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# Splitting property via shadow systems

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## Abstract

Let  $M_k^r$  denote the set of  $r$ -element multisets over the set  $\{1, \dots, k\}$ . We show that  $M_k^k$  has the so-called splitting property introduced by Ahlswede et al. Our approach gives a new interpretation of Sidorenko's construction appeared in [18] and is applicable to give an upper bound on weighted Turán numbers, matching previous bounds. We also show how these results are connected to Tuza's conjecture on minimum triangle covers.

**Keywords:** shadow systems, splitting property, Turán number, Tuza's conjecture

## 1 Introduction

Let  $\mathcal{P} = (P, \prec)$  be a finite partially ordered set. For a subset  $H \subseteq P$ , the sets  $\mathcal{U}(H) = \{x \in P : \exists h \in H : x \preceq h\}$  and  $\mathcal{L}(H) = \{x \in P : \exists h \in H : x \succeq h\}$  are called the upper and lower shadows of  $H$ , respectively. An antichain  $A \subseteq P$  is maximal if and only if  $\mathcal{U}(A) \cup \mathcal{L}(A) = P$ . We say that a maximal antichain  $A$  has the **splitting property** if it can be partitioned into two disjoint parts  $A_1 \cup A_2 = A$  such that  $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = P$ . This property was introduced and first studied by Ahlswede et al. [1]. They gave the following sufficient condition for the splitting property. A maximal antichain  $A \subseteq P$  is called **dense** if it satisfies the following: whenever  $x \prec a \prec y$  for some  $a \in A$  and  $x, y \in P$ , there exists an  $a' \in A \setminus \{a\}$  also satisfying  $x \prec a' \prec y$ . They proved the following theorem.

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**Theorem 1.1** (Ahlswede, Erdős and Graham). *Every dense maximal antichain in a finite poset satisfies the splitting property.*

The poset  $\mathcal{P}$  itself has the **splitting property** if every maximal antichain in  $\mathcal{P}$  satisfies the splitting property. The following negative result in [1] shows that this property is NP-hard to decide.

**Theorem 1.2** (Ahlswede, Erdős and Graham). *It is NP-hard to decide whether a given poset  $\mathcal{P} = (P, \prec)$  has the splitting property.*

On the other hand, Duffus and Sands [5] gave a complete characterization of finite distributive lattices with the splitting property.

**Theorem 1.3** (Duffus and Sands). *If  $\mathcal{P}$  is a finite distributive lattice with the splitting property, then it is either a Boolean lattice, or one of three other lattices.*

In this paper, we consider the poset of  $k$ -element multisets of  $k$  colours. Formally, let us use the elements of the group  $\mathbb{Z}_k$  as colours, denoted by  $\{1, \dots, k\}$ . We call the vectors  $\mathbb{Z}_k \rightarrow \mathbb{Z}$   **$k$ -colour vectors**, and denote their set by  $M_k$ . We can define a natural partial ordering on  $M_k$ : for  $a, c \in M_k$ ,  $a \prec c$  if  $a_i \leq c_i$  for every  $i \in \mathbb{Z}_k$  and  $a \neq c$ . If  $a \prec c$ , we also say that  $a$  is a **shadow** of  $c$ .  $(M_k, \prec)$  is a distributive lattice, however, it is not finite and therefore Theorem 1.3 is not applicable. Let

$$M_k^r = \{x \in M_k, \sum_{i \in \mathbb{Z}_k} x_i = r\}.$$

denote the set of  $k$ -colour vectors whose coordinates sum up to  $r$ . Our first result shows the splitting property of this antichain for  $r = k$ :

**Theorem 1.4.** *In the poset  $(M_k, \prec)$ , the maximal antichain  $M_k^k$  has the splitting property, that is,  $M_k$  can be partitioned into disjoint sets  $A_1$  and  $A_2$  such that  $\mathcal{U}(A_1) \cup \mathcal{L}(A_2) = M_k$ .*

It is easy to verify that  $M_k^k$  is not dense and therefore Theorem 1.1 does not imply our result. Take an arbitrary  $x \in M_k^{k-1}$  and let  $y_1 = x_1 + 2$  and  $y_i = x_i$  if  $i \neq 1$ . Then  $M_k^k$  contains exactly one element  $a$  with  $x \prec a \prec y$ .

In Theorem 1.4, the required property of  $A_1 \subset M_k$  is that for every vector  $c \in M_k^{k+1}$ ,  $A_1$  must contain at least one shadow of  $A_1$ . Generalizing this notion, for  $r < t$  we call  $A \subseteq M_k^r$  a  **$(t, r; k)$ -shadow system**, if for every colour vector  $c \in M_k^t$ ,  $A$  contains at least one shadow of  $c$ . With this terminology,  $A_1$  in Theorem 1.4 is a  $(k+1, k; k)$ -shadow system.

Shadow systems turn out to be closely related to Turán systems, a central notion of extremal combinatorics. For  $r \leq t \leq n$ , a **Turán  $(n, t, r)$ -system** is an  $r$ -uniform hypergraph on  $n$  nodes such that every  $t$ -element subset of the nodes spans at least one edge of the hypergraph. The Turán number  $T(n, t, r)$  asks for the minimum size of such a family; determining the exact values is a problem posed by Pál Turán [19]. The simplest case  $t = 3$ ,  $r = 2$  asks for the minimum number of edges of a graph such that every subset of 3 nodes contains at least one edge. This is equivalent to

determining the maximum number of edges in a triangle free graph on  $n$  nodes, a problem solved by Mantel in 1907. The optimal  $(n, 3, 2)$ -Turán system is the disjoint union of two cliques on node sets of size  $\lfloor \frac{n}{2} \rfloor$  and  $\lceil \frac{n}{2} \rceil$ .

The limit

$$t(t, r) = \lim_{n \rightarrow \infty} \frac{T(n, t, r)}{\binom{n}{r}}$$

expresses the fraction of all  $r$ -element subsets needed for a Turán  $(n, t, r)$ -system. No exact value is known for any  $t > r > 2$  - in 1981, Pál Erdős offered a bounty of \$500 for even a single special case and \$1000 for resolving the general case [6]. For surveys on Turán numbers, see [8, 15, 17]. De Caen [2] gave the lower bound  $t(t, r) \geq \frac{1}{\binom{t-1}{r-1}}$ .

The best currently known upper bound is due to Sidorenko [18].

**Theorem 1.5** (Sidorenko). *For any integers  $t > r$ ,*

$$t(t, r) \leq \left( \frac{r-1}{t-1} \right)^{r-1}. \quad (1)$$

We give a new interpretation of Sidorenko's construction in terms of shadow systems, and reprove the theorem using the following result. Consider a vector  $s \in \mathbb{Z}_k^r$ . The **colour profile**  $a = M(s) \in M_k^r$  can be naturally defined so that  $a_i$  equals the number of  $i$ 's in  $s$  for  $1 \leq i \leq k$ . We prove the following theorem.

**Theorem 1.6.** *For integers  $t > r$ , there exists a  $(t, r; t-1)$ -shadow system  $\mathcal{A}_r^t \subseteq M_{t-1}^r$  so that if we pick a vector  $s \in \mathbb{Z}_{t-1}^r$  uniformly at random, then the probability of  $M(s) \in \mathcal{A}_r^t$  equals  $\left( \frac{r-1}{t-1} \right)^{r-1}$ .*

*Proof of Theorem 1.5.* Let us take a uniform random colouring with  $t-1$  colours of a ground set  $V$  with  $|V| = n$  nodes. Consider a  $(t, r; t-1)$ -shadow system  $\mathcal{A}_r^t \subseteq M_{t-1}^r$  as in Theorem 1.6, and let the  $r$ -uniform hypergraph  $(V, \mathcal{E})$  contain those  $r$ -element subsets  $X$  whose colour profile is contained in  $\mathcal{A}_r^t$ . (An  $r$ -element set coloured by  $t-1$  colours naturally corresponds to a vector in  $\mathbb{Z}_{t-1}^r$ .) The  $(t, r; t-1)$ -shadow system property implies that every vector  $c \in M_{t-1}^t$  has a shadow in  $\mathcal{A}_r^t$ . Consequently, every  $t$ -element subset of  $V$  has a subset in  $\mathcal{E}$ , that is,  $\mathcal{E}$  is a Turán  $(n, t, r)$ -system. Theorem 1.5 follows since the expected size of  $\mathcal{E}$  is  $\left( \frac{r-1}{t-1} \right)^{r-1} \binom{n}{r}$  by Theorem 1.6.  $\square$

We introduce the natural weighted extension of the Turán numbers: we are given a nonnegative weight function  $w$  on the  $r$ -element subsets of  $V$ , and  $w^*$  denotes the total weight of all subsets. The **Turán weight**  $T_w(n, t, r)$  is the minimum weight of a Turán  $(n, t, r)$  system. Analogously to  $t(t, r)$  we may define

$$tw(t, r) = \lim_{n \rightarrow \infty} \sup_w \frac{T_w(n, t, r)}{w^*}.$$

Somewhat surprisingly, we show that  $tw(t, r) = t(t, r)$ , that is, the bound is not affected by the weight, and the bound on  $tw(t, r)$  can be derived from Theorem 1.6 the same way as the bound on  $t(t, r)$ .

The notion of weighted Turán numbers enables us to establish a connection between Turán systems and Tuza's [20] famous conjecture asserting that in every graph the

minimum number of edges covering every triangle is at most twice the maximum number of pairwise edge-disjoint triangles. Finding a minimum number of edges in a graph  $G = (V, E)$  covering every triangle is equivalent to computing the weighted Turán number  $T_w(n, 3, 2)$  with  $n = |V|$ , and  $w(e) = 1$  if  $e \in E$  and  $w(e) = 0$  otherwise. We propose a weighted hypergraphic version of Tuza's conjecture, and prove its fractional relaxation. This extends the result of Krivelevich [16] on the fractional version of Tuza's original conjecture and also makes use of our construction on shadow systems.

The rest of the paper is organized as follows. The proofs of Theorems 1.4 and 1.6 are given in Section 2. Here we also explain the equivalence of our construction in Theorem 1.6 to Sidorenko's construction [18]. The weighted Turán number is investigated in Section 3. Finally, we propose an extension of Tuza's conjecture about triangle covers in Section 4 and also prove its fractional version.

## 2 Shadow systems

Let  $x = (x_1, \dots, x_k) \in M_k$  be a  $k$ -colour vector. If  $x_j = 0$  and  $x_{j+1} \neq 0$  then  $x' = (x_1, \dots, x_{j-1}, x_{j+1} - 1, x_{j+2}, \dots, x_k) \in M_{k-1}$  is called **the reduction of  $x$  at the  $j$ th position** and is denoted by  $\text{red}[j](x)$  (indices are in a cyclic order, i.e.  $x_{k+1}$  refers to  $x_1$ ). A vector with no zero entries is called **irreducible**. Assume that a series of reduction steps at positions  $j_1, \dots, j_t$  is applied on vector  $x \in M_k$  which results in another vector  $x' \in M_m$  where  $t = k - m$ . We define the **ancestor**  $\text{anc}(i)$  of a position  $1 \leq i \leq m$  as the original position of that entry in the starting vector. Formally, these can be obtained by the algorithm on Figure 1. The following proposition unravels an important property of the reduction operation.

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anc(i) := i;
q := t;
while q > 0 do
  if j_q > anc(i) then
    | anc(i) := anc(i);
  else
    | anc(i) := anc(i) + 1;
  end
  q := q - 1;
end

```

Figure 1: Computing  $\text{anc}(i)$

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**Proposition 2.1.** *Let  $x \in M_k$  be a  $k$ -colour vector. Assume that after some reduction steps we obtain an irreducible vector  $x'$ . Then  $x'$  and the ancestors of its positions are independent from the choice of the reduction steps.*

*Proof.* For a contradiction, assume there exists a  $k$ -colour vector  $x \in M_k$  that can be reduced to two vectors  $x'$  and  $x''$  that are either different or are identical but one of the positions has different ancestors in them. Choose  $k$  as the minimum value where this may occur; clearly  $k > 2$ . By this minimal choice, the two reduction sequences must differ in the very first step. Assume the first sequence reduces at position  $j'$  and the second at position  $j''$ , resulting in  $y' = \text{red}[j'](x)$  and  $y'' = \text{red}[j''](x)$ . W.l.o.g. assume  $j' < j''$ ; then  $j'' > j' + 1$  follows as we cannot reduce at position  $j'$  if  $x_{j'+1} = 0$ . Consider now the reductions  $\text{red}[j'](y'')$  and  $\text{red}[j'' - 1](y')$ . These must be identical. Moreover, the ancestors of the positions in  $\text{red}[j'](y'')$  and  $\text{red}[j'' - 1](y')$  also coincide. However, by the minimal choice of  $k$ , any reduction sequence of  $y'$  and  $y''$  must result in the same vector  $z$  with the same ancestors, a contradiction.  $\square$

As an alternative proof, we can define the following quantity. Let  $\text{sum}(j, k) = \sum_{i=j}^{k-1} (x_i - 1)$  where indices are in cyclic order and  $\text{sum}(k, k)$  is defined as 0. Let  $x_i^{\text{red}} = \max\{0, x_i + \min_j \text{sum}(j, i)\}$ . Observe that the reduction stops with an  $x'$  which is obtained from  $x^{\text{red}}$  by deleting its zero entries. Moreover, the ancestor of position  $i$  is just the position of the corresponding nonzero entry in  $x^{\text{red}}$ .

The irreducible vector arising by applying a sequence of reductions on  $x$  is hence uniquely defined; it is called the **complete reduction** of  $x$  and is denoted by  $\text{red}(x)$ . The ancestor of position  $i$  in a complete reduction is denoted by  $\text{anc}(i)$ . Let us define the **rank** of  $x$ , denoted by  $rk(x)$ , as the length of the vector  $\text{red}(x)$ , and let

$$\mathcal{A}_k := \{x \in M_k^k : rk(x) = 1\}. \quad (2)$$

Note that reducing a vector in  $M_k^k$  gives a vector in  $M_{k-1}^{k-1}$  and the only irreducible vector in  $M_k^k$  is an all-one vector (that is, all its entries are 1). Consequently, the complete reduction of any vector in  $M_k^k$  is an all-one vector of dimension  $m \leq k$ , and  $x \in \mathcal{A}_k$  if and only if  $m = 1$ . Theorem 1.4 follows by the next lemma, showing that partitioning  $M_k^k$  to  $\mathcal{A}_k$  and  $M_k^k \setminus \mathcal{A}_k$  satisfies the splitting property.

**Lemma 2.2.** *Let  $\mathcal{B}_k = M_k^k \setminus \mathcal{A}_k$ . Then  $M_k = \mathcal{U}(\mathcal{A}_k) \cup \mathcal{L}(\mathcal{B}_k)$ .*

The proof needs one more operation. For  $x = (x_1, \dots, x_k) \in M_k$  we call  $x' = (x_1, x_2, \dots, x_{j-1}, 0, x_j + 1, x_{j+1}, \dots, x_k) \in M_{k+1}$  **the extension of  $x$  at the  $j$ th position** and denote it by  $\text{ext}[j](x)$ . The extension can be considered as a reverse counterpart of the reduction. However, there are no restrictions on the elements of  $x$  in this case and applying  $\text{ext}$  does not modify the result of  $\text{red}$ , namely  $\text{red}(x) = \text{red}(\text{ext}[j](x))$ .

*Proof of Lemma 2.2.* We have to show that (a) for every  $c \in M_k^{k+1}$ ,  $\mathcal{A}_k$  contains a shadow of  $c$ , that is,  $\mathcal{A}_k$  is a  $(k+1, k; k)$ -shadow system; and (b) for every  $d \in M_k^{k-1}$ , there exists a  $b \in \mathcal{B}_k$  such that  $d$  is a shadow of  $b$ .

Both statements are proved by induction on  $k$ . For  $k = 2$ ,  $\mathcal{A}_2 = \{(2, 0), (0, 2)\}$  and  $\mathcal{B}_2 = \{(1, 1)\}$ , and both statements clearly hold. Assume both (a) and (b) hold for all values strictly less than  $k$ .

For (a), consider an arbitrary vector  $c \in M_k^{k+1}$ . We distinguish two cases.

**Case 1.**  *$c$  is irreducible, that is, every entry is strictly positive.*

Since the sum of the elements of  $c$  is  $k+1$ , this is only possible if for some  $1 \leq p \leq k$ ,  $c_p = 2$  and  $c_i = 1$  for  $1 \leq i \leq k$ ,  $i \neq p$ . Consider the vector  $a \in M_k^k$  with  $a_p = 2$ ,  $a_{p+1} = 0$ ,  $a_i = 1$  for every other index  $i$ . Then  $a$  is a shadow of  $c$  and it is easy to verify that  $rk(a) = 1$ , that is,  $a \in \mathcal{A}_k$  as required.

**Case 2.** *There exists an index  $i$  with  $c_i = 0$ ,  $c_{i+1} \neq 0$ .*

Let  $c' = \mathbf{red}[i](c) \in M_{k-1}^k$ . By induction, there exists an  $a' \in \mathcal{A}_{k-1}^{k-1}$  that is a shadow of  $c'$ . Let  $a = \mathbf{ext}[i](a') \in M_k^k$ . Then  $rk(a) = rk(a') = 1$ , and therefore  $a \in \mathcal{A}_k$ . Now  $a$  is a shadow of  $c$ , completing the proof.

Let us now turn to statement (b). Consider an arbitrary colour vector  $d \in M_k^{k-1}$ . Since the sum of the elements of  $d$  is  $k-1$ , there is an index  $1 \leq i \leq k$  such that  $d_i = 0$  and  $d_{i+1} \neq 0$ . Let  $d' = \mathbf{red}[i](d)$  which is in  $M_{k-1}^{k-2}$ . By induction, there exists a  $b' \in \mathcal{B}_{k-1}$  such that  $d'$  is a shadow of  $b'$ . Let  $b = \mathbf{ext}[i](b') \in M_k^k$ . Since  $\mathbf{red}(b) = \mathbf{red}(b')$ , it follows that  $b \in \mathcal{B}_k$ , as required.  $\square$

The construction of the  $(t, r; t-1)$ -shadow system in Theorem 1.6 is also based on  $\mathcal{A}_k$ . We first need to define some further operations. For a vector  $x \in \mathbb{Z}_k^r$ , we obtain the vector  $x' = \delta x \in \mathbb{Z}_k^r$  by increasing every coordinate by 1:  $x'_i = x_i + 1$ . We call  $\delta$  the  **$k$ -shifting operator**; the  $j$ 'th power is denoted by  $\delta^j$ . Clearly  $\delta^k$  is the identity but  $\delta^j x \neq x$  for  $0 < j < k$ . The set  $\{x, \delta x, \delta^2 x, \dots, \delta^{k-1} x\}$  is called the  **$k$ -orbit** of  $x$ . Being in the same  $k$ -orbit defines an equivalence relation on  $\mathbb{Z}_k^r$ .

The  $k$ -shifting operation induces a natural operation on the colour vectors in  $M_k^r$ . For  $a \in M_k^r$ , let  $a' = \Delta a \in M_k^r$  be the vector with  $a'_i = a_{i-1}$  (with indices modulo  $k$ , i.e.  $a'_1 = a_k$ ). We call  $\Delta$  the **cyclic shifting operator**. Clearly,  $M(\delta x) = \Delta M(x)$  for every  $x \in \mathbb{Z}_k^r$  (recall that  $M(x)$  denotes the colour profile of  $x$ ). Again,  $\{a, \Delta a, \Delta^2 a, \dots, \Delta^{k-1} a\}$  defines the **cyclic orbits** of  $M_k^r$ , and being in the same orbit is again an equivalence relation. However, note that  $\Delta^j a = a$  may occur even for  $j < k$ . (For example, let  $k = 4$ ,  $r = 4$ ,  $j = 2$ ,  $a = (2020)$ .) If  $a$  and  $b$  are on the same cyclic orbits, then so are  $\mathbf{red}(a)$  and  $\mathbf{red}(b)$ . We denote the cyclic orbit of an  $a \in M_k^r$  by  $CO(a)$ . The above notions are illustrated on Figure 2.

**Remark 2.3.** It is worth mentioning that in Lemma 2.2, both sets  $\mathcal{A}_k$  and  $\mathcal{B}_k$  are closed under the operation  $\Delta$ .

We are ready to define  $\mathcal{A}_r^t$  as in Theorem 1.6. Consider  $\mathcal{A}_r$  as in (2), and let  $a \in \mathcal{A}_r$ . By definition,  $\mathbf{red}(a) = (1)$ . Let us call the ancestor of this single element the **tip** of the vector  $a$ . Let  $\mathbf{blow}(a) \in M_{t-1}^r$  denote the vector arising from  $a$  by inserting  $t-1-r$  zeros just after the tip of  $a$ . Define

$$\mathcal{A}_r^t := \bigcup_{a \in \mathcal{A}_r} CO(\mathbf{blow}(a)). \quad (\text{SHA})$$

For example, let  $r = 3$ ,  $t = 5$ , and  $a = (2, 0, 1) \in \mathcal{A}_3$ . The tip of  $a$  is the first element, and  $\mathbf{blow}(a) = (2, 0, 0, 0, 1)$ . Finally,  $CO(\mathbf{blow}(a)) = \{(2, 0, 0, 0, 1), (1, 2, 0, 0, 0), (0, 1, 2, 0, 0), (0, 0, 1, 2, 0), (0, 0, 0, 1, 2)\}$ . Also, note that if  $a' \in CO(a)$ , then  $CO(\mathbf{blow}(a)) = CO(\mathbf{blow}(a'))$ . Further,  $\cup_{a' \in CO(a)} \mathbf{blow}(a') \subsetneq CO(\mathbf{blow}(a))$ : in the above example,  $(0, 0, 0, 1, 2)$  is contained in the latter set but not in the first.

$\mathbb{Z}_3^2$	$M_3^2$	3-orbits of $\mathbb{Z}_3^2$	cyclic orbits of $M_3^2$
(1, 1)	(2, 0, 0)	{(1, 1), (2, 2), (3, 3)}	{(2, 0, 0), (0, 2, 0), (0, 0, 2)}
(1, 2)	(0, 2, 0)	{(1, 2), (2, 3), (3, 1)}	{(1, 1, 0), (0, 1, 1), (1, 0, 1)}
(1, 3)	(0, 0, 2)	{(1, 3), (2, 1), (3, 2)}	
(2, 1)	(1, 1, 0)		
(2, 2)	(1, 0, 1)		
(2, 3)	(0, 1, 1)		
(3, 1)			
(3, 2)			
(3, 3)			

Figure 2: The members and orbits of  $\mathbb{Z}_3^2$  and  $M_3^2$ .

We show that  $\mathcal{A}_r^t$  is a  $(t, r; t-1)$ -shadow system satisfying the requirement of Theorem 1.6. The shadow system property can be verified using an argument almost identical to that in the proof of Lemma 2.2.

**Lemma 2.4.** *For integers  $t > r$ ,  $\mathcal{A}_r^t \subseteq M_{t-1}^r$  defined by (SHA) is a  $(t, r; t-1)$ -shadow system.*

*Proof.* The proof is by induction on  $r$ . For  $r = 2$ ,  $\mathcal{A}_2 = \{(2, 0), (0, 2)\}$ , and for any  $t > r$ ,  $\mathcal{A}_2^t$  contains the vectors with one entry being 2 and all other entries 0. Every  $c \in M_{t-1}^t$  must contain at least one entry  $\geq 2$ , and therefore it has a shadow in  $\mathcal{A}_2^t$ . Assume we have proved the statement for all values strictly less than  $r$  and consider an arbitrary colour vector  $c \in M_{t-1}^t$ .

**Case 1.**  $c$  is irreducible, that is, every entry is strictly positive.

Since the sum of the elements of  $c$  is  $t$ , this is only possible if for some  $1 \leq p \leq t-1$ ,  $c_p = 2$  and  $c_i = 1$  for  $1 \leq i \leq t-1$ ,  $i \neq p$ . Consider the vector  $a \in M_{t-1}^r$  with

$$a_i = \begin{cases} 2 & \text{if } i = p, \\ 0 & \text{if } i = p+1, \dots, p+t-r, \\ 1 & \text{otherwise,} \end{cases}$$

where we use the indexing cyclically, i.e.  $t$  means 1. Clearly,  $a$  is a shadow of  $c$ , and  $a \in \mathcal{A}_r^t$  since removing  $t-1-r$  0's after the 2, we obtain  $a' = (1, \dots, 1, 2, 0, 1, \dots, 1) \in M_r^r$ , and it is easy to verify  $a' \in \mathcal{A}_r$ .

**Case 2.** *There exists an index  $i$  with  $c_i = 0$ ,  $c_{i+1} \neq 0$ .*

Let  $c' = \text{red}[i](c) \in M_{t-2}^{t-1}$ . By induction, there exists an  $a' \in \mathcal{A}_{t-2}^{r-1}$  that is a shadow of  $c'$ . Let  $a = \text{ext}[i](a') \in M_{t-1}^r$ . It is easy to verify  $a \in \mathcal{A}_r^t$ . Now  $a$  is a shadow of  $c$ , completing the proof.  $\square$

The following lemma considers elements of  $\mathbb{Z}_{t-1}^r$  instead of colour vectors, and gives the exact number of those having their colour profile in  $\mathcal{A}_r^t$ .

**Lemma 2.5.** *Let  $\mathcal{S} \subseteq \mathbb{Z}_{t-1}^r$  denote the set of vectors whose colour profile is in  $\mathcal{A}_r^t$ . Then  $|\mathcal{S}| = (r-1)^{r-1}(t-1)$ .*



Before proving the lemma, let us derive Theorem 1.6 as a consequence.

*Proof of Theorem 1.6.* We show that  $\mathcal{A}_r^t$  as defined by **(SHA)** satisfies the conditions. Lemma 2.4 shows that it is a  $(t, r; t-1)$ -shadow system. The total number of vectors in  $\mathbb{Z}_{t-1}^r$  is  $(t-1)^r$ . The probability that a randomly picked  $s \in \mathbb{Z}_{t-1}^r$  has its colour profile in  $\mathcal{A}_r^t$  is  $|\mathcal{S}|/(t-1)^r = \left(\frac{r-1}{t-1}\right)^{r-1}$  by Lemma 2.5 as required.  $\square$

By definition,  $\mathcal{A}_r^t$  is closed under the operation  $\Delta$ . While certain cyclic orbits may be shorter than  $t-1$ , the next claim shows this cannot be the case for orbits contained in  $\mathcal{A}_r^t$ .

**Claim 2.6.** *If  $a \in \mathcal{A}_r^t$ , then  $\Delta^j a \neq a$  for  $0 < j < t-1$ . Consequently, all cyclic orbits contained in  $\mathcal{A}_r^t$  have size exactly  $t-1$ .*

*Proof.* Every cyclic orbit in  $\mathcal{A}_r^t$  can be obtained as  $CO(\mathbf{blow}(a))$  for some  $a \in \mathcal{A}_r$ . It suffices to show that for any  $0 < j < t-1$ ,  $\Delta^j \mathbf{blow}(a) \neq \mathbf{blow}(a)$ . For a contradiction, assume there exists such a  $j$  and  $a$  for which  $\Delta^j \mathbf{blow}(a) = \mathbf{blow}(a)$ ; let  $b = \mathbf{blow}(a)$  and  $b' = \Delta^j \mathbf{blow}(a)$ . Without loss of generality, assume the tip of  $a$  is its first element.

As  $a \in \mathcal{A}_r$ , it can be reduced to  $(1)$ , which means that  $b$  can be reduced to  $(0, \dots, 0)$  consisting of  $t-r-1$  zeros and the ancestor of the  $i$ th zero is  $i$ . Recall that the complete reduction of  $b$  and the ancestors of the elements of  $\mathbf{red}(b)$  are uniquely defined by Proposition 2.1. By  $b' = b$ ,  $b'$  also has complete reduction  $(0, \dots, 0)$  consisting of  $t-r-1$  zeros where the ancestor of the  $i$ th zero is  $i$ . On the other hand, by  $b' = \Delta^j b$ , the ancestors of the elements of  $\mathbf{red}(b')$  are just the ancestors of the elements of  $\mathbf{red}(b)$  shifted by  $j$ , a contradiction as  $0 < j < t-1$ .  $\square$

*Proof of Lemma 2.5.* The cardinality of  $\mathbb{Z}_{r-1}^r$  is  $(r-1)^r$  and the number of  $(r-1)$ -orbits is  $(r-1)^{r-1}$ . Since  $\mathcal{A}_r^t$  is closed under  $\Delta$ , it follows that  $\mathcal{S}$  is closed under  $\delta$  and is hence a union of  $(t-1)$ -orbits. In what follows, we define a bijection  $\varphi$  between the  $(r-1)$ -orbits of  $\mathbb{Z}_{r-1}^r$  and the  $(t-1)$ -orbits of  $\mathcal{S}$ . Since every  $(t-1)$ -orbit has cardinality  $t-1$  by Lemma 2.5, this proves the lemma.

Consider a colour vector  $a \in M_{r-1}^r$ . It is easy to verify that its complete reduction has one entry that is 2 and all other entries are 1, that is  $\mathbf{red}(a) = (1, \dots, 1, 2, 1, \dots, 1)$ . Analogously as for elements of  $\mathcal{A}_r$ , we call the ancestor of the entry 2 the **tip** of  $a$ . Clearly, the tip of  $\Delta a$  is the tip of  $a$  plus one (in a cyclic sense).

Take an arbitrary  $(r-1)$ -orbit  $X$  in  $\mathbb{Z}_{r-1}^r$ . The colour profiles of the vectors in  $X$  map to a cyclic-orbit  $T$  of  $M_{r-1}^r$ .  $T$  must have an element  $a$  whose tip is the last  $((r-1)$ 'st) coordinate; pick an  $s \in X$  such that  $M(s) = a$ . Let us inject  $\mathbb{Z}_{r-1}$  into  $\mathbb{Z}_{t-1}$  by mapping  $i \in \mathbb{Z}_{r-1}$  to  $i \in \mathbb{Z}_{t-1}$  for  $1 \leq i \leq r-1$ , and let  $\bar{s} \in \mathbb{Z}_{t-1}^r$  be the image of  $s$  under this mapping. Let us define  $\varphi(X)$  as the  $(t-1)$ -orbit of  $\bar{s}$  in  $\mathbb{Z}_{t-1}^r$ . In what follows, we verify that  $\varphi$  is a good bijection.

**Well-defined.** We first have to show that  $\bar{s} \in \mathcal{S}$ , that is,  $M(\bar{s}) \in \mathcal{A}_r^t$ . Observe that  $\bar{a} = M(\bar{s}) \in M_{t-1}^r$  can be obtained from  $a = M(s) \in M_{r-1}^r$  by adding  $t-r$  zero coordinates at the  $(r-1)$ 'st position. The vector  $a$  can be reduced to  $(1, 1, \dots, 1, 2)$ ; apply the same reduction steps to  $\bar{s}$ . This gives a vector  $b = (1, 1, \dots, 1, 2, 0, \dots, 0)$  (with  $t-r$  zeros at the end), which can be further reduced to  $(1)$  after deleting the

last  $t - r - 1$  zeros.

**Injective.** Assume indirectly that  $X_1$  and  $X_2$  are different  $(r - 1)$ -orbits of  $\mathbb{Z}_{r-1}^r$ , such that  $\varphi(X_1) = \varphi(X_2)$ . For  $i = 1, 2$ , let  $T_i$  be the corresponding cyclic orbit,  $a^i \in T_i$  the element with tip  $(r - 1)$  and  $s^i \in X_i$  with  $M(s^i) = a^i$ . Define  $\bar{s}^i \in \mathcal{S}$  by mapping  $\mathbb{Z}_{r-1}$  to  $\mathbb{Z}_{t-1}$  and  $\bar{a}^i \in M_{t-1}^r$  as the colour profile of  $\bar{s}^i$ . Now  $s^1 \neq s^2$  are on different  $(r - 1)$ -orbits but  $\bar{s}^1 \neq \bar{s}^2$  are on the same  $(t - 1)$ -orbit. That means that there is a  $j$  such that  $\bar{s}^2 = \delta^j \bar{s}^1$ , and so  $\bar{a}^2 = \Delta^j \bar{a}^1$ .

We know that both  $\bar{a}^1$  and  $\bar{a}^2$  can be reduced to  $(1, \dots, 1, 2, 0, \dots, 0)$  (with  $t - r$  zeros at the end) by applying the same reductions steps as for  $a^1$  and  $a^2$ , and this vector can be further reduced to the all-zero  $(0, \dots, 0)$  vector consisting of  $t - r - 1$  zeros where the ancestor of the  $i$ th element is  $t - r$ . Again, the complete reduction of a vector and the ancestors of the elements of the reduction are uniquely defined by Proposition 2.1. We have seen that  $\bar{a}^1$  and  $\bar{a}^2$  has the same complete reduction. On the other hand, by  $\bar{a}^2 = \Delta^j \bar{a}^1$ , the ancestors of the elements of  $\text{red}(\bar{a}^2)$  are just the ancestors of the elements of  $\text{red}(\bar{a}^1)$  shifted by  $j$ , a contradiction as  $0 < j < t - 1$ .

**Surjective.** Consider any orbit  $Y$  of  $\mathcal{S}$ , and let  $a \in \mathcal{A}_r^t$  be the colour profile of an element  $s \in Y$ . We may choose  $s$  such that  $a_r = \dots = a_{t-1} = 0$ . This is since  $a$  is a vector in  $CO(\text{blow}(a_0))$  for some  $a_0 \in \mathcal{A}_r$ , that is, we insert  $t - 1 - r$  zeros after the tip of  $a_0$  and apply  $\Delta^j$  for some  $j$ . It is easy to verify that the element of  $a_0$  following the tip must be 0 because of  $rk(a_0) = 1$ .

Let us apply reduction steps on  $a$  avoiding the last  $t - r$  zeros but reducing all others. It is easy to verify that this reduces  $a$  to  $(1, \dots, 1, 2, 0, \dots, 0)$  (with  $t - r$  zeros at the end). Now let us map  $s \in \mathbb{Z}_{t-1}^r$  to  $s^* \in \mathbb{Z}_{r-1}^r$  by mapping  $i \in \mathbb{Z}_{t-1}$  to  $i \in \mathbb{Z}_{r-1}$  for  $1 \leq i \leq r - 1$  (this is well-defined as  $s$  does not contain colors  $r, \dots, t - 1$  by  $a_r = \dots = a_{t-1} = 0$ ). Observe that  $\varphi$  maps the orbit of  $s^*$  to  $Y$ , proving the claim.  $\square$

## 2.1 Relation to Sidorenko's construction

Sidorenko's construction is based on the following observation.

**Lemma 2.7.** *Let  $b_1, \dots, b_k$  be cyclically ordered reals, and  $b = \frac{b_1 + \dots + b_k}{k}$ . Then there exists an index  $m$  such that*

$$b_m + \dots + b_{m-s+1} \geq sb \quad \forall s = 1, \dots, k.$$

The construction is as follows: Divide the  $n$  elements into  $t - 1$  groups  $A_1, A_1, \dots, A_{t-1}$ . Let  $B$  be an  $r$ -element subset and  $b_i = |B \cap A_i|$ . Then set  $B$  is included into the set system  $\mathcal{T}$  if and only if there is an index  $m$  such that

$$\sum_{i=1}^s b_{m-i+1} \geq s + 1 \quad \forall s = 1, \dots, r - 1, \quad (3)$$

where indices are meant in cyclic order, that is,  $b_t = b_1$ . It follows from Lemma 2.7 that  $\mathcal{T}$  thus obtained is a Turán  $(n, t, r)$ -system.

The following lemma shows the connection between Sidorenko's construction and that of  $\mathcal{A}_r^t$ .

**Lemma 2.8.** *Assume that the  $n$  elements are divided into  $t-1$  groups  $A_1, A_1, \dots, A_{t-1}$ . An  $r$ -element subset  $B$  is included into  $\mathcal{T}$  if and only if  $(b_1, \dots, b_{t-1}) \in \mathcal{A}_r^t$ .*

*Proof.* Consider a set  $B$  with  $b = (b_1, \dots, b_{t-1}) \in \mathcal{A}_r^t$ . Then  $b \in CO(\mathbf{blow}(a))$  for some  $a \in \mathcal{A}_r$  where  $\mathcal{A}_r$  is defined by (2), say  $b = \Delta^j \mathbf{blow}(a)$ . Let  $p$  be the tip of  $a$  and define  $m = p + j$ . We claim that  $m$  and  $b$  satisfies (3). Indirectly, assume that there is an  $1 \leq s \leq r-1$  violating (3), that is,  $\sum_{i=1}^s b_{m-i+1} \leq s$ . From  $s \leq r-1$  and the definitions of  $b$  and  $m$ ,  $\sum_{i=1}^s b_{m-i+1} = \sum_{i=1}^s a_{p-i+1}$ . Choose  $s$  to be maximal. Then  $s < r-1$  as  $\sum_{i=1}^{r-1} a_{p-i+1} = r$ . Indeed,  $a \in \mathcal{A}_r$  so  $\sum_{i=1}^r a_{p-i+1} = r$ , and  $a \neq (1, \dots, 1)$  as it can be reduced to (1).

Recall that  $a' = \mathbf{red}(a)$  is obtained from  $a^{\mathbf{red}}$  by deleting its zero entries, where  $a_i^{\mathbf{red}} = \max\{0, a_i + \min_j \text{sum}(j, i)\}$  and  $\text{sum}(j, k) = \sum_{i=j}^{k-1} (a_i - 1)$  (we defined  $\text{sum}(k, k)$  as 0). However,  $\sum_{i=1}^s a_{p-i+1} \leq s$  means that in fact  $\sum_{i=1}^s a_{p-i+1} = s$ , otherwise  $a_p^{\mathbf{red}} = 0$  contradicting the tipness of  $p$ . The maximal choice of  $s$  implies  $\sum_{i=1}^q a_{p-s-i+1} \geq q$  for  $1 \leq q \leq r$  and  $\sum_{i=1}^{r-s} a_{p-s-i+1} = r-s > 0$ . Hence  $a_{r-s}^{\mathbf{red}} > 0$ , contradicting  $a \in \mathcal{A}_r$ .

Now take a  $B \in \mathcal{T}$  and an index  $m$  satisfying (3). W.l.o.g. assume that  $m = r$ . That is,  $\sum_{i=1}^s b_{r-i+1} \geq s+1$  for  $1 \leq s \leq r-1$ . As  $\sum_{i=1}^{t-1} b_{r-i+1} = r$ , we immediately have  $b_{r+1} = \dots = b_{t-1} = b_1 = 0$ . Let  $a = (a_1, \dots, a_r) = (b_1, \dots, b_r)$ . Then  $\sum_{i=1}^r a_{r-i+1} = r$  and  $\sum_{i=1}^s a_{r-i+1} \geq s+1$  for  $1 \leq s \leq r-1$ . We claim that  $a \in \mathcal{A}_r$ . To see this, it suffices to show that  $a_p^{\mathbf{red}} = 0$  for  $p = 1, \dots, r-1$ . Assume indirectly that  $a_p^{\mathbf{red}} > 0$  for some  $p$ . This implies  $\sum_{i=1}^q a_{p-i+1} \geq q$  for  $1 \leq q \leq r$ . We have  $r = \sum_{i=1}^r a_i = \sum_{i=1}^p a_{p-i+1} + \sum_{i=1}^{r-p} a_{r-i+1} \geq p + r - p + 1 = r + 1$ , a contradiction.  $\square$

In the proof of Theorem 1.5, we took a uniform random colouring of the ground set with  $t-1$  colours and showed that the expected number of  $r$ -element subsets whose colour profile is contained in  $\mathcal{A}_r^t$  is 'small enough'. Sidorenko's construction takes a deterministic colouring instead with almost equal groups, that is,  $||A_i| - |A_j|| \leq 1$  for  $1 \leq i < j \leq t-1$ , and shows that for such a colouring the number of  $r$ -element subsets with colour profile in  $\mathcal{A}_r^t$  does not exceeds the bound, thus proving (1).

### 3 Weighted Turán number

Recall the definition of the weighted Turán number  $tw(t, r)$  from the Introduction. The following easy observation shows that the presence of weights does not affect the upper bound.

**Theorem 3.1.** *For any integers  $t > r$ , we have  $tw(t, r) = t(t, r)$ , and therefore  $tw(t, r) \leq \binom{r-1}{t-1}^{r-1}$*

*Proof.* Clearly,  $tw(t, r) \geq t(t, r)$  as the unweighted Turán number corresponds to the special case  $w \equiv 1$ . To see the other direction, take an arbitrary Turán  $(n, t, r)$ -system (without taking weights into account). If we consider the weight of this system in a random permutation of the elements, then the expected value of its weight is exactly

$\frac{T(n,t,r)}{\binom{n}{r}} \cdot w^*$ , which means that there exists a Turán  $(n, t, r)$ -system with weight at most that, completing the proof. The second half follows by Theorem 1.5.  $\square$

Theorem 3.1 ensures the existence of a Turán  $(n, t, r)$ -system with ‘small’ weight. However, it is still not clear how to find and represent such a system. For  $t = 3$  and  $k = 2$ , Theorems 1.5 and 3.1 imply that in a weighted graph, we can choose a set of edges whose weight is at most the half of the total weight  $w^*$  covering every triangle. Indeed, the most simple maximum cut algorithm delivers such an edge set. Let us colour the nodes of the graph by two colours uniformly at random, and choose the set of edges whose two endpoints receive the same colour. Clearly, these edges must cover every triangle. Since every individual edge gets chosen by probability  $\frac{1}{2}$ , the expected cost of the chosen edge set will be  $\frac{w^*}{2}$ .

The proof of Theorem 1.5 using Theorem 1.6 presented in the Introduction also yields a simple randomized algorithm for finding an  $(n, t, r)$ -Turán system in question. We colour the nodes uniformly at random by  $(t - 1)$ -colours, and choose  $r$ -element subsets according to their colour profiles. Note that we must obtain a Turán system of cost at most  $\left(\frac{r-1}{t-1}\right)^{r-1} w^*$  with probability at least  $\left(\frac{r-1}{t-1}\right)^{r-1}$ . The construction of the  $(t, r; t - 1)$ -shadow system  $\mathcal{A}_r^t$  in Theorem 1.6 will give a simple and efficient way to decide whether a colour vector is contained in  $\mathcal{A}_r^t$ . Consequently, although the size of the construction is  $O(n^r)$ , the colouring provides a simple linear representation.

## 4 Tuza's conjecture

As outlined in the Introduction, the minimum number of edges covering all of the triangles in an arbitrary graph is the weighted Turán number  $T_w(n, 3, 2)$  for  $w_e = 1$  on the edges of the graph and  $w_e = 0$  otherwise. Given an undirected graph  $G = (V, E)$ , a set of pairwise edge-disjoint triangles is called a **triangle packing**, while a set of edges sharing an edge with all triangles is called a **triangle cover**. Let

$$\begin{aligned} \nu(G) &= \text{maximum cardinality of a triangle packing in } G, \\ \tau(G) &= \text{minimum cardinality of a triangle cover in } G. \end{aligned}$$

Hence the unweighted Turán number  $T(n, 3, 2)$  is the same as  $\tau(K_n)$ . The problem of determining the exact values of  $\nu(G)$  and  $\tau(G)$  is showed to be NP-complete by Holyer [14] and Yannakakis [22], respectively. Still, it would be interesting to give a connection between these parameters. Clearly,  $\nu(G) \leq \tau(G)$  holds so a natural approach would be to give an upper bound for  $\tau(G)$  as a function of  $\nu(G)$ . In [20], Tuza proposed the following conjecture.

**Conjecture 4.1** (Tuza).  $\tau(G) \leq 2\nu(G)$  for any simple undirected graph  $G$ .

It is worth mentioning that equality holds for infinitely many graphs. Indeed, take any graph with all maximal two-connected subgraphs isomorphic to either  $K_2$ ,  $K_4$  or  $K_5$ . That is, if Conjecture 4.1 is true then it is sharp.

The conjecture has been proved for various classes of graphs (see [4, 9, 11, 12, 13, 16, 21]). The first nontrivial bound for general graphs was given by Haxell by

proving that for any graph  $G$ , we have  $\tau(G) \leq (3 - \varepsilon)\nu(G)$ , where  $\varepsilon > \frac{3}{23}$  [10]. A fractional weakening of the conjecture was given by Krivelevich [16] who showed that  $\tau(G) \leq 2\tau^*(G)$  and  $\nu^*(G) \leq 2\nu(G)$  where  $\tau^*(G)$  and  $\nu^*(G)$  stand for the optimal fractional solutions of the corresponding covering and packing problems, respectively.

The problem of determining  $\nu(G)$  and  $\tau(G)$  can be generalized in two ways. In [7], Erdős and Tuza proposed a 'clique version' of the original problem by considering the covering of complete subgraphs with complete subgraphs, while in [3] Chapuy et al. studied an edge-weighted version of the conjecture, and weighted analogues of results of Tuza, Krivelevich and Haxell were proved. Putting together these two ideas, we formalize a more general version of the problem.

For an  $(r - 1)$ -uniform simple hypergraph  $H = (V, \mathcal{E})$ , an  $r$ -block is a subset of  $r$  nodes spanning a complete subhypergraph. The set of  $r$ -blocks is denoted by  $\mathcal{B}_r$ . A  $r$ -packing is a set of disjoint  $r$ -blocks, while an  $r$ -cover is a set of hyperedges such that each  $r$ -block spans at least one of them. Assume now that a weight function  $w : \mathcal{E} \rightarrow \mathbb{R}_+$  is also given. A **weighted  $r$ -packing** is a family of - not necessarily disjoint -  $r$ -blocks such that each hyperedge  $e$  is contained in at most  $w(e)$  of them. For the weighted case, let

$$\begin{aligned}\nu_w(H) &= \text{maximum cardinality of a weighted } r\text{-packing in } H, \\ \tau_w(H) &= \text{minimum weight of a } r\text{-cover in } H.\end{aligned}$$

Here  $\nu_w(H)$  and  $\tau_w(H)$  are called **weighted  $r$ -packing** and **weighted  $r$ -covering** numbers, respectively. These parameters can be interpreted as optimal solutions to the following integer programs. Let  $A$  be the hyperedge -  $r$ -block incidence matrix of  $H$ , that is,  $A_{e,R} = 1$  if  $e \in \mathcal{E}$  is spanned by  $r$ -block  $R$ , and 0 otherwise. Then

$$\begin{aligned}\nu_w(H) &= \max\{\mathbf{1} \cdot x \mid Ax \leq w, x \in \mathbb{Z}_+^{\mathcal{B}_r}\}, \\ \tau_w(H) &= \min\{w \cdot y \mid A^T y \geq \mathbf{1}, y \in \mathbb{Z}_+^{\mathcal{E}}\}.\end{aligned}$$

By relaxing the integrality constraints we get the following primal-dual pair of linear programs.

$$\begin{aligned}\nu_w^*(H) &= \max\{\mathbf{1} \cdot x \mid Ax \leq w, x \in \mathbb{R}_+^{\mathcal{B}_r}\}, \\ \tau_w^*(H) &= \min\{w \cdot y \mid A^T y \geq \mathbf{1}, y \in \mathbb{R}_+^{\mathcal{E}}\},\end{aligned}$$

where  $\nu_w^*(H)$  and  $\tau_w^*(H)$  are called the **weighted fractional  $r$ -packing** and **weighted fractional  $r$ -covering** numbers, respectively. The linear programming duality theorem gives

$$\nu_w(H) \leq \nu_w^*(H) = \tau_w^*(H) \leq \tau_w(H).$$

As a generalization of Tuza's, we propose the following conjecture.

**Conjecture 4.2.** *Let  $H = (V, \mathcal{E})$  be a simple  $(r - 1)$ -uniform hypergraph and  $w : \mathcal{E} \rightarrow \mathbb{R}_+$  a weight function. Then  $\tau_w(H) \leq \lceil \frac{r+1}{2} \rceil \nu_w(H)$ .*

Tuza's conjecture corresponds to the case when  $r = 3$ ,  $w \equiv 1$  and  $H$  is a simple graph. Similarly to the original conjecture, if Conjecture 4.2 is true then it is sharp.

Indeed, let  $w \equiv 1$  and take an  $(r - 1)$ -uniform complete hypergraph  $H = (V, \mathcal{E})$  on  $r + 1$  nodes. We claim that  $\nu_w(H) = 1$  and  $\tau_w(H) = \lceil \frac{r+1}{2} \rceil$ .

It is easy to see that  $\nu_w(H) = 1$  as the graph has only  $r + 1$  nodes, so any two  $r$ -blocks share  $r - 1$  nodes in common. As the graph is complete, there is a hyperedge spanned by these nodes, so  $w \equiv 1$  implies that at most one  $r$ -block is contained in any weighted  $r$ -packing.

To see  $\tau_w(H) \geq \lceil \frac{r+1}{2} \rceil$  it suffices to show that for any set  $\mathcal{C}$  of  $r$ -blocks with cardinality at most  $\lceil \frac{r+1}{2} \rceil - 1$  there exists a node  $v$  which is contained in all members of  $\mathcal{C}$ . That would clearly prove the lower bound as  $\mathcal{C}$  does not cover the  $r$ -block  $H - v$ . Assume indirectly that there is no such node, that is, each node is contained in at most  $|\mathcal{C}| - 1$  of them. We have

$$\sum_{v \in V} |\{e \in \mathcal{C} : v \in e\}| \leq (r + 1)(|\mathcal{C}| - 1).$$

On the other hand,

$$\sum_{v \in V} |\{e \in \mathcal{C} : v \in e\}| = \sum_{e \in \mathcal{C}} |e| = (r - 1)|\mathcal{C}|.$$

These together gives  $(r + 1)(|\mathcal{C}| - 1) \geq (r - 1)|\mathcal{C}|$ , hence  $|\mathcal{C}| \geq \lceil \frac{r+1}{2} \rceil$ , a contradiction.

It remains to show an  $r$ -cover with cardinality  $\lceil \frac{r+1}{2} \rceil$ . Let  $V = \{v_1, \dots, v_{r+1}\}$  and  $\mathcal{C} = \{V \setminus \{v_{2i-1}, v_{2i}\} \mid i = 1, \dots, \lceil \frac{r+1}{2} \rceil\}$  where indices are meant in cyclic order, so  $v_{r+2} = v_1$ . Then for any  $v \in V$  there is at least one  $e \in \mathcal{C}$  not containing  $v$ . Hence  $\mathcal{C}$  is an  $r$ -cover as for any  $r$ -block  $B$  there is an  $e \in \mathcal{C}$  not containing  $V \setminus B$ , thus  $e \subseteq B$ .

Conjecture 4.2 is widely open. With the help of the shadow system appearing in Theorem 1.6, we prove a fractional weakening of the conjecture which can be considered as a weighted counterpart of Krivelevich's result.

**Theorem 4.3.** *Let  $H = (V, \mathcal{E})$  be a simple  $(r - 1)$ -uniform hypergraph and  $w : \mathcal{E} \rightarrow \mathbb{R}_+$  a weight function. Then  $\tau_w(H) \leq (r - 1)\tau_w^*(H)$ .*

*Proof.* Suppose that the theorem does not hold and let  $H$  be a minimal counterexample, that is,  $\tau_w(H) > (r - 1)\tau_w^*(H)$  but  $\tau_w(H') \leq (r - 1)\tau_w^*(H')$  for every proper subhypergraph  $H'$  of  $H$ . This implies that each hyperedge  $e \in \mathcal{E}$  is contained in an  $r$ -block as otherwise it could be left out from  $H$  thus giving a smaller counterexample. Take a pair of optimal solutions of the weighted fractional  $r$ -packing and  $r$ -cover problems denoted by  $x^*$  and  $y^*$ , respectively.

**Case 1.**  $y_e^* \geq \frac{1}{r-1}$  for some  $e \in \mathcal{E}$ .

Let  $H'$  be the graph obtained by deleting the hyperedge  $e$  from  $H$ . Clearly,  $\tau_w(H') \geq \tau_w(H) - w(e)$ . On the other hand,  $z^*$  is a fractional  $r$ -cover in  $H'$  where  $z^*(e') = y^*(e')$  for  $e' \neq e$ . Hence  $\tau_w^*(H') \leq \tau_w^*(H) - \frac{w(e)}{r-1}$ . By the minimal choice of  $H$  we get

$$\tau_w(H) \leq \tau_w(H') + w(e) \leq (r - 1)\tau_w^*(H') + w(e) \leq (r - 1)\tau_w^*(H),$$

a contradiction.

**Case 2.**  $y_e^* < \frac{1}{r-1}$  for each  $e \in \mathcal{E}$ .

We claim that  $y_e^* > 0$  for each  $e \in \mathcal{E}$ . Indeed, an  $r$ -block spans  $r$  different hyperedges. If one of these hyperedges had  $y^*$  value 0 then the total  $y^*$  sum on them would be strictly smaller than 1, contradicting the assumption that  $y^*$  is a fractional  $r$ -cover. As mentioned earlier, each hyperedge is spanned by one of the  $r$ -blocks, hence the statement follows. By complementary slackness, we have

$$\sum_{\substack{B \in \mathcal{B}_r \\ B \text{ spans } e}} x^*(B) = w(e) \text{ for each } e \in \mathcal{E}.$$

That also implies that the exact value of the optimum for the fractional problem can be computed as

$$\tau_w^*(H) = \nu_w^*(H) = \sum_{B \in \mathcal{B}_r} x^*(B) = \frac{1}{r} \sum_{e \in \mathcal{E}} \sum_{\substack{B \in \mathcal{B}_r \\ B \text{ spans } e}} x^*(B) = \frac{1}{r} \sum_{e \in \mathcal{E}} w(e) = \frac{1}{r} w^*.$$

So it suffices to show that  $\tau_w(H) \leq \frac{r-1}{r} w^*$ . We do the same as in the proof of Theorem 3.1: colour the nodes uniformly at random with the colours  $1, \dots, r-1$  and define the  $r$ -cover as the set of hyperedges  $e$  with colour profile in  $\mathcal{A}_{r-1}^r$  defined in **(SHA)**. We have already seen that there exist a colouring of the nodes such that the total weight of the covering is at most  $\left(\frac{r-1}{r}\right)^{r-1} w^* \leq \frac{r-1}{r} w^*$ , and we are done.  $\square$

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