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Sparse hypergraphs with applications in combinatorial rigidity

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Abstract

A hypergraph $H = (V, E)$ is called $(1, k)$-sparse, for some integer k, if each subset $X \subseteq V$ with $|X| \geq k$ spans at most $|X| - k$ hyperedges. If we also have $|E| = |V| - k$ then H is $(1, k)$-tight. Hypergraphs of this kind occur in several problems of combinatorial rigidity, where the goal is to analyse the generic rigidity properties of point sets equipped with geometric constraints involving subsets of points. Motivated by this connection we develop a new inductive construction of 4-regular $(1, 3)$-tight hypergraphs and use it to deduce a Laman-type combinatorial characterization of generically minimally rigid projective frameworks on the projective line.

Hypergraphs with the same sparsity parameter show up in some key results of scene analysis, due to Whiteley, as well as in affine rigidity, introduced by Gortler et al. Thus our result implies a Henneberg-type inductive construction of generically minimally rigid affine frameworks in the plane. Based on the rank function of the corresponding count matroid on the edge set of H we also obtain purely combinatorial proofs for some results on generically affinely rigid hypergraphs.

1 Introduction

Given a set of objects (points, lines, bodies, etc.) in $\mathbb{R}^d$ satisfying certain geometric constraints (pairwise distances, directions, incidences, etc.), a basic question is whether (locally or globally) the given constraints uniquely determine the whole configuration up to trivial transformations (rigid motions, dilations, etc.) of the whole set. A well-studied example is the rigidity problem of d-dimensional bar-and-joint frameworks, where the objects are points and the constraints are pairwise distances. In several cases (local or global) uniqueness depends only on the the underlying combinatorial structure (for example, the graph of the framework) if the objects are in sufficiently general, or generic position.

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We shall study affine and projective constraints on (subsets of) a set of points, which lead to problems in which the underlying combinatorial structure is a hypergraph. By using a new inductive construction for a family of sparse hypergraphs as well as their matroidal properties we shall prove several results on the rigidity of generic affine and projective frameworks in low dimensional spaces.

1.1 Notation

Let $H = (V, E)$ be a hypergraph. For a set $X \subseteq V$ let $H[X]$ denote the subhypergraph induced by set X. The edge set of (the number of hyperedges in, resp.) $H[X]$ is denoted by $E_H(X)$ and $i_H(X)$, respectively. The degree of a vertex in H is denoted by $d_H(v)$. The set of neighbours of v in a graph G is denoted by $N_G(v)$. The subscripts may be omitted if the (hyper)graph is clear from the context.

2 Frameworks on hypergraphs

In this section we discuss three problems in combinatorial rigidity where the underlying combinatorial structure is a hypergraph: scene analysis, affine rigidity, and projective rigidity. Even though the first major results of scene analysis appeared in the 80's we start with affine rigidity, since this paper was motivated by a recent preprint on this topic [3].

2.1 Affine rigidity

In a recent paper Gortler, Gotsman, Liu, and Thurston [3] introduced the concept of affine rigidity, where affine constraints are imposed on sets of points, see also [10].

A d-dimensional *configuration* of a set V is a map $p : V \to \mathbb{R}^d$. A d-dimensional *affine framework* (H, p) is a pair, where H is a hypergraph and p is a d-dimensional configuration of its vertices. Two affine frameworks (H, p) and (H, q) are *affine equivalent* if for each hyperedge $e \in E(H)$ the positions of the vertices in p can be mapped to their positions in q by an affine map of $\mathbb{R}^d$. They are *affine congruent* if the positions of all the vertices in p can be mapped to their positions in q by a single affine map of $\mathbb{R}^d$.

A d-dimensional affine framework (H, p) is *globally affinely rigid* in $\mathbb{R}^d$ if for any other d-dimensional affine framework (H, q) which is equivalent to (H, p) we also have that (H, p) and (H, q) are affine congruent. A framework (H, p) is *locally affinely rigid* in $\mathbb{R}^d$ if there is a small neighbourhood in the configuration space so that for any (H, q) in this neighbourhood we have that if (H, q) is equivalent to (H, p) then (H, p) and (H, q) are congruent. It is not hard to see that a d-dimensional affine framework (H, p) is locally affinely rigid in $\mathbb{R}^d$ if and only if it is globally affinely rigid in $\mathbb{R}^d$ [3]. Thus we may omit the term local/global and simply call it *affinely rigid*.

A k-uniform hypergraph is a hypergraph $H = (V, E)$ where each hyperedge $e \in E$ contains exactly k vertices. For an integer k and hypergraph H let $B_k(H)$ denote the k-uniform hypergraph whose hyperedges are all those k-element subsets of the vertex set that are contained in some hyperedge of H. It suffices to consider uniform hypergraphs in the following sense.

Lemma 2.1. *[3] A framework (H,p) in general position is affinely rigid in $\mathbb{R}^d$ if and only if the associated framework $(B_{d+2}(H),p)$ is affinely rigid in $\mathbb{R}^d$.*

Gortler et. al. [3] define a family of matrices that encode the affine constraints of a framework. An *affinity matrix* of a d-dimensional affine framework (H,p) is a matrix in which each row encodes some affine relation between the coordinates of the vertices in a hyperedge of (H,p) as a homogeneous linear equation. That is, there are n columns in the matrix, the only non-zero entries in a row correspond to the vertices of some hyperedge, the sum of the entries in a row is 0, and each of the coordinates of p, thought of as a vector of length n, is in the kernel of the matrix. An affinity matrix is *strong* if it encodes all of the affinely independent relations in every hyperedge of (H,p). If (H,q) is affinely equivalent to (H,p) then the coordinates of q are in the kernel of any affinity matrix for (H,p). The converse is true if the affinity matrix is strong.

The kernel of an affinity matrix of a framework (H,p) always contains the subspace of $\mathbb{R}^n$ spanned by the coordinates of p along each axis and the vector of all 1's. If p is a *proper* d-dimensional configuration (i.e. with full d-dimensional affine span), these vectors are independent and span a $(d+1)$-dimensional space.

Theorem 2.2. *[3] Let H be a hypergraph with at least $d+1$ vertices. Let (H,p) be any proper d-dimensional affine framework and let M be any strong affinity matrix for (H,p). Then (H,p) is affinely rigid in $\mathbb{R}^d$ if and only if* $\dim(\ker(M)) = d+1$.

A configuration is *generic* if the set of the coordinates of the points is algebraically independent over $\mathbb{Q}$, i.e. they do not satisfy any non-zero polynomial with rational coefficients. A framework is *generic* if its configuration is generic. Since, as we shall see below, there is a strong affinity matrix whose entries are polynomials of the coordinates of the points with rational coefficients, it follows that affine rigidity in $\mathbb{R}^d$ is a generic property of a hypergraph. That is, either all generic affine frameworks on a hypergraph H are affinely rigid or there is no generic affine framework on H which is affinely rigid. In the former case we say that H is *generically affinely rigid*.

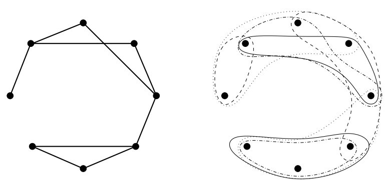

Figure 1: A graph and its neighbourhood hypergraph.

Two sufficient conditions were verified for generic affine rigidity. Given a graph G, define its *neighbourhood hypergraph*, denoted by $N(G)$, on the same set of vertices as follows: for each vertex v in G add a hyperedge to $N(G)$ consisting of v and its neighbours in G.

Theorem 2.3. *[3] Let G be a $(d+1)$-connected graph. Then the neighbourhood hypergraph of G is generically affinely rigid in $\mathbb{R}^d$.*

The other sufficient condition is due to Zha and Zhang [10]. We say that $H=(V,E)$ is *(d+1)-linked* if for each pair $h, h' \in E$ there is a sequence of hyperedges of H, starting at h and ending with h', such that consecutive pairs of hyperedges in the sequence share at least $d+1$ vertices.

Theorem 2.4. *[10] Suppose that H is $(d+1)$-linked. Then H is generically affinely rigid in $\mathbb{R}^d$.*

2.2 Scene analysis

It turns out that affine rigidity is equivalent to a concept in scene analysis, and in this context a necessary and sufficient condition for generic rigidity was obtained in an earlier paper by Whiteley [8]. A plane picture of a spatial polyhedral scene of plane faces and points of contact consists of the projections of the points and the combinatorial incidence structure of the vertices and faces. In scene analysis the (three-dimensional version of the) main question is whether a given plane picture has a corresponding scene, projecting to the picture, in which some (or all) faces are in distinct planes. It turns out that the space of scenes corresponds to the kernel of a matrix that can be obtained from the picture and the generic behaviour of a given incidence structure can be described by the generic rank of this matrix [8]. The incidence structure is naturally represented by a hypergraph in which the vertices are the points and the hyperedges are the faces.

Next we define this matrix, which turns out to be a strong affinity matrix of a generic affine framework. Let $H=(V,E)$ be a $(d+2)$-uniform hypergraph on n vertices and $p: V \to \mathbb{R}^d$ a map. We define a $(d+1)\times(d+2)$ matrix A_e for every $e = v_1 v_2 \ldots v_{d+2} \in E$ by letting

$$A_e := \begin{pmatrix} p_1^1 & p_2^1 & \cdots & p_{d+2}^1 \\ \vdots & \vdots & & \vdots \\ p_1^d & p_2^d & \cdots & p_{d+2}^d \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

where $p(v_i) = (p_i^1, p_i^2, \ldots, p_i^d)$ for $1 \le i \le n$.

Let A_e^l be the matrix obtained from A_e by deleting column l. A strong affinity matrix $A_d(H,p)$ of the affine framework (H,p) is then obtained as follows. The matrix has $|E|$ rows, indexed by the edges, and $|V|$ columns, indexed by the vertices in some fixed order. In the row corresponding to edge e (wlog. $e=\{v_1, v_2, ..., v_{d+2}\}$) the entry of column v_j is $(-1)^{j-1}\det A_e^j$, for $1 \le j \le d+2$, and the remaining entries are zeros. It is not difficult to prove the following observation:

Lemma 2.5. *$A_d(H,p)$ is a strong affinity matrix of (H,p).*

We shall call this matrix $A_d(H,p)$ the (d-dimensional) *affine rigidity matrix* of (H,p).

Let $H = (V, E)$ be a hypergraph and let k be an integer. We say that H is $(1, k)$-*sparse* if $i_H(X) \leq |X| - k$ for all $X \subseteq V$ with $|X| \geq k$. A $(1, k)$-sparse hypergraph with $|E| = |V| - k$ is called $(1, k)$-*tight*.

Whiteley [8] considered the same matrix in the context of scene analysis and determined its generic rank.

Theorem 2.6. *[8] Let $H = (V, E)$ be a $(d + 2)$-uniform hypergraph and let p be a generic configuration of V in $\mathbb{R}^d$. Then*
(i) the rows of $A_d(H, p)$ are independent if and only if H is $(1, d + 1)$-sparse,
(ii) $\mathrm{rank} A_d(H, p) = |V| - (d + 1)$ if and only if H has a $(1, d + 1)$-tight spanning subhypergraph.

We say that H is *generically minimally affinely rigid* in $\mathbb{R}^d$ if the rows of $A_d(H, p)$ of a generic realization of H are independent and $\mathrm{rank} A_d(H, p) = |V| - (d + 1)$. By Theorem 2.2, Lemma 2.5 and Theorem 2.6 we obtain:

Theorem 2.7. *Let H be a $d{+}2$-uniform hypergraph. Then H is generically minimally affinely rigid in $\mathbb{R}^d$ if and only if H is $(1, d + 1)$-tight.*

2.3 Projective rigidity

In a recent manuscript George and Ahmed [2] initiated the study of local and global rigidity properties of projective frameworks. A one-dimensional *projective framework* (H, p) is a pair, where H is a 4-uniform hypergraph and p is a map from $V(H)$ to distinct points of the one-dimensional projective space $\mathbb{P}^1$. They call a smooth deformation of the framework a flex if it preserves the cross ratio[1] for each 4-tuple that belongs to the edge set of H and call a framework rigid if it has only trivial flexes. As in the case of bar-and-joint frameworks with length constraints, one may define infinitesimal rigidity by considering the rank of the following *projective rigidity matrix* $Q(H, p)$ of the framework, in which the entries are obtained as partial derivatives of a smooth flex at time zero: let $Q(H, p)$ be a $|V| \times |V|$ matrix in which each row (resp. column) corresponds to an edge (resp. vertex) of H. The row corresponding to edge $v_i v_j v_k v_l$ is

$$\begin{pmatrix} 0...0 & \frac{(b-d)(c-d)}{(b-c)(a-d)^2} & 0...0 & \frac{(a-c)(d-c)}{(a-d)(b-c)^2} & 0...0 & \frac{(d-b)(b-a)}{(a-d)(b-c)^2} & 0...0 & \frac{(c-a)(a-b)}{(b-c)(a-d)^2} & 0...0 \end{pmatrix}$$

where $p_i = a, p_j = b, p_k = c, p_l = d$ and the non-zero entries are in the columns of v_i, v_j, v_k, v_l, respectively. Note that the entries of the matrix depend on the column labeling, i.e. the bijection between the vertices and columns, in a non-trivial way. (We shall prove later that the rank of the matrix does not.) They show that $\mathrm{rank} Q(H, p) \leq |V| - 3$ and, if p is generic, the framework is rigid if and only if equality holds. This leads to the problem of characterizing generically projectively rigid hypergraphs: the ones for which the generic rank of the projective rigidity matrix is $|V| - 3$.

[1] Recall that the *cross ratio* of four points a, b, c, d, in this order, is

$$R(ab, cd) = \frac{(a - c)(b - d)}{(a - d)(b - c)}.$$

2.4 New results

We develop a new inductive construction of 4-regular $(1,3)$-tight hypergraphs (Theorem 4.2 below) and use it to deduce a Laman-type combinatorial characterization of generically minimally rigid projective frameworks on the projective line (Theorem 5.3 below), verifying a conjecture from [2].

Our result implies a Henneberg-type inductive construction of generically minimally rigid affine frameworks in the plane (Theorem 4.10 below). Based on the rank function of the corresponding count matroid on the edge set of H we also obtain purely combinatorial proofs for (the two-dimensional version of) Theorem 2.3 and for Theorem 2.4.

3 Combinatorial properties of sparse hypergraphs

It is well-known that sparsity conditions define matroids on the edge set of a (hyper)graph and it is also known how to determine the corresponding rank functions, see e.g. [1, 9] as well as [7] for related algorithmic problems. Let $H = (V, E)$ be a hypergraph. We shall consider $(1, k)$-sparsity, leading us to matroid $\mathcal{M}_{1,k}(H)$ with groundset E and rank function $r_{1,k}$. A *cover* of $H = (V, E)$ is a collection $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V, each of size at least k, such that $E = \cup_{i=1}^{t} E_H(X_i)$. We say that a cover is *s-thin* if for each pair of distinct members $X_i, X_j \in \mathcal{X}$ we have $|X_i \cap X_j| \leq s$. We may restrict ourselves to $(k+1)$-uniform hypergraphs, for which the rank function can be expressed in the following simpler form. See [1, Section 13.5].

Theorem 3.1. *Let $H = (V, E)$ be a $(k+1)$-uniform hypergraph. Then $r_{1,k}(E) = \min \sum_{X\in\mathcal{X}}(|X| - k)$, where the minimum is taken over all $(k-1)$-thin covers $\mathcal{X}$ of H.*

The affine rigidity matrix of (H, p) defines the *affine rigidity matroid* of (H, p) on the ground set E where a set of edges $F \subseteq E$ is independent if and only if the rows of the affine rigidity matrix indexed by F are linearly independent. Any two generic d-dimensional frameworks (H, p) and (H, q) have the same affine rigidity matroid. We call this the *d-dimensional affine rigidity matroid* $\mathcal{A}_d(H)$ of hypergraph H. We denote the rank of $\mathcal{A}_d(H)$ by $a_d(H)$.

Theorem 2.6 and 3.1 now imply:

Theorem 3.2. *Let $H = (V, E)$ be a $(d+2)$-uniform hypergraph. Then H is generically affinely rigid in $\mathbb{R}^d$ if and only if for all d-thin covers $\mathcal{X}$ of H we have $\sum_{X\in\mathcal{X}}(|X| - (d+1)) \geq |V| - (d+1)$.*

To test whether a hypergraph is affinely rigid, we replace the hyperedges by $(d+2)$-hyperedges. It is enough to replace each hyperedge h by a minimally affinely rigid $(d+2)$-uniform hypergraph (with $|h| - (d+1)$ hyperedges on its vertices), showing that the reduction is polynomial.

The proof of Theorem 2.4 is quite easy, once we have Theorem 3.2 in hand.

Proof. *(of Theorem 2.4)* It is easy to see that H is $(d+1)$-linked if and only if $B_{d+2}(H)$ is $(d+1)$-linked. Thus, by Lemma 2.1, we may assume that $H=(V,E)$ is $(d+2)$-uniform. Suppose that H is not generically affinely rigid in $\mathbb{R}^d$. Then, by Theorem 3.2, there is a d-thin cover $\mathcal{X}=\{X_1,\ldots,X_k\}$ of H with $\sum_{i=1}^{k}(|X_i|-(d+1))<|V|-(d+1)$. We may suppose that $\mathcal{X}$ is chosen to minimize the left hand side. Clearly, $k\geq 2$ and we have a pair of hyperedges h_1,h_2 such that h_1 is a subset of X_1 but not of X_2 and vice versa. Then H cannot be $(d+1)$-linked, since $\mathcal{X}$ is d-thin. $\square$

4 Inductive constructions

We introduce a set of operations on hypergraphs which preserve $(1,k)$-sparsity and which can be used to generate all $(k+1)$-uniform $(1,k)$-tight hypergraphs from a single hyperedge, where $1\leq k\leq 3$.

Let $H=(V,E)$ be a $(k+1)$-uniform hypergraph, let j be an integer with $0\leq j\leq k-1$, and let $v\in V$ be a vertex with $d(v)\geq j$. The *j-extension operation at vertex v* picks j hyperedges $e_1,e_2,\ldots,e_j$ incident with v, adds a new vertex z to H as well as a new hyperedge e of size $k+1$ incident with both v and z, and replaces e_i by e_i-v+z for all $1\leq i\leq j$. Thus the new vertex z has degree $j+1$ in the extended hypergraph. See Figure 4. Note that a 0-extension operation simply adds a new vertex z and a new hyperedge of size $k+1$ incident with z.

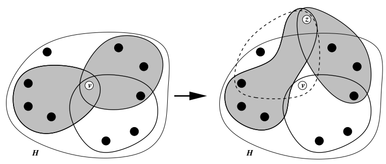

Figure 2: A 2-extension operation on a 4-uniform hypergraph.

The j-extension operation preserves sparsity in the following sense. The simple proof of the next lemma is omitted.

Lemma 4.1. *Let $H=(V,E)$ be an $(k+1)$-uniform $(1,k)$-sparse ($(1,k)$-tight) hypergraph and let H' be obtained from H by a j-extension operation, where $0\leq j\leq k-1$. Then H' is also $(1,k)$-sparse ($(1,k)$-tight, respectively).*

The main result of this section is the following constructive characterization.

Theorem 4.2. *Let $H=(V,E)$ be a 4-uniform hypergraph. H is (1,3)-tight if and only if it can be obtained from a single hyperedge of size four by a sequence of* 0*-extensions,* 1*-extensions, and* 2*-extensions.*

Before we prove the theorem we prove some preliminary lemmas about sparse hypergraphs. The next lemma is easy to verify by observing that the contribution of a hyperedge to the right hand side cannot be less than its contribution to the left hand side.

Lemma 4.3. *Let $H = (V, E)$ be a hypergraph and let $X, Y \subseteq V$ be subsets of vertices. Then*

$$i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y).$$

Let $H = (V, E)$ be an $(k+1)$-uniform $(1, k)$-tight hypergraph. We say that a subset $X \subseteq V$ is *critical* if $i(X) = |X| - k$ holds. A subset $Y \subseteq V$ is called *semi-critical* if $i(Y) \geq |Y| - k - 1$.

Lemma 4.4. *Let $H = (V, E)$ be an $(k+1)$-uniform $(1, k)$-sparse hypergraph and let $X, Y \subseteq V$ be subsets of vertices with $|X \cap Y| \geq k$. Then*
(i) if X and Y are both critical then $X \cup Y$ is also critical,
(ii) if X is critical and Y is semi-critical then $X \cup Y$ is semi-critical,
(iii) if X and Y are both semi-critical and $X \cap Y$ is not critical then $X \cup Y$ is semi-critical.
Furthermore,
(iv) if X and Y are both critical and $|X \cap Y| = k - 1$ then $X \cup Y$ is semi-critical.

Proof. Suppose that X and Y are both critical. Then, by using Lemma 4.3, we can deduce that

$$|X| - k + |Y| - k = i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y) \leq$$

$$\leq |X \cup Y| - k + |X \cap Y| - k = |X| - k + |Y| - k.$$

Thus we must have equality everywhere, which implies that $X \cup Y$ is also critical. This proves (i). The proofs of (ii), (iii), and (iv) are similar. □

We also need the following observation.

Lemma 4.5. *Let $H = (V, E)$ be a $(k+1)$-uniform $(1, k)$-tight hypergraph with $|V| \geq k + 1$. Then*
(i) $d(v) \geq 1$ for all $v \in V$, and
(ii) there is a vertex $z \in V$ with $d(z) \leq k$.

The inverse of the j-extension operation can be described as follows. Let z be a vertex with $d(z) = j+1$ and let v be a neighbour of z with $d(z, v) = 1$. Let $e_0, e_1, ..., e_j$ be the edges incident with z, where e_0 is the edge which is incident with v, too. The *j-reduction operation at vertex z with neighbour v* deletes e_0 and replaces e_i by $e_i - z + v$ for all $1 \leq i \leq j$.

We say that a j-reduction operation in an $(k+1)$-uniform $(1, k)$-sparse hypergraph H is *admissible* if the hypergraph obtained from H by the operation is also $(1, k)$-sparse. To prove our inductive construction by induction we need to show the existence of an admissible reduction. In what follows we shall consider the case when $k = 3$ and H is tight, that is, when H is a 4-uniform $(1, 3)$-tight hypergraph.

Theorem 4.6. *Let $H = (V, E)$ be a $(1, 3)$-tight 4-uniform hypergraph and let $z \in V$ be a vertex with $d(z) = j$ for some $1 \leq j \leq 3$. Then there is an admissible j-reduction at z.*

Proof. First suppose that $d(z) = 1$. Then the 1-reduction at z, which deletes the unique edge incident with z, is clearly admissible. Next suppose that $d(z) = 2$ holds and let e_1, e_2 be the hyperedges incident with z. The following property, which is implied by the sparsity of H, will be used several times in the proof. Let X be a subset of $V - z$. Then

(*) if X is critical then $e(z, X) \leq 1$ and if X is semi-critical then $e(z, X) \leq 2$ holds.

To show the existence of an admissible 2-reduction at z we have to show that for some neighbour v of z, for which $d(z, v) = 1$, the hypergraph obtained from H by deleting z and adding $e_2 - z + v$ is $(1, 3)$-sparse, where e_1 is the unique edge containing z and v. Observe that the addition of the new hyperedge $e_2 - z + v$ destroys $(1, 3)$-sparsity if and only if there is a critical set $X \subseteq V - z$ with $e_2 - z + v \subseteq X$.

Since H is $(1, 3)$-sparse and $d(z) = 2$, we have $4 \leq |N(z)| \leq 6$. Hence there exists a vertex $a \in N(z)$ with $d(z, a) = 1$. Let $e_1 = (a, b, c, z)$ and $e_2 = (d, e, f, z)$. If $|N(z)| = 4$ then the 2-reduction at z on neighbour a is admissible, for otherwise there exists a critical set X with $N(z) = \{a, d, e, f\} \subseteq X$ and $e(z, X) \geq 2$, contradicting (*). If $|N(z)| = 5$ then we may assume that $c = d$ and $e_1 - e_2 = \{a, b\}$. Hence $d(z, a) = d(z, b) = 1$. By assuming that the 2-reductions at a and b are both non-admissible we could deduce that there exist critical sets X, Y with $\{a, d, e, f\} \subseteq X$ and $\{b, d, e, f\} \subseteq Y$. Then, by Lemma 4.4(i), $X \cup Y$ would also be critical. Since $N(z) \subseteq X \cup Y$, this would again contradict (*). The case when $|N(z)| = 6$ is similar. Thus there is an admissible 2-reduction at z.

The last case to consider is when $d(z) = 3$. Let N_1 denote the set of neighbours x of z with $d(z, x) = 1$. Notice that

$$9 = \sum_{x \in N(z)} d(z, x) \geq 2|N(z) - N_1| + |N_1| = 2|N(z)| - |N_1|, \tag{1}$$

so $|N_1| \geq 2|N(z)| - 9$. Since H is $(1, 3)$-sparse and $d(z) = 3$, we have $5 \leq |N(z)| \leq 9$. Hence $N_1 \neq \emptyset$.

Let e_1, e_2, e_3 be the hyperedges incident with z. To show the existence of an admissible 3-reduction at z we have to show that for some neighbour v of z, for which $d(z, v) = 1$, the hypergraph obtained from H by deleting z and adding $e_2 - z + v$ and $e_3 - z + v$ is $(1, 3)$-sparse, where e_1 is the unique edge containing z and v. Observe that the addition of the new hyperedges $e_2 - z + v$ and $e_3 - z + v$ destroys $(1, 3)$-sparsity if and only if there is a critical set $X \subseteq V - z$ with $e_i - z + v \subseteq X$, for some $2 \leq i \leq 3$, or there is a semi-critical set $Y \subseteq V - z$ with $(e_2 - z + v) \cup (e_3 - z + v) \subseteq Y$. These critical or semi-critical sets X or Y, which show that the 3-reduction at z with v is non-admissible, are called the *blockers* of v.

For a contradiction suppose that there is no admissible 3-reduction at z. Then each vertex in N_1 has a blocker.

Claim 4.7. *Each blocker is critical.*

Proof. Let $x \in N_1$ and suppose, for a contradiction, that x has a semi-critical blocker Y. Let (z, x, a, b) be the unique edge containing z and x. Thus Y contains all neighbours of z, except, possibly, a and b. We may suppose that Y is a maximal semi-critical blocker of x. If $a, b \in Y$ then $N(z) \subseteq Y$ and $e(z, Y) \geq 3$ follow, contradicting (*). If, say, $a \notin Y$ then $a \in N_1$. Consider a blocker X of a. If X is critical then $X \cup Y$ is also a semi-critical blocker of x by Lemma 4.4(ii), contradicting the maximality of Y. If X is semi-critical then X contains all neighbours of z, except, possibly, x and b. Since $e(z, X \cap Y) \geq 2$, (*) implies that $X \cap Y$ is not critical. By using Lemma 4.4(iii) we conclude that $X \cup Y$ is semi-critical, contradicting the maximality of Y. □

Claim 4.8. *Every edge e incident with z contains a vertex w with $d(z, w) \geq 2$.*

Proof. For a contradiction suppose, without loss of generality, that $(e_1 - z) \cap (e_2 \cup e_3) = \emptyset$. Let $e_1 = (a, b, c, z)$. Then $a, b, c \in N_1$ and by the sparsity of H we also have $e_2 \cap N_1 \neq \emptyset$ and $e_3 \cap N_1 \neq \emptyset$. By symmetry may suppose that $e_2 - z \subseteq X_a \cap X_b$, where X_a and X_b are critical blockers of a and b, respectively. By Lemma 4.4(i) $X_a \cup X_b$ is also critical.

Let $f \in e_2 \cap N_1$ and let Z be a critical blocker of f. If $(e_1 - z) \subseteq Z$ then, by Lemma 4.4(i), $X_a \cup X_b \cup Z$ is also critical. Since $e(z, X_a \cup X_b \cup Z) \geq 2$, this contradicts (*). So we may suppose that the critical blocker Z_i of each vertex $f_i \in e_2 \cap N_1$ satisfies $(e_3 - z) \subseteq Z_i$. But, again by Lemma 4.4(i), this would imply that the union Z' of these sets Z_i is also critical. Since $e(z, Z') \geq 2$, this would contradict (*). This proves the claim. □

Claim 4.8 implies that $|N(z)| \leq 7$. First suppose that $|N(z)| = 5$. Let X be a critical blocker of some vertex $x \in N_1$. Then $|N(z) - X| \leq 1$ and hence $Y = N(z) \cup X$ is semi-critical. Since $e(z, Y) = 3$, this contradicts (*).

Next suppose that $|N(z)| = 6$. Then we have $|N_1| \geq 3$ by (1). Hence we can find two critical blockers X_a, X_b belonging to two distinct vertices $a, b \in N_1$. Each of these blockers contains at least four neighbours of z. If $N(v) \subseteq (X_a \cup X_b)$ then $|X_a \cap X_b| \geq 2$. Thus, by Lemma 4.4(i),(iv), it follows that $X_a \cup X_b$ is semi-critical. Since $e(z, X_a \cup X_b) = 3$, this contradicts (*). If $|(X_a \cup X_b) \cap N(v)| = 5$ then a similar argument, using Lemma 4.4(i) gives that $X_a \cup X_b$ is critical and hence $Y = X_a \cup X_b \cup N(z)$ is semi-critical, contradicting (*). If $|(X_a \cup X_b) \cap N(v)| = 4$ then $X_a \cup X_b$ is critical, with $e(z, X_a \cup X_b) = 2$, contradicting (*).

It remains to consider the case when $|N(v)| = 7$. First suppose that there is a vertex $w \in N(z)$ with $d(z, w) = 3$. Then $N_1 = N(z) - w$ must hold. By symmetry we may suppose that for some vertex $a \in e_1 \cap N_1$ and a critical blocker X_a of a we have $(e_2 - z) \subset X_a$. Let $e_2 = (z, w, c, d)$. Let X_c, X_d be critical blockers of c and d, respectively. If $(e_1 - z) \subset X_c$ then $X_a \cup X_b$ is critical, by Lemma 4.4(i), and has $e(z, X_a \cup X_b) \geq 2$, contradicting (*). A similar argument works for X_d. So we may assume that $(e_3 - z) \subseteq X_c \cap X_d$. But then, by Lemma 4.4(i), $X_c \cup X_d$ is a critical set with $e(z, X_c \cup X_d) \geq 2$, contradicting (*).

Next suppose that each vertex $w \in N(z)$ has $d(z, w) \leq 2$. Then we have two vertices $p, q \in N(z)$ with $d(z, p) = d(z, q) = 2$ and the other neighbours of z are all in N_1. Furthermore, by using Claim 4.8, we can deduce that the edges incident with z can be labeled as $e_1 = (p, a, b, z)$, $e_2 = (p, q, c, z)$, and $e_3 = (q, d, e, z)$. By symmetry we may suppose that a critical blocker X_c of c has $(e_1 - z) \subset X_c$. Let X_a and X_b be critical blockers of a and b, respectively. If, say, $(e_2 - z) \subset X_a$ holds then, by Lemma

4.4(i) it follows that $X_a \cup X_c$ is critical. Since $e(z, X_a \cup X_c) \geq 2$, this contradicts (*). A similar argument works for X_b. Thus we may assume that $(e_3 - z) \subseteq X_a \cap X_b$. This gives that $X_a \cup X_b$ is critical and $Y = X_a \cup X_b \cup X_c$ is semi-critical, by using Lemma 4.4(i) and (iv), respectively. Since $e(z, Y) = 3$, this contradicts (*). With this final contradiction the proof of the theorem is complete. $\square$

Proof. of Theorem 4.2 The 'if' part follows from Lemma 4.1. Theorem 4.6 implies the 'only if' part by induction on the number of vertices. $\square$

We have a similar result about j-reductions in 3-uniform hypergaphs, where $0 \leq j \leq 1$, which leads to the following inductive construction. The proof, which is similar to the first part of the proof of Theorem 4.6, where $d(z) \leq 2$, is omitted.

Theorem 4.9. *Let $H = (V, E)$ be a 3-uniform hypergraph. H is (1,2)-tight if and only if it can be obtained from a single hyperedge of size three by a sequence of* 0*-extensions and* 1*-extensions.*

The 2-uniform $(1, 1)$-tight hypergraphs are the trees, for which the existence of an admissible 0-reduction (leaf deletion) is straightforward. On the other hand, a statement similar to Theorem 4.6 does not hold for $(1, 4)$-tight 5-uniform hypergraphs. To see this consider a hypergraph on 7 vertices, $v_1, v_2, ..., v_7$, with edges $(v_1, v_2, v_3, v_4, v_7), (v_3, v_4, v_5, v_6, v_7), (v_1, v_2, v_5, v_6, v_7)$. Then each neighbour v_i of v_7 has $d(v_7, v_i) \geq 2$, showing that no 3-reduction can be performed. Hence an inductive construction for this family is probably much more difficult to obtain.

Theorem 4.2, together with Theorem 2.7, implies a Henneberg-type result for minimally affinely rigid hypergraphs in the plane.

Theorem 4.10. *Let $H = (V, E)$ be a* 4*-uniform hypergraph. Then H is generically minimally affinely rigid in $\mathbb{R}^2$ if and only if it can be obtained from a single hyperedge of size four by a sequence of* 0*-extensions,* 1*-extensions, and* 2*-extensions.*

Another application, leading to the characterization of generic projective rigidity on the projective line, is given in the next section. We close this section by noting that Theorem 4.2 was also used in the proof of a recent result in scene analysis [4].

5 Projective rigidity on the line

Let (H, p) be an affine framework in $\mathbb{P}^1$. We can easily see that $\text{rank}Q(H, p) \leq |V| - 3$, since the kernel of $Q(H, p)$ is at least three-dimensional, as it contains the vectors $(1, 1, ..., 1)$, $(p(v_1), p(v_2), ..., p(v_n))$, and $(1 + p(v_1)^2, 1 + p(v_2)^2, ..., 1 + p(v_n)^2)$. These vectors are linearly independent if there exist three vertices which are mapped to different points in the realization. It also follows that if the rows of $Q(H, p)$ are linearly independent then H is (1,3)-sparse.

A realization (H, p) of a 4-uniform hypergraph $H = (V, E)$ in $\mathbb{P}^1$ is *infinitesimally projectively rigid* if $\text{rank}Q(H, p) = |V| - 3$. We say that $H = (V, E)$ is *projectively rigid* in $\mathbb{P}^1$ if there exists an infinitesimally rigid realization of H in $\mathbb{P}^1$. A *minimally projectively rigid* hypergraph is a projectively rigid hypergraph with $|E| = |V| - 3$.

As we noted earlier, the entries of the projective rigidity matrix of (H,p) depend on the column labeling in a non-trivial way, just like the cross ratio. First we prove two lemmas that imply that the rank of the matrix does not, and hence we can indeed work with the above definitions.

Lemma 5.1. *Let (H,p) be a one-dimensional projective framework on n vertices. Suppose that* $\mathrm{rank}Q(H,p) = n-3$. *Then any set of $n-3$ columns of $Q(H,p)$ is linearly independent.*

Proof. Let C_i denote the column of $Q(H,p)$ that corresponds to vertex v_i, $1 \le i \le n$. Let us fix a triple $\{j,k,l\} \subseteq \{1,\dots,n\}$. We shall prove that C_l is spanned by the set of columns $\{C_t : 1 \le t \le n, t \ne j,k,l\}$, from which the lemma follows.

As mentioned above the vectors $\mathbf{1} = (1,1,\dots,1)$, $\mathbf{p} = (p(v_1), p(v_2), \dots, p(v_n))$ and $\mathbf{p}^2 = (p(v_1)^2, p(v_2)^2, \dots, p(v_n)^2)$ are in $\ker Q(H,p)$. Let $x = -p(v_j) - p(v_k)$ and $y = p(v_j)p(v_k)$. Then we have $p(v_j)^2 + xp(v_j) + y = p(v_k)^2 + xp(v_k) + y = 0$. Furthermore, $p(v_t)^2 + xp(v_t) + y = 0$ if and only if $t \in \{j,k\}$.

Consider the vector $\mathbf{p}^2 + x\mathbf{p} + y\mathbf{1}$, which is in the kernel of $Q(H,p)$. This gives rise to a linear combination of the columns of $Q(H,p)$, which gives the zero vector, and in which the coefficients of C_j, C_k are zeros and the coefficient of C_l is nonzero. Thus C_l is spanned by the set of columns $\{C_t : 1 \le t \le n, t \ne j,k,l\}$, as claimed. □

Lemma 5.2. *Let (H,p) be a one-dimensional projective framework. Let $Q(H,p)$ be the projective rigidity matrix in which the columns are labeled by vertices $v_1, v_2, \dots, v_n$, in this order. Suppose that* $\mathrm{rank}Q(H,p) = n-3$. *Let $Q'(H,p)$ be the projective rigidity matrix of (H,p) corresponding to the labeling $v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2} \dots, v_n$ for some $1 \le i \le n-1$. Then* $\mathrm{rank}Q'(H,p) = n-3$.

Proof. Let $Q_1(H,p)$ and $Q_1'(H,p)$ be the matrices that we get by deleting the columns of v_i and v_{i+i} from $Q(H,p)$ and $Q'(H,p)$, respectively. By Lemma 5.1 $\mathrm{rank}Q_1(H,p) = n-3$.

Next we will show that every row of $Q_1'(H,p)$ can be obtained from the corresponding row of $Q_1(H,p)$ by multiplying it with an appropriate scalar. First observe that if hyperedge e contains at most one of v_i and v_{i+i} then the rows corresponding to e in $Q_1(H,p)$ and $Q_1'(H,p)$ are equal. Now suppose that $e = v_jv_kv_iv_{i+1}$. We split the proof into three cases.

First suppose that $j < k < i$. Put $p(v_j) = a, p(v_k) = b, p(v_i) = c, p(v_{i+1}) = d$. With this notation the two nonzero entries of the row of e in $Q_1(H,p)$ are:

$$\frac{(b-d)(c-d)}{(b-c)(a-d)^2}, \frac{(a-c)(d-c)}{(a-d)(b-c)^2};$$

while the two entries in $Q_1'(H,p)$ are:

$$\frac{(b-c)(d-c)}{(b-d)(a-c)^2}, \frac{(a-d)(c-d)}{(a-c)(b-d)^2}.$$

Hence we can get the row of $Q_1'(H,p)$ by multiplying with the scalar $-\frac{(b-c)^2(a-d)^2}{(b-d)^2(a-c)^2}$.

If $j < i < k$ then denote $p(v_j) = a, p(v_i) = b, p(v_{i+1}) = c, p(v_k) = d$. Now the the two nonzero entries of the row of e in $Q_1(H,p)$ are:

$$\frac{(b-d)(c-d)}{(b-c)(a-d)^2}, \frac{(c-a)(a-b)}{(b-c)(a-d)^2}$$

and we get the row of $Q_1'(H,p)$ by multiplying with -1.

The last case is when $i < j < k$. Now put $p(v_i) = a, p(v_{i+1}) = b, p(v_j) = c, p(v_k) = d$. Here the two nonzero entries of the rows are

$$\frac{(d-b)(b-a)}{(a-d)(b-c)^2}, \frac{(c-a)(a-b)}{(b-c)(a-d)^2};$$

and

$$\frac{(a-d)(b-a)}{(b-d)(a-c)^2}, \frac{(b-c)(a-b)}{(a-c)(b-d)^2}.$$

In this case the scalar is $-\frac{(a-d)^2(b-c)^2}{(d-b)^2(a-c)^2}$.

Thus rank$Q'(H,p) \geq Q_1'(H,p) \geq Q_1(H,p) = n-3$, as required. $\square$

We are ready to prove the main result of this section.

Theorem 5.3. *Let $H = (V,E)$ be a 4-uniform hypergraph. Then H is minimally projectively rigid in $\mathbb{P}^1$ if and only if H is $(1,3)$-tight.*

Proof. We have to show that there exists a realization (H,p) of H in $\mathbb{P}^1$ with rank$Q(H,p) = |V| - 3$. We prove this by induction on $|V|$. In fact, we shall prove that p can be chosen so that the vertex coordinates are pairwise different. If $|V| = 4$ then any realization of type will do. Now suppose that $|V| \geq 5$ and the theorem holds for all 4-uniform hypergraphs with at most $|V| - 1$ vertices. By Theorem 4.2 H can be obtained from a 4-uniform hypergraph $H' = (V', E')$ by a j-extension operation at some vertex $v \in V'$, where $0 \leq j \leq 2$. Recall that the operation adds a new vertex z, replaces v by z in j edges incident with v and adds an additional edge e incident with v and z. Let F be the (possibly empty) set of the j edges replaced.

By induction, H' has a realization (H', p') without coincident points for which rank$Q(H', p') = |V'| - 3 = |V| - 4$ holds. We may suppose that the last j rows of $Q(H', p')$ correspond to the edges in F. By Lemma 5.2 we may also assume that the last column is indexed by v. Consider a realization (H,p) of H obtained from (H', p') by making z and v coincident, that is, let $p(z) = p'(v)$ and $p(w) = p'(w)$ for all $w \in V'$.

The matrix $Q(H,p)$ can be obtained from $Q(H', p')$ by inserting a new column, corresponding to z, next to the column of v and replacing the last j rows by $j+1$ new rows corresponding to the edges in $F \cup \{e\}$. Observe that by the choice of $p(z)$ we can also obtain $Q(H,p)$ from $Q(H', p')$ by inserting the column of z, moving the entries corresponding to F in the column of v to the column of z and then adding a new row corresponding to e. All entries of this new row will be zeros, except the two entries in the columns of v and z. Furthermore, these two entries are x and $-x$, for some non-zero real number x. Thus by adding the last column to the second last we obtain a block triangular matrix with $Q(H', p')$ in the upper left block and a non-zero number in the lower right block. Hence $\mathrm{rank}Q(H, p) = \mathrm{rank}Q(H', p') + 1 = |V| - 3$, as required. By perturbing the coordinates slightly, without decreasing the rank, we can then make sure that the vertex coordinates are pairwise different. □

6 Affine rigidity of neighbourhood hypergraphs

In this section we give a different, purely combinatorial proof for the two-dimensional case of Theorem 2.3. The original proof uses, among others, non-symmetric stress matrices and rubber band embeddings. Our proof relies on Theorem 3.2. The proof method is inspired by [5].

Theorem 6.1. *Let $G = (V, E)$ be a 3-connected graph. Then the neighbourhood hypergraph $N(G)$ is generically affinely rigid in $\mathbb{R}^2$.*

Proof. Suppose, for a contradiction, that there is a 3-connected graph G for which $N(G)$ is not generically affinely rigid in $\mathbb{R}^2$. Choose a counterexample $G = (V, E)$ for which $|V|$ is as small as possible and within the family of counterexamples of this size, the number of edges $|E|$ is as large as possible. Let $H = B_4(N(G))$. By Lemma 2.1 the 4-uniform hypergraph $H = (V, F)$ is not generically affinely rigid either. Hence, by Theorem 3.2, there is a 2-thin cover $\mathcal{X} = \{X_1, \ldots, X_k\}$ of H for which

$$\sum_{i=1}^{k}(|X_i| - 3) < |V| - 3 \tag{2}$$

holds. We say that a set $X_i \in \mathcal{X}$ is a *core* of some vertex $v \in V$ if $N_G(v) \cup \{v\} \subseteq X_i$.

Claim 6.2. *Each vertex $v \in V$ has a unique core.*

Proof. Since $\mathcal{X}$ covers H, each set $\{v, v_1, v_2, v_3\} \in F$ with $\{v_1, v_2, v_3\} \subseteq N_G(v)$ is covered by some $X_j \in \mathcal{X}$. By using the fact that $\mathcal{X}$ is 2-thin, we can deduce that there must be a unique set $X_i \in \mathcal{X}$ that contains v as well as all neighbours of v in G. □

We may also assume that $\mathcal{X}$ is chosen so that the left hand side of (2) is minimized. This implies that $G[X_i]$ is connected for all $1 \leq i \leq k$. (Observe that replacing X_i by the vertex sets of the components of $G[X_i]$ results in another cover since all hyperedges that induce connected subgraphs in G will be covered by the new smaller sets and for each hyperedge e, consisting of a subset of neighbours of v, say, which intersects at least two of the new smaller sets will remain covered by the core of v.)

For each $v \in V$ let $b(v)$ denote the number of those members of $\mathcal{X}$ that contain v.

Claim 6.3. $b(v) \geq 2$ *for every $v \in V$.*

Proof. Suppose, for a contradiction, that some vertex $v \in V$ is in X_1, say, but it is disjoint from X_i for all $2 \leq i \leq k$. It follows from Claim 6.2 that for each vertex $v_j \in N_G(v)$ we must have $N_G(v_j) \subseteq X_1$. This implies, by the maximality of $|E|$, that

$G[N_G(v)]$ is a complete subgraph of G. It also implies that $|X_1| \geq 5$ unless G is a complete graph on four vertices, for which the theorem is trivially true.

Let $\mathcal{X}' = \{X'_1, X'_2, \ldots, X'_k\}$, where $X'_1 = X_1 - v$ and $X'_i = X_i$ for all $2 \leq i \leq k$. Then $\mathcal{X}'$ is a cover of $B_4(N(G - v))$ satisfying

$$\sum_{i=1}^{k}(|X'_i| - 3) = \sum_{i=1}^{k}(|X_i| - 3) - 1 < |V| - 4 = |V(G - v)| - 3.$$

By the minimality of $|V|$ the graph $G - v$ is not a counterexample to the statement of the theorem, so it follows that $G - v$ is not 3-connected, that is, the graph $G - \{v, x, y\}$ is disconnected for some pair of vertices $x, y \in V$. But G is 3-connected, so v must have at least one neighbour in each connected component of $G - \{v, x, y\}$. This contradicts the fact that $G[N_G(v)]$ is complete. □

Claim 6.4. *Suppose that $b(v) \leq 3$ and let $X \in \mathcal{X}$ be the core of v. Then $|X| \geq 6$.*

Proof. First suppose $b(v) = 2$. Let Y be the other member of $\mathcal{X}$ containing v. Since $G[Y]$ is connected, it must contain a neighbour of v in G. The cover is 2-thin, so this implies that $X \cap Y = \{v, y\}$ for some $y \in N_G(v)$. It also follows that Y is the core of y. Since G is 3-connected, v has at least three neighbours in G. Suppose that $\{a, b\} \subseteq N_G(v) - Y$. Since $b(v) = 2$ the core of a and b must also be X. Using the facts that y cannot be a neighbour of a or b and that G is 3-connected we can deduce that $|X - \{v, y, a, b\}| \geq 2$, which gives $|X| \geq 6$.

Next suppose $b(v) = 3$. Let Y, Z be the other members of $\mathcal{X}$ containing v. As above, we obtain that $X \cap Y = \{v, y\}$ and $X \cap Z = \{v, z\}$ for some $y, z \in N_G(v)$ and that Y is the core of y and Z is the core of z. Let $a \in (N_G(v) - \{v, y, z\})$. Since $b(v) = 3$, X is the core of a. Using that a cannot be adjacent to y or z we get that a has at least two more neighbours in X and hence $|X| \geq 6$ follows. □

Claim 6.5. *Suppose that $b(v) \geq 3$. Then $\sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq 1$.*

Proof. Since $|X_i| \geq 4$ for all $1 \leq i \leq k$, the claim follows immediately if $b(v) \geq 4$. Now suppose that $b(v) = 3$. By Claim 6.4 we get $\sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq \left(1 - \frac{3}{4}\right) + \left(1 - \frac{3}{4}\right) + \left(1 - \frac{3}{6}\right) = 1$. □

To obtain a similar bound for the vertices with $b(v) = 2$, at least on average, we have to deal with them together and we need a more careful counting argument. Let $J = \{vx \in E : b(v) = 2,$ for some pair $X, Y \in \mathcal{X}$ we have $X \cap Y = \{v, x\}\}$, let $W = V(J)$ and $Z_1, Z_2, ..., Z_l$ be the vertex sets of the components of the graph $K = (W, J)$. Observe that each vertex with $b(v) = 2$ belongs to W and that each component of K is a star in which each leaf vertex v has $b(v) = 2$.

Claim 6.6.

$$\sum_{v \in W} \sum_{X_i : v \in X_i} \left(1 - \frac{3}{|X_i|}\right) \geq |W|.$$

Proof. It suffices to show that $\sum_{v\in Z_j}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq |Z_j|$ for all $1 \leq j \leq l$. Consider a component on Z_j. First suppose $|Z_j| \geq 4$. Then, by using Claim 6.4, we can give a lower bound on the contributions of the $|Z_j|-1$ leaves and the center vertex as follows:

$$\sum_{v\in Z_j}\sum_{X_i:v\in X_i\ni}\left(1-\frac{3}{|X_i|}\right) \geq (|Z_j|-1)\left(\frac{1}{2}+1-\frac{3}{|Z_j|}\right)+$$

$$+(|Z_j|-1)\frac{1}{2}+\left(1-\frac{3}{|Z_j|}\right) \geq |Z_j|,$$

as required.

Now suppose that $|Z_j| = 3$. If $b(c) \geq 4$ for the center vertex c of the star then, by using Claim 6.4 again, we obtain $\sum_{v\in Z_j}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq 2\left(\frac{1}{2}+\frac{1}{4}\right)+\left(2\cdot\frac{1}{2}+2\cdot\frac{1}{4}\right)=3$, as claimed. If $b(c)=3$ then $\sum_{v\in Z_j}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq 2\cdot 2\cdot\frac{1}{2}+3\cdot\frac{1}{2} > 3$ follows.

Finally, suppose that $|Z_j| = 2$. Then either both vertices in Z_j are contained by at most three sets of $\mathcal{X}$, in which case $\sum_{v\in Z_j}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq 2(\frac{1}{2}+\frac{1}{2})=2$, or one of them, say c, has $b(c) \geq 4$. In the latter case we get $\sum_{v\in Z_j}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq \frac{1}{2}+\frac{1}{4}+\frac{1}{2}+3\cdot\frac{1}{4}=2$. This completes the proof of the claim. □

The proof of the theorem follows by using Claims 6.5, 6.6 and the fact that $b(v) \geq 3$ for all $v \in V-W$:

$$\sum_{i=1}^{k}(|X_i|-3)=\sum_{i=1}^{k}|X_i|\left(1-\frac{3}{|X_i|}\right)=\sum_{v\in W}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right)+$$

$$+\sum_{v\in V-W}\sum_{X_i:v\in X_i}\left(1-\frac{3}{|X_i|}\right) \geq |W|+|V-W|=|V|,$$

contradicting (2). □

It may also be possible to use a similar method to deduce the higher dimensional versions of Theorem 2.3 but the proof gets more complicated. On the other hand, there is an even simpler combinatorial proof in the case when $d=1$. It is based on the fact that every 2-connected graph has an ear-decomposition and uses induction on the number of ears. Each new ear added to G generates a set of new hyperedges in $N(G)$. The (easy direction of) Theorem 4.9 can be used to show that a $(1,2)$-tight spanning subgraph can be maintained. We omit the details.

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References

[1] A. Frank, *Connections in Combinatorial Optimization*, Oxford University Press, 2011.

[2] S. George and F. Ahmed, Rigidity in the one-dimensional projective space, manuscript. See
`http://www.math.columbia.edu/~dpt/RigidityREU/AG11Rigidity1DProj.pdf`

[3] S.J. Gortler, C. Gotsman, L. Liu, and D.P. Thurston, On affine rigidity, arXiv:1011.5553v1, November 2010.

[4] V.E. Kaszanitzky and B. Schulze, Lifting symmetric pictures, manuscript, 2013.

[5] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* 3 (1982), no. 1, 91–98.

[6] J. G. Oxley, *Matroid Theory*, Oxford University Press, New York, 1992.

[7] I. Streinu and L. Theran, Sparse hypergraphs and pebble game algorithms, *European Journal of Combinatorics*, Volume 30, Issue 8, November 2009, Pages 1944–1964.

[8] W. Whiteley, A matroid on hypergraphs with appliations in scene analysis and geometry, *Discrete and Computational Geometry* 4, 1 (1989) 75-95.

[9] W. Whiteley, Some matroids from discrete applied geometry. Matroid theory (Seattle, WA, 1995), 171–311, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.

[10] H. Zha and Z. Zhang, Spectral properties of the alignment matrices in manifold learning, *SIAM Review* 51, 3 (2009) 545-566.