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Orientations and Detachments of Graphs with Prescribed Degrees and Connectivity

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Abstract

We give a necessary and sufficient condition for a graph to have an orientation that has k edge-disjoint arborescences rooted at a designated vertex s subject to lower and upper bounds on the in-degree at each vertex. The result is used to derive a characterization of graphs having a detachment that contains k edge-disjoint spanning trees. Efficient algorithms for finding those orientations and detachments are also described. In particular, the paper provides an algorithm for finding a connected (loopless) detachment in $O(nm)$ time, improving on the previous best running time bound, where n and m denote the numbers of vertices and edges, respectively.

1 Introduction

All graphs and digraphs considered are finite and may contain loops and multiple edges. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. For a function $r : V \to \mathbf{Z}_+$, an *r-detachment* of G is a graph H obtained by 'splitting' each vertex $v \in V$ into $r(v)$ vertices. The vertices $v_1, \ldots, v_{r(v)}$ obtained by splitting v are called the *pieces* of v in H. Every edge $vw \in E$ corresponds to an edge of H connecting some piece of v to some piece of w. For $v \in V$, we use $\deg(v)$ to denote the degree of v. An *r-degree specification* is a function f on V, such that, for each vertex $v \in V$, $f(v)$ is a sequence $d_1^v, ..., d_{r(v)}^v$ of positive integers so that $\sum_{i=1}^{r(v)} d_i^v = \deg(v)$. An *$f$-detachment* of G is an r-detachment in which the degrees of the pieces of each $v \in V$ are given by $f(v)$.

Crispin Nash-Williams [10] obtained the following necessary and sufficient condition for a graph to have a connected r-detachment or f-detachment. For X, Y disjoint subsets of $V(G)$, let $d(X, Y)$ be the number of edges of G from X to Y, and let $d(X) = d(X, V - X)$. For a single vertex $v \in V$ we shall simply write $d(v)$. We

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use $i(X)$ to denote the number of edges between the vertices of X. Thus $i(v)$ is the number of loops incident to v and $\deg(v) = d(v) + 2i(v)$. Let $e(X) = i(X) + d(X)$, $r(X) = \sum_{x \in X} r(x)$, and $c(X)$ be the number of components of $G - X$.

Theorem 1.1 ([10]). *Let $G = (V, E)$ be a graph and $r : V \to \mathbf{Z}_+$. Then G has a connected r-detachment if and only if $r(X) + c(X) \leq e(X) + 1$ for every $X \subseteq V$. Furthermore, if G has a connected r-detachment, then G has a connected f-detachment for every r-degree specification f.*

The original proof of Theorem 1.1 was based on the matroid intersection theorem of Edmonds [2]. A subsequent paper of Nash-Williams [11] contains an alternative proof using orientations. Recently, Hiroshi Nagamochi [8] presented an interesting application of connected detachments to molecular structure analysis. He also presented an efficient algorithm for finding a connected loopless r-detachment in $O(\min\{r(V)^{3.5} + m, r(V)^{1.5} m r_{\max}\})$ time, where $r_{\max} = \max\{r(v) \mid v \in V\}$, relying on matroid intersection algorithms. In this paper, we present an improved $O(nm)$ algorithm for finding connected loopless detachments via orientations.

The structure of the paper is as follows. In Section 2 we give a necessary and sufficient condition for a graph to have an orientation that has k edge-disjoint arborescences rooted at a designated vertex s subject to lower and upper bounds on the in-degree at each vertex. In Section 3 we discuss algorithms for finding such an orientation and analyse their running time. In Section 4 we use these results to derive a characterization of graphs having a detachment that contains k edge-disjoint spanning trees and to obtain efficient algorithms for finding such detachments. These results will lead to the above mentioned improved running time bound for finding connected loopless detachments. Concluding remarks are given in Section 5.

2 Orientations of graphs

Let $D = (V, E)$ be a digraph. We shall use $\varrho_D(X)$ to denote the number of edges entering a subset $X \subseteq V$ of D. For a singleton $X = \{v\}$ we simply write $\varrho_D(v)$. We may omit the subscript if D is clear from the context.

Consider a graph $G = (V, E)$ and two functions $l : V \to \mathbf{Z}_+$ and $u : V \to \mathbf{Z}_+ \cup \{\infty\}$. We call an orientation D of G an *l-orientation* (resp. *u-orientation*) if $l(v) \leq \varrho(v)$ (resp. $\varrho(v) \leq u(v)$) holds for every $v \in V$. We say that D is an *(l, u)-orientation* if it simultaneously satisfies the lower and upper bounds on the in-degrees.

For a function $a : V \to \mathbf{Z}_+$ and $X \subseteq V$ we use $a(X)$ to denote $\sum_{v \in X} a(v)$. Frank and Gyárfás [4] gave the following theorem on the existence of an (l, u)-orientation of a graph.

Theorem 2.1 ([4]). *Let $G = (V, E)$ be a graph and let $l : V \to \mathbf{Z}_+$ and $u : V \to \mathbf{Z}_+ \cup \{\infty\}$ be two functions with $l \leq u$. Then G has an (l, u)-orientation if and only if*

$$e(X) \geq l(X) \text{ for all } X \subseteq V, \tag{1}$$

and

$$i(X) \leq u(X) \text{ for all } X \subseteq V. \tag{2}$$

Let $D = (V, E)$ be a digraph and let $s \in V$. We say that D is *k-edge-connected from s* if $\varrho(X) \geq k$ for all $X \subseteq V - s$. Note that, by Menger's theorem, D is k-edge-connected from s if and only if $\lambda(s, v; D) \geq k$ for all $v \in V - s$, where $\lambda(x, y; D)$ denotes the maximum number of pairwise edge-disjoint directed paths from x to y in digraph D. The following theorem, characterizing the existence of an (l, u)-orientation which is k-edge-connected from a designated vertex s was implicit in a paper of András Frank [5], where he extended Theorem 2.1 to various directions[1]. We describe here a direct proof, which also leads to an efficient algorithm for finding such an orientation. In the proof, we repeatedly use the well-known fact that in a directed graph the following submodular inequality holds for all pairs $X, Y \subseteq V$:

$$\varrho(X) + \varrho(Y) \geq \varrho(X \cap Y) + \varrho(X \cup Y) \tag{3}$$

For a partition $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of V, let $e(\mathcal{P})$ denote the number of edges connecting distinct members of $\mathcal{P}$.

Theorem 2.2. *Let $G = (V, E)$ be a graph, let $s \in V$, and let $l : V \to \mathbf{Z}_+$ and $u : V \to \mathbf{Z}_+ \cup \{\infty\}$ be two functions with $l \leq u$. Then G has an (l, u)-orientation which is k-edge-connected from s if and only if*

$$e(\mathcal{P}) \geq \sum_{i=1}^{t} h(X_i) \tag{4}$$

for all partitions $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of V, where $h(X) = k$ for all $X \subseteq V - s$ with $|X| \geq 2$, $h(v) = \max\{l(v), k\}$ for all $v \in V - s$, $h(s) = l(s)$, and $h(X) = 0$ otherwise, and

$$i(X) + k\epsilon(X) \leq u(X) \tag{5}$$

for all $X \subseteq V$, where $\epsilon(X) = 1$ if $s \notin X$ and $\epsilon(X) = 0$ otherwise.

Proof. It is easy to see that (4) and (5) are both necessary for the existence of the required orientation.

To see that these conditions together are sufficient, first we show that G has an (l, u)-orientation. Consider a set $X = \{x_1, x_2, ..., x_m\} \subseteq V$. By applying (4) to the partition $\mathcal{P}_X = \{\{x_1\}, \{x_2\}, ..., \{x_m\}, V - X\}$ we can deduce that $e(X) \geq l(X)$. Condition (5) implies that $u(X) \geq i(X)$. Thus, by Theorem 2.1, G has an (l, u)-orientation D.

Let k' be the largest integer for which D is k'-edge-connected from s. If $k' \geq k$, we are done. Otherwise, when $k' \leq k-1$, we shall prove that by reversing the orientations of all edges on an appropriately chosen directed path in D, we may obtain another

[1]A pair of subsets $X, Y \subseteq V$ is *crossing* if none of $X \cap Y$, $X \setminus Y$, $Y \setminus X$, and $V \setminus (X \cup Y)$ are empty. A set function $q : 2^V \to \mathbf{Z}_+$ with $q(\emptyset) = q(V) = 0$ is said to be *crossing G-supermodular* if it satisfies

$$q(X) + q(Y) \leq q(X \cap Y) + q(X \cup Y) + d(X, Y)$$

for all crossing pairs $X, Y \subseteq V$. Frank proved that for a given graph $G = (V, E)$ and crossing G-supermodular function $q : 2^V \to \mathbf{Z}_+$ there exists an orientation D of G in which $\varrho(X) \geq q(X)$ holds for all $X \subseteq V$ if and only if $e(\mathcal{P}) \geq \sum_{i=1}^{t} q(X_i)$ and $e(\mathcal{P}) \geq \sum_{i=1}^{t} q(V \setminus X_i)$ hold for every partition $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of V.

(l,u)-orientation D^* of G, which is also k'-edge-connected from s and such that $\{v \in V-s : \lambda(s,v;D) \geq k'+1\}$ is a proper subset of $\{v \in V-s : \lambda(s,v;D^*) \geq k'+1\}$. This will complete the proof, since it implies that D can be made k-edge-connected from s by reversing at most $(k-k')(|V|-1)$ directed paths without violating the in-degree bounds.

We say that a set $X \subseteq V-s$ is *tight* (resp. *critical*) if $\varrho(X)=k'$ (resp. $\varrho(X)=k'+1$). It follows from (3) that the maximal (w.r.t. inclusion) tight sets $T_1,T_2,...,T_p$ are pairwise disjoint. Let $T=\cup_{i=1}^p T_i$ and let Q be the set of vertices reachable from T in D. Note that $p \geq 1$ and $T \neq \emptyset$. Furthermore, we also have $Q-T \neq \emptyset$, for otherwise we had

$$e(\{T_1,...,T_p,V-T\}) = \sum_{i=1}^{p} \varrho(T_i) + \varrho(V-T) = pk' < pk \leq \sum_{i=1}^{p} h(T_i), \tag{6}$$

contradicting (4).

Claim 2.3. *Let $a \in T$, $b \in Q-T$ with $\varrho(a) < u(a)$ and $\varrho(b) > l(b)$ and let P be a directed path from a to b. Suppose that there is no tight or critical set X with $b \in X$ and $a \in V-X$. Let D^* be the digraph obtained from D by reversing all edges in P. Then D^* is an (l,u)-orientation of G which is k'-edge-connected from s and for which $\{v \in V-s : \lambda(s,v;D) \geq k'+1\}$ is a proper subset of $\{v \in V-s : \lambda(s,v;D^*) \geq k'+1\}$.*

Proof. Clearly, D^* is an (l,u)-orientation of G. Since the reversal of P decreases the in-degree of a set $X \subseteq V-s$ (by one) if and only if $b \in X$ and $a \in V-X$, it follows from the hypotheses of the claim that D^* is k'-edge-connected from s. Moreover, the in-degree of all tight sets containing a is increased by one after reversing P. This proves the last inequality. □

It remains to prove that there is a directed path P in D satisfying the hypotheses of Claim 2.3.

Claim 2.4. *Let X be a critical set with $X \cap T = \emptyset$ and $X \cap (Q-T) \neq \emptyset$. Then $X \subseteq Q-T$.*

Proof. Suppose that $X-Q \neq \emptyset$. Then $\varrho(X-Q) \geq k'+1$. Furthermore, since there is no edge from Q to $V-Q$ and since there is a path from T to each vertex of $Q-T$ in $D[Q]$, we have $\varrho(X) \geq \varrho(X-Q)+\varrho(X \cap Q) \geq k'+1+1 = k'+2$, a contradiction. □

For each vertex $b \in Q-T$ let C_b denote the smallest critical set X with $b \in X$ (if there is no critical set X with $b \in X$ then let $C_b = V$). It follows from (3) that C_b is indeed unique.

Claim 2.5. *There exists a vertex $b \in Q-T$ with $\varrho(b) > l(b)$ and $C_b \cap T \neq \emptyset$.*

Proof. Consider the family $\mathcal{X}$ of all critical sets X with $X \subseteq Q-T$ and let $R_1,R_2,...,R_q$ be the vertex sets of the connected components of $\mathcal{X}$ (viewed as a hypergraph on ground-set $Q-T$). By submodularity, $R_i \cap R_j = \emptyset$ for $1 \leq i < j \leq q$. Suppose, for a contradiction, that for all vertices $b \in Q-T$ with $\varrho(b) > l(b)$ we have $C_b \cap T = \emptyset$.

Then we also have $C_b \subseteq Q - T$ by Claim 2.4, and hence $b \in R_i$ for some $1 \le i \le q$. Let $Y = Q - T - \cup_{i=1}^{q} R_i$.

Consider the partition $\mathcal{F}$ of V consisting of the sets $T_1, T_2, ..., T_p$, and the non-empty sets among $R_1, R_2, ..., R_q$, the singletons of Y, and the set $V - Q$. Note that $\varrho(V - Q) = 0$. We have

$$e(\mathcal{F}) = \sum_{X \in \mathcal{F}} \varrho(X) = pk' + q(k'+1) + \sum_{y \in Y} l(y) + \varrho(V - Q) < \sum_{X \in \mathcal{F}} h(X),$$

where the strict inequality follows from the facts that $p \ge 1$ and $h(T_i) \ge k \ge k' + 1$ for $1 \le i \le p$. This contradicts (4). $\square$

Let $Z = \{a \in T : \varrho(a) < u(a)\}$.

Claim 2.6. *Let X be a tight set. Then $Z \cap X \neq \emptyset$. In particular, $Z \cap T_i \neq \emptyset$ and all vertices in T_i can be reached from $Z \cap T_i$ in $D[T_i]$, for all $1 \le i \le p$.*

Proof. Suppose that $Z \cap X = \emptyset$. Then $i(X) = \sum_{v \in X} \varrho(v) - \varrho(X) = u(X) - k' > u(X) - k\epsilon(X)$, contradicting (5). This implies the second part of the claim by observing that (i) T_i is tight, (ii) if the set Z_i of vertices reachable from $Z \cap T_i$ in $D[T_i]$ was a proper subset of T_i then $T_i - Z_i$ would also be tight. $\square$

To complete the proof let us take a vertex $b \in Q - T$ with $\varrho(b) > l(b)$ and $C_b \cap T_i \neq \emptyset$ for some $1 \le i \le p$. Such a vertex exists by Claim 2.5. First suppose that $C_b = V$ (in which case there is no critical set containing b). By the definition of Q there is directed path P from some vertex $a \in T$ to b. By Claim 2.6 we may suppose that $a \in Z$. Then P satisfies the hypotheses of Claim 2.3, as required. Next suppose that C_b is critical. Observe that, by the minimality of C_b, b is reachable from any other vertex of C_b in $D[C_b]$. Furthermore, we have $k' + 1 + k' = \varrho(C_b) + \varrho(T_i) \ge \varrho(C_b \cap T_i) + \varrho(C_b \cup T_i) \ge k' + k' + 1$, which implies that $C_b \cap T_i$ is tight. Thus there is a vertex $a \in Z$ in $C_b \cap T_i$ by Claim 2.6. Then a directed path P from a to b satisfies the hypotheses of Claim 2.3, as required. This proves the theorem. $\square$

By letting $u \equiv \infty$ (resp. $l \equiv 0$) in Theorem 2.2 we can deduce from the proof that G has an l-orientation (u-orientation, resp.) which is k-edge-connected from s if and only if (4) holds (resp. G has k edge-disjoint spanning trees and (5) holds).

For $k = 1$ and lower bounds we can simplify the result as follows, see also [4].

Theorem 2.7. *Let $G = (V, E)$ be a graph with $s \in V$ and $l : V \to \mathbf{Z}_+$. Then G has an l-orientation which contains an s-arborescence if and only if*

$$e(X) \ge l(X) + c(X) - \epsilon(X) \tag{7}$$

for all $X \subseteq V$, where $\epsilon(X) = 1$ if $s \notin X$ and $\epsilon(X) = 0$ otherwise.

3 Algorithms

In this section we discuss various algorithms, and their running time bounds, for testing the existence of an (l,u)-orientation that is k-edge-connected from s.

The proof of Theorem 2.2 naturally leads to the following polynomial time algorithm. The algorithm starts with finding an (l,u)-orientation D. This can be done in $O(nm)$ time by solving a maximum flow problem. Then it computes the edge-connectivity λ of D from s and finds λ edge-disjoint s-arborescences in D. This can be done in $O(\lambda^2 n^2)$ time by the algorithm of Gabow [6]. To identify the set T, the algorithm checks if there exist $\lambda+1$ edge-disjoint directed paths from s to each $v \in V-s$. Since λ edge-disjoint s-arborescences are already obtained, we can accomplish this for each v in $O(m)$ time by a simple path search after reversing λ directed paths from s to v in the arborescences. Thus it takes $O(\lambda^2 n^2 + nm)$ time to identify T. The algorithm then identifies the set Z and tries to find a reversible path from Z to $b \in V \setminus T$ with $\varrho(b) > l(b)$. To do this efficiently, the algorithm picks up each such vertex b and tests if b is reachable from Z after reversing the λ directed paths from s to b. If so, then the algorithm reverses the directed path from Z to b in D and goes to the next iteration with the modified D. If it finds that b is not reachable, then b will never become reachable until λ increases. After $O(n)$ iterations, D becomes $(\lambda+1)$-edge-connected from s. The algorithm terminates when D becomes k-edge-connected from s. Thus the algorithm runs in $O(k^3 n^3 + kn^2 m)$ time.

This is not the best algorithm for general k. In fact, the following alternative algorithm is faster. The algorithm starts with finding k edge-disjoint spanning trees in G. This can be done in $O(kn\sqrt{m + kn\log n})$ time. Orient all the edges in these spanning trees so that they form k edge-disjoint s-arborescences. The other edges can be oriented arbitrarily. The resulting orientation D is k-edge-connected from s, but it may not satisfy the lower or upper bounds on the in-degrees.

The algorithm then identifies the set $W = \{a \in V : \varrho(a) < u(a)\}$. For each vertex v that violates the upper bound, the algorithm checks if there is a reversible path from W to v. To do this efficiently, the algorithm searches for an edge-disjoint collection of k directed paths from s to v and one directed path from W to v, which takes $O(km)$ time. If such edge-disjoint paths are found, the algorithm reverses the one from W to v in D. If no such paths exist, it turns out that v is contained in a subset $X \subseteq V-s$ with $\varrho(X) = k$ and $X \cap W = \emptyset$, which implies that X violates (5), and hence there exists no (l,u)-orientation which is k-edge-connected from s. The algorithm repeats this until all the vertices satisfy the upper bounds.

Then it identifies the set L of vertices that violate the lower bounds. For each vertex b with $\varrho(b) > l(b)$, the algorithm checks if there exists a reversible path from L to b. To do this efficiently, the algorithm searches for an edge-disjoint collection of k directed paths from s to b and one directed path from L to b, which also takes $O(km)$ time. If such edge-disjoint paths are found, the algorithm reverses the one from L to b in D. If no such paths exist, it turns out that b is contained in a subset $X \subseteq V-s$ with $\varrho(X) = k$ and $X \cap L = \emptyset$. (If this holds for all such b then the maximal sets X with these properties - which are pairwise disjoint - and the remaining vertices as singleton sets violate (4).) The algorithm repeats this until D satisfies the lower bounds.

Each time the algorithm reverses a directed path, it reduces the total amount of violation $\Phi := \sum_{v \in V} \max\{0, \varrho(v) - u(v)\} + \sum_{v \in V} \max\{0, l(v) - \varrho(v)\}$ by at least one. Since the initial value of Φ is at most m, the algorithm performs $O(m)$ searches. Thus the algorithm runs in $O(km^2)$ time.

3.1 The special case $k = 1$

When it comes to the case of $k = 1$, we have a more efficient algorithm, which is in fact a specialized version of our first algorithm. This algorithm starts with an (l, u)-orientation D, which can be found in $O(nm)$ time. Then it checks if all the vertices are reachable from s. If so, then the algorithm terminates. Otherwise, let R be the set of vertices reachable from s in D. The algorithm then identifies the set $Q \subseteq R$ of vertices reachable from $T = V \setminus R$. A vertex $v \in Q$ is called *active* if $\varrho(v) > l(v)$ holds. (If there is no active vertex then T, $V - T - Q$, and the singletons of Q violate (4).) For each active vertex $v \in Q$, the algorithm tests if D has a pair of edge-disjoint directed paths P_{sv} from s to v and P_{tv} from some $t \in T$ with $\varrho(t) < u(t)$ to v. (If $\varrho(t) \geq u(t)$ for all $t \in T$ then T violates (5).) If such a pair exists, then the algorithm reverses the directed path P_{tv} in D. If no such pair exists, then v is contained in a subset $Y \subseteq V - s$ with $\varrho(Y) = 1$. In this case, the algorithm proceeds to the next vertex, if it exists. The algorithm eventually finds an active vertex with such a pair of edge-disjoint paths (and increases the set R), or it terminates with no active vertex left. In the latter case, we may assert that G has no (l, u)-orientation that contains an s-arborescence. (This can be seen by observing that $Y \subseteq R$ must hold - otherwise $Y \cap T$ violates (5) -, $Y \subseteq Q$ may be assumed, and the maximal such sets Y with these properties, the remaining singletons in Q, T, and $V - T - Q$ violate (4).)

We now discuss the complexity of this algorithm. The test for the existence of a pair of edge-disjoint paths can be done in $O(m)$ time. It finds a pair of edge-disjoint paths or a subset $Y \subseteq V - s$ with $\varrho(Y) = 1$. In the former case, the algorithm successfully extends the set R of vertices rechable from s in D. This can happen at most n times throughout the algorithm. In the latter case, the set Y continues to satisfy $\varrho(Y) = 1$ until the end of the algorithm. Therefore, the algorithm omits the test for v in the subsequent iterations. Thus the number of searches is $O(n)$, and the total running time bound is $O(nm)$.

Theorem 3.1. *Given a graph $G = (V, E)$ with $s \in V$ and functions $l, u : V \to \mathbf{Z}_+$ with $l \leq u$ that satisfy (4) and (5), an (l, u)-orientation of G which contains an s-arborescence can be found in $O(nm)$ time.*

4 Detachments

In this section we consider detachments of graphs containing k edge-disjoint spanning trees. We shall apply Theorem 2.2 to create such detachments.

We shall use the following operation to adjust the degree sequence in a detachment of a graph. For vertices x, y, z of G with $xz \in E(G)$ we define the graph $G(xz \to yz)$

obtained by *flipping* xz to yz by putting $G(xz \to yz) = G - xz + yz$. A graph G containing k edge-disjoint spanning trees is called *k-partition-connected*.

Theorem 4.1. *Let $G = (V, E)$ be a graph and k be a positive integer. Then G has a k-partition-connected r-detachment if and only if*

$$i(X_0) + e(\mathcal{P}) \geq k(t-1) + kr(X_0) \tag{8}$$

for all partitions $\mathcal{P} = \{X_0, X_1, ..., X_t\}$ of V, where X_0 may be empty or $t = 0$ may hold. Furthermore, if G has a k-partition-connected r-detachment then G has a k-partition-connected f-detachment for every r-degree specification f in which $d_i^v \geq k$ for all $v \in V$ and $1 \leq i \leq r(v)$.

Proof. Consider a k-partition-connected detachment H of G and let

$$\mathcal{P}' = \{\{v_1\}, \{v_2\}, ..., \{v_{r(X_0)}\}, X_1', ..., X_t'\}$$

be the partition of $V(H)$, where v_i denotes a piece of a vertex in X_0 for $1 \leq i \leq r(X_0)$, and X_j' denotes the union of the pieces of the vertices in X_j, $1 \leq j \leq t$. Then $i(X_0) + e(\mathcal{P}) \geq e(\mathcal{P}') \geq k(r(X_0) + t - 1) = k(t-1) + kr(X_0)$. This proves that (8) is necessary.

To see sufficiency let us consider $l : V \to \mathbf{Z}_+$ with $l(v) = kr(v)$ for $v \neq s$ and $l(s) = k(r(s) - 1)$, where $s \in V$ is an arbitrarily specified vertex. Let $u \equiv \infty$. It is easy to verify that (8) implies (4) for this function l. It is clear that (5) holds. Hence, by Theorem 2.2, G has an l-orientation D which is k-edge-connected from s. By Edmonds' branching theorem [3] it follows that D has k edge-disjoint s-arborescences $T_1, T_2, ..., T_k$. Now let us 'detach' the heads of all edges which enter vertex v and which do not belong to any T_i and create $r(v) - 1$ new pieces of v, each of in-degree at least k, for each $v \in V$. The in-degree lower bounds guarantee that this is possible. The underlying graph of this detachment is the required k-partition-connected r-detachment H of G.

The second part of the theorem follows by using edge flippings to adjust the degrees. If, for some vertex v, the degree sequence of the pieces is different from $f(v)$ then it has two pieces v_i, v_j with $d(v_i) > d_i^v$ and $d(v_j) < d_j^v$. Let $v_i w$ be an edge in H which does not belong to the unique $v_i v_j$ path in T_h, for any $1 \leq h \leq k$. It is easy to see that $H(v_i w \to v_j w)$ is k-partition-connected and has an 'improved' degree sequence. □

Theorem 4.1 implies Theorem 1.1 (by observing that when $k = 1$, condition (8) holds if and only if it holds for all $\mathcal{P}$ in which $d(X_i, X_j) = 0$ for $1 \leq i < j \leq t$) and the characterization of k-partition-connected graphs by Tutte and Nash-Williams [9, 13] (by taking $r \equiv 1$). By using a similar edge flipping argument we obtain the following loopless version.

Theorem 4.2. *Let $G = (V, E)$ be a graph, let $r : V \to \mathbf{Z}_+$, and let f be an r-degree specification. Suppose that G has a k-partition connected r-detachment (f-detachment, respectively). Then*
(i) *G has a k-partition connected loopless r-detachment if and only if $r(v) \geq 2$ for all*

$v \in V$ *with* $i(v) \geq 1$,
(ii) *G has a k-partition connected loopless f-detachment if and only if* $d_i^v \leq d(v) + i(v)$ *for all* $v \in V$ *and* $1 \leq i \leq r(v)$.

Proof. (i) Let H be a k-partition connected r-detachment and suppose that there is a loop on some piece v_i of v. Then $r(v) \geq 2$, and hence the loop can be eliminated by flipping it to some other piece of v.
(ii) Let H be a k-partition connected f-detachment and suppose that there is a loop on some piece v_i of v. Since $\deg(v_i) = d_i^v \leq d(v) + i(v)$, it follows that there is an edge $v_j v_h$ connecting two pieces of v with $i \neq j, h$. Hence the loop on v_i can be eliminated by two edge flippings (which delete the loop and add two new edges $v_i v_j$ and $v_i v_h$) preserving the k-partition connectivity of H as well as all degrees. □

The special case $k = 1$ of Theorem 4.2, with a different proof, appeared in [8]. The following recent result of Gu, Lai, and Liang [7] turns out to be another special case of Theorem 4.2.

Theorem 4.3 ([7])**.** *Let* $d = (d_1, d_2, ..., d_n)$ *be a nonincreasing sequence of nonnegative integers with* $n \geq 2$, $d_1 \leq d_2 + ... + d_n$, *and for which* $\sum_{i=1}^n d_i$ *is even. Then there exists a loopless k-partition connected graph with degree sequence d if and only if*
(i) $d_n \geq k$, *and*
(ii) $\sum_{i=1}^n d_i \geq 2k(n-1)$.

Proof. Necessity is easy to see. Sufficiency follows by applying Theorems 4.1 and 4.2 to the graph G consisting of a single vertex v incident to $(\sum_{i=1}^n d_i)/2$ loops, by putting $r(v) = n$ and $f(v) = (d_1, d_2, ..., d_n)$. □

Theorem 3.1, together with Theorems 2.7 and 4.2, implies that a connected loopless r-detachment (or f-detachment), if it exists, can be found in $O(nm)$ time. This gives an improvement on the running time of Nagamochi's algorithm, as mentioned in Introduction.

5 Concluding remarks

In this section we first provide an alternative proof for Theorem 2.2 from the viewpoint of supermodular functions. A set function $g : 2^V \to \mathbf{R}$ is called *fully supermodular* if

$$g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y) \tag{9}$$

holds for all pairs $X, Y \subseteq V$. A pair of subsets $X, Y \subseteq V$ is *intersecting* if none of $X \cap Y$, $X \setminus Y$, and $Y \setminus X$ are empty. A set function g is *intersecting supermodular* if it satisfies (9) for all intersecting pairs $X, Y \subseteq V$. With an intersecting supermodular function g, we associate the *extended contrapolymatroid*

$$\mathrm{P}(g) = \{x \mid x \in \mathbf{R}^V,\ x(X) \geq g(X),\ \text{for each } X \subseteq V\}.$$

The *Dilworth truncation* $\hat{g} : 2^V \to \mathbf{R}$ of an intersecting supermodular function g is defined by

$$\hat{g}(X) = \max_{\mathcal{P}(X)} \sum_{X_i \in \mathcal{P}(X)} g(X_i),$$

where the maximum is taken over all partitions of X. It is well-known that $\hat{g}$ is fully supermodular and $\mathrm{P}(g) = \mathrm{P}(\hat{g})$, see e.g. [12, Chapters 48,49]. With function $\hat{g}$, we also associate the *base polyhedron*

$$\mathrm{B}(\hat{g}) = \{x \mid x \in \mathrm{P}(\hat{g}), x(V) = \hat{g}(V)\}.$$

For a vector $z \in \mathbf{Z}^V$, let $\hat{g}_z$ be the set function defined by

$$\hat{g}_z(X) = \max\{z(Y) + \hat{g}(V \setminus Y) \mid Y \subseteq X\}.$$

Then we have

$$\mathrm{P}(\hat{g}_z) = \mathrm{P}(\hat{g}) \cap \{y \mid y \in \mathbf{R}^V,\ y \geq z\}.$$

This implies that there exists a base vector $y \in \mathrm{B}(\hat{g})$ with $z \leq y$ if and only if $\hat{g}_z(V) = \hat{g}(V)$. If, in addition, we have $u \in \mathrm{P}(\hat{g})$ with $z \leq u$, then there exists a base vector $y \in \mathrm{B}(\hat{g})$ with $z \leq y \leq u$. Thus we obtain the following theorem.

Theorem 5.1. *Let $g : 2^V \to \mathbf{R}$ be an intersecting supermodular function. For a given pair of vectors $l, u \in \mathbf{Z}^V$ with $l \leq u$, there exists a base vector $y \in \mathrm{B}(\hat{g}) \cap \mathbf{Z}^V$ with $l \leq y \leq u$ if and only if*

$$l(X_0) + \sum_{i=1}^{t} g(X_i) \leq \hat{g}(V) \tag{10}$$

holds for all partitions $\mathcal{P} = \{X_0, X_1, \ldots, X_t\}$ of V, and

$$u(X) \geq g(X) \tag{11}$$

holds for all $X \subseteq V$.

For a graph $G = (V, E)$ with $s \in V$, $|V| = n$, and $|E| = m$, consider the set function $g : 2^V \to \mathbf{R}$ defined by $g(\emptyset) = 0$, $g(V) = m$, and $g(X) = i(X) + k\epsilon(X)$ for proper nonempty subsets X of V. Then g is an intersecting supermodular function that satisfies $\hat{g}(V) = g(V) = m$. The graph G has an orientation which is k-edge-connected from s and satisfies $y(v) = \varrho(v)$ for all $v \in V$ if and only if $y \in \mathrm{B}(\hat{g}) \cap \mathbf{Z}^V$. Applying Theorem 5.1 to this function g gives Theorem 2.2.

5.1 Detachments of directed graphs

We may also ask whether there is a directed version of Theorem 4.1, in which we look for a detachment of a given digraph, which is k-edge-connected from a specified vertex s. This version happens to be easier to handle.

Let $D = (V, E)$ be a digraph. For two disjoint subsets X, Y of V let $\varrho(X, Y)$ denote the number of edges from Y to X and let $\varrho(X) = \varrho(X, V - X)$. Let $\delta(X, Y) = \varrho(Y, X)$

and $\delta(X) = \varrho(V - X)$. A digraph $D = (V, E)$ is *k-edge-connected* if $\varrho(X) \geq k$ for every proper subset $\emptyset \neq X \subset V$. Let $d(X, Y) = \varrho(X, Y) + \delta(X, Y)$. We use $i(v)$ to denote the number of loops incident to a vertex $v \in V$ and we let $\varrho^*(v) = \varrho(v) + i(v)$ and $\delta^*(v) = \delta(v) + i(v)$ denote the *in-degree* and the *out-degree* of a vertex $v \in V$, respectively.

The definition of an r-detachment H of a digraph D is similar to the undirected case. An r-degree specification of D is a function f on V, such that for each vertex $v \in V$, $f(v)$ is a sequence of ordered pairs $(\varrho_i^v, \delta_i^v)$, $1 \leq i \leq r(v)$ of positive integers so that $\sum_{i=1}^{r(v)} \varrho_i^v = \varrho^*(v)$ and $\sum_{i=1}^{r(v)} \delta_i^v = \delta^*(v)$. An f-detachment of D is an r-detachment in which the in- and out-degrees of the pieces of each $v \in V$ are given by the pairs of $f(v)$.

We shall characterize when a digraph has a detachment which is k-edge-connected from a given vertex s. To this end, we first recall the following result of Berg, Jackson, and Jordán [1].

Theorem 5.2 ([1]). *Let $D = (V, E)$ be a digraph and let $r : V \to \mathbf{Z}_+$. Then D has a k-edge-connected r-detachment if and only if*
(a) *D is k-edge-connected, and*
(b) *$\varrho^*(v) \geq kr(v)$ and $\delta^*(v) \geq kr(v)$ for all $v \in V$.*
Furthermore, if D has a k-edge-connected r-detachment then D has a k-edge-connected f-detachment for any r-degree specification f for which $\varrho_i^v \geq k$ and $\delta_i^v \geq k$ for every $v \in V$ and every $1 \leq i \leq r(v)$.

The rooted k-edge-connected version is as follows.

Theorem 5.3. *Let $D = (V, E)$ be a digraph, $s \in V$, and let $r : V \to \mathbf{Z}_+$ with $r(s) = 1$. Then D has an r-detachment which is k-edge-connected from s if and only if*
(a) *D is k-edge-connected from s, and*
(b) *$\varrho^*(v) \geq kr(v)$ for all $v \in V - s$.*
Furthermore, if D has an r-detachment which is k-edge-connected from s then D has an f-detachment which is k-edge-connected from s for any r-degree specification f for which $\varrho_i^v \geq k$ for every $v \in V - s$ and every $1 \leq i \leq r(v)$.

Proof. Necessity is obvious. To prove sufficiency add $kr(v)$ parallel edges with tail v and head s for all $v \in V - s$ and apply Theorem 5.2. The second part of the statement is obtained by defining the out-degree specification so that $\delta_i^v \geq k$ for every $v \in V - s$ and every $1 \leq i \leq r(v)$. □

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