Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2014-08. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

On minimally highly vertex-redundantly rigid graphs

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September 2014

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Abstract

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ if $|V| \geq k + 1$ and after deleting any set of at most $k - 1$ vertices the resulting graph is rigid in $\mathbb{R}^d$. A k-rigid graph G is called *minimally k-rigid* if the omission of an arbitrary edge results in a graph that is not k-rigid. B. Setvatius showed that a 2-rigid graph in $\mathbb{R}^2$ has at least $2|V| - 1$ edges and this bound is sharp. We extend this lower bound for arbitrary values of k and d and show its sharpness for the cases where $k = 2$ and d is arbitrary and where $k = d = 3$. We also provide a sharp upper bound for the number of edges of minimally k-rigid graphs in $\mathbb{R}^d$ for all k.

1 Introduction

A graph $G = (V, E)$ is called *k-rigid* in $\mathbb{R}^d$ or simply $[k, d]$-rigid if $|V| \geq k + 1$ and for any $U \subseteq V$ with $|U| \leq k - 1$ the graph $G - U$ is rigid in $\mathbb{R}^d$. In this context we will call graphs that are rigid in $\mathbb{R}^d$ $[1, d]$-rigid. Every $[k, d]$-rigid graph is $[l, d]$-rigid by definition for $1 \leq l \leq k$. We remark that another equivalent definition of $[k, d]$-rigidity is also used in the literature. By this equivalent definition a graph is $[k, d]$-rigid if $|V| \geq k + 1$ and the deletion of any set of $k - 1$ vertices results in a graph that is rigid in $\mathbb{R}^d$. The following well-known lemma shows the equivalence of these two definitions.

Lemma 1.1. *A graph $G = (V, E)$, with $|V| \geq k + 1$, is $[k, d]$-rigid if and only if the deletion of any set of $k - 1$ vertices results in a graph that is rigid in $\mathbb{R}^d$.* □

In this paper we will use both definitions.

G is called *minimally $[k, d]$-rigid* if it is $[k, d]$-rigid but $G - e$ fails to be $[k, d]$-rigid for every $e \in E$. G is said to be *strongly minimally $[k, d]$-rigid* if it is minimally $[k, d]$-rigid and there is no (minimally) $[k, d]$-rigid graph on the same vertex set with less edges. If G is minimally $[k, d]$-rigid but not strongly minimally $[k, d]$-rigid then it is called *weakly minimally $[k, d]$-rigid.*

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The investigation of $[k,d]$-rigid graphs was commenced in the plane by B. Servatius [11] and was continued recently in higher dimensions by Anderson, Montevallian, Summers and Yu [9, 10, 12, 13] motivated by multi-agent formations and sensor networks.

The following theorem gives a formula for the edge number of minimally rigid graphs.

Theorem 1.2 ([17]). *Let $G=(V,E)$ be minimally $[1,d]$-rigid. If $|V|\geq d+1$ then $|E|=d|V|-\binom{d+1}{2}$.* □

We note that the proof of this theorem follows from the fact, that the edge set of a minimally $[1,d]$-rigid graph corresponds to a base of a matroid, called the rigidity matroid of the graph. Hence it is not surprising that the edge sets of minimally $[1,d]$-rigid graphs on the same node set have the same cardinality. However, as we will see later, this is not true for $[k,d]$-rigid graphs when $k\geq 2$, there are minimally $[k,d]$-rigid graphs for any $k\geq 2$ and $d\geq 1$ with different edge numbers, that is, the set of weakly minimally $[k,d]$-rigid graphs is not empty for any d if $k\geq 2$.

To see a simple example consider the case where $d=1$. It is well known that G is rigid in $\mathbb{R}^1$ if and only if G is connected. Hence G is minimally $[k,1]$-rigid if and only if it is minimally k-connected. It is easy to construct minimally k-connected graphs with different edge-numbers, for example, the complete bipartite graph $K_{n-k,k}$ is minimally k-connected with $k(n-k)$ edges and there are k-regular k-connected graphs that must be minimal and have $kn/2$ edges.

It was shown by B. Servatius [11] that the smallest possible number of edges in a $[2,2]$-rigid graph is $2|V|-1$ and this bound is sharp. Later, lower and upper bounds were provided for the edge number of minimally $[k,d]$-rigid graphs for $d=2$ and 3 in [9, 10, 12, 13] for some other values of $[k,d]$. We summarize these results in the following theorem.

Theorem 1.3. *Let $G=(V,E)$ be a minimally $[k,d]$-rigid graph. Then*

(i) $|E|\geq 2|V|-1$ if $k=d=2$ and $|V|$ is sufficiently large. This bound is sharp.

(ii) $|E|\geq 2|V|+2$ if $k=3$, $d=2$ and $|V|$ is sufficiently large. This bound is sharp.

(iii) $|E|\geq \left\lceil \frac{k+1}{2}|V|\right\rceil$ if k is arbitrary, $d=2$ and $|V|$ is sufficiently large.

(iv) $|E|=\Omega\left(\binom{k+d}{2}n\right)$ if k is arbitrary and $d=2$ or 3. □

The main result of the present paper is a sharp upper bound for the number of edges of minimally $[k,d]$-rigid graphs for every pair $[k,d]$. We provide a lower bound for the number of edges of minimally $[k,d]$-rigid graphs which is sharp for $k=2$ for all d and for $k=3$, $d\leq 3$. We also show that weakly minimally $[k,d]$-rigid graphs exist for every pair $[k,d]$ and we disprove a conjecture of Summers, Yu and Anderson [12, 13].

1.1 Notation

In this paper, we skip the basic definitions and theorems of rigidity theory. We refer the reader to the book of Graver et al. [3] for more details.

$\mathcal{R}_d(G)$ denotes the d-dimensional rigidity matroid of G. We call an edge of G an M-bridge if the deletion of e reduces the rank of $\mathcal{R}_d(G)$. We call a set of edges C of G an M-circuit, if C is a circuit (that is, a minimal dependent set) in $\mathcal{R}_d(G)$. We shall also use some standard notation from graph theory. $\Delta(G)$ denotes the maximum degree in G. K_n is the complete graph with n vertices. C_n denotes the cycle on n vertices. We will use the notation $V(C_n) = \{v_1, \dots, v_n\}$ and $E(C_n) = \{v_iv_{i+1} : 1 \leq i \leq n\}$ where $v_{n+1} := v_1$. C_n^d is the dth power of C_n, or equivalently $E(C_n^d) = \{v_iv_j : i-d \leq j \leq i+d\}$ where $v_{n+i} := v_i$. P_n denotes the path on n vertices. We will use the notation $V(P_n) = \{v_1, \dots, v_n\}$ and $E(P_n) = \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. P_n^d is the dth power of P_n, or equivalently $E(P_n^d) = \{v_iv_j : \min\{1, i-d\} \leq j \leq \max\{n, i+d\}\}$.

2 Preliminaries – Operations preserving rigidity

Constructive characterizations are useful tools in combinatorial rigidity. Even though we do not have a constructive characterization theorem for the class of rigid graphs for $d \geq 3$ it can be very useful to find operations that preserve rigidity. In this section we mention some of these operations.

The *d-dimensional Henneberg-0 extension*, or simply *0-extension* on G adds a new vertex and connects it to d distinct vertices of G. The *d-dimensional 1-extension*, or simply *1-extension* deletes an edge $uw \in E$, adds a new vertex v and connects it to u, v and $d-1$ other vertices of G. The d-dimensional 0-extension is also called *d-valent vertex addition* and the d-dimensional 1-extension is also called $d+1$-*valent edge split*.

Theorem 2.1 ([14])**.** *If G is rigid in $\mathbb{R}^d$ and G' is obtained from G by a d-dimensional 0-extension or 1-extension operation then G' is rigid in $\mathbb{R}^d$.* □

As d-dimensional 0- and 1-extensions are used when we are in $\mathbb{R}^d$, we will simply call them 0- and 1-extensions if d is clear from context. It is well known that a graph is minimally $[1,2]$-rigid graph if and only if it can be built up from the graph K_2 by a sequence of 0- and 1-extensions. However, this is not the case for $d = 3$, moreover there is no similar constructive characterization result for minimally $[1,3]$-rigid graphs. Although, there are some operations that are known to preserve rigidity in higher dimensions. In this paper, we will use the following that we call a *(d-dimensional) simplex-based X-replacement*. Let $d \geq 2$ and let $a, b, w_1, \dots, w_{d-2}$ be a complete subgraph of G and $cd \in E$ an edge which is node-disjoint from the simplex. The d-dimensional simplex-based X-replacement extension deletes ab, cd, adds a new vertex v and connects it to $a, b, c, d, w_1, \dots, w_{d-2}$. When $d = 2$ or 3, we call a d-dimensional simplex-based X-replacement a *2-dimensional X-replacement* or a *triangle-based X-replacement*, respectively. It is folklore that these operations preserve rigidity as the following lemma shows. For completeness, we give a proof to this lemma in the Appendix.

Lemma 2.2. *Let G be rigid in $\mathbb{R}^d$ and let G' be the result of a d-dimensional simplex-based X-replacement applied to G. Then G' is rigid in $\mathbb{R}^d$.*

3 The effect of coning on $[k, d]$-rigid graphs

We shall also use another type of operation that not only preserves rigidity of graph but augments a $[1, d]$-rigid graph to a $[1, d+1]$-rigid one. The *cone graph* of G is the graph that arises from G by adding a new vertex v and edges vu for every $u \in V$. We will denote this graph by $G * v$. The operation that creates the cone graph of G is called *coning*.

Theorem 3.1 (Whiteley [15]). *A graph G is $[1, d]$-rigid if and only if the cone graph $G * v$ is $[1, d+1]$-rigid.* □

Next, we prove some important consequences of Theorem 3.1 that will be useful throughout this paper.

Lemma 3.2. *Let $e \in E$ be an M-bridge in $\mathcal{R}_d(G)$. Then e is a M-bridge in $\mathcal{R}_{d+1}(G * v)$.*

Proof. We can assume that G is rigid in $\mathbb{R}^d$. (If it is not rigid then we add a minimum set of edges that makes it rigid and so e is still a bridge.) Then by Theorem 3.1 $G * v$ is rigid is $\mathbb{R}^{d+1}$ but $(G - e) * v = (G * v) - e$ is not. Hence e is a bridge in $\mathcal{R}_{d+1}(G * v)$ as we claimed. □

We remark that Theorem 3.1 cannot be generalized to k-rigid graphs. That is, if G is $[k, d]$-rigid for some $k \geq 2$, then $G * v$ is not necessarily $[k, d+1]$-rigid. For example C_n is $[2, 1]$-rigid, but $C_n * v$ (which is the wheel graph with $n + 1$ vertices) is not $[2, 2]$-rigid. However, the following results show that coning can be used to construct $[k, d]$-rigid graphs.

Lemma 3.3. *Let G be a $[k, d+1]$-rigid graph. Then G is $[k+1, d]$-rigid.*

Proof. Let G' be a $[1, d+1]$-rigid graph that we obtain from G by deleting $k - 1$ arbitrary vertices. Suppose, for a contradiction, that there is a vertex $u \in V(G')$ such that $G' - u$ is not $[1, d]$-rigid. Then $(G' - u) * u$ is not $[1, d+1]$-rigid by Theorem 3.1 which contradicts the $[1, d+1]$-rigidity of $G' \subseteq (G' - u) * u$. □

Lemma 3.4. *Let $k \geq 2$ and $d \geq 1$ be integers and let $G = (V, E)$ be a $[k-1, d]$-rigid graph. Then $G * v$ is $[k, d]$-rigid.*

Proof. We need to show that after deleting $k-1$ vertices $G*v$ remains $[1, d]$-rigid. If v is omitted, then we are done by the $[k-1, d]$-rigidity of G. Otherwise, let $u_1, ..., u_{k-1}$ be the omitted vertices. $G - \{u_1, ..., u_{k-2}\}$ is $[1, d]$-rigid and v is connected to every neighbor of v_{k-1}. Hence $(G * v) - \{u_1, ..., u_{k-1}\}$ has a subgraph isomorphic to the $[1, d]$-rigid graph $G - \{u_1, ..., u_{k-2}\}$ showing that it is $[1, d]$-rigid. □

4 Lower bounds for the number of edges in $[k,d]$-rigid graphs

In this section, we present several lower bounds for the number of edges in $[k,d]$-rigid graphs for arbitrary positive integers k and d. Theorem 1.3 (i)-(iii) summarizes the lower bounds that were known earlier. First we improve (i) and (ii) and extend them to every dimension d.

Theorem 4.1. *If a graph $G=(V,E)$ is $[k,d]$-rigid with $|V|\geq d^2+d+k$ then*

$$|E|\geq d|V|-\binom{d+1}{2}+(k-1)d+\max\left\{0,\left\lceil k-1-\frac{d+1}{2}\right\rceil\right\}. \tag{1}$$

Note that the bound given in (1) coincides with the bounds given in Theorem 1.3 (i)-(ii) for $[k,d]=[2,2]$, $[3,2]$ hence it is sharp for these values of k and d. In Sections 6 and 7, we show that this lower bound is sharp for $[2,d]$ where d is arbitrary, and for $[k,d]=[3,3]$.

Proof. We prove this theorem by induction on k. For $k=1$ the theorem immediately follows by Theorem 1.2.

Now, let $G=(V,E)$ be a $[k,d]$-rigid graph for $k\geq 2$ with $|V|\geq d^2+d+k$ and assume that the theorem is true for $k-1$. Let $v\in V$ be a node of maximum degree in G. As $G-v$ is $[k-1,d]$-rigid with at least $d^2+d+k-1$ nodes,

$$|E(G-v)|\geq d(|V|-1)-\binom{d+1}{2}+(k-2)d+\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}$$

by induction. Using this inequality, we have

$$\begin{aligned}|E| &\geq d(|V|-1)-\binom{d+1}{2}+(k-2)d+\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}+\Delta(G)\\ &= d|V|-\binom{d+1}{2}+(k-1)d+\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}+(\Delta(G)-2d)\end{aligned}$$

Here, $\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}=0=\max\left\{0,\left\lceil k-1-\frac{d+1}{2}\right\rceil\right\}$ if $k-1\leq\frac{d+1}{2}$ and $\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}+1=\left\lceil k-2-\frac{d+1}{2}\right\rceil+1=\left\lceil k-1-\frac{d+1}{2}\right\rceil=\max\left\{0,\left\lceil k-1-\frac{d+1}{2}\right\rceil\right\}$ if $k-1>\frac{d+1}{2}$. Therefore, we need to prove that $\Delta(G)\geq 2d$ for all k and $\Delta(G)\geq 2d+1$ also holds if $k-1>\frac{d+1}{2}$.

To prove that $\Delta(G)\geq 2d$ for all k, let us observe that if a graph $H=(V',E')$ is $[1,d]$-rigid with $|V'|\geq d^2+d+2$ then $\Delta(H)\geq 2d$. (To see this suppose that $\Delta(H)\leq 2d-1$. Then $|E'|\leq|V'|d-\frac{|V'|}{2}<|V'|d-\binom{d+1}{2}$ which contradicts Theorem 1.2.) Since a $[k,d]$-rigid graph is also $[1,d]$-rigid and we have $|V|\geq d^2+d+k$, we get that $\Delta(G)\geq 2d$. But then

$$|E|\geq d|V|-\binom{d+1}{2}+(k-1)d+\max\left\{0,\left\lceil k-2-\frac{d+1}{2}\right\rceil\right\}$$

and hence $|E|>d|V|$ if $k-1>\frac{d+1}{2}$. Therefore we get $\Delta(G)\geq 2d+1$ if $k-1>\frac{d+1}{2}$ as we wanted. □

The following theorem gives a better lower bound if k is large compared to d. This result extends Theorem 1.3 (iii) for higher dimensions.

Theorem 4.2. *Let $k \geq d+2$ and let $G = (V, E)$ be a $[k, d]$-rigid graph with $|V| \geq d+k$. Then $|E| \geq \lceil \frac{d+k-1}{2} |V| \rceil$.*

Proof. If we delete $k-1$ neighbors of a node v we get a $[1, d]$-rigid graph with at least $d+1$ nodes. Since the minimum degree of such a graph is at least d, we get $d_G(v) \geq k-1+d$. Thus the minimum degree in G is at least $k-1+d$ hence $|E| \geq \lceil \frac{d+k-1}{2} |V| \rceil$. □

5 Upper bound for the number of edges in minimally $[k, d]$-rigid graphs

In this section, we give an upper bound for the number of edges of minimally $[k, d]$-rigid graphs. First we prove the following lemma.

Lemma 5.1. *Suppose that G is a minimally $[k, d]$-rigid graph. Then G is independent in $\mathcal{R}_{d+k-1}(G)$.*

Proof. By the minimality of G, for each e, there is a set $U_e \subseteq V$ such that $|U_e| = k-1$ and $G - U_e - e$ is not rigid. ($G - U_e$ is rigid by the $[k, d]$-rigidity of G.) Then e is an M-bridge in $\mathcal{R}_d(G - U_e)$. By Lemma 3.2 e is an M-bridge in $\mathcal{R}_{d+k-1}((\dots(G * v_1) * \dots) * v_{k-1})$ and so it is an M-bridge in $\mathcal{R}_{d+k-1}(G)$. □

By combining Lemma 5.1 and Theorem 1.2 we immediately get the following upper bound.

Theorem 5.2. *Let $G = (V, E)$ be a minimally $[k, d]$-rigid graph. Then*

$$|E| \leq (d+k-1)|V| - \binom{d+k}{2}.$$

□

The sharpness of this bound for $d \geq 2$ will be proved later in Lemma 8.4. As a graph is $[k, 1]$-rigid if and only if it is k-connected Mader's sharp upper bound for the edge number of minimally k-connected graphs can be applied for the edge number of minimally $[k, 1]$-rigid graphs, see [8]. This gives us the following.

Theorem 5.3. *Let $G = (V, E)$ be a minimally $[k, 1]$-rigid graph with $|V| \geq 3k-1$. Then*

$$|E| \leq k|V| - k^2$$

and this bound is sharp. □

6 Minimally $[2, d]$-rigid graphs

In this section, we consider the case where $k = 2$. First we show that the lower bound given in Theorem 4.1 is sharp for $k = 2$ in any dimension and next we disprove Conjecture 6.3.

Consider graph C_n^d and its subgraph L_d induced by vertices $v_{n-d+1}, \dots, v_n$. (Note that L_d is isomorphic to K_d.) $H_{n,2}^d = C_n^d - E(L_d)$ denotes the graph we get from C_n^d after deleting the edge set of L_d. First we prove that $H_{n,2}^d$ is $[2, d]$-rigid.

Lemma 6.1. *$H_{n,2}^d$ is $[2, d]$-rigid if $n \geq 3d$.*

Proof. Let $v_i \in V(H_{n,2}^d)$ be arbitrary. We will prove that $H_{n,2}^d - v_i$ is $[1, d]$-rigid by constructing it from a subgraph isomorphic to K_d using (d-dimensional) 0- and 1-extensions. (See Figure 1.)

First suppose that $v_i \notin V(L_d)$. For simplicity, we can assume that $\lfloor \frac{n-d+1}{2} \rfloor \leq i \leq n-d$. Since $n \geq 3d$ we have $i \geq d+1$. Vertices $v_1, \dots, v_d$ induce a subgraph isomorphic to K_d hence we can add $v_{d+1}, \dots, v_{i-1}$ in this order using 0-extensions which connect v_j to vertices $v_{j-d+1}, \dots, v_{j-1}$ for every $d+1 \leq j \leq i-1$. Therefore $v_1, \dots, v_{i-1}$ induce a $[1, d]$-rigid subgraph.

Now we will add vertices $v_{i+1}, \dots, v_{i+d}$ in this order using 0-extensions. If $j \leq n-d$ then the extension connects v_j to vertices $v_{j-d}, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}$ and to v_1. Note that $v_j v_1$ is not an edge of $H_{n,2}^d - v_i$ if $j \leq n-d$. We will apply 1-extensions on these extra edges. If $j > n-d$ then it will be connected to $v_{j-d}, \dots, v_{i-1}, v_{i+1}, \dots, v_{n-d}$ and to $v_1, \dots, v_{d-n+j}$ all of which are edges of $H_{n,2}^d - v_i$.

From now on we will use 1-extensions only for adding vertices $v_{i+d+1}, \dots, v_n$ in this order. When adding v_j for $j \leq n-d$ we apply the 1-extension on edge $v_{j-d}v_1$ that connects v_j to $v_{j-d+1}, \dots, v_{j-1}$. In this case we remove the extra edge $v_{j-d}v_1$ and add a new one $v_j v_1$. If $j > n-d$ then similarly we apply the 1-extension on edge $v_{j-d}v_1$ but we connect v_j to $v_{j-d}, \dots, v_{n-d}$ and to $v_2, \dots, v_{d-n+j}$ and all of these edges are present in $H_{n,2}^d - v_i$. In this case the number of extra edges decreased by one.

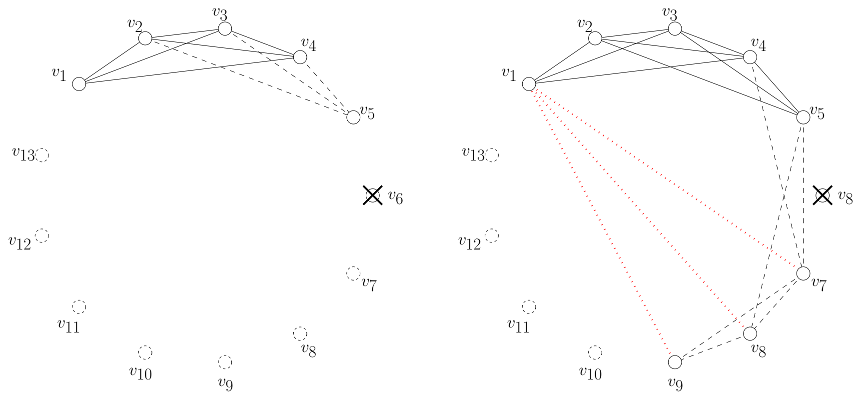
If $v \in V(L_d)$, then it is easy to see that $H_{n,2}^d$ has a subgraph that can be built up using 0-extensions only (we first build up the subgraph induced by vertices of $H_{n,2}^d$ and then we add the nodes in $V(L_d) - v$). $\square$

If $G = (V, E)$ is $[2, d]$-rigid then $|E| \geq d|V| - \binom{d+1}{2} + d = d|V| - \binom{d}{2}$ if $|V| \geq d^2 + d + 2$ by Theorem 4.1. $|E(H_{n,2}^d)| = dn - \binom{d}{2}$ since C_n^d has dn edges if $n \geq 2d + 1$ and the deleted edges form a complete subgraph with d vertices. Hence by Lemma 6.1 we get the main result of this section:

Theorem 6.2. *If $G = (V, E)$ is a strongly minimally $[2, d]$-rigid graph with $|V| \geq d^2 + d + 2$ then $|E| = d|V| - \binom{d}{2}$.*

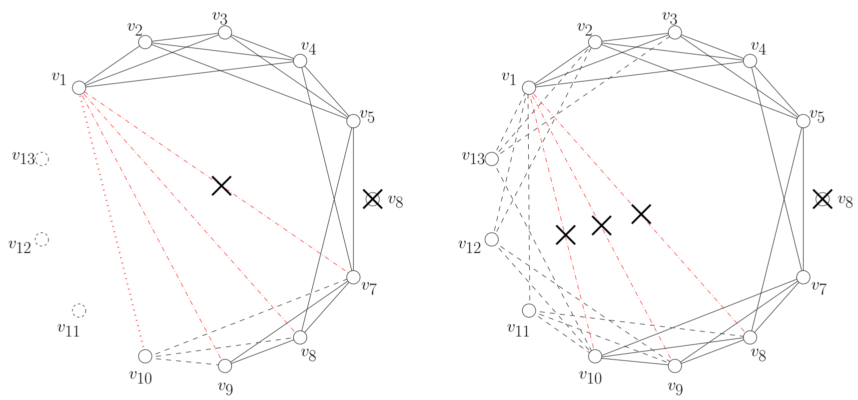
6.1 A counterexample for a conjecture of Summers et al.

B. Servatius proved a constructive characterization theorem for the class of strongly minimally $[2, 2]$-rigid graphs that only uses 1-extensions in [11]. As far as we know



(a) Add v_5 with a 0-extension.

(b) Add v_7, v_8, v_9 with 0-extensions by adding the extra (dotted red) edges $v_j v_1$ for $7 \leq j \leq 9$.



(c) Add v_{10} with 1-extension on the (crossed red) extra edge $v_7 v_1$ by adding the extra (dotted red) edge $v_{10} v_1$

(d) Add v_{11}, v_{12}, v_{13} with 1-extensions on the (red) edges $v_8 v_1, v_9 v_1, v_{10} v_1$, respectively.

Figure 1: Building up $C_{13}^3 - E(L_3) - v_5$ using Henneberg operations.

finding an inductive construction for the class of minimally $[2,2]$-rigid graphs is an open problem. It was observed in [13] that the 2-dimensional X-replacement preserves minimally $[2,2]$-rigidity in specific cases. Summers, Yu and Anderson conjectured that the 3-valent vertex addition and the 2-dimensional X-replacement operations are sufficient to build up every weakly minimally $[2,2]$-rigid graph with at least nine vertices.

Conjecture 6.3 ([12, 13])**.** *Let $G = (V, E)$ be a minimally $[2,2]$-rigid graph with at least nine vertices. Then there exists either (a) a degree 4 vertex on which a reverse X-replacement operation can be performed to obtain a weakly minimal $[2,2]$-rigid graph or (b) there exists a degree three vertex on which a reverse 3-valent vertex addition can be performed to obtain a weakly minimally $[2,2]$-rigid graph.*

Now we disprove Conjecture 6.3 by constructing minimally [2,2]-rigid graphs that do not have a vertex at which the reverse degree 3 vertex addition or the reverse X-replacement can be performed. To give such an example we will need the following simple observation.

We define an operation called K_4*-extension* that preserves $[2,2]$-rigidity. Let $G = (V, E)$ be a graph with $|V| \geq 4$, and let $v_1, v_2, v_3, v_4 \in V$ be four distinct vertices. The K_4-extension adds four new vertices u_1, u_2, u_3, u_4 to G, connects v_i to u_i for every $1 \leq i \leq 4$ and u_k to u_l for every pair $1 \leq k, l \leq 4$.

Claim 6.4. *If* $G = (V, E)$ *is* $[2,2]$*-rigid then* $G' = (V', E')$ *obtained by a* K_4*-extension is also* $[2,2]$*-rigid. Furthermore* $G' - e$ *is not* $[2,2]$*-rigid for any* $e \in E' - E$.

Proof. Clearly, $G' - v$ is rigid for any $v \in V'$.

Consider the graph $G' - e$ for some $e \in E' - E$. Let $u_i \in V' - V$ be such that e is not incident to u_i. We claim that $G'' = G' - u_i - e$ is not rigid. G'' consist of G and a set of three vertices that is incident to five edges only. Hence there are only $2|V| - 3 + 5 = 2|V'| - 4$ independent edges in G'' thus G'' is not rigid as we claimed. $\square$

Now let $G_0 = (V_0, E_0)$ be a $[2,2]$-rigid graph with $V_0 \geq 4$. Apply some K_4-extensions to vertices of V_0, let the resulting graph be $G_1 = (V_1, E_1)$ (see Figure 2). Suppose that every vertex in V_0 is incident to at least five edges from $E_1 - E_0$. After the extensions delete edges from E_1 (if necessary) to obtain a minimally $[2,2]$-rigid graph $G_2 = (V_1, E_2)$. By Claim **??** deleting any edge from $E_1 - E_0$ results in a graph that is not $[2,2]$-rigid hence the minimum degree in G_2 is four and all the degree four vertices are in $V_1 - V_0$. Clearly we cannot perform the reverse degree 3 vertex addition in G_2. Every vertex in $V_1 - V_0$ is contained in a K_4 subgraph of G_2 and every reverse X-replacement on one of these vertices creates a parallel pair of edges. Thus no reverse X-replacement operation preserves minimal $[2,2]$-rigidity of G_2. This disproves Conjecture 6.3.

We remark that for any positive integer t graph G_1 can be constructed such that every vertex in V_0 is incident to at least t edges from $E_1 - E_0$. Hence the minimum degree in G_2 is four and the vertices in V_0 have degree at least t. Since t can be arbitrarily large this example shows that it may not be easy to find a constructive characterization that only uses operations that add low-degree vertices.

7 Strongly minimally $[3,3]$-rigid graphs

In this section, we show that the lower bound given in Theorem 4.1 is sharp when $k = d = 3$.

Lemma 7.1. C_n^3 *is* $[3,3]$*-rigid if* $n \geq 9$.

Proof. Let $v_i, v_j \in V(C_n^3)$ be arbitrary. We will prove that $C_n^3 - \{v_i, v_j\}$ is $[1,3]$-rigid by constructing it from a subgraph isomorphic to K_4 using 3-dimensional 0- and 1-extensions and simplex-based X-replacements.

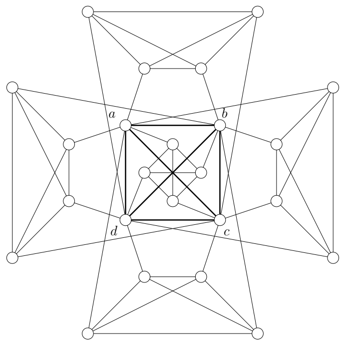


Figure 2: A counterexample H for Conjecture 6.3 that we get by performing five K_4-extensions on the subgraph induced by vertices a, b, c, d. K_4 is minimally $[2, 2]$-rigid hence G_c is $[2, 2]$-rigid by Claim 6.4. It can be easily seen that deleting any of the edges bc, cd, db from graph $G_c - a$ results in a non-rigid graph. By symmetry the deletion of any edge of the starting graph results in a graph that is not $[2, 2]$-rigid. This implies that G_c is minimally $[2, 2]$-rigid.

We can assume that $j = n$ and $i \geq \lceil \frac{n}{2} \rceil$. $n \geq 9$ hence $i \geq 5$ and as in the proof of Lemma 6.1 it can be seen easily that the subgraph induced by $v_1, \dots, v_{i-1}$ is rigid.

Let $\ell = n - i - 1$. We have to perform ℓ more extensions to add the remaining vertices. We split the proof into two cases depending on ℓ.

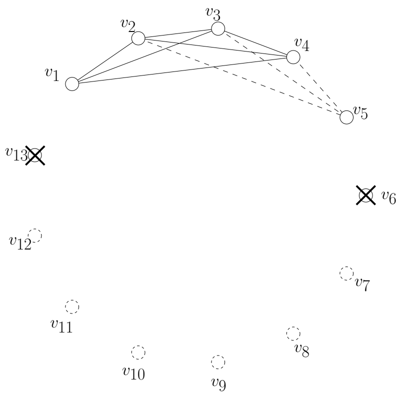
If $1 \leq \ell \leq 3$, we add v_{i+1} and connect it to v_1, v_{i-2}, v_{i-1}. If $\ell \geq 2$ then we add v_{i+2} and connect it to v_1, v_{i-1}, v_{i+1}. If $\ell = 3$ then we can add v_{i+3} performing a 1-extension on edge $v_{i+1}v_1$ and connecting v_{i+3} to v_{i+2} and v_2.

If $\ell \geq 4$ then we will need a simplex-based X-replacement on edges v_2v_{n-3}, v_1v_{n-4}. In this case we will add vertices $v_{i+1}, v_{i+2}, v_{i+3}$ by 0-extensions, $v_{i+4}, \dots, v_{n-2}$ by 1-extensions. We will perform these operations such that after adding v_{n-2} edges $v_2v_{n-3}, v_1v_{n-2}, v_1v_{n-4}, v_{n-2}v_{n-4}$ will be present in the resulting graph.

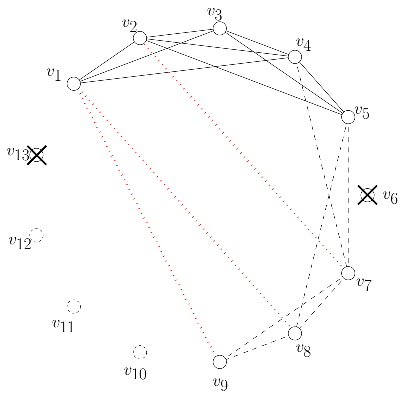
Let $\sigma : \mathbb{Z} \to \{1, 2\}$ be a function with $\sigma(t) := 2$ if $t \equiv \ell - 2 \pmod 3$ and $\sigma(t) := 1$ otherwise. We add v_{i+1} with 0-extension that connects it to $v_{i-2}, v_{i-1}, v_{\sigma(1)}$. Then add v_{i+2} with a 0-extension that connects it to $v_{i-1}, v_{i+1}, v_{\sigma(2)}$. Next, we add v_{i+3} with a 0-extension that connects it to $v_{i-1}, v_{i-2}, v_{\sigma(3)}$. Then we add v_{i+m} for $4 \leq m \leq \ell - 1$ in sequence with 1-extension on $v_{i+m-3}v_{\sigma(m-3)}$ that connects it to v_{i+m-2}, v_{i+m-1}. Finally, we add v_{n-1} with a simplex-based X-replacement on edges v_2v_{n-3}, v_1v_{n-4} as $v_{n-2}v_1v_{n-1}$ is a triangle. $\square$

We have proved that C_n^3 is $[3, 3]$-rigid. It is easy to see that C_n^3 has $3n$ edges if $n \geq 7$. These together with Theorem 4.1 gives the following:

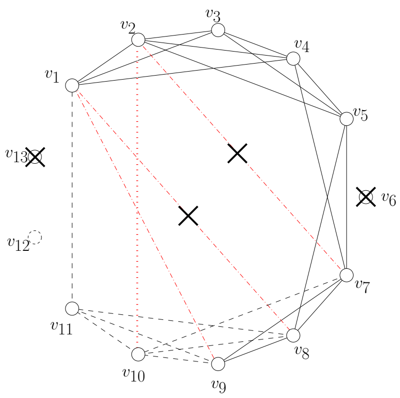
Theorem 7.2. *If $G = (V, E)$ is a strongly minimally $[3, 3]$-rigid graph with $|V| \geq 15$, then $|E| = 3|V|$.*



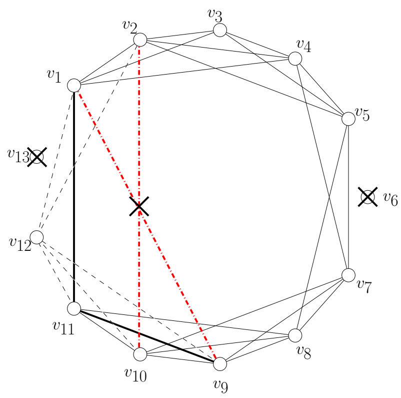
(a) Add u_5 with a 0-extension.



(b) Add v_7, v_8, v_9 with 0-extensions by adding extra edges $v_{i+m}v_{\sigma(i)}$.



(c) Add v_{10} and v_{11} with 1-extensions on edges v_7v_2, v_8v_1, respectively. By performing the first of these extensions we create the extra edge $v_{10}v_2$.



(d) Add v_{12} with triangle-based X-replacement on edges v_2v_{10}, v_1v_9. (Note that $v_1v_9v_{11}$ is a triangle.)

Figure 3: Building up $C^3_{12} - \{u, v\}$.

8 Higher dimensions revisited

It remains open whether the lower bounds given in Theorem 4.1 and 4.2 are tight for some other pairs $[k, d]$ different from $[2, d]$, $[3, 2]$ and $[3, 3]$. This question seems to be more complicated for larger values of k and d as there are just a few operations known that preserve rigidity in higher dimensions.

We note that a proof similar to that of Lemma 7.1 works if one wants to prove that C^d_n is $[3, d]$-rigid for any other $d \geq 4$. As the edge number of these graphs does not coincide with the bound given in Theorem 4.1, we skip the details. However, we conjecture that the lower bound given in Theorem 4.1 is sharp for $k = 3$ for all $d \geq 3$. To formulate the conjecture more precisely, recall that L_d denotes the complete subgraph of C^d_n spanned by vertices $v_{n-d+1}, \dots, v_n$. Let L'_d denote the graph that we get from L_d by deleting the Hamiltonian cycle that consists of edges v_iv_{i+1} for $n-d+1 \leq i \leq n-1$ and $v_{n-d+1}v_n$. Note that L'_3 is the empty graph on three vertices. Lemma 7.1 states that $C_n^d - L'_d$ is strongly minimally $[3,3]$-rigid. $|E(C_n^d - L'_d)| = dn - \binom{d}{2} + d = dn - \binom{d+1}{2} + 2d$ which motivates the following conjecture.

Conjecture 8.1. *$C_n^d - L'_d$ is a strongly minimally $[3,d]$-rigid graph if n is sufficiently large. Thus the lower bound given in Theorem 4.1 is sharp for $k=3$ and $d \geq 3$.*

Now we turn to the case where $k \geq d+2$. Our conjecture is that the bound given in Theorem 4.2 is tight in these cases. We formulate this conjecture precisely only for $d=2$ and $k=4$ and 5.

Let $C_n^{[1,2,4]}$ be the graph that we get from C_n by adding the edges between the nodes with distances 2 and 4 and let $C_n^{[1,2,\frac{5}{2}]}$ be the graph that we get from C_n by adding the edges between the nodes with distance 2 and between every second pair of nodes with distance 5 (in this case, we can complete the graph only with some different edges if n is odd: the last two node pairs will have distance 2 and we connect all distance 5 neighbors to the first node.)

Conjecture 8.2. *$C_n^{[1,2,\frac{5}{2}]}$ is a strongly minimally $[2,4]$-rigid graph and $C_n^{[1,2,4]}$ is a strongly minimally $[2,5]$-rigid graph if n is sufficiently large. Thus the lower bound given in Theorem 4.2 is sharp for $d=2$ and $k=4$ and 5.*

8.1 Examples for minimally $[k,d]$-rigid graphs

The question whether weakly minimally $[k,d]$-rigid graphs exist for every pair $[k,d]$ can still be solved without knowing the edge count of strongly minimally $[k,d]$-rigid graphs. There are examples for weakly minimally $[2,2]$-rigid graphs in [11, 12, 13] but the existence of weakly minimally $[k,d]$-rigid graphs for other values of k and d was open so far. In this section, we will give examples for minimally $[k,d]$-rigid graphs with the same number of vertices but with different number of edges. Such a pair of graphs shows that the graph with the larger number of edges has to be weakly minimally $[k,d]$-rigid.

Let $H_{n,i}^d$ denote the cone graph of $H_{n,(i-1)}^d$ for $i \geq 3$. (For the definition of $H_{n,2}^d$ see Section 6.) By Lemma 3.4 and Lemma 6.1, we can get get a minimally $[k,d]$-rigid graph by deleting some edges of $H_{t,k}^d$ (to obtain minimality).

Corollary 8.3. *Let n, d and k be three positive integers such that $t \geq 3d$ and $k \geq 2$. Then there exists a minimally $[k,d]$-rigid graph $H_{t,k,\text{reduced}}^d$ with $n = t+k-2$ vertices and at most $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2)$ edges.* □

We shall use Lemma 3.3 in the proof of the following lemma that also shows that the upper bound given in Theorem 5.2 is sharp for $d \geq 2$.

Lemma 8.4. *Let $t \geq 1$, $k \geq 1$ and $d \geq 2$ be three integers. There exists a minimally $[k,d]$-rigid graph with $n = t+k+d-1$ vertices and $(k+d-1)n - \binom{k+d}{2}$ edges.*

Proof. Define graph Y_t^c as follows for any integers c and t. Take the disjoint union of an independent set I_t of t nodes (on the vertex set $\{v_1, \dots, v_t\}$) and a complete graph K_c (on the vertex set $\{w_1, \dots, w_c\}$) and add edges $v_i w_j$ for every pair $1 \le i \le t, 1 \le j \le c$ (see Figure 4).

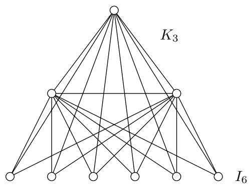


Figure 4: Y_6^3.

Y_t^1 is minimally $[1,1]$-rigid as it is a tree. Hence by Theorem 3.1 we get that Y_t^c is minimally $[1,c]$-rigid as we get this graph after using the coning operation $c-1$ times on Y_t^1. Thus Y_t^{k+d-1} is $[1,k+d-1]$-rigid and hence it is $[k,d]$-rigid by Lemma 3.3.

Next we show that Y_t^{k+d-1} is *minimally* $[k,d]$-rigid. We have seen this for $k=1$. Now let $k,d \ge 2$. Let $uv \in E(Y_t^{k+d-1})$ be an arbitrary edge. By symmetry, we can assume that $u,v \in \{v_1, v_2, w_1, w_2\}$. Observe that after the omission of the $k-1$ nodes $v_{d+1}, \dots, v_{k+d+1}$ from Y_t^{k+d-1} we get Y_t^d that is a minimally $[1,d]$-rigid graph as we observed before. Since $d \ge 2$, $uv \in E(Y_t^d)$ also holds. But $Y_t^d - uv$ is not $[1,d]$-rigid by the minimally $[1,d]$-rigidity of Y_t^d, hence $Y_t^{k+d-1} - uv$ is not $[k,d]$-rigid. Therefore, Y_t^{k+d-1} is minimally $[k,d]$-rigid.

Clearly, $|V(Y_t^{k+d-1})| = t+k+d-1 =: n$ and $|E(Y_t^{k+d-1})| = \binom{k+d-1}{2} + (k+d-1)t = (k+d-1)(t+k+d-1) - (k+d-1)^2 + \binom{k+d-1}{2} = (k+d-1)n - \binom{k+d}{2}$. □

Some other examples for minimally $[k,d]$-rigid graphs can be found in a preliminary version of this paper (see [6]). Now, we are ready to prove the following theorem.

Theorem 8.5. *Let d and k be positive integers with $k \ge 2$. Then there are weakly minimally $[k,d]$-rigid graphs, that is, there are minimally $[k,d]$-rigid graphs that are not strongly minimally $[k,d]$-rigid.*

Proof. By Corollary 8.3, there exists a minimally $[k,d]$-rigid graph on n nodes with at most $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2)$ edges if $n \ge 3d+k-2$. By Lemma 8.4, $Y_{n-k-d+1}^{k+d-1}$ is a minimally $[k,d]$-rigid graph on n nodes with at most $(k+d-1)n - \binom{k+d}{2}$ edges if $n \ge k+d$. Since $(d+k-2)n - \binom{d}{2} + \binom{k-2}{2} - (d+k-2)(k-2) < (k+d-1)n - \binom{k+d}{2}$ if n is sufficiently large, $Y_{n-k-d+1}^{k+d-1}$ is a weakly minimally $[k,d]$-rigid graph for $k,d \ge 2$ if n is sufficiently large. □

9 Concluding remarks

The results presented in this paper are about the edge numbers of minimally $[k,d]$-rigid graphs. Similar questions were asked about minimally globally $[k,d]$-rigid graphs in [10, 12] where $G=(V,E)$ is globally $[k,d]$-rigid if $|V| \ge k+1$ and after deleting any set of at most $k-1$ vertices the resulting graph is globally rigid in $\mathbb{R}^d$. Other version of the problem is $[k,d]$-edge-rigidity (and global $[k,d]$-edge-rigidity) where instead of any set of at most $k-1$ vertices we delete any set of at most $k-1$ edges of the graph. Proving similar results on these variants of the problem is a possible direction of future research. Some of our methods (for example our lower bound for large k in Theorem 4.2) can be used easily for these graph classes. For example, as rigidity is a necessary condition for global rigidity, all our lower bounds are valid for globally $[k,d]$-rigid graphs. We note that a sharp upper bound for the edge number of minimally [2,2]-edge-rigid graphs was recently given by Jordán [5].

A different direction is to characterize inductively the class of graphs mentioned above for some values of $[k,d]$ which seems to be an interesting and difficult open question.

Acknowledgments

The authors received a grant (no. CK 80124) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund. Research was supported by the MTA-ELTE Egerváry Research Group and by the Hungarian Scientific Research Fund – OTKA, K109240.

The authors thank Zsuzsanna Jankó and János Geleji for the inspiring discussions and Tibor Jordán for posing the interesting questions solved in this paper.

10 Appendix

In this section, we prove Lemma 2.2. In the proof we will use special (non-generic) realizations of graphs. Again, we refer to the book of Graver et. al [3] for definitions.

It is well-known (as it is used for the proof of the Henneberg-1 part of Theorem 2.1) that for a 0-extension we do not really need a generic realization, that is, the following lemma holds for not necessarily generic frameworks.

Lemma 10.1. *Let (G,p) be independent in the d-dimensional rigidity matroid and let G' be the graph that arises from G by a d-dimensional 0-extension such that $V(G') = V(G) + v$ and let p' be a realization of G' in $\mathbb{R}^d$ such that $p(u) = p'(u)$ for every $u \in V$. Suppose that $p'(v)$ and its d neighbors have full affine span. Then (G',p') is independent in the d-dimensional rigidity matroid.* □

Now we prove Lemma 2.2.

Proof of Lemma 2.2. We may assume that G is minimally rigid in $\mathbb{R}^d$ by deleting some redundant edges of G other than those we use for the extension. Let (G,p) be a generic realization of G. Let S be the hyperplane that contains the $(d-1)$ dimensional simplex spanned by $p(a), p(b), p(w_1), \ldots, p(w_{d-2})$ and let ℓ be the line of $p(c), p(d)$. Put $p(v) = S \cap \ell$ and let $G_0 = (V+v, E \cup \{va, vc\} \cup \{vw_i : 1 \leq i \leq d-2\})$. By Lemma 10.1, the framework (G_0,p) is independent and hence minimally rigid.

Now we construct framework (G',p) from (G_0,p) by replacing edges ab and cd with vb and vd, respectively. We shall prove that (G',p) is rigid. First add vb, let

$G_1 = G_0 + vb$. There is a unique M-circuit in (G_1, p) in the d-dimensional rigidity matroid which is the K_{d+1} induced by $v, a, b, w_1, \ldots, w_{d-2}$. (Note that points $p(v), p(a), p(b), p(w_1), \ldots, p(w_{d-2})$ lie on a hyperplane.) Thus with the notation $G_1 - ab = G_2$ framework (G_2, p) is independent.

Similarly, with the notation $G_3 = G_2 + vd$ the unique M-circuit in framework (G_3, p) in the the d-dimensional rigidity matroid is the triangle spanned by v, c, d. Again, with removing cd we get an independent framework, equivalently (G', p) is rigid as we claimed. □

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