $\beta = \sum_{a \in A} c(a) + 1$. Let (D^+, c^+) be the (α, β)-extension of (D, c). In the instance of Problem 2 given by (D^+, k, c^+, s), the minimum cost s-rooted k-arborescences naturally correspond to minimum c-cost k-arborescences in D (since they contain exactly k arcs leaving s because of the value of β). Moreover, the minimum size of a transversal is at most $|A|$ as A itself is a transversal. This shows that every minimum transversal is a subset of A. $\qquad\square$

To describe the structure of minimum cost k-arborescences, we introduce the notion of a k-arborescence being tight for some laminar family of node subsets. Given a digraph $D = (V, A)$ and a laminar family $\mathcal{L} \subseteq 2^V$, a k-arborescence $F \subseteq A$ is called **$\mathcal{L}$-tight** if $F[W]$ is a k-arborescence in $D[W]$ for every $W \in \mathcal{L}$. Note that if $\mathcal{L} \subseteq 2^{V-s}$, then an s-rooted k-arborescence $F \subseteq A$ is $\mathcal{L}$-tight if and only if $\varrho_F(W) = k$ for every $W \in \mathcal{L}$. The link between $\mathcal{L}$-tight s-rooted k-arborescences and minimum cost s-rooted k-arborescences is provided by the following theorem.

Theorem 3. *[12, Corollary 53.6a] Given a digraph $D = (V, A)$ and a node $s \in V$, the system (1)–(2) below is TDI, and it describes the convex hull of subsets of A containing an s-rooted k-arborescence.*

$$0 \le x(a) \le 1 \text{ for every } a \in A \tag{1}$$
$$\varrho_x(Z) \ge k \text{ for ever non-empty } Z \subseteq V - s. \tag{2}$$

If a cost function $c : A \to \mathbb{R}$ is also given and we consider the problem of minimizing cx under the conditions above, then there is an optimal dual solution where the dual variables corresponding to (2) have laminar support.

Complementary slackness conditions imply the following.

Corollary 4. *Given a digraph $D = (V, A)$, a cost function $c : A \to \mathbb{R}_+$, a node $s \in V$ and a positive integer k, one can find a laminar family $\mathcal{L} \subseteq 2^{V-s}$ and two disjoint arc-sets $A_0, A_1 \subseteq A$ with the property that an s-rooted k-arborescence $F \subseteq A$ has minimum cost if and only if $A_1 \subseteq F \subseteq A - A_0$ and F is $\mathcal{L}$-tight.*

Proof. Consider the LP $\min\{cx : x \in \mathbb{R}^A, 0 \le x \le 1, \varrho_x(Z) \ge k$ for ever non-empty $Z \subseteq V - s\}$. By Theorem 3, this has an integer optimal solution, which is a minimum cost s-rooted k-arborescence. Let y^*, z^* be an optimal solution of the dual

$$\max \sum_{\emptyset \neq Z \subseteq V - s} k y_Z - \sum_{a \in A} z_a$$
$$y \in \mathbb{R}_+^{2^{V-s} - \{\emptyset\}}, z \in \mathbb{R}_+^A$$
$$\sum_{Z : a \in \delta^{in}(Z)} y_Z - z_a \le c_a \text{ for every } a \in A.$$

We can assume that the support of y^* is a laminar family $\mathcal{L} \subseteq 2^V$ by Theorem 3. The complementary slackness conditions show that a feasible primal solution x^* is optimal if and only if the following three conditions hold.

1. $x_a^* = 0$ for every $a \in A$ with $\sum_{Z:a \in \delta^{in}(Z)} y_Z^* - z_a^* < c_a$ (**forbidden** arcs),

2. $\varrho_{x^*}(W) = k$ for every $W \in \mathcal{L}$, and

3. $x_a^* = 1$ for every $a \in A$ with $z_a^* > 0$ (**mandatory** arcs).

By denoting the forbidden arcs by A_0 and the mandatory arcs by A_1 we obtain the required structure. □

Theorem 5. *Problem 2 can be reduced to the following Problem 3 in polynomial time.*

Problem 3. *Given a digraph $D = (V, A)$, a root s, and a laminar family $\mathcal{L} \subseteq 2^{V-s}$, find a minimum cardinality transversal of the family of $\mathcal{L}$-tight s-rooted k-arborescences.*

Proof. Given a digraph $D = (V, A)$, a cost function $c : A \to \mathbb{R}_+$, a node $s \in V$ and a positive integer k, we consider A_0, A_1, and $\mathcal{L}$ as in Corollary 4. If there exists a mandatory arc, then it is a singleton transversal of the family of optimal s-rooted k-arborescences. If $A_1 = \emptyset$, then the problem is equivalent to finding a minimum transversal of the family of $\mathcal{L}$-tight s-rooted k-arborescences in $A - A_0$. □

Note that we can decide in polynomial time whether an $\mathcal{L}$-tight s-rooted k-arborescence exists by finding a minimum cost s-rooted k-arborescence for the cost function $c(e) = |\{W \in \mathcal{L} : e \in \delta_D^{in}(W)\}|$.

3   Matroid-restricted k-arborescences

In this section we introduce matroid-restricted k-arborescences, a notion that will be useful in describing the structure of $\mathcal{L}$-tight k-arborescences. Let $D = (V, A)$ be a digraph, and for every $v \in V$ let $M_v = (\delta_D^{in}(v), r_v)$ be a matroid. Let furthermore $\mathcal{M} = \{M_v : v \in V\}$ be the family of these matroids. A k-arborescence $F \subseteq A$ is said to be $\mathcal{M}$-**matroid-restricted** (or **matroid-restricted** for short) if $F \cap \delta_D^{in}(v)$ is independent in M_v for every $v \in V$. Similarly, an s-rooted k-arborescence $F \subseteq A$ is said to be $\mathcal{M}$-matroid-restricted if $F \cap \delta_D^{in}(v)$ is independent for every $v \in V - s$ (note that the matroid M_s does not play a role here). The notion of matroid-restricted s-rooted k-arborescence was introduced by Frank [5] in a slightly more general setting, where there is an additional matroid on the set of arcs leaving s. Our definition corresponds to the case where this is a free matroid. Some of the results of this section could be derived from [5, Theorem 4.5]; however, since the context is different, it is easier to include self-contained proofs.

Let us define the matroid $M^\oplus = (A, r^\oplus)$ as the direct sum of the matroids M_v ($v \in V$). The following theorem is an easy consequence of the matroid intersection theorem.

Theorem 6. *Given a digraph $D = (V, A)$ and matroids $M_v = (\delta_D^{in}(v), r_v)$ for every $v \in V$, there exists a matroid-restricted k-arborescence in D if and only if the following inequality holds for every subpartition $\mathcal{X}$ of V:*

$$\sum \{r^{\oplus}(\delta_D^{in}(X)) : X \in \mathcal{X}\} \geq k(|\mathcal{X}| - 1). \tag{3}$$

Proof of Theorem 6. The necessity of (3) is clear: if $F \subseteq A$ is a matroid-restricted k-arborescence and $\mathcal{X}$ is a subpartition of V, then $k(|\mathcal{X}| - 1) \leq \sum_{X \in \mathcal{X}} \varrho_F(X) \leq \sum_{X \in \mathcal{X}} r^{\oplus}(\delta_D^{in}(X))$. In order to prove sufficiency, let $M_1 = (A, r_1)$ be k times the circuit matroid of the underlying undirected graph of D. Note that condition (3) implies that D contains k edge-disjoint spanning trees, thus $r_1(A) = k(|V| - 1)$. For every $v \in V$, let $M_v' = (\delta_D^{in}(v), r_v')$ be the k-shortening of M_v, that is $r_v'(E) = \min\{r_v(E), k\}$ for every $E \subseteq \delta_D^{in}(v)$. Let furthermore $M_2 = (A, r_2)$ be the direct sum of the matroids M_v'. Observe that $F \subseteq A$ is a matroid-restricted k-arborescence in D if and only if F is a common independent set of M_1 and M_2 and has size $k(|V| - 1)$. By Edmonds' matroid intersection theorem [3], such an F exists if and only if

$$r_1(E) + r_2(A - E) \geq k(|V| - 1) \text{ for every } E \subseteq A. \tag{4}$$

We show that condition (3) implies (4). Suppose that (4) fails for some E. Clearly, we can assume that E is closed in M_1 and $M_1|E$ does not contain bridges (a **bridge** in a matroid is an element that is contained in every base).

Claim 7. *If $E \subseteq A$ is closed in M_1 and $M_1|E$ does not contain bridges, then there exists a partition $\mathcal{Y}$ of V such that $r_1(D[Y]) = k(|Y| - 1)$ for every $Y \in \mathcal{Y}$ and $E = \cup_{Y \in \mathcal{Y}} D[Y]$.*

Proof. We say that a non-empty $Y \subseteq V$ is **tight** (with respect to E) if $r_1(E[Y]) = k(|Y| - 1)$. In other words, Y is tight if $E[Y]$ contains k edge-disjoint trees, each spanning Y. For example, sets of size 1 are tight. If Y_1, Y_2 are both tight and $Y_1 \cap Y_2 \neq \emptyset$ then $Y_1 \cup Y_2$ is tight, too. To prove this, let $T_1 \subseteq E$ be a tree spanning Y_1 and $T_2 \subseteq E$ be a tree spanning Y_2, and observe that T_1 can be extended to a tree spanning $Y_1 \cup Y_2$ using the edges of $T_2 - E[Y_1]$. Therefore let $\mathcal{Y}$ be the partition of V consisting of the maximal tight sets. Since E is closed in M_1, it contains every arc of D that is induced in some $Y \in \mathcal{Y}$. Let $G' = (V', E')$ be the graph obtained from (V, E) after contracting every $Y \in \mathcal{Y}$ into a node y. We claim that $i_{G'}(Z) < k(|Z| - 1)$ for every $Z \subseteq V'$ with $|Z| \geq 2$. Assume not and take an inclusionwise minimal set Z with $i_{G'}(Z) \geq k(|Z| - 1)$. Then $G'[Z]$ contains k edge-disjoint spanning trees by the theorem of Tutte and Nash-Williams [13], which contradicts the maximality of the tight sets in $\mathcal{Y}$. This implies that the bases of $M_1|E$ contain every arc of E going between different members of the partition $\mathcal{Y}$. But since $M_1|E$ does not contain bridges, $E = \cup_{Y \in \mathcal{Y}} D[Y]$, as claimed. □

Consider the partition $\mathcal{Y}$ in the above claim and observe that $r_1(\cup_{Y \in \mathcal{Y}} D[Y]) + r_2(\cup_{Y \in \mathcal{Y}} \delta_D^{in}(Y)) = k(|V| - |\mathcal{Y}|) + \sum_{Y \in \mathcal{Y}} r_2(\delta_D^{in}(Y)) < k(|V| - 1)$, thus $\sum_{Y \in \mathcal{Y}} r_2(\delta_D^{in}(Y)) < k(|\mathcal{Y}| - 1)$. Let $\mathcal{X} = \{Y \in \mathcal{Y} : r_2(Y) < k\}$ and note that $\sum_{X \in \mathcal{X}} r_2(\delta_D^{in}(X)) < k(|\mathcal{X}| - 1)$ holds as well. But $r_2(\delta_D^{in}(X)) = r^{\oplus}(\delta_D^{in}(X))$ for every $X \in \mathcal{X}$, thus we get a contradiction with (3). □

Let us fix some $s \in V$. From now on we are interested in matroid-restricted s-rooted k-arborescences, and we assume $r_v(\delta_D^{in}(v)) = k$ for every $v \in V - s$. Let

$$\mathcal{B}^s = \{I \subseteq \delta_D^{out}(s) : |I| = k \text{ and}$$
$$\exists \text{ matroid-restricted } s\text{-rooted } k\text{-arborescence } F \subseteq A \text{ s.t. } I = F \cap \delta_D^{out}(s)\}. \quad (5)$$

Our aim below is to show that $\mathcal{B}^s$ is the family of bases of a matroid on ground set $\delta_D^{out}(s)$. For an arc set $I \subseteq \delta_D^{out}(s)$, we use the notation $I \cup D[V - s]$ for the digraph obtained from D by deleting the edges of $\delta_D^{out}(s) - I$.

Lemma 8. *Let $D = (V, A)$ be a digraph, let $s \in V$, and let $M_v = (\delta_D^{in}(v), r_v)$ be matroids of rank k for every $v \in V - s$. The following properties are equivalent for $I \subseteq \delta^{out}(s)$.*

(i) $I \in \mathcal{B}^s$,

(ii) $|I| = k$ and I satisfies $r^{\oplus}(\delta_{I \cup D[V-s]}^{in}(X)) \geq k$ for every non-empty $X \subseteq V - s$,

(iii) $|I| = k$ and I satisfies $|I \cap E| + r^{\oplus}(\delta_{D-E}^{in}(X)) \geq k$ for every $E \subseteq \delta_D^{out}(s)$ and non-empty $X \subseteq V - s$.

Proof. It is clear that (i) implies (ii). Let us prove that (ii) implies (i). Let $D' = I \cup D[V - s]$. We will prove that there exists a matroid-restricted k-arborescence in D' by applying Theorem 6. Suppose that $\sum \{r^{\oplus}(\delta_{D'}^{in}(X)) : X \in \mathcal{X}\} < k(|\mathcal{X}| - 1)$ for some subpartition $\mathcal{X}$. Note that we can assume $r^{\oplus}(\delta_{D'}^{in}(X)) < k$ for every member X of $\mathcal{X}$, and clearly $|\mathcal{X}| > 1$ has to hold. Therefore there must exist a member $X \in \mathcal{X}$ with $s \notin X$ and $r^{\oplus}(\delta_{D'}^{in}(X)) < k$, contradicting (ii).

Next we show that (i) implies (iii). If $F \subseteq A$ is a matroid-restricted s-rooted k-arborescence with $I = F \cap \delta_D^{out}(s)$, $E \subseteq \delta_D^{out}(s)$, and $X \subseteq V - s$, then $k \leq \varrho_F(X) = \varrho_{F \cap E}(X) + \varrho_{F-E}(X) \leq |F \cap E| + r^{\oplus}(\delta_{D-E}^{in}(X)) = |I \cap E| + r^{\oplus}(\delta_{D-E}^{in}(X))$. Finally, we show that (iii) implies (ii). Take some non-empty $X \subseteq V - s$, let $E = (\delta_D^{out}(s) \cap \delta_D^{in}(X)) - I$ and apply the property in (iii) for X and E to obtain (ii). $\qquad\square$

Consider the following polyhedron.

$$P = \{x \in \mathbb{R}^{\delta^{out}(s)} : \; x \geq 0, \quad (6)$$
$$x(E) \geq k - r^{\oplus}(\delta_{D-E}^{in}(X)) \text{ for every } E \subseteq \delta_D^{out}(s) \text{ and } \emptyset \neq X \subseteq V - s\}. \quad (7)$$

Clearly, P is non-empty if and only if $r^{\oplus}(\delta_D^{in}(X)) \geq k$ for every non-empty $X \subseteq V - s$ (the condition is necessary because otherwise (7) does not hold for $E = \emptyset$; on the other hand, if this condition holds, then $k\mathbf{1} \in P$). Furthermore, it is enough to require (7) for non-empty subsets X that contain the head of every arc of E. We can also observe that non-negativity of x is implied by (7) in the definition of P. Indeed, let $st \in A$ be arbitrary and apply (7) for $E = \{st\}$ and $X = \{t\}$ to get $x(st) \geq k - r_t(\delta^{in}(t) - st) \geq 0$.

From now on we assume that P is non-empty. Define the set function $p : 2^{\delta_D^{out}(s)} \to \mathbb{R}$ as

$$p(E) = \max\{k - r^{\oplus}(\delta_{D-E}^{in}(X)) : \emptyset \neq X \subseteq V - s\}. \quad (8)$$

Note that $p \leq k$ and $p(\delta_D^{out}(s)) = k - r^\oplus(\delta_{D-\delta_D^{out}(s)}^{in}(V-s)) = k$. Furthermore, $p(\emptyset) = 0$ ($p(\emptyset) \leq 0$ by the non-emptiness of P, and take any $v \in V - s$ and use $r_v(\delta_D^{in}(v)) = k$ to obtain $p(\emptyset) \geq k - r^\oplus(\delta_D^{in}(v)) = 0$), and p is monotone increasing. With this definition, P is described as

$$P = \{x \in \mathbb{R}^{\delta_D^{out}(s)} : x(E) \geq p(E) \text{ for every } E \subseteq \delta_D^{out}(s)\}.$$

Recall that a function $p : 2^S \to \mathbb{R}$ is **near supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ holds for every intersecting pair $X, Y \subseteq V$ of non-separable sets, where a set X is separable if there exists a partition $X_1, X_2, \ldots, X_t$ of X such that $p(X) \leq \sum_i p(X_i)$.

Theorem 9. *The function p defined in* (8) *is near supermodular.*

For the proof of Theorem 9 we need the following claims.

Claim 10. *Let $E_1, E_2 \subseteq \delta_D^{out}(s)$ and $X_1, X_2 \in V - s$ be arbitrary, then*

$$r^\oplus(\delta_{D-E_1}^{in}(X_1)) + r^\oplus(\delta_{D-E_2}^{in}(X_2)) \geq r^\oplus(\delta_{D-(E_1 \cup E_2)}^{in}(X_1 \cup X_2)) + r^\oplus(\delta_{D-(E_1 \cap E_2)}^{in}(X_1 \cap X_2)). \tag{9}$$

Proof. By the properties of the direct sum, it is enough to show the following for an arbitrary $v \in V$, where Δ denotes $\delta_D^{in}(v)$.

$$r_v(\delta_{\Delta-E_1}^{in}(X_1)) + r_v(\delta_{\Delta-E_2}^{in}(X_2)) \geq r_v(\delta_{\Delta-(E_1 \cup E_2)}^{in}(X_1 \cup X_2)) + r_v(\delta_{\Delta-(E_1 \cap E_2)}^{in}(X_1 \cap X_2)). \tag{10}$$

If $v \notin X_1 \cup X_2$, then there is nothing to prove, every term is zero on both sides of (10). If $v \in X_1 - X_2$, then the second term is zero on both sides of (10), and the inequality $r_v(\delta_{\Delta-E_1}^{in}(X_1)) \geq r_v(\delta_{\Delta-(E_1 \cup E_2)}^{in}(X_1 \cup X_2))$ is implied by the mononicity of r_v. Clearly, the case $v \in X_2 - X_1$ is analogous, therefore assume $v \in X_1 \cap X_2$. Observe that (11) and (12) holds. For an illustration, see Figure 1.

$$\delta_{\Delta-E_1}^{in}(X_1) \cap \delta_{\Delta-E_2}^{in}(X_2) = \delta_{\Delta-(E_1 \cup E_2)}^{in}(X_1 \cup X_2) \tag{11}$$

$$\delta_{\Delta-E_1}^{in}(X_1) \cup \delta_{\Delta-E_2}^{in}(X_2) = \delta_{\Delta-(E_1 \cap E_2)}^{in}(X_1 \cap X_2). \tag{12}$$

This, together with the submodularity of r_v, finishes the proof. $\square$

Let us introduce the following notation. For a set $E \subseteq \delta_D^{out}(s)$, let $X_E \subseteq V - s$ be an arbitrary subset that attains the maximum in the definition (8) of $p(E)$ (that is, $X_E \neq \emptyset$ and $p(E) = k - r^\oplus(\delta_{D-E}^{in}(X_E))$).

Claim 11. *If $E \subseteq \delta_D^{out}(s)$ is non-separable, then X_E contains the head of every arc of E.*

Proof. Suppose not and let $E_1 \subsetneq E$ be the subset of those arcs which have their head in X_E. Then $p(E) = k - r^\oplus(\delta_{D-E}^{in}(X_E)) = k - r^\oplus(\delta_{D-E_1}^{in}(X_E)) \leq p(E_1)$. But then $p(E) \leq p(E_1) + p(E - E_1)$ by the non-negativity of p, contradicting the non-separability of E. $\square$
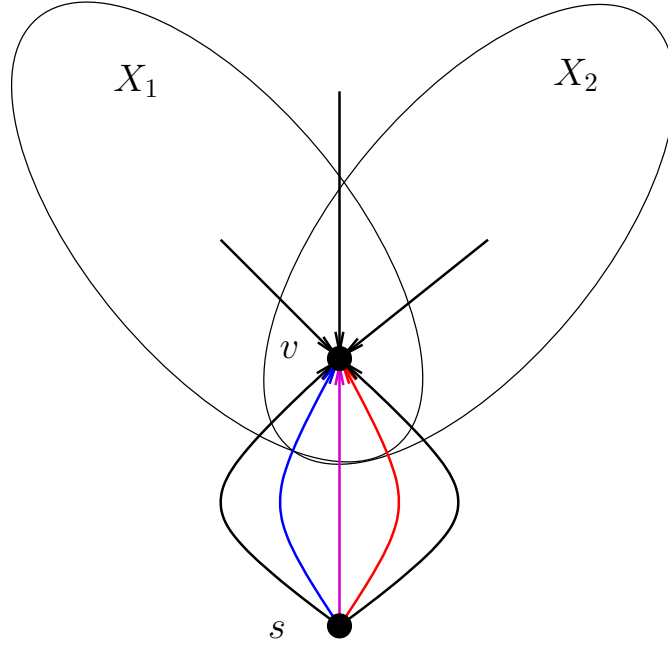
Figure 1: An illustration for proving (11) and (12). The arcs of $(E_1 - E_2) \cap \delta_D^{in}(v)$ are coloured blue, those in $(E_2 - E_1) \cap \delta_D^{in}(v)$ are red, and those in $(E_1 \cap E_2) \cap \delta_D^{in}(v)$ are magenta. That is, $\Delta - E_1$ is the set of arcs in the figure that are neither blue, nor magenta, etc.

Proof of Theorem 9. Let $E_1, E_2 \subseteq \delta_D^{out}(s)$ be non-separable sets so that $E_1 \cap E_2 \neq \emptyset$. By Claim 11, $X_i = X_{E_i}$ contains the head of each arc of E_i for both $i = 1, 2$. This implies that $X_1 \cap X_2 \neq \emptyset$, and Claim 10 gives

$$p(E_1) + p(E_2) = \sum_{i=1,2} k - r^{\oplus}(\delta_{D-E_i}^{in}(X_i)) \leq$$

$$2k - \left(r^{\oplus}(\delta_{D-(E_1 \cup E_2)}^{in}(X_1 \cup X_2)) + r^{\oplus}(\delta_{D-(E_1 \cap E_2)}^{in}(X_1 \cap X_2)) \right) \leq$$
$$p(E_1 \cap E_2) + p(E_1 \cup E_2).$$

$\square$

Theorems 1 and 9 imply that P is an integer polyhedron. It is also easy to see the following.

Corollary 12. *The polyhedron $B = \{x \in P : x(\delta_D^{out}(s)) = k\}$ (if not empty) is a base polyhedron of a matroid. It is the convex hull of incidence vectors of members of $\mathcal{B}^s$.*

Proof. We show that $x \in B$ implies $x \leq 1$. This, together with Theorems 1 and 9 and Lemma 8, proves the corollary. Take $x \in B$ and $st \in A$. Let $E = \delta^{out}(s) - st$ and $X = V - s$. By (7), we have $k - x(st) = x(E) \geq k - r^{\oplus}(\delta_{st}^{in}(V - s)) = k - r_t(\{st\}) \geq k - 1$. $\square$

The following claim describes the (fully supermodular) truncation of p.

Claim 13. *For any $E \subseteq \delta_D^{out}(s)$,*

$$p^\wedge(E) = \max \left\{ \sum_{X \in \mathcal{X}} (k - r^\oplus(\delta_{D-E}^{in}(X))) : \mathcal{X} \text{ is a subpartition of } V - s \right\}. \quad (13)$$

Proof. Let $E \subseteq \delta_D^{out}(s)$ and let $\mathcal{H}$ be a partition of E that gives $p^\wedge(E) = \sum\{p(H) : H \in \mathcal{H}\}$ and, subject to this, $|\mathcal{H}|$ is minimal. Clearly, every $H \in \mathcal{H}$ is non-separable. We claim that $\{X_H : H \in \mathcal{H}\}$ is a subpartition of $V - s$. If there exist $H_1, H_2 \in \mathcal{H}$ so that $X_{H_1} \cap X_{H_2} \neq \emptyset$, then (by Claim 10) $p(H_1) + p(H_2) \leq 2k - (r^\oplus(\delta_{D-(H_1 \cap H_2)}^{in}(X_{H_1} \cap X_{H_2})) + r^\oplus(\delta_{D-(H_1 \cup H_2)}^{in}(X_{H_1} \cup X_{H_2}))) \leq p(H_1 \cup H_2) + p(\emptyset) = p(H_1 \cup H_2)$, therefore $\mathcal{H}' = \mathcal{H} - \{H_1, H_2\} + \{H_1 \cup H_2\}$ also gives $p^\wedge(E) = \sum\{p(H) : H \in \mathcal{H}'\}$, contradicting our choice of $\mathcal{H}$. $\square$

Corollary 14. *Let $D = (V, A)$ be a digraph, let $s \in V$, and let $M_v = (\delta_D^{in}(v), r_v)$ $(v \in V - s)$ be matroids of rank k. The family $\mathcal{B}^s$ defined in (5), if non-empty, defines the family of bases of a matroid M^s on ground set $\delta_D^{out}(s)$. The family is not empty if and only if*

(a) $r^\oplus(\delta_D^{in}(X)) \geq k$ for every non-empty $X \subseteq V - s$, and

(b) $\sum\{k - r^\oplus(\delta_{D[V-s]}^{in}(X)) : X \in \mathcal{X}\} \leq k$ for every subpartition $\mathcal{X}$ of $V - s$.

The rank function of M^s is given by the following formula for any $E \subseteq \delta_D^{out}(s)$:

$$r^s(E) = \min \left\{ \sum_{X \in \mathcal{X}} r^\oplus(\delta_{E \cup D[V-s]}^{in}(X)) - k(|\mathcal{X}| - 1) : \mathcal{X} \text{ is a subpartition of } V - s \right\}.$$

Proof. Consider the function $p^\wedge$ defined by (13). By Theorem 1, $p^\wedge$ is monotone increasing and supermodular, and $P = \{x \in \mathbb{R}^{\delta^{out}(s)} : x(E) \geq p^\wedge(E)$ for every $E \subseteq \delta_D^{out}(s)\}$ if P is non-empty. Thus $B = \{x \in P : x(\delta_D^{out}(s)) = k\}$ is not empty if and only if $P \neq \emptyset$ and $p^\wedge(\delta_D^{out}(s)) = k$, that is, if and only if both (a) and (b) hold. Since the fully supermodular function describing the base polyhedron B is $p^\wedge$, it is the co-rank function of the matroid M^s, and its rank function is given by the formula

$$r^s(E) = p^\wedge(\delta_D^{out}(s)) - p^\wedge(\delta_D^{out}(s) - E) = k - p^\wedge(\delta_D^{out}(s) - E)$$

$$= \min\{\sum_{X \in \mathcal{X}} r^\oplus(\delta_{E \cup D[V-s]}^{in}(X)) - k(|\mathcal{X}| - 1) : \mathcal{X} \text{ is a subpartition of } V - s\}.$$

$\square$

4 Matroidal description of $\mathcal{L}$-tight k-arborescences

Let $D = (V, A)$ be a digraph, let $\mathcal{L} \subseteq 2^V$ be a laminar family, and assume that there exists an $\mathcal{L}$-tight k-arborescence in D. Without loss of generality, we also assume that

V and all singletons are in $\mathcal{L}$. Let furthermore D^+ denote the $(|A| + k)$-extension of D. The $\mathcal{L}$-tight k-arborescences in D^+ are all rooted at s and, since $V \in \mathcal{L}$, there is a natural (though not one-to-one) correspondence between $\mathcal{L}$-tight k-arborescences in D and those in D^+. For $W \in \mathcal{L}$, let D_W denote the digraph obtained from D^+ by contracting $V + s - W$ to a single node s_W and removing the loops that arise. Note that there is a natural bijection between $\delta_{D_W}^{out}(s_W)$ and $\delta_{D^+}^{in}(W)$; we will basically identify these two arc-sets in the discussion below. The main theorem of this section is the following.

Theorem 15. *The family $\mathcal{B}_W = \{I \subseteq \delta_{D_W}^{out}(s_W) : |I| = k$ and I can be extended to an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence in $D_W\}$ forms the family of bases of a matroid $M_W = (\delta_{D_W}^{out}(s_W), r_W)$.*

Proof. We recursively show that the family $\mathcal{B}_W$ indeed defines a matroid M_W for every $W \in \mathcal{L}$. For the singletons $\{v\} \in \mathcal{L}$ it is clear that $M_{\{v\}}$ is the uniform matroid of rank k on ground set $\delta_{D^+}^{in}(v)$. Let $W \in \mathcal{L}$ be a non-singleton, and assume that $M_{W'}$ has already been defined for every $W' \in \mathcal{L}$ that is a proper subset of W. Let $W_1, W_2, \ldots, W_l$ be the maximal members of $\mathcal{L}[W] - W$, and let us contract each W_i into a single node w_i ($i = 1, 2, \ldots, l$). Let $\hat{W} = W/\{W_1, W_2, \ldots, W_l\}$ be the set obtained from W by these contractions, and similarly, for a subgraph $(W + s_W, E)$ of D_W we use the notation $\hat{E} = E/\{W_1, W_2, \ldots, W_l\}$ to mean the graph obtained from $(W + s_W, E)$ by the contractions (and deletion of the loops that arise). In particular, let $\hat{D} = D_W/\{W_1, W_2, \ldots, W_l\}$. The matroids M_{W_i} naturally give rise to matroids $M_{w_i} = (\delta_{\hat{D}}^{in}(w_i), r_{w_i})$ for every i; let $\mathcal{M} = \{M_{w_1}, \ldots, M_{w_l}\}$.

Claim 16. *If $F \subseteq A(D_W)$ is an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence, then $\hat{F}$ is $\mathcal{M}$-matroid-restricted. Conversely, if $F' \subseteq \hat{D}$ is an $\mathcal{M}$-matroid-restricted s_W-rooted k-arborescence in $\hat{D}$ and $|\delta_{F'}^{out}(s_W)| = k$, then there exists an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence $F \subseteq A(D_W)$ such that $\hat{F} = F'$.*

Proof. The first statement is clear from the definition of the matroids M_{w_i}. For the other direction, let $F' \subseteq \hat{D}$ be an $\mathcal{M}$-matroid-restricted s_W-rooted k-arborescence in $\hat{D}$, such that $|\delta_{F'}^{out}(s_W)| = k$. Consider F' as a subgraph of D_W, and note that $\delta_{F'}^{in}(W_i)$ is a base of M_{W_i} for every i. By the definition of M_{W_i}, $\delta_{F'}^{in}(W_i)$ can be extended to an $\mathcal{L}[W_i]$-tight arborescence F_i in D_{W_i} for every i. The s_W-rooted k-arborescence $F = F' \bigcup \cup_i F_i$ is $\mathcal{L}[W]$-tight and $\hat{F} = F'$, as required. $\square$

The claim implies that $\mathcal{B}_W$ consists of the arc sets of size k that can be obtained as the arcs incident to s_W of an $\mathcal{M}$-matroid-restricted s_W-rooted k-arborescence, so the statement of the theorem follows from Corollary 14. $\square$

Corollary 17. *The matroids defined in Theorem 15 have the property that a k-arborescence $F \subseteq A(D^+)$ is $\mathcal{L}$-tight if and only if $F \cap \delta_{D^+}^{in}(W)$ is a base of M_W for every $W \in \mathcal{L}$.*

A recursive formula for the rank function r_W of the matroid M_W defined in Theorem 15 can be deduced from Corollary 14. We state this recursive formula expicitly below
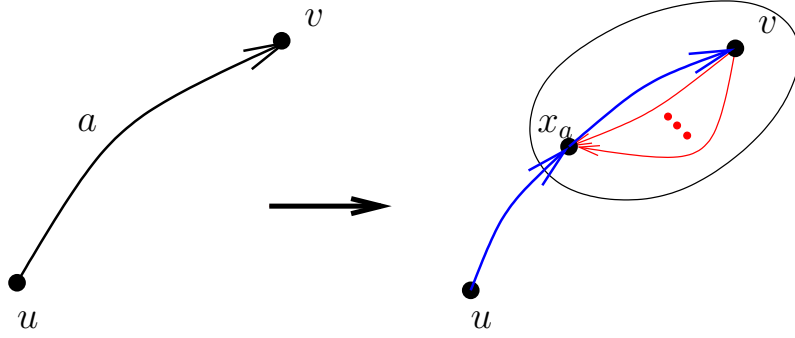
Figure 2: An illustration for the MANDATORY ARC TRANSFORMATION.

because it will be used extensively. Let $W_1, \ldots, W_l$ denote the maximal members of $\mathcal{L}[W] - W$. For an arc set $E \subseteq \bigcup_{i=1}^{l} \delta_{D^+}^{in}(W_i)$, we use the notation $r_W^\oplus(E) = \sum_{i=1}^{l} r_{W_i}(E \cap \delta_{D^+}^{in}(W_i))$. A subset X of W is called $\mathcal{L}[W]$-**compatible** if it is the union of some maximal members of $\mathcal{L}[W] - W$. A subpartition $\mathcal{P}$ of W is $\mathcal{L}[W]$-**compatible** if every member of $\mathcal{P}$ is $\mathcal{L}[W]$-compatible.

Corollary 18. *Let $W \in \mathcal{L}$ and $E \subseteq \delta_{D^+}^{in}(W)$. If $|W| = 1$, then $r_W(E) = \min\{k, |E|\}$; otherwise*

$$r_W(E) = \min\{\sum_{X \in \mathcal{X}} r_W^\oplus(\delta_{E \cup D[W]}^{in}(X)) - k(|\mathcal{X}| - 1) :$$

$$\mathcal{X} \text{ is an } \mathcal{L}[W]\text{-compatible subpartition of } W\}.$$

Theorem 15 for $W = V$ gives the following corollary.

Corollary 19. *The convex hull of root vectors of $\mathcal{L}$-tight k-arborescences in D is a base polyhedron.*

Theorem 15 in itself does not imply that the root vectors of minimum-cost k-arborescences also determine a base polyhedron, because we have to deal with mandatory arcs, i.e. the arcs of A_1 in Corollary 4. The following transformation solves this issue.

MANDATORY ARC TRANSFORMATION   Given a digraph $D = (V, A)$, a node $s \in V$, an arc $a = uv$ (where $u, v \in V - s$), and a laminar family $\mathcal{L} \subseteq 2^{V-s}$, we construct a digraph $D' = (V + x_a, A - a + B_a)$, where $B_a = \{ux_a, x_a v\} \cup \{k - 1 \text{ parallel copies of } vx_a\}$. Let furthermore $\mathcal{L}' \subseteq 2^{V+x_a}$ be defined as $\mathcal{L}' = \{W \in \mathcal{L} : v \notin W\} \cup \{W + x_a : v \in W \in \mathcal{L}\}$ (note that $\{v\} \in \mathcal{L}$ implies that $\{x_a, v\} \in \mathcal{L}'$). See Figure 2 for an illustration. It is easy to check that $\mathcal{L}'$ is laminar.

Claim 20. *For an $\mathcal{L}$-tight s-rooted k-arborescence $F \subseteq A$ containing $a = uv$, let $\phi(F) = F - a + B_a$. Then ϕ is a bijection between $\mathcal{L}$-tight s-rooted k-arborescences containing a in D and $\mathcal{L}'$-tight s-rooted k-arborescences in D'.*

Proof. First we show that if $F \subseteq A$ is an $\mathcal{L}$-tight s-rooted k-arborescence containing a, then $\phi(F)$ is an $\mathcal{L}'$-tight s-rooted k-arborescence in D'. Let $F_1, F_2, \ldots, F_k$ be a decomposition of F into k s-rooted arborescences and assume that $a \in F_1$. Let $F_1' = F_1 - a + \{ux_a, x_a v\}$, and let $F_i' = F_i$ plus a copy of the arc vx_a for every $i = 2, \ldots, k$. Then $F_1', F_2', \ldots, F_k'$ is a decomposition of $\phi(F)$ into k s-rooted arborescences in D', so $\phi(F)$ is indeed an s-rooted k-arborescence in D'. Furthermore, $\phi(F)$ is $\mathcal{L}'$-tight, as $\phi(F)[\{x_a, v\}]$ is a k-arborescence, and the indegree of any other set $W \in \mathcal{L}'$ in the subgraph $\phi(F)$ is k.

For the other direction, let $F' \subseteq A'$ be an arbitrary $\mathcal{L}'$-tight s-rooted k-arborescence in D'. Since $F'[\{x_a, v\}]$ is a k-arborescence and $\varrho_{F'}(x_a) = k$, $B_a \subseteq F'$ must hold. Let $F = F' - B_a + a$; we show that F is a $\mathcal{L}$-tight s-rooted k-arborescence in D – since $a \in F$ and $F' = \phi(F)$, this completes the proof. Let $F_1', F_2', \ldots, F_k'$ be a decomposition of F' into k s-rooted arborescences in D', and assume that $ux_a \in F_1'$. Then clearly $x_a v$ is in F_1' too, so $F_1' - \{ux_a, x_a v\} + a, F_2' - vx_a, F_3' - vx_a, \ldots, F_k' - vx_a$ is a decomposition of F into k s-rooted arborescences in D. The $\mathcal{L}$-tightness of F can be shown similarly. $\qquad\square$

Using this transformation we can now prove the following.

Theorem 21. *The convex hull of the root vectors of optimal k-arborescences is a base polyhedron.*

Proof. Given a digraph $D = (V, A)$ and a cost function $c : A \to \mathbb{R}$, let $\alpha = k + 1$, $\beta = \sum_{a \in A} c(a) + 1$, and let (D^+, c^+) be the (α, β)-extension of (D, c). By previous remarks, optimal k-arborescences in D and optimal k-arborescences in D^+ correspond to each other in a natural way (and k-arborescences in D^+ are rooted at s). By Corollary 4, there exists a laminar family $\mathcal{L} \subseteq 2^V$ and two disjoint sets $A_0, A_1 \subseteq A^+$, such that a k-arborescence $F \subseteq A^+$ is optimal if and only if $A_1 \subseteq F \subseteq A^+ - A_0$ and F is $\mathcal{L}$-tight. Due to symmetry, A_1 contains either all or none of the parallel arcs between s and a given node $v \in V$. Since there are $k + 1$ parallel arcs, the former is impossible, so $A_1 \subseteq A$.

Starting with $D^+ - A_0$, repeat the MANDATORY ARC TRANSFORMATION above for every $a \in A_1$, to obtain $D' = (V + s + \{x_a : a \in A_1\}, A^+ - (A_0 \cup A_1) + \cup_{a \in A_1} B_a))$ and the laminar family $\mathcal{L}' \subseteq 2^{V + \{x_a : a \in A_1\}}$. For any $\mathcal{L}$-tight s-rooted k-arborescence $F \subseteq A^+$ with $A_1 \subseteq F \subseteq A^+ - A_0$, let $\phi(F) = F - A_1 + \cup_{a \in A_1} B_a$. By Claim 20, ϕ defines a bijection between $\mathcal{L}$-tight s-rooted k-arborescences in $D^+ - A_0$ containing A_1 and $\mathcal{L}'$-tight s-rooted k-arborescences in D'. By Corollary 17, the family $\{I \subseteq \delta_{D'}^{out}(s) : |I| = k$ and I is contained in a $\mathcal{L}'$-tight s-rooted k-arborescence of $D'\}$ is the family of bases of a matroid. This implies that the convex hull of root vectors of optimal k-arborescences in D is a base polyhedron. $\qquad\square$

5 Blocking $\mathcal{L}$-tight k-arborescences

In this section we show that if k is fixed, then there is a polynomial-time algorithm that finds a minimum transversal of the family of $\mathcal{L}$-tight k-arborescences. Let $D = (V, A)$

be a digraph and let $\mathcal{L} \subseteq 2^V$ be a laminar family. We assume that $\mathcal{L}$ contains V and all the singletons, and that D contains an $\mathcal{L}$-tight k-arborescence. Let D^+ be the α-extension of D, where $\alpha = |A| + k$. The minimum transversals for D and D^+ are the same because the arcs sv have $|A| + k$ copies each, so these arcs never appear in a minimum transversal. Recall that for $W \in \mathcal{L}$, the digraph D_W is obtained by contracting $V + s - W$ in D^+ to a single root node s_W.

In what follows, we will often use the matroids $M_W = (\delta_{D^+}^{in}(W), r_W)$ for $W \in \mathcal{L}$, as defined in Theorem 15. Furthermore, we will often remove some subset of arcs $H \subseteq A$ from D^+ and we will usually denote $D^+ - H$ by D'. Thus D'_W for some $W \in \mathcal{L}$ will denote the digraph obtained from $D^+ - H$ by contracting $V + s - W$ into a single node s_W. If D'_W contains an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence for some $W \in \mathcal{L}$, then we can consider the modified matroid obtained by using D'_W in place of D_W in Theorem 15. To emphasize the dependence of this matroid on D', we denote it by $M_{D',W}$, and its rank function by $r_{D',W}$. Likewise, we use the notation $r_{D',W}^{\oplus}(E)$ in place of $r_W^{\oplus}(E)$ if we refer to the direct sum defined using D'.

For a non-singleton $W \in \mathcal{L}$ and an arc set $E \subseteq \delta_{D^+}^{in}(W)$, we say that an $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W **determines** $r_W(E)$ if $r_W(E) = \sum_{X \in \mathcal{X}} r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X)) - k(|\mathcal{X}| - 1)$. By Corollary 18, such a subpartition exists. Notice that if $\mathcal{X}$ determines $r_W(E)$, then $r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X)) \leq k$ for every $X \in \mathcal{X}$. Moreover, if $r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X)) = k$ for some $X \in \mathcal{X}$, then $\mathcal{X} - X$ also determines $r_W(E)$. In particular, if $r_W(E) = k$, then $r_W(E)$ is determined by the empty subpartition.

Our first lemma shows that the rank of an arc set cannot decrease by more than one if we remove only one arc from D.

Lemma 22. *Let $E \subseteq \delta_{D^+}^{in}(W)$, and let $D' = D^+ - e$ for an arbitrary arc $e \in D_W$ (not necessarily in E). If D'_W contains an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence, then*

$$r_W(E) - 1 \leq r_{D',W}(E - e) \leq r_W(E).$$

Proof. Let $E' = E - e$. The inequalities $r_{D',W}(E') \leq r_W(E') \leq r_W(E)$ follow from the definition of the rank. We prove the remaining inequality by induction on the size of $\mathcal{L}[W]$; it is clearly true if W is a singleton. Otherwise, by Corollary 18, there is an $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W that determines $r_{D',W}(E')$, i.e. $r_{D',W}(E') = \sum_{X \in \mathcal{X}} r_{D',W}^{\oplus}(\delta_{E' \cup D'[W]}^{in}(X)) - k(|\mathcal{X}| - 1)$. We know by induction that $r_{D',W}^{\oplus}(\delta_{E' \cup D'[W]}^{in}(X)) \geq r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X)) - 1$ for every $X \in \mathcal{X}$, and the ranks are different for at most one member of $\mathcal{X}$, since $e \in D_{W_i}$ for at most one W_i. This proves the inequality because $r_W(E) \leq \sum_{X \in \mathcal{X}} r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X)) - k(|\mathcal{X}| - 1)$. $\square$

The next result is a characterization of inclusionwise minimal transversals lying inside A.

Theorem 23. *Let $H \subseteq A$ be an inclusionwise minimal transversal of the family of $\mathcal{L}$-tight k-arborescences in D^+. Let $D' = D^+ - H$ and let $W \in \mathcal{L}$ be an inclusionwise minimal member of $\mathcal{L}$ for which D'_W does not contain an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence. Then $H \subseteq D[W]$, and there is an $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W such that $\sum_{X \in \mathcal{X}} r_{D',W}^{\oplus}(\delta_{D'[W]}^{in}(X)) = k(|\mathcal{X}| - 1) - 1$.*

Proof. First note that $|W| > 1$, since $H \subseteq A$. As $H \cap D_W$ is a transversal of $\mathcal{L}[W]$-tight s_W-rooted k-arborescences (and hence of $\mathcal{L}$-tight k-arborescences), minimality of H implies that $H \subseteq D_W$. Let $W_1, \ldots, W_l$ be the maximal members of $\mathcal{L}[W] - W$. By the choice of W, D'_{W_i} contains an $\mathcal{L}[W_i]$-tight s_{W_i}-rooted k-arborescence for every i, thus $r^{\oplus}_{D',W}$ is well-defined.

Since D'_W does not contain an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence, (a) or (b) fails to hold in Corollary 14 for $r^{\oplus}_{D',W}$. Suppose that $r^{\oplus}_{D',W}(\delta^{in}_{D'_W}(X)) < k$ for some $\mathcal{L}[W]$-compatible subset $X \subseteq W$. Then there is a set W_i such that $r_{D',W_i}(\delta^{in}_{D'}(W) \cap \delta^{in}_{D'}(W_i)) < k$. However, since we did not delete any arc leaving s, and already the arcs going from s to W_i have rank k in M_{W_i}, we get (by monotonicity of r_{W_i}) that $r_{W_i}(\delta^{in}_{D'}(W) \cap \delta^{in}_{D'}(W_i)) = k$, a contradiction.

Thus (b) fails to hold in Corollary 14, that is, $\sum_{X \in \mathcal{X}} r^{\oplus}_{D',W}(\delta^{in}_{D'[W]}(X)) < k(|\mathcal{X}| - 1)$ for some $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W. As H is inclusionwise minimal and the removal of an arc can decrease a rank by at most one according to Lemma 22, the left hand side must be equal to $k(|\mathcal{X}| - 1) - 1$. Since the formula involves only arcs in $D'[W]$, minimality also implies that $H \subseteq D[W]$. $\square$

The characterization in the theorem does not lead automatically to an efficient algorithm for finding a transversal of minimum size. In fact, for a given X with $r^{\oplus}_W(\delta^{in}_{D[W]}(X)) = k$, it is not clear how to compute the minimmum number of arcs that have to be removed in order to decrease the rank by one. However, the following lemma implies that if the rank is strictly smaller than k, then we can decrease it by removing only one arc.

Lemma 24. *Let $W \in \mathcal{L}$ and $E \subseteq \delta^{in}_{D^+}(W)$ such that $0 < r_W(E) < k$. Then there exists an arc $e \in E \cup D[W]$ such that either $D_W - e$ does not contain an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence, or $r_{D',W}(E - e) = r_W(E) - 1$, where $D' = D^+ - e$.*

Proof. The proof is by induction on $|W|$; the claim is clearly true if W is a singleton. Let $W_1, \ldots, W_l$ be the maximal members of $\mathcal{L}[W] - W$. By Corollary 18, there exists an $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W that determines $r_W(E)$. We can choose a set $X \in \mathcal{X}$ and an index i for which $W_i \subseteq X$ and

$$0 < r_{W_i}(\delta^{in}_{E \cup D[W]}(X) \cap \delta^{in}_{E \cup D[W]}(W_i)) < k.$$

Let Δ denote $\delta^{in}_{E \cup D[W]}(X) \cap \delta^{in}_{E \cup D[W]}(W_i)$. By induction, there is an arc $e \in \Delta \cup D[W_i]$ such that either D'_{W_i} does not contain an $\mathcal{L}[W_i]$-tight s_{W_i}-rooted k-arborescence (where D' is the digraph obtained by removing e), or $r_{D',W_i}(\Delta - e) = r_{W_i}(\Delta) - 1$. The latter possibility means that $r_{D',W}(E - e) < r_W(E)$; on the other hand, the rank can decrease by at most one by Lemma 22. $\square$

We can formulate a similar statement for an $\mathcal{L}[W]$-compatible subset of W, which easily follows from the previous lemma.

Lemma 25. *Let $W \in \mathcal{L}$, let $X \subseteq W$ be an $\mathcal{L}[W]$-compatible set, and let $E \subseteq \delta^{in}_{D^+}(X)$ such that $0 < r^{\oplus}_W(E) < k$. Then there exists an arc $e \in E \cup D[X]$ such that either $D_W - e$ does not contain an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence, or $r^{\oplus}_{D',W}(E - e) = r^{\oplus}_W(E) - 1$, where $D' = D^+ - e$.* $\square$

Let γ be the minimum size of a transversal of the family of $\mathcal{L}$-tight k-arborescences. Using the above lemma, we will show that if $\gamma \geq k$, then there exists a minimum transversal having a special structure. This will lead to a polynomial algorithm for fixed k the following way: first we check every arc subset of size at most $k-1$; if none of these is a transversal, then we look for a minimum transversal among those having the special structure. As we will see, this can be done in polynomial time using the results in [1].

We start with an easy corollary of Lemma 25 that describes a case that cannot happen when $\gamma \geq k$; the proof is left to the reader.

Corollary 26. *If there exists $W \in \mathcal{L}$ and two nonempty disjoint $\mathcal{L}[W]$-compatible sets $X_1, X_2 \subseteq W$ with $r_W^{\oplus}(\delta_{D[W]}^{in}(X_j)) < k$ for both $j = 1, 2$, then $\gamma < k$.* $\qquad\square$

To describe the special structure of the minimum transversal that we are looking for, we use a set function that also played a crucial role in the $k = 1$ case that was solved in [1]. For $W \in \mathcal{L}$ and $Z \subseteq W$, we define

$$f_W(Z) := |\{e \in D[W] : e \in \delta^{in}(Z), \ e \notin \delta^{out}(W') \text{ if } W' \in \mathcal{L}[W] \text{ and } W' \cap Z \neq \emptyset \}|.$$

If D' is a digraph different from D, then we use $f_{D',W}(Z)$ to denote the analogous set function for D'. The following claim was proved for $k = 1$ in [1, Lemma 3].

Claim 27. *Let $D = (V, A)$ be a digraph and $\mathcal{L} \subseteq 2^V$ a laminar family. If there exists an $\mathcal{L}$-tight k-arborescence in D, then $f_W(Z_1) + f_W(Z_2) \geq k$ for any $W \in \mathcal{L}$ and nonempty disjoint sets $Z_1, Z_2 \subseteq W$.*

Proof. Suppose for contradiction that there exists an $\mathcal{L}$-tight k-arborescence in D and there exist $W \in \mathcal{L}$ and nonempty disjoint sets $Z_1, Z_2 \subseteq W$ such that $f_W(Z_1) + f_W(Z_2) \leq k - 1$. Consider the digraph D' obtained from D the following way: for every arc $e \in \delta_{D[W]}^{in}(Z_j)$ for which there exists $W' \in \mathcal{L}[W]$ such that $W' \cap Z_j \neq \emptyset$ and $e \in \delta_{D[W]}^{out}(W')$, we change the tail of e to an arbitrary node in $W' \cap Z_j$ ($j = 1, 2$). This is the **tail-relocation operation** introduced in [1]. The following can be seen easily:

- If F is an $\mathcal{L}[W]$-tight k-arborescence in D, then the corresponding arc set in D' is also an $\mathcal{L}[W]$-tight k-arborescence;

- $f_W(Z_j) = f_{D',W}(Z_j) = \varrho_{D'[W]}(Z_j)$ ($j = 1, 2$).

This contradicts $f_W(Z_1) + f_W(Z_2) \leq k - 1$, because the existence of an $\mathcal{L}[W]$-tight k-arborescence implies $\varrho_{D'[W]}(Z_1) + \varrho_{D'[W]}(Z_2) \geq k$. $\qquad\square$

Note that in the case $k = 1$, [1, Lemmas 3, 4] state that there exists an $\mathcal{L}$-tight arborescence in D if and only if $f_W(Z_1) + f_W(Z_2) \geq 1$ for any $W \in \mathcal{L}$ and nonempty disjoint sets $Z_1, Z_2 \subseteq W$. Unfortunately, the analogous statement is not true for $k > 1$, as illustrated in Figure 3.

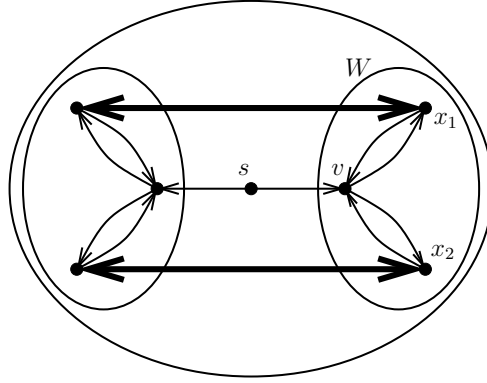The following upper bound on the rank can be proved similarly to Claim 27.

Figure 3: A digraph that does not admit an $\mathcal{L}$-tight 2-arborescence. Bold arcs are bidirected and have multiplicity 2, and $\mathcal{L}$ has 3 members, indicated by ellipses. There is no $\mathcal{L}$-tight 2-arborescence, although $\sum_{X \in \mathcal{X}} f_W(X) \geq k(|\mathcal{X}| - 1)$ holds for every $W \in \mathcal{L}$ and every $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W. Note that the arc sv is a loop in the matroid M_W, and $r_W(\{sv\})$ is determined by the subpartition $\{\{x_1\}, \{x_2\}\}$.

Lemma 28. *If $W \in \mathcal{L}$ and $E \subseteq \delta_{D^+}^{in}(W)$, then $r_W(E) \leq f_W(Z) + \varrho_E(Z)$ for every non-empty $Z \subseteq W$.*

Proof. By the definition of the rank, there is an $\mathcal{L}[W]$-tight s_W-rooted k-arborescence F such that $|F \cap E| = r_W(E)$. We apply the tail-relocation operation described in the proof of Claim 27; let D' be the modified digraph, and let F' be the $\mathcal{L}[W]$-tight s_W-rooted k-arborescence obtained from F. On one hand, $f_W(Z) = f_{D',W}(Z) = \varrho_{D'[W]}(Z)$. On the other hand,

$$k \leq \varrho_{F'}(Z) \leq \varrho_{D'[W]}(Z) + \varrho_{E \cap F}(Z) + \varrho_{F-E}(W) \leq \varrho_{D'[W]}(Z) + \varrho_E(Z) + (k - r_W(E)),$$

so $r_W(E) \leq \varrho_{D'[W]}(Z) + \varrho_E(Z) = f_W(Z) + \varrho_E(Z)$, as required. $\qquad\square$

Our next observation is that for some special arc sets the above formula is tight. To describe these special arc sets, we use a recursive definition. For $W \in \mathcal{L}$ and $E \subseteq \delta_{D^+}^{in}(W)$, we say that E is W**-elementary** if $r_W(E) < k$ and

- either $|W| = 1$

- or there exists an $\mathcal{L}[W]$-compatible set $X \subseteq W$ such that the subpartition $\{X\}$ determines $r_W(E)$, and $\delta_{E \cup D[W]}^{in}(X) \cap \delta_{E \cup D[W]}^{in}(W')$ is W'-elementary for every maximal member W' of $\mathcal{L}[W] - W$.

Intuitively, an arc set is elementary if only subpartitions of cardinality 1 occur in its recursive rank formula. Note that $E = \emptyset$ is W-elementary for every W, since $\{W\}$ determines $r_W(E)$.

Lemma 29. *Let $W \in \mathcal{L}$ and $E \subseteq \delta_{D^+}^{in}(W)$. If E is W-elementary, then $r_W(E) = \min\{f_W(Z) + \varrho_E(Z) : \emptyset \neq Z \subseteq W\}$.*

Proof. By Lemma 28, $r_W(E) \leq \min\{f_W(Z) + \varrho_E(Z) : \emptyset \neq Z \subseteq W\}$. We prove the other direction by induction on the size of W. If $|W| = 1$, then equality holds for $Z = W$, because we assumed that $r_W(E) < k$. If $|W| > 1$, then let $W_1, \ldots, W_l$ be the maximal members of $\mathcal{L}[W] - W$. Since E is W-elementary, there is a $\mathcal{L}[W]$-compatible set $\emptyset \neq X \subseteq W$ such that $r_W(E) = r_W^{\oplus}(\delta_{E \cup D[W]}^{in}(X))$ and $E_i := \delta_{E \cup D[W]}^{in}(X) \cap \delta^{in}(W_i)$ is W_i-elementary for every i. We may assume that $X = \cup_{i=1}^{t} W_i$ for some $1 \leq t \leq l$, and thus $r_W(E) = \sum_{i=1}^{t} r_{W_i}(E_i)$. By induction, there exist nonempty $Z_i \subseteq W_i$ ($i = 1, \ldots, t$) such that $r_{W_i}(E_i) = f_{W_i}(Z_i) + \varrho_{E_i}(Z_i)$. Let $Z = \cup_{i=1}^{t} Z_i$. Observe that an arc entering W_i but not entering X does not contribute to $f_W(Z) + \varrho_E(Z)$, thus $f_W(Z) + \varrho_E(Z) = \sum_{i=1}^{t}(f_{W_i}(Z_i) + \varrho_{E_i}(Z_i)) = r_W(E)$. $\qquad\square$

If a digraph D' is considered instead of D, then we speak of (D', W)-elementary arc sets. We also extend the notion to arc sets in $D[W]$ entering a specified $\mathcal{L}[W]$-compatible subset. For $W \in \mathcal{L}$ and an $\mathcal{L}[W]$-compatible subset X of W, we say that a set $E \subseteq \delta_{D[W]}^{in}(X)$ is X-**elementary** if $r_W^{\oplus}(E) < k$ and $E \cap \delta^{in}(W')$ is W'-elementary for every maximal member W' of $\mathcal{L}[W] - W$. The following is an easy consequence of Lemma 29.

Lemma 30. *Let $W \in \mathcal{L}$ and let $E \subseteq \delta_{D[W]}^{in}(X)$ for some nonempty $\mathcal{L}[W]$-compatible subset X of W. If E is X-elementary, then $r_W^{\oplus}(E) = \min\{f_W(Z) : \emptyset \neq Z \subseteq X\}$.* $\quad\square$

Using this lemma, we can finally prove our main result on the minimum size of transversals.

Theorem 31. *If the minimum size of a transversal is $\gamma \geq k$, then γ equals*

$$\min_{W \in \mathcal{L}} \min\{f_W(Z_1) + f_W(Z_2) - k + 1 : Z_1, Z_2 \text{ are disjoint subsets of } W\}. \qquad (14)$$

Proof. By Claim 27, if $W \in \mathcal{L}$ and Z_1, Z_2 are nonempty disjoint subsets of W, then there is a transversal of size $f_W(Z_1) + f_W(Z_2) - k + 1$, thus γ is at most (14) (this is true even if $\gamma < k$).

To show that equality holds for some $W \in \mathcal{L}$, let H be a minimum transversal, and let $D' = D^+ - H$. By Theorem 23, there exists $W \in \mathcal{L}$ and an $\mathcal{L}[W]$-compatible subpartition $\mathcal{X}$ of W such that $H \subseteq D[W]$ and $\sum_{X \in \mathcal{X}} r_{D',W}^{\oplus}(\delta_{D'[W]}^{in}(X)) = k(|\mathcal{X}| - 1) - 1$. Let us choose a minimum transversal H for which W is the smallest possible, and (subject to that) $\mathcal{X}$ has the smallest possible cardinality; this implies that $r_{D',W}^{\oplus}(\delta_{D'[W]}^{in}(X)) < k$ for every $X \in \mathcal{X}$.

Claim 32. $|\mathcal{X}| = 2$.

Proof. Suppose for contradiction that $|\mathcal{X}| \geq 3$. Then $0 < r_{D',W}^{\oplus}(\delta_{D'[W]}^{in}(X)) < k$ for every $X \in \mathcal{X}$; furthermore, by the assumption $\gamma \geq k$ and Corollary 26, all of these ranks except for at most one were originally k in D. Let X_0 be one of the members of $\mathcal{X}$ for which $r_{D,W}^{\oplus}(\delta_{D[W]}^{in}(X_0)) = k$, and let X_1 be another member. Let $E_1 = \delta_{D'[W]}^{in}(X_1)$, and consider the following arc exchange operation.

(X_0, E_1)-EXCHANGE   By Lemma 24, there is an arc $e \in D'[W]$ such that $r^{\oplus}_{D'-e,W}(E_1 - e) < r^{\oplus}_{D',W}(E_1)$. Choose an arbitrary arc $e_0 \in H$ whose head is in X_0 (such an arc exists because $r^{\oplus}_{D',W}(\delta^{in}_{D'[W]}(X_0)) < r^{\oplus}_{D,W}(\delta^{in}_{D[W]}(X_0)))$. Let $H_1 = H - e_0 + e$.

By the choice of H, there is no $W' \subset W$ such that H_1 is a transversal of $\mathcal{L}[W']$-tight k-arborescences in $D'_{W'}$. By the choice of e, H_1 is still a transversal of $\mathcal{L}[W]$-tight k-arborescences, so it is a minimum transversal. We can apply the exchange operation repeatedly until we obtain a minimum transversal H'' for which $r^{\oplus}_{D'',W}(\delta^{in}_{D''[W]}(X_0)) = k$, where $D'' = D^+ - H''$. At this point, $\mathcal{X} - X_0$ is a good subpartition for H'' that has fewer members than $\mathcal{X}$, in contradiction to the choice of H and $\mathcal{X}$. $\qquad\square$

We obtained that $\mathcal{X}$ is a subpartition with two members, so $\mathcal{X} = \{X_1, X_2\}$ and $r^{\oplus}_{D',W}(\delta^{in}_{D'[W]}(X_1)) + r^{\oplus}_{D',W}(\delta^{in}_{D'[W]}(X_2)) = k - 1$. The next claim shows that H can be modified so that the arc sets in the formula become elementary.

Claim 33. *There is a minimum transversal H^* of $\mathcal{L}[W]$-tight k-arborescences such that $\delta^{in}_{D^*[W]}(X_j)$ is (D^*, X_j)-elementary and $r^{\oplus}_{D^*,W}(\delta^{in}_{D^*[W]}(X_j)) = r^{\oplus}_{D',W}(\delta^{in}_{D'[W]}(X_j))$ for $j = 1, 2$ (where D^* denotes $D^+ - H^*$).*

Proof. If $\delta^{in}_{D'[W]}(X_j)$ is (D', X_j)-elementary for $j = 1, 2$, then H has the required properties. Suppose that $\delta^{in}_{D'[W]}(X_j)$ is not (D', X_j)-elementary . This means that if we recursively compute the rank of $\delta^{in}_{D'[W]}(X_j)$, then at some point we have to compute a rank $r_{D',W'}(E')$ for some $W' \in \mathcal{L}[W] - W$ and some $E' \subseteq \delta^{in}_{D'}(W')$, but the smallest $\mathcal{L}[W']$-compatible subpartition $\mathcal{Y}$ that determines $r_{D',W'}(E')$ has at least two members.

Since $0 < r_{D',W'}(E') < k$, we have $0 < r^{\oplus}_{D',W'}(\delta^{in}_{E' \cup D'[W']}(Y)) < k$ for every $Y \subseteq \mathcal{Y}$. Let $E = E' \cup (H \cap \delta^{in}_D(W'))$. By the assumption $\gamma \geq k$ and Corollary 26, we know that $r^{\oplus}_{W'}(\delta^{in}_{E \cup D[W']}(Y)) = k$ for all but at most one member of $\mathcal{Y}$; let Y_0 be a member for which it is k, and let Y_1 be another member. Let $E_1 = \delta^{in}_{E' \cup D'[W']}(Y_1)$. By the same argument as in the proof of Claim 32, a (Y_0, E_1)-EXCHANGE operation results in a transversal of the same size as H, for which $r^{\oplus}_{D',W'}(\delta^{in}_{E' \cup D[W']}(Y_0))$ increases by one. By applying the exchange operation repeatedly, we eventually obtain a transversal H'' such that $|H''| = |H|$ and $r^{\oplus}_{D'',W'}(\delta^{in}_{E'' \cup D''[W']}(Y_0)) = k$, where $E'' = E - H''$ and $D'' = D^+ - H''$. At this point, $\mathcal{Y} - Y_0$ also determines the rank $r_{D'',W'}(E'') = r_{D',W'}(E')$, and has fewer members than $\mathcal{Y}$.

By repeating this procedure, we eventually obtain a transversal H^* which satisfies the claimed properties. $\qquad\square$

Let H^* be the minimum transversal given by Claim 33. By Lemma 30, there is a nonempty set $Z_j \subseteq X_j$ such that $r^{\oplus}_{D^*,W}(\delta^{in}_{D^*[W]}(X_j)) = f_{D^*,W}(Z_j)$, for both $j = 1, 2$. Thus $f_{D^*,W}(Z_1) + f_{D^*,W}(Z_2) = k - 1$. Since the removal of an arc from D can decrease $f_W(Z_1) + f_W(Z_2)$ by at most one, we have $\gamma = |H^*| \geq f_W(Z_1) + f_W(Z_2) - k + 1$. As the reverse inequality has already been proved, this completes the proof of the theorem. $\qquad\square$

The theorem not only characterizes the minimum size of transversals if $\gamma \geq k$, but also guarantees the existence of minimum transversals that have a special structure.

Corollary 34. *Suppose that $\gamma \geq k$, and let (W, Z_1, Z_2) be minimizers of (14). For $j = 1, 2$, let*

$$E_j = \{e \in D[W] : e \in \delta^{in}(Z_j), \ e \notin \delta^{out}(W') \text{ if } W' \in \mathcal{L}[W] \text{ and } W' \cap Z_j \neq \emptyset\}.$$

Then every arc set $H \subseteq E_1 \cup E_2$ of size $|E_1 \cup E_2| - k + 1$ is a minimum transversal of the family of $\mathcal{L}$-tight k-arborescences.

Proof. By Theorem 31, $\gamma = f_W(Z_1) + f_W(Z_2) - k + 1$, so $|H| = \gamma$. Let $D' = D - H$; by definition, $f_{D',W}(Z_1) + f_{D',W}(Z_2) = f_W(Z_1) + f_W(Z_2) - |H|$, thus $f_{D',W}(Z_1) + f_{D',W}(Z_2) = k - 1$. According to Claim 27, no $\mathcal{L}$-tight k-arborescence exists in D', so H is a transversal. $\square$

Using this, we can give a polynomial time algorithm if k is fixed. We check if there is a transversal of size at most $k - 1$ by brute force search. If there is none, then we can use the algorithm `covering_tight_arborescences` in [1] to compute $\min_{W \in \mathcal{L}}(\min\{f_W(Z_1) + f_W(Z_2) : Z_1, Z_2 \text{ are nonempty, disjoint subsets of } W\})$ and minimizers (W, Z_1, Z_2) in polynomial time. We can also determine the arc sets E_1, E_2 as in Corollary 34, so we can find a transversal of minimum size.

6 Conclusion

As the example in Figure 3 shows, the minimum size of a transversal can be smaller than (14). To make further progress on the problem, this case should be better understood. As mentioned at the end of Section 2, it can be decided in polynomial time using a weighted matroid intersection algorithm whether there is an $\mathcal{L}$-tight k-arborescence; in this sense, the case $\gamma = 0$ is well-understood in terms of general matroid techniques. However, such techniques do not suffice for higher γ, as the transversal problem for general matroid intersection (and even for general matroids) is NP-hard. The algorithm presented in Section 5 sidesteps this problem by simply checking for every arc subset of size at most k whether it is a transversal; this of course means that the algorithm is not even fixed-parameter tractable for the parameter k. One possible approach to improve this would be to generalize the subpartition-finding algorithms of [2] to laminar families.

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