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An extension of Lehman's theorem and ideal set functions

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Abstract

Lehman's theorem on the structure of minimally nonideal clutters is one of the fundamental results of polyhedral combinatorics. One approach to extend it has been to give a common generalization with the characterization of minimally imperfect clutters [11, 4]. We give a new generalization of this kind, which combines two types of covering inequalities and works well with the natural definition of minors. We also show how to extend the notion of idealness to unit-increasing set functions, in a way that is compatible with minors and blocking operations.

1 Introduction

A set family $\mathcal{C}$ on a ground set V of size n is called a *clutter* if no set in $\mathcal{C}$ is a subset of another. Let $\mathcal{C}^\uparrow$ denote the uphull of $\mathcal{C}$, that is, $\{U \subseteq V : \exists C \in \mathcal{C} : C \subseteq U\}$. The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is defined as the family of the (inclusionwise) minimal sets that intersect each member of $\mathcal{C}$. It is easy to check that $b(b(\mathcal{C})) = \mathcal{C}$.

One of the most well-studied objects of polyhedral combinatorics is the *covering polyhedron* of a clutter, which we consider in the following bounded version:

$$P(\mathcal{C}) = \{x \in \mathbb{R}^V : 0 \le x \le \mathbf{1},\ x(C) \ge 1 \text{ for every } C \in \mathcal{C}\},$$

where $x(C)$ denotes $\sum_{v \in C} x_v$. The integer points of $P(\mathcal{C})$ correspond to the sets in $b(\mathcal{C})^\uparrow$. A clutter $\mathcal{C}$ is called *ideal* if the polyhedron $P(\mathcal{C})$ is integer. By a result of Lehman [5], a clutter is ideal if and only if its blocker is.

Deciding whether a clutter is ideal is hard. However, interesting structural properties can be proved for clutters which are minimally nonideal (*mni*) in the sense that any facet of P defined by setting a variable to 0 or 1 is integer. The following theorem of Lehman [6], which is the basis of these structural results, is considered to be one of the fundamental results on covering polyhedra.

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Theorem 1.1 (Lehman [6]). *Let $\mathcal{C}$ be a minimally nonideal clutter nonisomorphic to a finite degenerate projective plane. Then $P(\mathcal{C})$ has a unique noninteger vertex, namely $\frac{1}{r}\mathbf{1}$, where r is the minimal size of an edge in $\mathcal{C}$. There are exactly n sets of size r in $\mathcal{C}$ and each element of V is contained in exactly r of them. The blocker $b(\mathcal{C})$ also has exactly n sets of minimum size, which correspond to the vertices of $P(\mathcal{C})$ adjacent to the noninteger vertex.*

An important consequence of the theorem, observed by Seymour [12], is that the problem of deciding idealness of a clutter is in co-NP, provided that we have a membership oracle for $\mathcal{C}^{\uparrow}$. After Lehman's groundbreaking result, there have been several attempts to better understand the structure of minimally nonideal clutters (see [8] for enumeration of mni matrices of small dimension, [2] for a survey, and [3, 14] for more recent developments).

There have been successful efforts to combine Lehman's theorem with another fundamental result, Lovász' co-NP characterization of minimally imperfect clutters [7]. Sebő [11], and Gasparyan, Preissmann and Sebő [4] considered polyhedra defined by both packing and covering constraints, and gave an extension of Lehman's theorem. An inconvenience of their approach is that the class of polyhedra they consider is not closed under taking facets defined by setting variables to 0 or 1, and there is no natural way to define a blocker.

In this paper we present two different approaches that address these issues. In the first part of the paper, in Sect. 2, we prove an extension of Lehman's theorem to another class of polyhedra that includes both packing and covering polyhedra as a subclass. Let $\mathcal{C}$ and $\mathcal{D}$ be clutters on the same ground set V. We consider polyhedra of the form

$$P(\mathcal{C},\mathcal{D}) = \{x \in \mathbb{R}^V : \ 0 \le x \le \mathbf{1},\ x(C) \ge 1 \text{ for every } C \in \mathcal{C},\\ x(D) \ge |D| - 1 \text{ for every } D \in \mathcal{D}\}. \quad (1)$$

We will see that facets defined by setting a variable to 0 or 1 are also in this class. If $\mathcal{D}$ is empty, then this is the same as $P(\mathcal{C})$. On the other hand, if $\mathcal{C}$ is empty, then $\mathbf{1} - P(\mathcal{C},\mathcal{D})$ is the packing polyhedron of $\mathcal{D}$. Clearly, $\mathbf{1} - P(\mathcal{C},\mathcal{D})$ is integral if and only if $P(\mathcal{C},\mathcal{D})$ is integral. Our main result is that if $P(\mathcal{C},\mathcal{D})$ is minimally nonideal, then it has a unique non-integer vertex, that is simple (i.e. it is on n facets), and all of its components are the same, except for the case of finite degenerate projective planes.

An integer-valued set function f on ground set V is *unit-increasing* if $f(U) \le f(U+v) \le f(U)+1$ for every $U \subseteq V$ and $v \notin U$. In the second part of the paper, in Sect. 3, we extend the notion of idealness to unit-increasing set functions. To a clutter $\mathcal{C}$ we can associate the unit-increasing function

$$f_{\mathcal{C}}(U) = \begin{cases} 1 & \text{if } U \in \mathcal{C}^{\uparrow}, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

We show that it is possible to associate $(n+1)$-dimensional polyhedra to unit-increasing set functions in such a way that the notions of minor, blocker, and idealness are natural extensions of these notions for clutters, so the blocker of $f_\mathcal{C}$ is $f_{b(\mathcal{C})}$, and $f_\mathcal{C}$ is ideal if and only if $\mathcal{C}$ is ideal. Furthermore, the property that idealness is equivalent to the idealness of the blocker remains true for any unit-increasing function. For matroids this means that both the rank function (which is submodular) and the co-rank function (which is supermodular) are ideal.

Another attractive characteristic of this approach is the existence of a "twisting" operation on unit-increasing set functions that preserves idealness. For example, the degenerate projective plane on n elements (that can be considered as the exceptional case in Lehman's theorem) is nothing else but a twisting of the set function corresponding to the exceptional non-Helly clutter in the theorem of Lovász.

One caveat is that in this setting there is no direct analogue of packing polyhedra. However, we will show that to a clutter $\mathcal{D}$ one can associate a set function $g_\mathcal{D}$ such that $g_\mathcal{D}$ is minimally nonideal if and only if $\mathcal{D}$ is minimally imperfect.

It seems that Lehman's theorem cannot be directly generalized to this setting, and this gives rise to several open question that are presented in Sect. 4. We will show an example of an mni function where the fractional vertex of the polyhedron is not simple. However, we are unaware of any example where the polyhedron of an mni set function has more than one non-integer vertex. Note that Lehman's theorem implies that idealness of clutters (i.e. functions of type $f_\mathcal{C}$) is in co-NP if we have a function evaluation oracle. An interesting open question is whether this is true for arbitrary unit-increasing set functions.

1.1 Preliminaries on clutters

As several definitions in the paper are derived from the same notions used in the theory of clutters, it is useful to desribe the clutter versions first. There are two types of minor operations for a clutter $\mathcal{C}$ on ground set V, corresponding to including or excluding an element $v \in V$ in the blocker:

- the *deletion minor* is the clutter $\mathcal{C}\backslash v$ on ground set $V - v$ with members $\{C \in \mathcal{C} : v \notin C\}$,
- the *contraction minor* is the clutter $\mathcal{C}/v$ on ground set $V - v$ whose members are the inclusionwise minimal sets in $\{C - v : C \in \mathcal{C}\}$.

A *minor* of $\mathcal{C}$ is a clutter obtained by repeated application of these two operations – it is easy to see that the order of the operations does not matter. The covering polyhedron of a minor is the face of $P(\mathcal{C})$ obtained by setting the deleted variables to 1 and the contracted variables to 0. A clutter is *minimally nonideal* (or *mni* for short) if it is not ideal but all of its minors are ideal.

For an integer $t \geq 2$, the clutter $\mathcal{J}_t = \{\{1, 2, \dots t\}, \{0, 1\}, \{0, 2\}, \dots \{0, t\}\}$ on ground set $\{0, 1, \dots t\}$ is called the finite degenerate projective plane. It is known that $\mathcal{J}_t$ is an mni clutter whose blocker is itself.

2 Generalization of Lehman's theorem to clutter pairs

Let $\mathcal{C}$ and $\mathcal{D}$ be clutters on ground set V of size n. We consider the polyhedron

$$P(\mathcal{C},\mathcal{D}) = \{x \in \mathbb{R}^V : x(C) \geq 1 \ \forall C \in \mathcal{C},\ x(D) \geq |D|-1 \ \forall D \in \mathcal{D},\ 0 \leq x \leq \mathbf{1}\}.$$

Without loss of generality, we can assume that every set in $\mathcal{D}$ has size at least 3, and that $|C \cap D| \leq 1$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

As mentioned in the introduction, we would like minors to correspond to faces obtained by fixing some variables to 0 or 1. This can be achieved by defining minors of a pair $(\mathcal{C},\mathcal{D})$ the following way:

- The *deletion* minor for $v \in V$ is $(\mathcal{C} \setminus v, \mathcal{D} \setminus v)$, where $\mathcal{C} \setminus v = \{C \in \mathcal{C} : v \notin C\}$ and $\mathcal{D} \setminus v$ consists of the maximal members of $\{D - v : D \in \mathcal{D}\}$.
- The *contraction* minor for $v \in V$ is $(\mathcal{C}/v, \mathcal{D}/v)$, where $\mathcal{C}/v$ consists of the minimal members of $\{C - v : C \in \mathcal{C}\} \cup \{w : \exists D \in \mathcal{D} : v, w \in D\}$, and $\mathcal{D}/v = \{D \in \mathcal{D} : v \notin D\}$.

We call a pair $(\mathcal{C},\mathcal{D})$ *ideal* if $P(\mathcal{C},\mathcal{D})$ is an integer polyhedron. Thus $(\mathcal{C},\emptyset)$ is ideal if and only if $\mathcal{C}$ is an ideal clutter, and $(\emptyset,\mathcal{D})$ is ideal if and only if $\mathcal{D}$ is a perfect clutter. The pair $(\mathcal{C},\mathcal{D})$ is *minimally nonideal* if every minor is ideal but $(\mathcal{C},\mathcal{D})$ itself is not. Equivalently, every non-integer vertex has only non-integral components.

Let $(\mathcal{C},\mathcal{D})$ be a minimally nonideal pair, and let $\mathbf{0} < x^* < \mathbf{1}$ be a non-integral vertex of $P(\mathcal{C},\mathcal{D})$. We introduce the following notation.

$$\begin{aligned}
\mathcal{C}^* &= \{C \in \mathcal{C} : x^*(C) = 1\},\\
\mathcal{D}^* &= \{D \in \mathcal{D} : x^*(D) = |D|-1\},\\
\mathcal{C}^*_v &= \{C \in \mathcal{C}^* : v \notin C\},\\
\mathcal{D}^*_v &= \{D \in \mathcal{D}^* : v \notin D\}.
\end{aligned}$$

Before proving the main theorem of this section, we prove a sequence of propositions that are analogous to ones used in various proofs of Lehman's theorem (see e.g. [12]). By a slight abuse of notation, we sometimes identify sets and their characteristic vectors, so if $\mathcal{F}$ is a family of sets, then $\langle \mathcal{F} \rangle$ denotes the subspace generated by the characteristic vectors of the members.

Proposition 2.1. *For any $v \in V$ and any $Z \in \mathcal{C}^*_v \cup \mathcal{D}^*_v$ we have* $\dim\langle \mathcal{C}^*_v \cup \mathcal{D}^*_v \rangle \leq n - |Z|$.

Proof. Let $C \in \mathcal{C}^*_v$, and let x^*_{-v} denote the vector x^* restricted to $V - v$. This vector is in the integer polyhedron $P(\mathcal{C} \setminus v, \mathcal{D} \setminus v)$, so x^*_{-v} is a convex combination $\sum \lambda_j X_j$ of integer vertices. For every $u \in C$ we have $x^*_u > 0$, so there exists j_u such that $u \in X_{j_u}$. Since $x^*(C) = 1$, we have $X_{j_u} \cap C = \{u\}$. Therefore the vectors $\{X_{j_u} : u \in C\}$ are linearly independent. Since $x^* \notin \langle X_{j_u} : u \in C \rangle$, we have $\dim\langle X_{j_u} - x^* : u \in C \rangle = |C|$.

On the other hand, each X_j is tight for the inequalities corresponding to sets in $\mathcal{C}^*_v \cup \mathcal{D}^*_v$. This means that $(X_{j_u} - x^*)(C') = 0$ for every $u \in C$ and every $C' \in$

$\mathcal{C}_v^*$. Furthermore, $(X_{j_u} - x^*)(D') = 0$ for every $u \in C$ and every $D' \in \mathcal{D}_v^*$. Thus $\dim\langle \mathcal{C}_v^* \cup \mathcal{D}_v^* \rangle \leq n - |C|$.

Now let $D \in \mathcal{D}_v^*$. We have $x_u^* < 1$ for every $u \in D$, so there exists j_u such that $u \notin X_{j_u}$. Since $x^*(D) = |D| - 1$ and $|X_{j_u} \cap D| \geq |D| - 1$ (the latter is because $D \in \mathcal{D} \setminus i$), we have $X_{j_u} \cap D = D - u$. Therefore the vectors $\{X_{j_u} : u \in D\}$ are affine independent. Since $x^* \notin \text{aff}(X_{j_u} : u \in D)$, we have $\dim\langle X_{j_u} - x^* : u \in D\rangle = |D|$.

Here too we have $(X_{j_u} - x^*)(D') = 0$ for every $u \in D$ and every $D' \in \mathcal{D}_v^*$, and $(X_{j_u} - x^*)(C') = 0$ for every $u \in D$ and every $C' \in \mathcal{C}_v^*$. Thus $\dim\langle \mathcal{C}_v^* \cup \mathcal{D}_v^* \rangle \leq n - |D|$. □

Proposition 2.2. *The size of $\mathcal{C}^* \cup \mathcal{D}^*$ is n, and $|Z| = n - |\mathcal{C}_v^* \cup \mathcal{D}_v^*|$ for every $v \in V$ and every $Z \in \mathcal{C}_v^* \cup \mathcal{D}_v^*$. Every vertex of $P(\mathcal{C}, \mathcal{D})$ adjacent to x^* is integer.*

Proof. Let $\mathcal{B}$ be a base chosen from $\mathcal{C}^* \cup \mathcal{D}^*$. The size of $\mathcal{B}$ is n, and by Proposition 2.1 we have $|\mathcal{B}_v| \leq n - |Z|$ for every $Z \in \mathcal{C}_v^* \cup \mathcal{D}_v^*$ and for every v. Let $U = \{u \in V : \exists B \in \mathcal{B} \text{ s.t. } u \notin B\}$. We can write

$$n = \sum_{B \in \mathcal{B}} 1 = \sum_{B \in \mathcal{B}} \sum_{v \in V \setminus B} \frac{1}{n - |B|} = \sum_{u \in U} \sum_{B \in \mathcal{B}_u} \frac{1}{n - |B|}$$
$$\leq \sum_{u \in U} \sum_{B \in \mathcal{B}_u} \frac{1}{|\mathcal{B}_u|} = \sum_{u \in U} 1 = |U| \leq n.$$

Therefore there is equality throughout, so $U = V$, and $|B| = n - |\mathcal{B}_v|$ for every v and every $B \in \mathcal{B}_v$.

Let $H = (V, \mathcal{E})$ be the hypergraph with hyperedges $\mathcal{E} = \{V \setminus Z : Z \in \mathcal{C}^* \cup \mathcal{D}^*\}$, and let $H' = (V, \mathcal{E}')$ be the subhypergraph corresponding to $\mathcal{B}$. Let $H_1 = (V_1, \mathcal{E}_1), \dots, H_k = (V_k, \mathcal{E}_k)$ denote the components of H'. By the above, H' has no isolated node, and there are numbers $r_1, \dots, r_k$ such that component H_j is r_j-regular and r_j-uniform. If $H \neq H'$, then there is a set $B' \in \mathcal{B}$ and a set $B'' \in (\mathcal{C}^* \cup \mathcal{D}^*) \setminus \mathcal{B}$ such that $\mathcal{B}'' = \mathcal{B} - B' + B''$ is also a base. Let H'' be the corresponding sub-hypergraph. This must also have regular and uniform components, but since we replaced only one hyperedge, this is only possible if $B' = B''$, a contradiction. Thus we have $H = H'$, and $|\mathcal{E}| = n$.

We can also show by a similar argument that every vertex of $P(\mathcal{C}, \mathcal{D})$ adjacent to x^* is an integer vertex. Indeed, a non-integer adjacent vertex would satisfy with equality all but one of the inequalities corresponding to $\mathcal{C}^* \cup \mathcal{D}^*$. Furthermore, together with a new tight inequality we would obtain a hypergraph with the same kind of structure (because what we proved up to now is true for any non-integer vertex). This is impossible because we cannot have regular and uniform components after replacing a single hyperedge. □

Now we are ready to prove the main theorem of this section.

Theorem 2.3. *If the pair $(\mathcal{C}, \mathcal{D})$ is mni, then one of $\mathcal{C}^*$ and $\mathcal{D}^*$ is empty, and the other one is uniform and regular (except for the case $\mathcal{C} = \mathcal{J}_{n-1}, \mathcal{D} = \emptyset$). The polyhedron $P(\mathcal{C}, \mathcal{D})$ has a single non-integer vertex that is simple.*

Proof. Let x^* be a non-integer vertex. As in the proof of Proposition 2.2, $H = (V, \mathcal{E})$ denotes the hypergraph with hyperedges $\mathcal{E} = \{V \setminus Z : Z \in \mathcal{C}^* \cup \mathcal{D}^*\}$, and its components are $H_1, \dots, H_k$, where H_i is r_i-uniform and r_i-regular. We assume that $r_1 \le r_2 \le \cdots \le r_k$. The vertex x^* is simple because $|\mathcal{C}^* \cup \mathcal{D}^*| = n$ by Proposition 2.2. The proof of the other properties is divided into three cases.

Case 1: $\mathcal{D}^* = \emptyset$. It can be seen that

$$x^*_v = \frac{1}{(-1 + \sum_{j=1}^{k} \frac{|V_j|}{r_j}) r_l} \text{ if } v \in V_l,$$

because this is the unique solution of the equation system given by $\mathcal{C}^*$. If $k = 1$ or $k \ge 3$ then $x^*_v \le \frac{1}{2}$ for every v, which implies that $\mathcal{D}$ is empty. If $k = 2$, then $x^*_v \le \frac{1}{2}$ for every v unless $|V_1| = 1$. In this case $x^*_v = \frac{r_2}{n-1}$ if $v = V_1$ and $x^*_v = \frac{1}{n-1}$ otherwise, which implies that $x^*(Z) < |Z| - 1$ for every set Z of size at least 3. Thus $\mathcal{D}$ is empty again. The statements of the theorem follow from Theorem 1.1.

Case 2: $\mathcal{C}^* = \emptyset$. Since $|C \cap D| \le 1$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, $\mathcal{C}$ must be empty in case of $k \ge 3$ because every pair of elements is in some $D \in \mathcal{D}^*$. This is also true for $k = 2$ unless $|V_1| = 1$, when there may be size 2 sets in $\mathcal{C}$. But those can also be considered as members of $\mathcal{D}$, so we have a minimally imperfect clutter, and the properties in the theorem follow from [7] and [9] (of course even stronger properties follow from the Strong Perfect Graph Theorem [1]).

If $k = 1$, then $\mathcal{D}^*$ is r-regular and r-uniform, so $x^*_v = \frac{r-1}{r}$ for every $v \in V$. In addition, the vertex x^* is optimal for the all-1 cost vector because of the regularity of $\mathcal{D}^*$, and any optimal vertex must satisfy with equality the inequalities corresponding to sets in $\mathcal{D}^*$. Since $\mathcal{D}^*$ is a base, x^* is the only such vertex. This proves that there cannot be another non-integer vertex x' for which Case 2 holds; on the other hand, there is no other non-integer vertex for which Case 1 holds either because we have seen that $\mathcal{D}$ is empty in Case 1. Therefore we are done if we prove that Case 3 below is impossible.

Case 3: $\mathcal{C}^* \ne \emptyset$, $\mathcal{D}^* \ne \emptyset$. Consider a set $D \in \mathcal{D}^*$. If $k \ge 3$, then any $C \in \mathcal{C}^*$ intersects D in at least 2 elements, which is impossible because $|C \cap D| \le 1$ for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$. If $k = 2$, then for the same reason the only possibility is that $|V_1| = 1$, $\mathcal{D}^* = \{V_2\}$ and $\mathcal{C}$ has only sets of size 2. As in Case 2, we can argue that this must correspond to a minimally imperfect clutter (actually it is not hard to see that this case is impossible). Therefore we can assume that $k = 1$, and $\mathcal{C}^* \cup \mathcal{D}^*$ is r-regular and r-uniform, where $r \ge 3$.

We now prove that it is impossible to have $\mathcal{C}^* \ne \emptyset$ and $\mathcal{D}^* \ne \emptyset$. Let $D \in \mathcal{D}^*$ and let X be the vertex adjacent to x^* that is not tight for D. This means that $D \subseteq X$, and $|X| = \mathbf{1}x^* + \frac{1}{r}$ because X is tight for all other inequalities corresponding to $\mathcal{C}^* \cup \mathcal{D}^*$. Thus the fractional part of $\mathbf{1}x^*$ is $\frac{r-1}{r}$. If $x^*_v > \frac{r-1}{r}$ for some $v \in V$, then $\mathbf{1}x^*_{-v} < \lfloor \mathbf{1}x^* \rfloor = |X| - 1$, which is impossible because $P(\mathcal{C} \setminus v, \mathcal{D} \setminus v)$ has no integer vertex Y with $|Y| < |X| - 1$. Thus $x^*_v \le \frac{r-1}{r}$ for every v, which implies that for every $D \in \mathcal{D}^*$ and $v \in D$ we have $x^*_v = \frac{r-1}{r}$.

Suppose that there exist $C \in \mathcal{C}^*$ and $D \in \mathcal{D}^*$ such that $|C \cap D| = 1$, and let v be the intersection. Since $\mathbf{1}x^*_{-v} = \lfloor \mathbf{1}x^* \rfloor$, it must be a convex combination of integer vertices $X_1,\dots,X_t$ of $P(\mathcal{C}\setminus v,\mathcal{D}\setminus v)$ that all satisfy $|X_j| = \mathbf{1}x^*_{-v}$. This means that $|X_j{+}v| = |X|$, so $X_j{+}v$ satisfies all but one of the inequalities corresponding to $\mathcal{C}^*\cup\mathcal{D}^*$ with equality, and the slack of the remaining inequality is 1. Consequently, $X_j + v$ is a vertex adjacent to x^* in $P(\mathcal{C},\mathcal{D})$ for every j. We can now get a contradiction using the fact that $|C| \geq 3$. Indeed, x^*_{-v} is positive on the vertices of $C - v$, and each X_j contains at most 1 such element, so there are at least two sets X_{j_1} and X_{j_2} containing an element of $C - v$. Thus $X_{j_1} + v$ and $X_{j_2} + v$ are not tight for C, hence they are tight for all other members of $\mathcal{C}^*\cup\mathcal{D}^*$. But this is impossible because there is only one integer vertex of $P(\mathcal{C},\mathcal{D})$ that is tight for all of those sets.

The only remaining case is when $C\cap D = \emptyset$ for any $C\in\mathcal{C}^*$ and $D\in\mathcal{D}^*$. Let $U_1 = \cup\mathcal{C}^*$ and $U_2 = \cup\mathcal{D}^*$. Since $\mathcal{C}^*\cup\mathcal{D}^*$ is a base, $\mathcal{C}^*$ must be a base on U_1, thus $x^*|_{U_1}$ is a vertex of $P(\mathcal{C}\setminus U_2,\mathcal{D}\setminus U_2)$, contradicting the assumption that this polyhedron is integer. We obtained that this case is also impossible. □

3 Ideal set functions

The aim of this section is to extend the notions of the blocking relation and idealness from clutters to unit-increasing set functions. We show that several properties of ideal clutters can be maintained: idealness is preserved for taking minors and blockers. We also show that new types of minimally nonideal structures emerge. In addition, we describe a transformation, called twisting of the set function at a subset, that preserves idealness.

Let V be a finite ground set, and let $f : 2^V \to \mathbb{Z}$ be an integer-valued unit-increasing set function. The *blocker* $b(f) : 2^V \to \mathbb{Z}$ of f is the unit-increasing set function defined by

$$b(f)(X) = -f(V\setminus X)$$

for any set $X\subseteq V$. Obviously, $b(b(f)) = f$. We define the following two minor operations on unit-increasing functions for a given $v\in V$:

- the *deletion minor* is the function on ground set $V - v$, denoted by $f\backslash v$, for which $f\backslash v(X) = f(X)$ for every $X\subseteq V - v$,
- the *contraction minor* is the function on ground set $V - v$, denoted by f/v, for which $f/v(X) = f(X + v)$ for every $X\subseteq V - v$.

A function f' is a *minor* of f if it can be obtained from f by deletions and contractions. It is easy to see that the order of the operations does not affect the minor we get, the minors are unit-increasing functions, and $b(f\backslash v) = b(f)/v$ and $b(f/v) = b(f)\backslash v$.

We call functions f_1 and f_2 *equivalent* if there is a constant c such that $f_2(X) = f_1(X) + c$ for every $X\subseteq V$; we will use the notation $f_1 \cong f_2$.

3.1 Polyhedra and idealness

We now show that it is possible to associate polyhedra to unit-increasing set functions in such a way that minors correspond to faces, blockers to integer vertices, and the notion of idealness can be defined in terms of integrality of polyhedra. The trick is to move to $(n+1)$-dimensional space. For a function f, let

$$P(f) = \{(y,\beta) \in \mathbb{R}^{n+1} :\ 0 \le y \le \mathbf{1},\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\}.$$

Proposition 3.1. *The following hold for the minors of f:*

$$P(f\backslash v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,1,\beta) \in P(f)\}, \text{ and}$$

$$P(f/v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,0,\beta) \in P(f)\},$$

that is, both $P(f\backslash v)$ and $P(f/v)$ are facets of $P(f)$.

Proof. It is easy to see that for a vector $(y,1,\beta) \in P(f)$, (y,β) satisfies the inequalities of $P(f\backslash v)$, since they are present in the system of $P(f)$ too.

If $(y,\beta) \in P(f\backslash v)$ and $X \subseteq V - v$, then on one hand we have $(y,1)(X) - \beta = y(X) - \beta \ge f\backslash v(X) = f(X)$, and on the other hand $(y,1)(X+v) - \beta = y(X)+1-\beta \ge f\backslash v(X) + 1 = f(X) + 1 \ge f(X+v)$, since f is unit-increasing. So $(y,1,\beta) \in P(f)$.

It is easy to see that for a vector $(y,0,\beta) \in P(f)$, (y,β) satisfies the inequalities of $P(f/v)$, since $y(X) - \beta = (y,0)(X+v) - \beta \ge f(X+v) = f/v(X)$.

If $(y,\beta) \in P(f/v)$ and $X \subseteq V - v$, then on one hand we have $(y,0)(X) - \beta = y(X) - \beta \ge f/v(X) = f(X+v) \ge f(X)$, since f is unit-increasing, and on the other hand $(y,0)(X+v) - \beta = y(X) - \beta \ge f/v(X) = f(X+v)$, thus $(y,0,\beta) \in P(f)$. □

The unit-increasing set function f is called *ideal* if the polyhedron $P(f)$ is integral. As expected, idealness is preserved under taking minors.

Proposition 3.2. *If f is ideal, then any minor of it is also ideal.*

Proof. It follows from Proposition 3.1. □

This enables us to call a unit-increasing function f *minimally nonideal* (*mni*) if it is not ideal but every minor is ideal. Before showing that this is a direct extension of the same notion for clutters, we note that we get the same notion of idealness if we remove the upper bound or both bounds on y in the polyhedron. Let

$$Q(f) = \{(y,\beta) \in \mathbb{R}^{n+1} :\ y \ge 0,\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\},$$
$$R(f) = \{(y,\beta) \in \mathbb{R}^{n+1} :\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\}.$$

Proposition 3.3. *If f is unit-increasing, then $P(f)$ is integral $\Leftrightarrow$ $Q(f)$ is integral $\Leftrightarrow$ $R(f)$ is integral.*

Proof. See the proof of Theorem 3.6 in the Appendix. □

Recall that to a clutter $\mathcal{C}$ we associate the set function (2). It is easy to check that this works well with the minor operations: for any $v \in V$, $f_{\mathcal{C}\backslash v} = f_{\mathcal{C}}\backslash v$ and $f_{\mathcal{C}/v} = f_{\mathcal{C}}/v$. Likewise, one can check that the blocker $b(f_{\mathcal{C}})$ is equivalent the set function corresponding to the blocker of $\mathcal{C}$ (they differ by 1).

Proposition 3.4. *A clutter $\mathcal{C}$ is ideal if and only if $f_{\mathcal{C}}$ is ideal.*

Proof. It is easy to see that

$$\begin{aligned} Q(f_{\mathcal{C}}) &= \{(y,\beta) \in \mathbb{R}^{n+1} : \ y \geq 0, \ \ y(X) - \beta \geq f_{\mathcal{C}}(X) \ \ \forall X \subseteq V\} = \\ &= \{(y,\beta) \in \mathbb{R}^{n+1} : \ y \geq 0, \ \ \beta \leq 0, \ y(X) - \beta \geq 1 \ \ \forall X \in \mathcal{C}\}. \end{aligned}$$

So the face of $Q(f_{\mathcal{C}})$ in the $\beta = 0$ hyperplane is the covering polyhedron of $\mathcal{C}$, thus if $f_{\mathcal{C}}$ is ideal then $\mathcal{C}$ is ideal too.

To see the other direction, note that in the above description all inequalities but $\beta \leq 0$ are incident to the vector $(\mathbf{0}, -1)$. Therefore $Q(f_{\mathcal{C}})$ is the intersection of a cone pointed at $(\mathbf{0}, -1)$ and the halfspace $\{(y,\beta) : \ \beta \leq 0\}$. It follows that if $\mathcal{C}$ is ideal then $f_{\mathcal{C}}$ is also ideal. □

Corollary 3.5. *A clutter $\mathcal{C}$ is mni if and only if $f_{\mathcal{C}}$ is mni.*

We note that Lehman's Theorem 1.1 has the consequence that if $\mathcal{C}$ is mni, then the polyhedron $P(f_{\mathcal{C}})$ has a unique fractional vertex and it is simple (here a vertex is simple if it lies on $n+1$ facets).

3.2 Blockers and idealness

As in Sect. 2, we will abuse notation by identifying sets and their characteristic vectors. For a unit-increasing set function f, let us define the following finite set of vectors in $\mathbb{R}^{n+1}$:

$$S(f) = \{(X, f(X)) : \ X \subseteq V\}.$$

We denote the set $S(f) - \text{cone}\{(\mathbf{0}, -1)\}$ by $S^{\downarrow}(f)$. We note that the idealness of f is equivalent to $P(f) = \text{conv}\{S^{\downarrow}(b(f))\}$.

Theorem 3.6. *For a unit-increasing set function, the following are equivalent:*

(i) *f is ideal, that is,* $P(f) = \text{conv}\{S^{\downarrow}(b(f))\}$

(ii) *$b(f)$ is ideal, that is,* $P(b(f)) = \text{conv}\{S^{\downarrow}(f)\}$

(iii) *$R(f)$ is an integer polyhedron*

(iv) *$R(b(f))$ is an integer polyhedron*

(v) *$Q(f)$ is an integer polyhedron*

(vi) *$Q(b(f))$ is an integer polyhedron*

Proof. See the Appendix, Sect. 5.1 □

As an example, we show that for a matroid $M = (V, r)$, both its rank function r and its co-rank function q are ideal functions. It is known that these are unit-increasing functions, and the rank function is submodular, while q is supermodular.

Proposition 3.7. *Both the rank function r and the corank function q of a matroid are ideal functions.*

Proof. First we prove that q is ideal. It is enough to show that the polyhedron

$$R(q) = \{(y, \beta) \in \mathbb{R}^{n+1} : \ y(X) - \beta \geq q(X) \ \forall X \subseteq V\}$$

is integer. Since q is supermodular, a standard uncrossing proof gives that this system is TDI, hence the polyhedron is integral.

The blocker of r is $b(r)(X) = -r(V - X) = q(X) - r(V)$, which is equivalent to q, thus $b(r)$ is ideal, and r is also ideal by Theorem 3.6. □

3.3 Twisting

In this section we introduce the twisting operation that preserves indealness. Let f be a unit-increasing set function on ground set V, and let U be a subset of V. The *twisting of f at U* is the set function f^U on ground set V defined by

$$f^U(X) = f(X \Delta U) + |X \cap U|.$$

It is easy to see that f^U is a unit-increasing set function. The interaction with minors is the following.

Proposition 3.8. *For a set $U \subseteq V$ and an element $v \in V$ the following hold.*

(i)

$$(f \backslash v)^{U-v} \cong \begin{cases} f^U/v & \text{if } v \in U, \\ f^U \backslash v & \text{if } v \notin U, \end{cases}$$

(ii)

$$(f/v)^{U-v} \cong \begin{cases} f^U \backslash v & \text{if } v \in U, \\ f^U/v & \text{if } v \notin U. \end{cases}$$

Proof. See Sect. 5.2 in the Appendix. □

Proposition 3.9. *Every twisting of an ideal set function is also ideal.*

Proof. See Appendix, Sect. 5.2. □

Corollary 3.10. *Every twisting of an mni set function is also mni.*

Proof. This follows from Propositions 3.8 and 3.9. □

As an example, consider the following set function on ground set V of size n:

$$\theta_n(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ n-2 & \text{if } X = V, \\ |X| - 1 & \text{otherwise.} \end{cases}$$

This set fuction is equivalent to a twisting of the function corresponding to the degenerate projective plane:

$$\theta_n \cong f_{\mathcal{J}_{n-1}}^{V \backslash \{0\}}.$$

3.4 Further mni functions

It is a natural question whether the idealness introduced in this section generalizes the notion of idealness of clutter pairs used in Sect. 2. The answer is no; in fact, it does not even generalize perfectness of clutters. It would be natural to associate to a clutter $\mathcal{D}$ the unit-increasing set function

$$g_{\mathcal{D}}(X) = \max\{0, \max\{|X \cap D| - 1 : D \in \mathcal{D}\}\}. \tag{3}$$

The problem is that $P(g_{\mathcal{D}})$ is not necessarily integral if $\mathcal{D}$ is the set of inclusionwise maximal cliques of a perfect graph. Also, the minors of $g_{\mathcal{D}}$ do not necessarily belong to this class.

In this light, it is somewhat surprising that the following is true. An interesting question is whether one can prove it without using the Strong Perfect Graph Theorem.

Theorem 3.11. *The function $g_{\mathcal{D}}$ is minimally nonideal if and only if $\mathcal{D}$ is minimally imperfect.*

Proof. If $g_{\mathcal{D}}$ is ideal, then $\mathcal{D}$ is perfect, because the packing polyhedron of $\mathcal{D}$ is the same as the facet of $P(g_{\mathcal{D}})$ given by $\beta = 0$. Another observation is that if $\mathcal{D}$ is perfect, then $g_{\mathcal{D}}$ cannot be minimally non-ideal. Indeed, The point $(\mathbf{1}, 1)$ is not in $P(g_{\mathcal{D}})$, but satisfies with equality all facet-defining inequalities of $P(g_{\mathcal{D}})$ except for $\beta \leq 0$ and $y \geq 0$. This means that any all-fractional vertex of $P(g_{\mathcal{D}})$ must satisfy $\beta = 0$. However, the face $\beta = 0$ is the same as the packing polyhedron of $\mathcal{D}$, so it has only integer vertices.

These observations together imply that if $g_{\mathcal{D}}$ is minimally non-ideal, then $\mathcal{D}$ is minimally imperfect. To prove the other direction, we resort to the characterization of Lovász and the Strong Perfect Graph Theorem. According to these, $\mathcal{D}$ is minimally imperfect if and only if it is the non-Helly clutter (consisting of the complements of singletons), or the clutter of inclusionwise maximal cliques of an odd hole or odd antihole.

The function associated to the non-Helly clutter is θ_n, which is mni. If $\mathcal{D}$ is the odd hole clutter, then $g_{\mathcal{D}} = f_{\mathcal{D}}$, so it is mni because the clutter is mni. The proof that the function associated to an odd antihole is mni is contained in Sect. 5.3 of the Appendix. □

We present one more mni function that shows the difficulty of extending Lehman's theorem. So far all mni functions we have seen satisfied the property that $P(f)$ has a unique non-integer vertex that is simple. The following mni set function f on ground set $\{1, 2, 3, 4, 5\}$ is an example where the unique fractional vertex of $P(f)$ is not simple. The properties were checked using the software Polymake.

$$f(X) = \begin{cases} 0 & \text{if } X = \emptyset, \\ 1 & \text{if } |X| = 1 \text{ or } X \in \{\{1,2\}, \{2,3\}, \{3,4\}\{4,5\}\}, \\ 2 & \text{if } |X| = 3 \text{ or } X \in \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,5\}\} \\ & \quad \text{or } X \in \{\{1,3,4,5\}, \{1,2,4,5\}, \{1,2,3,5\}\}, \\ 3 & \text{if } X \in \{\{1,2,3,4\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}. \end{cases}$$

4 Open questions

There are several questions about ideal unit-increasing functions that we think may lead to better insights into the structure of 0-1 polyhedra, in particular concerning classes larger than packing and covering polyhedra.

- We are not aware of an example of an mni function that has more than one non-integer vertex, so one is tempted to conjecture that the non-integer vertex is always unique. In addition, in all known examples there is a value λ such that every component of the unique non-integer vertex (except for the last one) is either λ or $1-\lambda$.
- Can one define a class of functions that contains all functions of type $f_{\mathcal{C}}$ and $g_{\mathcal{C}}$, is closed under taking minors, blockers, and twisting, and has the property that any minimally non-ideal member of the class has a unique fractional vertex that is simple?
- In a model where functions are given by an evaluation oracle, is it in co-NP to decide if a unit-increasing function is ideal?

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5 Appendix

5.1 Proof of Theorem 3.6

First we prove some basic properties of the polyhedra $P(f)$, $Q(f)$, and $R(f)$.

Proposition 5.1. *If f is a unit-increasing set function, then $Q(f) = P(f) + \mathbb{R}^n_+$.*

Proof. The $Q(f) \supseteq P(f) + \mathbb{R}^n_+$ inclusion is easy, since the describing matrix of $Q(f)$ has nonnegative coefficients in the first n variables.

For the $Q(f) \subseteq P(f) + \mathbb{R}^n_+$ inclusion, let $(y, \beta) \in Q(f)$. We want to show that there is a $(y', \beta) \in P(f)$ for which $y' \leq y$. Let $y'_i = \min(y_i, 1)$. Then $y' \leq y$ and $0 \leq y' \leq 1$ hold, so it remains to show that $y'(X) - \beta \geq f(X)$ for each $X \subseteq V$. We have $y'(X) = |X \cap \{i : y_i > 1\}| + y(X \cap \{i : y_i \leq 1\}) \geq |X \cap \{i : y_i > 1\}| + f(X \cap \{i : y_i \leq 1\}) + \beta \geq f(X) + \beta$, since f is unit-increasing. □

Let C be the cone generated by $\{e_i : i \in [n]\} \cup \{-e_i - e_{n+1} : i \in [n]\}$. We call a set X *tight* with respect to f and a vector (y, β) if $y(X) - \beta = f(X)$.

Proposition 5.2. *Every vertex (y^*, β^*) of $R(f)$ satisfies $0 \leq y^* \leq 1$, and the characteristic cone of $R(f)$ is C, hence $R(f) = P(f) + C$.*

Proof. First let us show that the characteristic cone is C. It is easy to see that all the vectors e_i and $-e_i - e_{n+1}$ are in the characteristic cone of $R(f)$. If a vector (z, γ) is in the characteristic cone of $R(f)$, then for every $X \subseteq V$, $z(X) - \gamma \geq 0$ holds. For $X = \{i : z_i < 0\}$ we have $(z, \gamma) = \sum_{i \in X} -z_i(-e_i - e_{n+1}) + (z', \gamma')$, where $z' \geq 0$ and $\gamma' \leq 0$, and it is easy to see that $(z', \gamma') \in C$.

Now let (y^*, β^*) be a vertex, and suppose that $y^*_v < 0$. Then every tight set X contains v, because otherwise the inequality for $X + v$ would be violated since $f(X + v) \geq f(X)$. Now, if every tight set X contains v, then $(y^*, \beta^*) + \varepsilon(\chi_v, 1)$ is in $R(f)$ for some positive ε. This contradicts the fact that (y^*, β^*) is a vertex and $(-\chi_v, -1)$ is an extreme direction.

Now suppose that $y^*_v > 1$ for a vertex (y^*, β^*). Then no tight set contains v, since otherwise the inequality for $X - v$ would be violated: $y^*(X-v) - \beta < y^*(X) - 1 - \beta = f(X) - 1 \leq f(X-v)$, a contradiction. This implies that for some positive ε, the vector $(y^*, \beta^*) - \varepsilon(\chi_v, 0)$ is in $R(f)$, which contradicts the fact that (y^*, β^*) is a vertex and e_v is an extreme direction. □

Corollary 5.3. *If f is a unit-increasing set function, then $R(f) = Q(f) + C$.*

Proof. It follows from Propositions 5.1 and 5.2 and that $\mathbb{R}^n_+ \subset C$. □

For a polyhedron P, let $\mathrm{vert}(P)$ denote the set of its vertices.

Corollary 5.4. *For a unit-increasing function f it is always true that* $\mathrm{vert}(P(f)) \supseteq \mathrm{vert}(Q(f)) \supseteq \mathrm{vert}(R(f))$.

In the proof of Theorem 3.6 we will use an operation B on polyhedra in $\mathbb{R}^{n+1}$, which is similar to taking the blocker of a polyhedron, it differs only in the last coordinate. For a polyhedron $P \subseteq \mathbb{R}^{n+1}$, let us define $B(P)$ as follows:

$$B(P) = \{(y, \beta) \in \mathbb{R}^{n+1} : x^\mathsf{T} y \geq \alpha + \beta \text{ for every } (x, \alpha) \in P\}.$$

Note that $B(P)$ is indeed a polyhedron, since using standard polyhedral techniques one can prove that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$ for finite vector sets S and T in $\mathbb{R}^{n+1}$, then

$$B(P) = \{(y, \beta) \in \mathbb{R}^{n+1} : s_{[n]}^\mathsf{T} y \geq s_{n+1} + \beta \ \forall s \in S \text{ and } t_{[n]}^\mathsf{T} y \geq t_{n+1} \ \forall t \in T\}. \tag{4}$$

Suppose that the polyhedron $P \subset \mathbb{R}^{n+1}$ has the following properties:

(a) $\exists \bar{\alpha} : (\mathbf{0}, \bar{\alpha}) \in P$

(b) P is bounded from above in the last coordinate

(c) $(\mathbf{0}, -1)$ is in the characteristic cone of P

Proposition 5.5. *If P satisfies properties (a)-(c) then so does $B(P)$.*

Proof. To see property (a), we can observe that if $P = \mathrm{conv}\{S\} + \mathrm{cone}\{T\}$, then from (4) we get that for $\bar{\beta} = \min(-s_{n+1} : s \in S)$, $(\mathbf{0}, \bar{\beta}) \in P$. For property (b) we can take an $\bar{\alpha}$ such that $(\mathbf{0}, \bar{\alpha}) \in P$ which implies that $\beta \leq \mathbf{0}^\mathsf{T} y - \bar{\alpha} = -\bar{\alpha}$. For property (c) we need that $x^\mathsf{T}\mathbf{0} \geq -1$ which is obvious, and that $B(P)$ is nonempty which follows from (a). □

Lemma 5.6. *If P satisfies properties (a)-(c) then $B(B(P)) = P$.*

Proof. For every $(x, \alpha) \in P$ and $(y, \beta) \in B(P)$ we have $x^\mathsf{T} y \geq \alpha + \beta$ which shows that $P \subseteq B(B(P))$.

Suppose that there is a vector $(x^*, \alpha^*) \in B(B(P))$ which is not in P. Then there is a vector (z, γ) and a number ξ such that ${x^*}^\mathsf{T} z + \alpha^* \gamma < \xi$, but for every $(x, \alpha) \in P$, $x^\mathsf{T} z + \alpha\gamma \geq \xi$. From (c) it follows that $\gamma \leq 0$.

Case 1: $\gamma = 0$. We show that there is an $\varepsilon > 0$ such that $x^{*\mathsf{T}}z + \alpha^*(-\varepsilon) < x^\mathsf{T}z + \alpha(-\varepsilon)$ for each $(x,\alpha) \in P$. Because of (b) we know that there is an $a \in \mathbb{R}$ such that $\alpha \le a$ for every $(x,\alpha) \in P$. We can assume that $a > \alpha^*$. If $\varepsilon < \frac{\xi - x^{*\mathsf{T}}z}{a-\alpha^*}$, then for every $(x,\alpha) \in P$, $\varepsilon(\alpha-\alpha^*) \le \varepsilon(a-\alpha^*) < \xi - x^{*\mathsf{T}}z \le x^\mathsf{T}z - x^{*\mathsf{T}}z$. Since $x^\mathsf{T}z + \alpha(-\varepsilon)$ attains its minimum on P, we have an instance of Case 2.

Case 2: $\gamma < 0$. We can assume that $\gamma = -1$, since we can scale the inequalities with a positive multiplier. So we have $x^{*\mathsf{T}}z - \alpha^* < \xi$, and for each $(x,\alpha) \in P$, $x^\mathsf{T}z - \alpha \ge \xi$. That means the vector $(z,\xi) \in B(P)$ but for this vector (x^*,α^*) does not fulfil the required inequality to be in the blocker of $B(P)$, which contradicts $(x^*,\alpha^*) \in B(B(P))$. □

Notice that for a unit-increasing function f, the polyhedron $P(f)$ satisfies properties (a)-(c).

Proposition 5.7. *$B(P(f)) = \mathrm{conv}\{S(f)\} + C$ and $B(R(f)) = \mathrm{conv}\{S^{\downarrow}(f)\}$.*

Proof. First we prove that $B(\mathrm{conv}\{S(f)\} + C) = P(f)$, by Lemma 5.6 this implies the first equation. Using (4), we have

$$\begin{aligned} B(\mathrm{conv}\{S(f)\}+C) = \{\ (y,\beta) \in \mathbb{R}^{n+1}:\ & y(X) \ge f(X) + \beta \ \ \forall X \subseteq V, \\ & y_i \ge 0 \ \ \forall i \in [n],\ -y_i \ge -1 \ \ \forall i \in [n]\ \}, \end{aligned}$$

which is equal to $P(f)$.

Now let us prove that $B(\mathrm{conv}\{S^{\downarrow}(f)\}) = R(f)$, which implies the second equation. Using (4), we have

$$B(\mathrm{conv}\{S^{\downarrow}(f)\}) = \{\ (y,\beta) \in \mathbb{R}^{n+1}:\ y(X) \ge f(X) + \beta \ \forall X \subseteq V\},$$

which is $R(f)$. □

Proof of Theorem 3.6. Using Propositions 5.2 and 5.7 and Lemma 5.6 we have

$$\begin{aligned} & P(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\} \overset{+C}{\Longrightarrow} R(f) = \mathrm{conv}\{S^{\downarrow}(b(f))\} + C \overset{B(.)}{\Longrightarrow} \\ & \overset{B(.)}{\Longrightarrow} \mathrm{conv}\{S^{\downarrow}(f)\} = P(b(f)) \overset{+C}{\Longrightarrow} \mathrm{conv}\{S^{\downarrow}(f)\} + C = R(b(f)), \end{aligned}$$

which shows the equivalence of (i)-(iv). Corollary 5.4 implies that if $P(f)$ is integral then so is $Q(f)$, and if $Q(f)$ is integral then so is $R(f)$, which together with the above equivalences imply the equivalence of (v) (and also (vi)) and the other statements. □

5.2 Proofs about twisting

Proof of Proposition 3.9. Suppose that $v \in U$ and take a set $X \subseteq V - v$. Then

$$\begin{aligned} (f\backslash v)^{U-v}(X) &= f\backslash v(X\Delta(U-v)) + |X \cap (U-v)| = \\ &= f((X+v)\Delta U) + |(X+v) \cap U| - 1 = \\ &= f^U(X+v) - 1 = f^U/v(X) - 1, \ \text{and} \end{aligned}$$

$$(f/v)^{U-v}(X) = f/v(X\Delta(U-v)) + |X \cap (U-v)| =$$
$$= f(X\Delta U) + |X \cap U| = f^U(X) = f^U \backslash v(X).$$

The other cases are similar. □

Proof of Proposition 3.9. Let f be an ideal set function on V, and U be a subset of V. Consider the following $(|V|+1)\times(|V|+1)$ matrix:

$$M_U = \begin{array}{cl} \begin{pmatrix} -1 & & & & & & & & \\ & -1 & & & & & & & \\ & & \ddots & & & & 0 & & \\ & & & -1 & & & & & \\ & & & & 1 & & & & \\ & & 0 & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \\ -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} & \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} U \\ \\ \\ \\ \\ \\ \end{array} \end{array}$$

It is easy to check that $M_U^{-1} = M_U$, so M_U is unimodular. We claim that

$$R(f) = M_U R(f^U) + (U, |U|).$$

Indeed, if we denote by A the describing matrix of $R(f)$ (i.e. the matrix whith rows $(X,-1)^\mathsf{T}$), then by $(X,-1)^\mathsf{T} M_U^{-1} = (X\Delta U, -1)^\mathsf{T}$, we have

$$\begin{aligned} &M_U R(f^U) + (U,|U|) = \{M_U(y,\beta):\ A(y,\beta) \geq f^U\} + (U,|U|) = \\ &= \{(z,\gamma):\ AM_U^{-1}(z,\gamma) \geq f^U + AM_U^{-1}(U,|U|)\} = \\ &= \{(z,\gamma):\ (X\Delta U,-1)^\mathsf{T}(z,\gamma) \geq f^U(X) + (X\Delta U,-1)^\mathsf{T}(U,|U|) \quad \forall X \subseteq V\} = \\ &= \{(z,\gamma):\ z(X\Delta U) - \gamma \geq f(X\Delta U) + |X\cap U| + |U\setminus X| - |U| \quad \forall X \subseteq V\} = \\ &= \{(z,\gamma):\ z(Y) - \gamma \geq f(Y) \quad \forall Y \subseteq V\} = R(f). \end{aligned}$$

Hence we also have $R(f^U) = M_U^{-1}(R(f) - (U,|U|)) = M_U R(f) + (U,0)$. Therefore $R(f)$ is integer if and only if $R(f^U)$ is integer. □

5.3 Proof that odd antihole functions are mni

Theorem 5.8. *If $\mathcal{D}$ is the clutter of inclusionwise maximal cliques of an odd antihole, then $g_\mathcal{D}$ (as defined in* (3)*) is minimally nonideal.*

The polyhedron $P(g_\mathcal{D})$ can be written as:

$$P(g_\mathcal{D}) = \{(y,\beta) \in \mathbb{R}^{n+1}:\ 0 \leq y \leq \mathbf{1},\ \beta \leq 0,\ y(K) - \beta \geq |K| - 1\ \forall \text{ clique } K\}.$$

It will be more convenient to consider a transformed polyhedron for the complement graph and packing type constraints. For a graph $G=(V,E)$, let

$$P(G) = \{(x,t) \in \mathbb{R}^{|V|+1} : 0 \leq x \leq \mathbf{1},\ t \geq 0,\ x(S) \leq 1 + t \text{ for every stable set } S\}.$$

Clearly $P(G)$ is integer if and only if $g_\mathcal{D}$ is ideal for the clutter $\mathcal{D}$ of inclusionwise maximal stable sets of G. Thus the following proposition implies Theorem 5.8.

Proposition 5.9. *If G is a path, then $P(G)$ is an integral polyhedron. If G is an odd cycle, then $P(G)$ has a unique non-integral vertex.*

Proof. We use induction on $|V|$ and we consider both cases simultaneously. Let (x^*, t^*) be a non-integer vertex of $P(G)$.

First we claim that $\mathrm{supp}(x^*) = V$. Suppose indirectly that $x^*(v) = 0$ for some $v \in V$. If G is a path, then let G_1 and G_2 be the two paths of $G - v$, and let $x_i = x^*|_{V(G_i)}$ (for $i = 1, 2$). Let t_1 and t_2 be minimal such that $(x_i, t_i) \in P(G_i)$. Then $t_1 + t_2 + 1 \leq t^*$, since there are stable sets S_1 and S_2 which are tight, so $t^* + 1 \geq x^*(S_1 \cup S_2) = x_1(s_1) + x_2(S_2) = t_1 + t_2 + 2$.

By induction, (x_1, t_1) and (x_2, t_2) can be written as convex combination of integer points in $P(G_1)$ and $P(G_2)$, respectively: $(x_1, t_1) = \sum \lambda_i(a_i, b_i)$, $(x_2, t_2) = \sum \mu_i(c_i, d_i)$. Then the convex combination $\sum_{i,j} \lambda_i \mu_j (a_i, 0, c_j, b_i + d_j + 1)$ (where the 0 component corresponds to v) produces $(x^*, t_1 + t_2 + 1)$, and it is easy to see that every vector used in the combination is in $P(G)$. This and $t^* \geq t_1 + t_2 + 1$ implies that (x^*, t^*) can not be a vertex.

In the case that G is a cycle, the proof is a similar reduction to the path case.

Next, suppose that x^* has an interval of consecutive ones, with odd length (and the neighboring values are smaller than 1). Let u and v be the neighboring nodes. Then every tight set S contains every other node in the interval (1st, 3rd etc.), and does not contain u or v (because otherwise we could obtain a stable set S' with $x^*(S') > x^*(S)$ by moving more elements of S to the interval). But then $(x^*, t^*) \pm \epsilon(\chi_u - \chi_v)$ would be still in the polyhedron $P(G)$, which contradicts that (x^*, t^*) is a vertex. In the case that G is a path and the interval of ones is at the beginning, we get a similar contradiction.

Now consider the case that (x^*, t^*) is such that every consecutive interval of ones is of even length. Let I denote the set of nodes in V where x^* is one and let $2k$ be its cardinality. We write (x^*, t^*) as the following convex combination for some λ close to 1:

$$(x^*, t^*) = (1 - \lambda)(\chi_I, k - 1) + \lambda \left(\max(\chi_I, \frac{x^*}{\lambda}), \frac{t^* - (1 - \lambda)(k - 1)}{\lambda} \right).$$

The vector $(\chi_I, k - 1)$ is in $P(G)$, because of the evenness property of I. Let (x', t') denote the second vector, about which we want to show that it is in $P(G)$ for λ close enough to 1. The nonnegativity constraints and the $x' \leq \mathbf{1}$ constraint hold around 1.

Let S be an arbitrary stable set. If $|S \cap I| < k$, then there is another stable set S' for which $x'(S') > x'(S)$, so we can assume that $|S \cap I| = k$. Then

$$x'(S) = k + \frac{x^*(S \setminus I)}{\lambda} = k + \frac{x^*(S) - k}{\lambda} \leq k + \frac{1 + t^* - k}{\lambda} = 1 + \frac{t^* - (1 - \lambda)(k - 1)}{\lambda},$$

which proves that $(x', t') \in P(G)$.

We remain with the case when x^* has only non-integer values. In this case, every node v has to be in a tight set. The vector $(\mathbf{0}, -1)$ satisfies all of these tight inequalities with equality too, except for $t \geq 0$. Thus $t^* = 0$, and x^* is a vertex of $\mathrm{QSTAB}(\overline{G})$. If G is a path, then $\mathrm{QSTAB}(\overline{G})$ is integer, while for odd circuits it has a unique non-integer vertex. This concludes the proof. $\square$