Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-17. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

# Gain-sparsity and Symmetry-forced Rigidity in the Plane

Tibor Jordán, Viktória E. Kaszanitzky, and Shin-ichi Tanigawa

December 2012

# Gain-sparsity and Symmetry-forced Rigidity in the Plane

Tibor Jordán*, Viktória E. Kaszanitzky**, and Shin-ichi Tanigawa***

**Abstract**

We consider planar bar-and-joint frameworks with discrete point group symmetry in which the joint positions are as generic as possible subject to the symmetry constraint. We provide combinatorial characterizations for symmetry-forced rigidity of such structures with cyclic or odd-order dihedral symmetry, unifying and extending previous work on this subject.

We also explore the matroidal background of our results and show that the matroids induced by the row independence of the orbit matrices of the symmetric frameworks are isomorphic to gain sparsity matroids defined on the quotient graph of the framework, whose edges are labeled by elements of the corresponding symmetry group.

The proofs are based on new Henneberg type inductive constructions of the gain graphs that correspond to the bases of the matroids in question, which can also be seen as symmetry preserving graph operations in the original graph.

## 1 Introduction

This paper deals with planar bar-and-joint frameworks with point group symmetry and provides combinatorial characterizations for symmetry-forced rigidity of such structures with cyclic or dihedral symmetry, unifying and extending previous work on this subject.

Frameworks can be used to model various structures with pairwise distance constraints and are useful in applications ranging from civil engineering [16, 28] and crystallography [31] to sensor network localization [12] and biochemistry [34]. In several applications the model frameworks have symmetry, which makes it interesting to explore the impact of symmetry on the flexibility and rigidity of the framework.

In the past ten years this research area has received an ever increasing attention which has led to rigorous definitions, a clear separation of different directions and a

*Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `jordan@cs.elte.hu`

**Department of Operations Research, Eötvös University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary. e-mail: `viktoria@cs.elte.hu`

***Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan. e-mail: `tanigawa@kurims.kyoto-u.ac.jp`

number of new results [8, 6, 24, 27, 15]. Similar questions have been identified in the study of infinite periodic frameworks along with similar definitions and methods to attack fundamental problems [4, 5, 22, 17, 21, 13, 14].

Our goal is to extend Laman's classical theorem on generically rigid planar frameworks (with no symmetry conditions), as well as its matroidal background and algorithmic implications, to planar frameworks with cyclic or dihedral symmetry, assuming that the joint positions are as generic as possible subject to the symmetry conditions. In our symmetry-forced setting a framework is said to be flexible (non-rigid) if it has a non-trivial *symmetric* infinitesimal motion. For the generic frameworks that we consider this is equivalent to the existence of a non-trivial symmetry preserving flex [26]. By using the orbit rigidity matrix, introduced by Schulze and Whiteley [27], we can reformulate our problems in terms of the generic rank of a matrix in which each row corresponds to an edge orbit and each vertex orbit has two columns. This in turn is equivalent to characterizing independence in a matroid defined on the edge set of the group-labeled quotient graph of the framework, in which vertices and edges correspond to vertex and edge orbits, respectively, and which concisely represents the graph structure with the corresponding symmetry. Our main results characterize these matroids in the case of cyclic or odd-order dihedral symmetry. In the case of cyclic symmetry, the matroid turns out to be a $(k, l)$*-gain-count matroid*, in which independence is defined by imposing certain sparsity conditions on the edge sets of a graph, whose edges are labeled by group elements. In the dihedral case of odd order the matroid arises by a related, but more general construction.

Matroids of the former type can be obtained by matroidal operations (e.g. matroid union and Dilworth truncation) from matroids that have been studied before and are called frame matroids (or bias matroids) in the literature [35, 36]. These matroids, and their relatives, which also play a role in the theory of infinite periodic frameworks, have been generalized in a recent paper [29] which unified most of the existing results on symmetric and periodic frameworks, including our cyclic case. However, the matroid of the dihedral case does not fit this general class.

We prove our results by developing Henneberg type inductive constructions for the bases of our matroids and show that these operations preserve the row-independence of the orbit rigidity matrix. This approach, which has been used in many combinatorial characterizations of rigidity theory, leads to the desired result. In our problems, due to the more complex sparsity conditions and the group labeling, we also need some new operations and extended geometric arguments, to handle the symmetry constraints.

The complete answer in the case of dihedral symmetry remains open. However, most of our inductive steps (extending or reducing a symmetric framework or a labeled graph, respectively) are valid also for dihedral groups of even order, and can be used to show that in the even case the irreducible graphs (frameworks), where our reduction operations are not applicable, are very special. Interestingly, the smallest such framework, which is predicted to be rigid by the matroidal count but is flexible is the Bottema mechanism, a well-known mechanism in the engineering literature.

The structure of the paper is as follows. In the rest of this section we introduce some basic notation. In Section 2 we define and investigate gain graphs, which are directed multigraphs with edges labeled by elements of a group. Gain count matroids, defined on gain graphs by sparsity conditions, are introduced in Section 3 along with the necessary matroidal background. In Section 4 we develop our inductive construction for the bases of a specific gain count matroid by using three operations and a single base graph. In Section 5 we recall the basic definitions and results needed to study symmetric frameworks, including the orbit rigidity matrix and the necessary count conditions. In Section 6 we prove the first geometic lemmas and use them, together with results of Section 4, to complete the characterization of rigid frameworks with cyclic symmetry. In Section 7 we prove the inductive construction for the bases of our second matroid by using five operations and four types of base graphs. In this case we may need to handle graphs of minimum degree four and hence we need more operations and longer arguments. To make the paper more readable, the lengthy case, when the graph is four-regular, is moved to the end of the paper, to Section 9. In Section 8 we prove additional geometic lemmas and use them, together with the inductive construction of Section 7, to prove the second main result, the characterization of rigid frameworks with dihedral symmetry of odd order. We also present frameworks that meet the sparsity requirements but are dependent and flexible when the underlying dihedral group has even order. In Section 10 we briefly discuss the algorithmic implications and make some further remarks.

In the rest of the introduction, let us introduce notations used thoughout the paper.

Let $E$ be a finite set. A *partition* $\mathcal{P}$ of $E$ is a family of nonempty subsets of $E$ such that each element of $E$ belongs to exactly one member of $\mathcal{P}$. If $E = \emptyset$, the partition of $E$ is defined as the empty set. A *subpartition* of $E$ is a partition of a subset of $E$.

Let $G = (V, E)$ be an undirected graph. For $v \in V$, let $d_G(v)$ be the degree of $v$ in $G$ and $N_G(v)$ be the set of neighbors of $v$ in $G$. For $F \subseteq E$, $V_G(F)$ denotes the set of endvertices of edges in $F$, and let $G[F] = (V(F), F)$, that is, the graph edge-induced by $F$. If the graph is clear from the context, the subsript $G$ may be dropped. For $F \subseteq E$ and $v \in V(F)$, let $d_F(v) = d_{G[F]}(v)$.

A vertex subset $X \subset V(G)$ (resp., an edge subset $X \subset E(G)$ is called a *separator* (resp., a *cut*) if the removal of $X$ disconnects $G$. A separator $X$ with $|X| = 1$ is called a *cut-vertex*. $G$ is called *$k$-connected* (resp., *$k$-edge-connected*) if the size of any separator (resp., any cut) is at least $k$. A separator (resp., a cut) is called *nontrivial* if its removal disconnects $G$ into at least two nontrivial connected components, where a connected component is called trivial if it consists of a single vertex. $G$ is called *essentially $k$-connected* (resp., *essentially $k$-edge-connected*) if the size of any nontrivial separator (resp., any nontrivial cut) is at least $k$.

For simplicity, some properties of edge-induced subgraphs will be associated with the corresponding edge sets as follows. Let $F \subseteq E$. $F$ is called *connected* if $G[F]$ is connected. A *connected component* of $F$ is the edge set of a connected component of $G[F]$. $C(F)$ denotes the partition of $F$ into connected components of $F$, and let $c(F) = |C(F)|$. $F$ is called a *forest* if it contains no cycle and called a *tree* if it is a connected forest. $F$ is called a *spanning tree* of a graph $G = (V, E)$ if $F$ is a tree with $F \subseteq E$ and $V(F) = V$.

Let $G = (V, E)$ be a directed graph. A *walk* in $G$ is a sequence $W = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ of vertices and edges such that $v_{i-1}$ and $v_i$ are the endvertices of $e_i$ for every $1 \leq i \leq k$. The reversed walk of $W$ is $W^{-1} = v_k, e_k, \ldots, e_1, v_0$.

We often denote a walk as a sequence of edges implicitly assuming the incidence at each vertex. For two walks $W$ and $W'$ for which the end vertex of $W$ and the starting vertex of $W'$ coincide, we denote the concatenation of $W$ and $W'$ (that is, the walk $W$ followed by $W'$) by $W * W'$. A walk is called *closed* if the starting vertex and the end vertex coincide.

It is sometimes convenient to regard the empty set as a subgroup of a group. Let $\mathcal{D}$ be a dihedral group. For a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$, $\bar{\mathcal{C}}$ denotes the maximal cyclic subgroup containing $\mathcal{C}$.

# 2 Gain Graphs

In this section we shall review some basic properties of gain graphs. We refer the reader to [11, 35, 36] for more details.

Let $G = (V, E)$ be a directed graph which may contain multiple edges and loops, and let $\mathcal{S}$ be a group. An $\mathcal{S}$*-gain graph* $(G, \phi)$ is a pair, in which each edge is associated with an element of $\mathcal{S}$ by a *gain function* $\phi : E \to \mathcal{S}$. See Figure 1 for an example. The orientation of $G$ is, in some sense, arbitrary, and is used only as a reference orentation: the orientation of each edge may be changed, provided that we also modify $\phi$ such that if the edge has gain $g$ in one direction then it has gain $g^{-1}$ in the other direction. Therefore we often do not distinguish between $G$ and the underlying undirected graph and use notations introduced in §1, implicitly referring to the underlying graph.

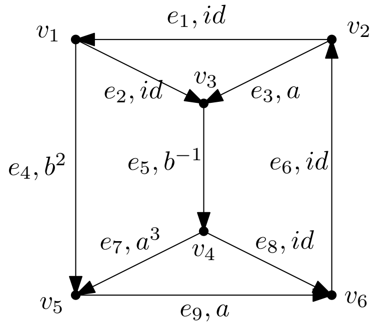



Figure 1: An example of an $\mathcal{S}$-gain graph, where $\mathcal{S}$ is a group generated by $a$ and $b$.

Let $W$ be a walk in $(G, \phi)$. The *gain* of $W$ is defined as $\phi(W) = \phi(e_1) \cdot \phi(e_2) \cdots \phi(e_k)$ if each edge is oriented in the forward direction through $W$, and for a backward edge $e_i$ we replace $\phi(e_i)$ with $\phi(e_i)^{-1}$ in the product. For example, in Figure 1, $W = e_2, e_5, e_7, e_4$ is a closed walk starting at $v_1$ and its gain is $b^{-1}a^3b^{-2}$. Note that $\phi(W^{-1}) = \phi(W)^{-1}$.

Let $(G, \phi)$ be a gain graph. For $v \in V(G)$ we denote by $\pi_1(G, v)$ the set of closed walks starting at $v$. Similarly, for $X \subseteq E(G)$ and $v \in V(G)$, $\pi_1(X, v)$ denotes the set of closed walks starting at $v$ and using only edges of $X$, where $\pi_1(X, v) = \emptyset$ if $v \notin V(X)$.

Let $X \subseteq E(G)$. The subgroup induced by $X$ relative to $v$ is defined as $\langle X \rangle_{\phi,v} = \{\phi(W) \mid W \in \pi_1(X, v)\}$. The subscript $\phi$ of $\langle X \rangle_{\phi,v}$ is sometimes omitted if it is clear from the context.

**Proposition 2.1.** *For any connected $X \subseteq E(G)$ and two vertices $u, v \in V(X)$, $\langle X\rangle_u$ is conjugate to $\langle X\rangle_v$.*

*Proof.* Since $X$ is connected, there is a path $P$ starting at $u$ and ending at $v$. Then, for all $W \in \pi_1(X, u)$, $P^{-1} * W * P \in \pi_1(X, v)$ and hence $\phi(P)^{-1}\phi(W)\phi(P) \in \langle X\rangle_v$. □

## 2.1 The switching operation

For $v \in V(G)$ and $g \in \mathcal{S}$, a *switching operation at $v$ with $g$* changes the gain function $\phi$ on $E(G)$ as follows.

$$\phi'(e) = \begin{cases} g \cdot \psi(e) \cdot g^{-1} & \text{if } e \text{ is a loop incident with } v \\ g \cdot \phi(e) & \text{if } e \text{ is a non-loop edge and is directed from } v \\ \phi(e) \cdot g^{-1} & \text{if } e \text{ is a non-loop edge and is directed to } v \\ \phi(e) & \text{otherwise.} \end{cases} \tag{1}$$

We say that a gain function $\phi'$ on edge set $E(G)$ is *equivalent* to another gain function $\phi$ on $E(G)$ if $\phi'$ can be obtained from $\phi$ by a sequence of switching operations.

**Proposition 2.2.** *Let $(G, \phi)$ be a gain graph. Let $\phi'$ be the gain function obtained from $\phi$ by a switching operation. Then, for any $X \subseteq E(G)$ and $u \in V(G)$, $\langle X\rangle_{\phi',u}$ is conjugate to $\langle X\rangle_{\phi,u}$.*

*Proof.* Suppose the switching operation is performed at $v \in V(G)$ with $g \in \mathcal{S}$. Notice that $\phi'(e)\phi'(f) = \phi(e)\phi(f)$ for any incoming edge $e$ to $v$ and any outgoing edge $f$ from $v$. Also, $\phi'(e) = \phi(e)$ for any edge $e$ not incident to $v$. Hence, for any closed walk $W$ starting at $u \in V(G)$, we have $\phi'(W) = \phi(W)$ if $u \neq v$ and $\phi'(W) = g \cdot \phi(W) \cdot g^{-1}$ if $u = v$. Thus, for any $X \subseteq E(G)$, we have $\langle X\rangle_{\phi',u} = \langle X\rangle_{\phi,u}$ if $u \neq v$, and $\langle X\rangle_{\phi',v} = g \cdot \langle X\rangle_{\phi,v} \cdot g^{-1}$ if $u = v$. □

Proposition 2.2 implies the following useful property.

**Proposition 2.3.** *Let $(G, \phi)$ be a gain graph. Then, for any forest $F \subseteq E(G)$, there is a gain function $\phi'$ equivalent to $\phi$ such that $\phi'(e) = \mathrm{id}$ for every $e \in F$.*

*Proof.* Suppose that $G$ is connected. Let $T$ be a spanning tree of $G$ with $F \subseteq T$. Take a vertex $u$ as a root and consider $T$ as a rooted tree, i.e., edges of $T$ are oriented from the root to descendants. We then perform switching operations from the root to descendants so that $\phi(e) = \mathsf{id}$ for $e \in T$.

More precisely, we first take a child $v$ of the root $u$ and perform a switching operation at $v$ with $\phi(uv)$. We then take a child $w$ of $v$ and perform a switching operation at $w$ with $\phi''(vw)$, where $\phi''$ is the gain function obtained by the first switching operation. We perform this process from the root to all leaves. Each operation makes the gain of an edge $e$ of $T$ identity, and after that the gain of $e$ is never changed. Therefore, for the final gain function $\phi'$, we have $\phi'(e) = \mathsf{id}$ for all $e \in T$. See Figure 2 for an example.

If $G$ is not connected, we can apply this argument to each connected component. □

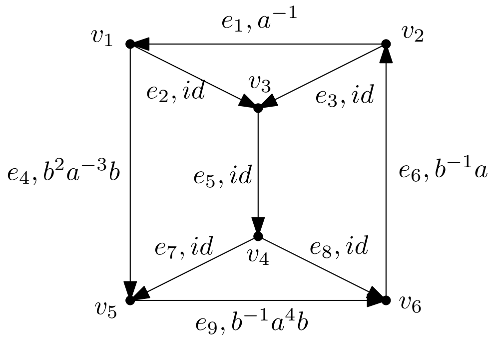



Figure 2: An equivalent gain function for the graph of Figure 1, where the gain of each edge in $T = \{e_2, e_3, e_5, e_7, e_8\}$ is identity.

## 2.2 Balanced and cyclic sets of edges

As we shall see, the subgroup $\langle X\rangle_v$ itself will not be important, when we define our matroids induced by gains. We only need to know whether $\langle X\rangle_v$ is trivial or not, or whether it is cyclic or not. We now introduce notions to describe these properties.

Let $(G, \phi)$ be a gain graph. A connected edge subset $F \subseteq E(G)$ is called *balanced* if $\langle F\rangle_v = \{\text{id}\}$ for some $v \in V(F)$. $F$ is called *unbalanced* if it is not balanced. By Proposition 2.1, this property is invariant under the choice of the base vertex $v \in V(F)$, and $F$ is unbalanced if and only if $F$ contains an unbalanced cycle. Thus we can extend this notion to any (possibly disconnected) $F \subseteq E(G)$, and say that $F$ is *unbalanced* if and only if $F$ contains an unbalanced cycle.

In the same way, a connected edge subset $F \subseteq E(G)$ is called *cyclic* if $\langle F\rangle_v$ is a cyclic subgroup of $\mathcal{S}$ for some $v \in V(F)$. (Note that the terms balanced and cyclic are not exclusive.) As above, this property is invariant under the choice of the base vertex $v \in V(F)$. However, $F$ may be cyclic even if $F$ contains two closed walks $W_1$ and $W_2$ such that the group generated by $\phi(W_1)$ and $\phi(W_2)$ is not cyclic, if the starting vertices of the walks are distinct. In general, a (possibly disconnected) edge subset $F \subseteq E(G)$ is called *cyclic* if every connected component of $F$ is cyclic.

A gain graph $(G, \phi)$ is called *balanced* and *cyclic* if $E(G)$ is balanced and cyclic, respectively.

Consider two closed walks $W, W' \in \pi_1(F, v)$ for which $W$ first walks through a path $P$ starting from $v$ and $W'$ walks through $P^{-1}$ at the end as shown in Figure 3. If we omit $P^{-1}$ and $P$ in $W' * W$, we obtain a closed walk with base vertex $v$. This walk is denoted by $W' \circ W$. Notice that

$$\phi(W' \circ W) = \phi(W') \cdot \phi(W). \tag{2}$$

Proposition 2.3 suggests a simple way to check the above introduced properties of $X$, in analogy with the fact that the cycle space of a graph is spanned by fundamental cycles. For a connected $X \subseteq E(G)$, take a spanning tree $T$ of the edge induced graph $G[X]$. By Proposition 2.3 we can convert the gain function to an equivalent gain function such that $\phi(e) = \text{id}$ for all $e \in T$. Then, observe that any closed walk $W \in \pi_1(X, v)$ can be decomposed into $W = W_1 \circ W_2 \circ \cdots \circ W_k$ such that $W_i$ is a closed

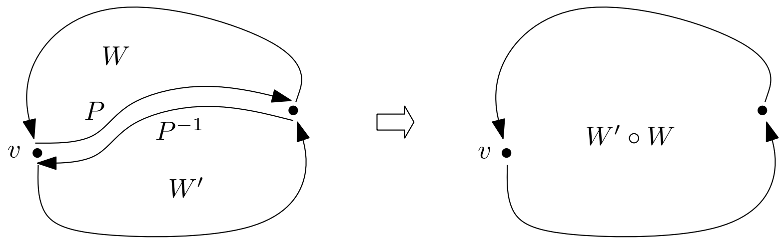



Figure 3

walk in $\pi_1(X, v)$ that passes through only one edge of $X \setminus T$. To see this, denote $W$ by $W = v_1v_2, v_2v_3, \ldots, v_kv_{k+1}$, and let $W_i = P_i * \{v_iv_{i+1}\} * P_{i+1}^{-1}$ for $1 \leq i < k$, where $P_i$ denotes the path from $v$ to $v_i$ in $T$.

By (2) and $\phi(e) = \mathrm{id}$ for all $e \in T$, we deduce that $\phi(W)$ is a product of elements in $\{\phi(e) : e \in X \setminus T\}$, implying that $\langle X \rangle_{\phi,v} \subseteq \langle \phi(e) : e \in X \setminus T \rangle$, where $\langle \phi(e) : e \in X \setminus T \rangle$ is the group generated by $\{\phi(e) : e \in X \setminus T\}$. Conversely, $\phi(e)$ is contained in $\langle X \rangle_{\phi,v}$ for all $e \in X \setminus T$. Thus, $\langle X \rangle_{\phi,v} = \langle \phi(e) : e \in X \setminus T \rangle$. In particular, we proved the following.

**Lemma 2.4.** *For a connected $X \subseteq E(G)$ and a spanning tree $T$ of $G[X]$, suppose that $\phi(e) = \mathrm{id}$ for all $e \in T$. Then, $\langle X \rangle_{\phi,v} = \langle \phi(e) \colon e \in X \setminus T \rangle$. In particular, the following hold.*

**(i)** *$X$ is unbalanced if and only if there is an edge in $X \setminus T$ whose gain is non-identity.*

**(ii)** *$X$ is cyclic if and only if all gains of $X \setminus T$ are contained in a cyclic subgroup of $\mathcal{S}$.*

The following technical lemmas will be used in the proof of our main theorem.

**Lemma 2.5.** *Let $(G, \phi)$ be a $\mathcal{S}$-gain graph, and $X$ and $Y$ be connected edge subsets such that the graph $(V(X) \cap V(Y), X \cap Y)$ is connected.*

**(1)** *If $X$ and $Y$ are balanced, then $X \cup Y$ is balanced.*

**(2)** *If $X$ is balanced and $Y$ is cyclic, then $X \cup Y$ is cyclic.*

**(3)** *If $X, Y$ are cyclic and $X \cap Y$ is unbalanced, then $X \cup Y$ is cyclic, provided that for every non-trivial cyclic subgroup $\mathcal{C}$ of $\mathcal{S}$ there is a unique largest cyclic subgroup $\bar{\mathcal{C}}$ of $\mathcal{S}$ containing $\mathcal{C}$.*

*Proof.* Since the graph $(V(X) \cap V(Y), X \cap Y)$ is connected, there is a spanning tree $T$ in $G[X \cup Y]$ such that $T \cap X$ is a spanning tree of $G[X]$, $T \cap Y$ is a spanning tree of $G[Y]$, and $T \cap X \cap Y$ is a spanning tree of $G[X \cap Y]$. By Proposition 2.3, there is a gain function $\phi'$ equivalent to $\phi$ such that $\phi'(e) = \mathsf{id}$ for each $e \in T$.

If $X$ and $Y$ are balanced, Lemma 2.4 implies that $\phi'(e) = \mathsf{id}$ for all $e \in X \cup Y$. Thus (1) holds.

If $X$ is balanced, then every label in $X \cup Y$ is contained in $\langle Y \rangle_{\phi',v}$ by Lemma 2.4, and hence $X \cup Y$ is cyclic if $Y$ is cyclic. This implies (2).

If $X, Y$ are cyclic and $X \cap Y$ is unbalanced, then there is an edge $e \in X \cap Y$ for which $\phi'(e)$ is non-identity. Let $\mathcal{C}$ be a cyclic subgroup of $\mathcal{S}$ generated by $\phi'(e)$ and $\bar{\mathcal{C}}$ be the largest cyclic subgroup containing $\mathcal{C}$. Since $X$ and $Y$ are cyclic, Lemma 2.4 implies that $\phi'(e) \in \bar{\mathcal{C}}$ holds for every $e \in X$ and for every $e \in Y$. Therefore $X \cup Y$ is cyclic. □

**Lemma 2.6.** *Let $(G, \phi)$ be a gain graph, and $X$ and $Y$ be connected balanced edge subsets. If the number of connected components of the graph $(V(X) \cap V(Y), X \cap Y)$ is two, then $X \cup Y$ is cyclic.*

*Proof.* We take a spanning tree $T$ of $G[X \cup Y]$ such that $T \cap X$ is a spanning tree of $G[X]$. Since the number of connected components of $(V(X) \cap V(Y), X \cap Y)$ is two, $T \cap Y$ consists of two connected components, denoted $T_1$ and $T_2$. $\{V(T_1), V(T_2)\}$ partitions $Y$ into three subsets $\{Y_1, Y_2, Y_3\}$ such that $Y_i = \{e \in Y : V(\{e\}) \subseteq V(T_i)\}$ for $i = 1, 2$ and $Y_3 = Y \setminus (Y_1 \cup Y_2)$.

By Proposition 2.3, we can take a gain function $\phi'$ equivalent to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in T$. Since $X$ and $Y$ are balanced, we have $\phi'(e) = \mathsf{id}$ for $e \in X \cup Y_1 \cup Y_2$. Moreover, assuming that every edge in $Y_3$ is oriented toward $V(Y_1)$, we have $\phi'(e) = \phi'(f)$ for all $e, f \in Y_3$, since otherwise $T_1 \cup T_2 \cup \{e, f\}$ contains an unbalanced cycle, contradicting the fact that $Y$ is balanced. Therefore $X \cup Y$ is cyclic. □

*Remark* 2.1. By Proposition 2.1, for each $X \subseteq E(G)$, the property of being balanced is invariant under the choice of the base vertex $v \in V(X)$ and hence is simply determined by the *homology* of $X$ rather than $\pi_1(X, v)$, see e.g., [35]. For the property of being cyclic or non-cyclic, we need $\pi_1(X, v)$.

# 3 Gain Count Matroids

## 3.1 Matroids induced by submodular functions

Let $E$ be a finite set. A function $\mu : 2^E \to \mathbb{R}$ is called *submodular* if $\mu(X) + \mu(Y) \geq \mu(X \cup Y) + \mu(X \cap Y)$ for every $X, Y \subseteq E$. $\mu$ is *monotone* if $\mu(X) \leq \mu(Y)$ for any $X \subseteq Y$. A monotone submodular function $\mu : 2^E \to \mathbb{Z}$ *induces* a matroid on $E$, where $F \subseteq E$ is independent if and only if $|I| \leq \mu(I)$ for every nonempty $I \subseteq F$. See e.g. [9, Section 13.4]. This matroid is denoted by $\mathcal{M}(\mu)$.

For a monotone submodular function $\mu$, let $\nu = \mu - 1$. Then, $\nu$ is monotone submodular and induces the matroid $\mathcal{M}(\nu)$. This matroid is referred to as the *Dilworth truncation* of $\mathcal{M}(\mu)$. Although the details are omitted here, the name of Dilworth truncation is justified from a connection with Dilworth truncation for general matroids, see [9, 23] for more details.

Now we consider the union of two matroids induced by monotone submodular functions $\mu_1$ and $\mu_2$. Since monotonicity and submodularity are both preserved under the sum operation, $\mu_1 + \mu_2$ is monotone and submodular. In general, the union of $\mathcal{M}(\mu_1)$ and $\mathcal{M}(\mu_2)$ is not equal to $\mathcal{M}(\mu_1 + \mu_2)$. We do have equality in some special cases, for example, when $\mu_1 = \mu_2$ or when both $\mu_1$ and $\mu_2$ are nonnegative.

As an example, consider the union of two copies of the graphic matroid of a graph $G = (V, E)$. It is the matroid induced by $f_{2,2}$ defined by $f_{2,2}(F) = 2|V(F)| - 2$ on $E$, as $f_{2,2}/2$ induces the graphic matroid on $G$. The 2-dimensional generic rigidity matroid is the one induced by $f_{2,2} - 1$, and hence it is the Dilworth truncation of the union of two copies of the graphic matroid.

In general, for a graph $G = (V, E)$ and two integers $k$ and $l$ with $k \geq 1$ and $l \leq 2k - 1$, let

$$f_{k,l}(F) = k|V(F)| - l \qquad (F \subseteq E).$$

$G$ is called *$(k, l)$-sparse* if $|F| \leq f_{k,l}(F)$ for any nonempty $F \subseteq E$. The matroid induced by $f_{k,l}$ is called the *$(k, l)$-count matroid* on $G$. If $l \geq 0$, $\mathcal{M}(f_{k,l})$ is indeed the one induced by $f_{k,0}$, truncated $l$ times. See e.g. [9] for more detail. Below we shall apply the same construction to the union of some copies of a frame matroid to define gain-count matroids.

## 3.2 Gain-count matroids

Let $\Theta$ be the graph with two vertices $u$ and $v$ and three parallel edges. A subdivision of $\Theta$ is called a *theta graph*. So a theta graph consists of three openly disjoint paths between $u$ and $v$, and contains three cycles.

Let $G = (V, E)$ be an undirected multigraph which may contain loops and parallel edges. A family $\mathcal{C}$ of cycles of $G$ is called a *linear class* if it satisfies the following property: if two cycles in $\mathcal{C}$ form a theta subgraph, then the third cycle of the theta subgraph is also contained in $\mathcal{C}$. For a graph $G = (V, E)$ and a linear class $\mathcal{C}$ of cycles, the *frame matroid* (sometimes called *bias matroid*) $\mathcal{F}(G, \mathcal{C})$ is defined such that $F \subseteq E$ is independent if and only if each connected component of $F$ contains no cycle or just one cycle, which is not included in the linear class $\mathcal{C}$ [36]. Therefore, the rank of $F \subseteq E$ in $\mathcal{F}(G, \mathcal{C})$ is equal to

$$g_{\mathcal{C}}(F) := \sum_{F_i \in C(F)} (|V(F_i)| - 1 + \alpha_{\mathcal{C}}(F_i))$$

where

$$\alpha_{\mathcal{C}}(F) = \begin{cases} 1 & \text{if } F \text{ contains a cycle not included in } \mathcal{C} \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $g_{\mathcal{C}}$ is monotone and submodular.

In this paper we shall consider frame matroids on gain graphs. For a group $\mathcal{S}$ and an $\mathcal{S}$-gain graph $(G, \phi)$, let $\mathcal{C}$ be the set of balanced cycles. It is easy to check that $\mathcal{C}$ forms a linear class, and the associated frame matroid can be defined with respect to $\mathcal{C}$. This matroid is called the *frame matroid of* $(G, \phi)$ [36]. If we define $g_{\mathcal{S}} : 2^E \to \mathbb{Z}$ by

$$g_{\mathcal{S}}(F) = \sum_{F_i \in C(F)} (|V(F_i)| - 1 + \alpha_{\mathcal{S}}(F_i)) \tag{3}$$

where

$$\alpha_{\mathcal{S}}(F) = \begin{cases} 1 & \text{if } F \text{ is unbalanced} \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

then the frame matroid is the matroid induced by $g_{\mathcal{S}}$. We omit the subscript $\mathcal{S}$ from $\alpha_{\mathcal{S}}$ if it is clear from the context.

For an $\mathcal{S}$-gain graph and two positive integers $k$ and $l$ with $k \leq l$, we define $g_{k,l} : 2^E \to \mathbb{Z}$ by

$$g_{k,l}(F) = kg_{\mathcal{S}}(F) - (l-k). \tag{5}$$

We call the matroid $\mathcal{M}(g_{k,l})$ induced by $g_{k,l}$ a $(k,l)$*-gain-count matroid* or *g-count matroid* for short. This matroid is the union of $k$ copies of the frame matroid, followed by $l-k$ Dilworth truncations. In this paper, we shall investigate the $(2,3)$-g-count matroid and its variants.

The independence of $\mathcal{M}(g_{k,l})$ can be described in a compact form.

**Lemma 3.1.** *Let $(G,\phi)$ be an $\mathcal{S}$-gain graph. Then $E(G)$ is independent in $\mathcal{M}(g_{k,l})$ if and only if $|F| \leq k|V(F)| - l + k\alpha(F)$ for any nonempty $F \subseteq E(G)$.*

*Proof.* "If"-part: Suppose that $|F| \leq k|V(F)| - l + k\alpha(F)$ for any nonempty $F \subseteq E$. Note that $g_{k,l}(F) = k|V(F)| - l + k\alpha(F)$ if $F$ is connected. Thus, for any $F \subseteq E$, we have $|F| = \sum_{F_i \in C(F)} |F_i| \leq \sum_{F_i \in C(F)} (k|V(F_i)| - l + k\alpha(F_i)) = \sum_{F_i \in C(F)} k(|V(F_i)| - 1 + \alpha(F_i)) - (l-k)c(F) \leq g_{k,l}(F)$ by $c(F) \geq 1$. Thus $E(G)$ is independent in $\mathcal{M}(g_{k,l})$.

"Only if"-part: If $E(G)$ is independent in $\mathcal{M}(g_{k,l})$, then for each connected $F$ we have $|F| \leq g_{k,l}(F) = k|V(F)| - l + k\alpha(F)$. Therefore,

$$|F| = \sum_{F_i \in C(F)} |F_i| \leq k|V(F)| - lc(F) + k \sum_{F_i \in C(F)} \alpha(F_i). \tag{6}$$

Since $\alpha$ is a monotone 0-1 valued function, $\sum_{F_i \in C(F)} \alpha(F_i) \leq \alpha(F) + c(F) - 1$. Combining this with (6) and $k \leq l$, we get $|F| \leq k|V(F)| - lc(F) + k\alpha(F) + k(c(F)-1) \leq k|V(F)| - l + k\alpha(F)$ for any nonempty $F \subseteq E(G)$. ☐

In this sense, we may define $(k,l)$*-gain-sparsity* as in the case of $(k,l)$-sparsity of undirected graphs as follows.

**Definition 3.1.** Let $k$ and $l$ be positive integers with $k \leq l$ and $(G,\phi)$ be an $\mathcal{S}$-gain graph with a graph $G = (V,E)$ and a group $\mathcal{S}$. An edge set $X \subseteq E$ is called $(k,l)$*-gain-sparse* (or $(k,l)$*-g-sparse* for short) if $|F| \leq g_{2,3}(F)$ for any nonempty $F \subseteq X$, i.e.,

- $|F| \leq k|V(F)| - l$ for every nonempty balanced $F \subseteq X$;
- $|F| \leq k|V(F)| - l + k$ for every nonempty unbalanced $F \subseteq X$,

and it is clalled $(k,l)$*-gain-tight* (or $(k,l)$*-g-tight* for short) if it is $(k,l)$-g-sparse with $|X| = g_{k,l}(X)$.

$(G,\phi)$ is called $(k,l)$*-g-sparse* if so is $E(G)$, and it is called *maximum* $(k,l)$*-g-tight* if it is $(k,l)$-g-sparse with $|E(G)| = k|V(G)| - l + k$.

*Remark* 3.1. Note that the value of $g_{k,l}$ is invariant under switching operations, and thus the induced matroid is uniquely determined up to equivalence of gain functions.

*Remark* 3.2. We can further consider the union of frame matroids of gain graphs $(G,\phi_1)$ and $(G,\phi_2)$ with the same underlying graph but distinct gain functions. We should remark that both graphic matroids and bicircular matroids are special cases of frame matroids. The union of copies of graphic, frame and bicircular matroids on an $\mathcal{S}$-gain graph, followed by Dilworth truncations, can be described as the matroid induced by a counting condition. For example, in the union of the graphic matroid and the frame matroid of a gain graph $(G,\phi)$, followed by a single Dilworth truncation, $E(G)$ is independent if and only if $|F| \leq 2|V(F)|-3$ for any balanced set $F \subseteq E(G)$ and $|F| \leq 2|V(F)|-2$ for any nonempty $F \subseteq E(G)$. This matroid was used by Ross [22] for characterizing the generic rigidity of bar-joint frameworks on a torus. In [29] Tanigawa proposed a more general class of gain graphs extending matroid union operations.

# 4 Constructive Characterization of Maximum $(2,3)$-g-tight Graphs

## 4.1 Operations preserving $(2,3)$-g-sparsity

In this section we define three operations, called *extensions*, that preserve $(2,3)$-g-sparsity. The first two operations generalize the well-known Henneberg operations [30, 32] to gain graphs.

Let $(G,\phi)$ be an $\mathcal{S}$-gain graph. The *0-extension* adds a new vertex $v$ and two new non-loop edges $e_1$ and $e_2$ to $G$ such that the new edges are incident to $v$ and the other endvertices are two not necessarily distinct vertices of $V(G)$. If $e_1$ and $e_2$ are not parallel then their labels can be arbitrary. Otherwise the labels are assigned such that $\phi(e_1) \neq \phi(e_2)$, assuming that $e_1$ and $e_2$ are directed to $v$.

The *1-extension* first chooses an edge $e$ and a vertex $z$, where $e$ may be a loop and $z$ may be an endvertex of $e$. It subdivides $e$, with a new vertex $v$ and new edges $e_1, e_2$ such that the tail of $e_1$ is the tail of $e$ and the tail of $e_2$ is the head of $e$. The labels of the new edges are assigned such that $\phi(e_1)\cdot\phi(e_2)^{-1} = \phi(e)$. The 1-extension also adds a third edge $e_3$ oriented to $v$. The label of $e_3$ is assigned so that it is *locally unbalanced*, i.e., every two-cycle $e_ie_j$, if exists, is unbalanced.

The *loop 1-extension* adds a new vertex $v$ to $G$ and connects it to a vertex $z \in V(G)$ by a new edge with any label. It also adds a new loop $l$ incident to $v$ with $\phi(l) \neq \mathrm{id}$.

**Lemma 4.1.** *Let $(G,\phi)$ be a (2,3)-g-sparse $\mathcal{S}$-gain graph. Applying the 0-extension, 1-extension or loop 1-extension to $G$ results in a (2,3)-g-sparse graph $(G',\phi')$ with $|V(G')| = |V(G)|+1$ and $|E(G')| = |E(G)|+2$.*

*Proof.* For a contradiction, suppose that $G'$ contains an edge set $F \subseteq E(G')$ for which $|F| > 2|V(F)|-3+2\alpha(F)$. Let $v$ be the new vertex added by the extension, and let $E_v$ be the set of edges incident to $v$. Since $E(G')\setminus E_v \subseteq E(G)$, $E_v \cap F \neq \emptyset$. In particular, $v \in V(F)$. Also, since the new labeling is assigned to be locally unbalanced, $F$ is not contained in $E_v$.

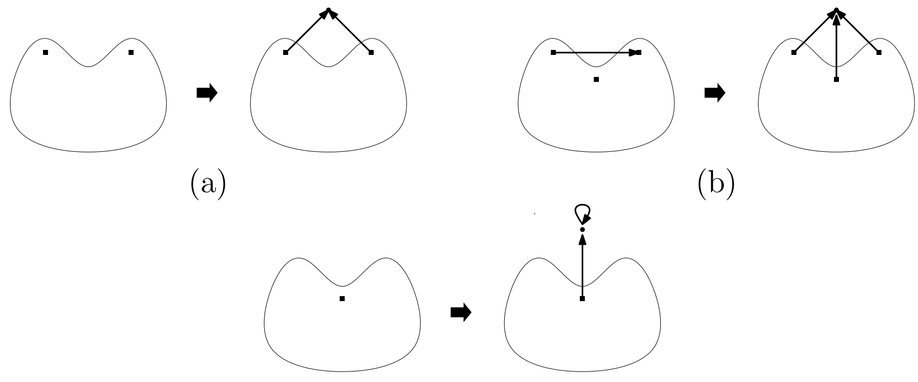


Figure 4: (a) 0-extension, where the new edges may be parallel. (b) 1-extension, where the removed edge may be a loop and the new edges may be parallel. (c) loop-1-extension.

If $G'$ is constructed by a 1-extension then let $e$ be the subdivided edge of $G$ and let $e_1$ and $e_2$ be the resulting two new edges.

Let $F' = F \setminus E_v$. If $G'$ is constructed by a 1-extension and $\{e_1, e_2\} \subseteq F$, then we further insert $e$ to $F'$. We then have $|F'| \geq |F| - 2$, $|V(F')| = |V(F)| - 1$, and $\alpha(F') \leq \alpha(F)$ in each case. These imply $|F'| \geq |F| - 2 > 2|V(F)| - 5 + 2\alpha(F) \geq 2|V(F')| - 3 + 2\alpha(F')$, contradicting the $(2, 3)$-g-sparsity of $G$ as $\emptyset \neq F' \subseteq E(G)$. □

We shall define the inverse moves of the operations above, which are called *reductions*. For a vertex $v$ and two incoming non-loop edges $e_1 = (u, v)$ and $e_2 = (w, v)$, we denote by $e_1 \cdot e_2^{-1}$ a new edge from $u$ to $w$ with label $\phi(e_1) \cdot \phi(e_2)^{-1}$ (by extending $\phi$). If $u = w$ then $e_1 \cdot e_2^{-1}$ is a loop. Each reduction corresponds to one of the following operations on a gain graph $(G, \phi)$.

The *0-reduction* chooses a degree two vertex and deletes it from $G$.

The *1-reduction* chooses a vertex $v$ with $d(v) = 3$ that is not incident to a loop. Let $e_1, e_2, e_3$ be the edges incident to $v$. Without loss of generality we may assume that each $e_i$ is oriented to $v$. The 1-reduction deletes $v$ with the incident edges and adds one of $e_1 \cdot e_2^{-1}$, $e_2 \cdot e_3^{-1}$ and $e_3 \cdot e_1^{-1}$ as a new edge.

The *loop 1-reduction* chooses a vertex incident to exactly one loop and one non-loop edge and deletes the chosen vertex with the incident edges.

A 1-reduction may destroy the $(2, 3)$-g-sparsity of a graph. We say that a reduction (at a vertex $v$) is *admissible* if the resulting graph is (2,3)-g-sparse.

## 4.2 Constructive characterization

**Lemma 4.2.** *Let $(G, \phi)$ be a (2,3)-g-sparse graph and $v \in V(G)$ a vertex not incident to a loop with $d(v) = 3$. Then there is an admissible 1-reduction at $v$.*

*Proof.* Let $E = E(G)$, $G' = G - v$ and $E' = E(G')$. Let $e_1, e_2, e_3$ be the edges incident to $v$ in $G$. Without loss of generality we may assume that each $e_i$ is oriented to $v$. For simplicity we put $e_{i,j} = e_i \cdot e_j^{-1}$.

Suppose for a contradiction that there is no admissible splitting at $v$, that is, none of $E' + e_{1,2}$, $E' + e_{2,3}$ and $E' + e_{3,1}$ is independent in $\mathcal{M}(g_{2,3})$. Equivalently, $e_{1,2}, e_{2,3}, e_{3,1} \in \mathrm{cl}_g(E')$, where $\mathrm{cl}_g$ denotes the closure operator of $\mathcal{M}(g_{2,3})$. Let $X = \{e_1, e_2, e_3, e_{1,2}, e_{2,3}, e_{3,1}\}$.

**Claim 4.3.** $e_1 \in \mathrm{cl}_g(X - e_1)$.

*Proof.* We split the proof into three cases depending on the cardinality of $N(v)$.

If $|N(v)| = 3$ then, by Proposition 2.3, we may assume $\phi(e_1) = \phi(e_2) = \phi(e_3) = \mathrm{id}$. We then have $\phi(e_{1,2}) = \phi(e_{2,3}) = \phi(e_{3,1}) = \mathrm{id}$. Therefore $X$ forms a balanced $K_4$, which is a circuit of $\mathcal{M}(g_{2,3})$. Thus, $e_1 \in \mathrm{cl}_g(X - e_1)$ holds.

If $|N(v)| = 2$ then we may assume that $e_1$ and $e_2$ are parallel. By Proposition 2.3, we may assume that $\phi(e_2) = \phi(e_3) = \mathsf{id}$. This implies $\phi(e_{1,3}) = \phi(e_1)$ and $\phi(e_{2,3}) = \mathsf{id}$. Since $G$ is $(2,3)$-g-sparse, we have $\phi(e_1) \neq \mathsf{id}$ by $\phi(e_2) = \phi(e_3) = \mathsf{id}$, which implies that $e_{1,2}$ is an unbalanced loop with $\phi(e_{1,2}) = \phi(e_1)$. Thus, it can be easily checked, by counting, that $X$ is indeed a circuit in $\mathcal{M}(g_{2,3})$. Thus, $e_1 \in \mathrm{cl}_g(X - e_1)$ holds.

If $|N(v)| = 1$ then let $X' = \{e_1, e_2, e_3, e_{1,2}\}$. We have $|X'| = 2|V(X')|$ and $X'$ is a circuit of $\mathcal{M}(g_{2,3})$. Therefore $e_1 \in \mathrm{cl}_g(X' - e_1) \subset \mathrm{cl}_g(X - e_1)$. ☐

Since $e_{1,2}, e_{2,3}, e_{3,1} \in \mathrm{cl}_g(E')$, by Claim 4.3, we have $e_1 \in \mathrm{cl}_g(X - e_1) \subseteq \mathrm{cl}_g(E' + X - e_1) = \mathrm{cl}_g(E' + e_2 + e_3) = \mathrm{cl}_g(E - e_1)$, which contradicts the $(2,3)$-g-sparsity of $G$. ☐

We are now ready to show a constructive characterization of maximum $(2,3)$-g-tight graphs.

**Theorem 4.4.** *An $\mathcal{S}$-gain graph $(G, \phi)$ is maximum (2,3)-g-tight if and only if it can be built up from an $\mathcal{S}$-gain graph with one vertex and an unbalanced loop incident to it with a sequence of 0-extensions, 1-extensions, and loop-1-extensions.*

*Proof.* By Lemma 4.1, by applying any of the extension operations we obtain a maximum (2,3)-g-tight graph from a maximum $(2,3)$-g-tight graph.

To prove the other direction it is sufficient to show that $G$ can be reduced to a smaller $(2,3)$-g-tight graph. Since $|E(G)| = 2|V(G)| - 1$, the average degree is less than 4, which implies that there is a vertex $v$ of degree at most 3. If $d(v) = 2$, the 0-reduction can be applied at $v$ which is always admissible. If $d(v) = 3$, we have two cases depending on whether $v$ is incident to a loop or not. If $v$ is incident to a loop, the loop-1-reduction, which is always admissible, can be applied at $v$ to obtain a smaller $(2,3)$-g-tight graph. Otherwise, by Lemma 4.2, there is an admissible 1-reduction at $v$. ☐

# 5 Symmetry-forced Rigidity

In this section we define the notion of symmetry-forced infinitesimal rigidity, introduced by Schulze and Whiteley [27]. In §5.1, we first introduce $\mathcal{S}$-symmetric graphs, whose automorphism group has a subgroup isomorphic to $\mathcal{S}$. In §5.2 we shall review the conventional notion of infinitesimal rigidity. In §5.3 we introduce symmetry-forced infinitesimal rigidity, which is only concerned with infinitesimal motions invariant under the underlying symmetry. In §5.4 we introduce the orbit rigidity matrix, which is the main tool for investigating symmetry-forced infinitesimal rigidity in the subsequent sections. In §5.5 we prove a necessary condition for symmetric frameworks to be symmetry-forced infinitesimally rigid.

## 5.1 $\mathcal{S}$-symmetric graphs

Let $H$ be a simple graph. An *automorphism* of $H$ is a permutation $\pi : V(H) \to V(H)$ such that $\{u, v\} \in E(H)$ if and only if $\{\pi(u), \pi(v)\} \in E(H)$. The set of all automorphisms of $H$ forms a subgroup of the symmetric group of $V(H)$, known as the *automorphism group* $\mathrm{Aut}(H)$ of $H$.

Let $\mathcal{S}$ be a group. An *action* of $\mathcal{S}$ on $H$ is a group homomorphism $\rho : \mathcal{S} \to \mathrm{Aut}(H)$. An action $\rho$ is called *free* if $\rho(g)(v) \neq v$ for any $v \in V$ and any non-identity $g \in \mathcal{S}$. We say that a graph $H$ is $(\mathcal{S}, \rho)$*-symmetric* if $\mathcal{S}$ acts on $H$ by $\rho$. If $\rho$ is clear from the context, we will simply denote $\rho(g)(v)$ by $g \cdot v$ or $gv$. Note that, for $g \in \mathcal{S}$ and $u, v \in V$, $\{u, v\} \in E(H)$ if and only if $\{gu, gv\} \in E(H)$, and hence there is an induced action of $\mathcal{S}$ on $E(H)$ defined by $g \cdot \{u, v\} = \{gu, gv\}$.

Let $H$ be an $(\mathcal{S}, \rho)$-symmetric graph. The *quotient graph* $H/\mathcal{S}$ of $H$ is a multigraph on the set $V(H)/\mathcal{S}$ of vertex orbits, together with the set $E(H)/\mathcal{S}$ of edge orbits as the edge set. An edge orbit may be represented by a loop in $H/\mathcal{S}$. Figure 5 provides an example when $\mathcal{S}$ is the dihedral group of order 4.

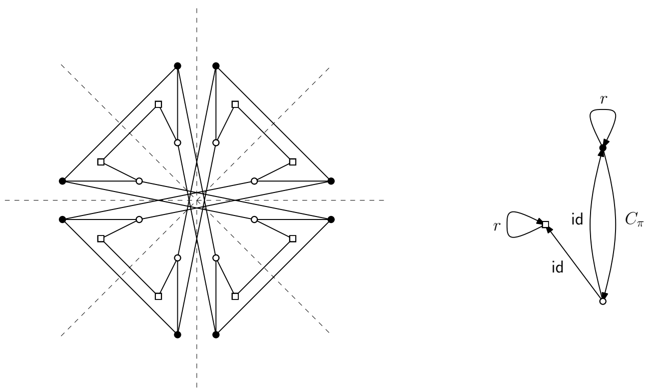



Figure 5: A $\mathcal{D}_4$-symmetric graph and the quotient gain graph.

Different graphs may have the same quotient graph. However, if we assume that $\rho$ is free, then a gain labeling makes the relation one-to-one. To see this, we arbitrarily choose a vertex $v$ as a representative vertex from each vertex orbit. Then, each orbit is written by $\mathcal{S}v = \{gv : g \in \mathcal{S}\}$. If $\rho$ is a free action, an edge orbit connecting $\mathcal{S}u$ and $\mathcal{S}v$ in $H/\mathcal{S}$ can be written by $\{\{gu, ghv\} : g \in \mathcal{S}\}$ for a unique $h \in \mathcal{S}$. We then orient the edge orbit from $\mathcal{S}u$ to $\mathcal{S}v$ in $H/\mathcal{S}$ and assign to it the gain $h$. In this way, we obtain *the quotient* $\mathcal{S}$*-gain graph*, denoted $(H/\mathcal{S}, \phi)$.

Conversely, any $\mathcal{S}$-gain graph $(G, \phi)$ can be "lifted" as an $(\mathcal{S}, \rho)$-symmetric graph with a free action $\rho$. To see this, we simply denote the pair $(g, v)$ of $g \in \mathcal{S}$ and $v \in V(G)$ by $gv$. The *covering graph* (also known as the derived graph) of $(G, \phi)$ is the simple graph with vertex set $\mathcal{S} \times V(G) = \{gv : g \in \mathcal{S}, v \in V(G)\}$ and the edge set $\{\{gu, g\phi(e)v\} : e = (u, v) \in E(G), g \in \mathcal{S}\}$. Clearly, $\mathcal{S}$ freely acts on the covering graph, under which the quotient gain graph comes back to $(G, \phi)$. For more properties of covering graphs, see e.g. [3, 11].

## 5.2 Bar-joint frameworks and infinitesimal rigidity

Before we investigate the rigidity theory of symmetric graphs we review the basic notions of the conventional rigidity of graphs.

A *d-dimensional bar-joint framework* (or simply a framework) is a pair $(H, p)$ of a simple graph $H$ and a mapping $p : V(H) \to \mathbb{R}^d$, called a *joint-configuration*. We denote the set $\{p(v) : v \in V(H)\}$ of points by $p(H)$.

Infinitesimal rigidity is concerned with the dimension of the space of infinitesimal motions. An *infinitesimal motion* of a framework $(H, p)$ is defined as an assignment $m : V(H) \to \mathbb{R}^d$ such that

$$\langle m(u) - m(v), p(u) - p(v)\rangle = 0 \qquad \text{for all } \{u, v\} \in E(H) \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in the $d$-dimensional Euclidean space. The set of infinitesimal motions forms a linear space, denoted $L(H, p)$.

In general, for a set $P \subseteq \mathbb{R}^d$ of points, an *infinitesimal isometry* of $P$ is defined by $m : P \to \mathbb{R}^d$ such that

$$\langle m(x) - m(y), x - y\rangle = 0 \qquad \text{for all } x, y \in P.$$

The set of infinitesimal isometries forms a linear space, denoted by $\mathrm{iso}(P)$. Notice that, for a skew-symmetric matrix $S$ and $t \in \mathbb{R}^d$, a mapping $m : P \to \mathbb{R}^d$ defined by

$$m(x) = Sx + t \qquad (x \in P)$$

is an infinitesimal isometry of $P$. Indeed, it is well-known that any infinitesimal isometry can be described in this form, and

$$\dim \mathrm{iso}(P) = d(k + 1) - \binom{k+1}{2}, \tag{8}$$

where $k$ denotes the affine dimension of $P$.

*Example* 5.1. Let us consider the infinitesimal isometries of a point set $P$ in the plane. According to (8), we have

$$\dim \mathrm{iso}(P) = \begin{cases} 3 & \text{if } |P| \geq 2 \\ 2 & \text{if } |P| = 1. \end{cases}$$

For $t \in \mathbb{R}^2$, let $m_t(x) = t$ $(x \in P)$. Then, $m_t$ is an infinitesimal isometry, called a *translation*. On the other hand, let $m_r(x) = C_{\pi/2}x$ $(x \in P)$, where $C_{\pi/2}$ denotes the $2 \times 2$ orthogonal matrix representing the 4-fold rotation around the origin. Then, $m_r$ is also an infinitesimal isometry, which we call an *infinitesimal rotation*. It is well known that $\mathrm{iso}(P)$ is spanned by $\{m_t, m_{t'}, m_r\}$ for two linearly independent vectors $t, t' \in \mathbb{R}^2$. See Figure 6 for examples.

An infinitesimal motion $m : V(H) \to \mathbb{R}^d$ of a framework $(H, p)$ is said to be *trivial* if $m$ can be expressed by

$$m(v) = Sp(v) + t \qquad (v \in V(H)) \tag{9}$$

for some skew-symmetric matrix $S$ and $t \in \mathbb{R}^d$. The set of all trivial motions forms a linear subspace of $L(H, p)$, denoted by $\mathrm{tri}(H, p)$. By definition, $\mathrm{tri}(H, p)$ is isomorphic to $\mathrm{iso}(p(H))$, and hence (8) gives the exact dimension of $\mathrm{tri}(H, p)$. $(H, p)$ is called *infinitesimally rigid* if $L(H, p) = \mathrm{tri}(H, p)$.

## 5.3 Symmetric frameworks and symmetric infinitesimal rigidity

A *discrete point group* (or simply a *point group*) is a finite discrete subgroup of $\mathcal{O}(\mathbb{R}^d)$, the *orthogonal group* of dimension $d$, i.e., the set of $d \times d$ orthogonal matrices over $\mathbb{R}$. For $d = 2$, point groups are classified into two classes, *groups* $\mathcal{C}_k$ *of* $k$*-fold rotations* and *dihedral groups* $\mathcal{D}_k$ of order $k$. For a special case, $\mathcal{D}_1$ consists of a mirror-reflection and the identity. In the subsequent discussion of this section, $\mathcal{S}$ denotes a point group.

Suppose that $H$ is $(\mathcal{S}, \rho)$-symmetric for a point group $\mathcal{S}$. A joint-configuration $p$ is said to be $(\mathcal{S}, \rho)$*-symmetric* (or, simply, $\mathcal{S}$-symmetric) if

$$gp(v) = p(gv) \qquad \text{for all } g \in \mathcal{S} \text{ and for all } v \in V(H). \tag{10}$$

Such a pair $(H, p)$ is called an $(\mathcal{S}, \rho)$*-symmetric framework* (or simply an $\mathcal{S}$-symmetric framework or a symmetric framework).

We shall consider "symmetry-preserving" infinitesimal motions of symmetric frameworks. We say that an infinitesimal motion $m : V(H) \to \mathbb{R}^d$ is *symmetric* if

$$gm(v) = m(gv) \qquad \text{for all } g \in \mathcal{S} \text{ and for all } v \in V(H). \tag{11}$$

The set of $\mathcal{S}$-symmetric infinitesimal motions and the set of trivial ones form linear subspaces of $L(H, p)$ and $\mathrm{tri}(H, p)$, denoted $L_{\mathcal{S}}(H, p)$ and $\mathrm{tri}_{\mathcal{S}}(H, p)$, respectively. We say that $(H, p)$ is *symmetry-forced infinitesimally rigid* if $L_{\mathcal{S}}(H, p) = \mathrm{tri}_{\mathcal{S}}(H, p)$.

A set $P$ of points is called $\mathcal{S}$*-symmetric* if $gP = \{gp\colon p \in P\} = P$ for all $g \in \mathcal{S}$. An infinitesimal isometry $m : P \to \mathbb{R}^d$ of an $\mathcal{S}$-symmetric point set $P$ is called $\mathcal{S}$*-symmetric* if $gm(x) = m(gx)$ for all $x \in P$ and $g \in \mathcal{S}$. The set of $\mathcal{S}$-symmetric infinitesimal isometries forms a linear subspace of $\mathrm{iso}(P)$, denoted $\mathrm{iso}_{\mathcal{S}}(P)$. Clearly, $\mathrm{tri}_{\mathcal{S}}(H, p)$ is isomorphic to $\mathrm{iso}_{\mathcal{S}}(p(H))$.

*Example* 5.2. Let us consider point groups in $\mathcal{O}(\mathbb{R}^2)$, which will be mainly discussed in §6 and §8. Let $P$ be an $\mathcal{S}$-symmetric point set in $\mathbb{R}^2$. See Figure 6 for examples of $\mathcal{C}_k$-symmetric infinitesimal isometries. In general, if $|P| > 1$,

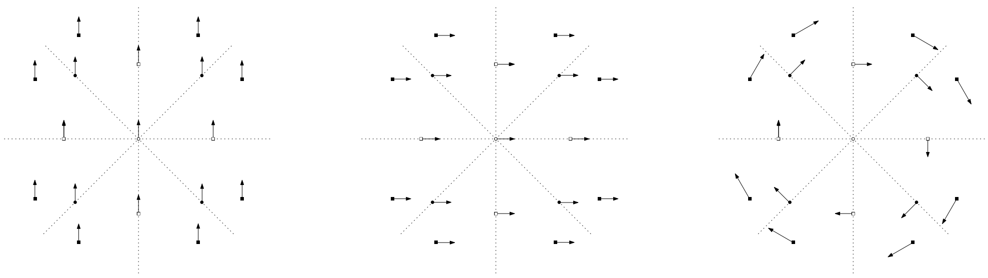
$$\dim \mathrm{iso}_{\mathcal{C}_k}(P) = \begin{cases} 3 & \text{if } k = 1 \\ 1 & \text{if } k \geq 2, \end{cases}$$

and if $P = \{x\}$,

$$\dim \mathrm{iso}_{\mathcal{C}_k}(P) = \begin{cases} 2 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \text{ (where } x \text{ should be the origin)} \end{cases}$$

Similarly, for the dihedral group $\mathcal{D}_k$ of order $k$,

$$\dim \mathrm{iso}_{\mathcal{D}_k}(P) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2, \end{cases}$$



(a) (b) (c)

Figure 6: Three independent infinitesimal isometries in the plane, among which (a) is symmetric with respect to the group of a vertical reflection, (b) is symmetric with respect to the group of a horizontal reflection, and (c) is symmetric with respect to the group of rotations.

A result of Schulze [26] motivates us to look at $\mathcal{S}$-symmetric infinitesimal rigidity, which states that if $(H, p)$ is not symmetry-forced infinitesimally rigid on an $\mathcal{S}$-generic $p$, then $(H, p)$ has a nontrivial continuous motion that preserves the $(\mathcal{S}, \rho)$-symmetry.

## 5.4 The orbit rigidity matrix

Let $(H, p)$ be an $(\mathcal{S}, \rho)$-symmetric framework in $\mathbb{R}^d$. Due to (11), the system (7) of linear equations (with respect to $m$) is redundant. Schulze and Whiteley [27] pointed out that the system can be reduced to $|E(H)/\mathcal{S}|$ linear equations.

To see this, we first define a *joint-configuration* $\tilde{p}$ of vertex orbits by $\tilde{p} : V(H)/\mathcal{S} \to \mathbb{R}^d$. By taking a representative vertex $v$ from each vertex orbit $\mathcal{S}v$, we shall fix a one-to-one correspondence by $\tilde{p}(\mathcal{S}v) = p(v)$. (Then, the locations of the other non-representative vertices are uniquely determined by (10).)

In a similar way, we define an *infinitesimal motion* of $(H/\mathcal{S}, \tilde{p})$ by $\tilde{m} : V(H)/\mathcal{S} \to \mathbb{R}^d$. By using the representative vertices determined above, we fix a one-to-one correspondence between $\mathcal{S}$-symmetric infinitesimal motions of $V(H)$ and infinitesimal motions of $V(H)/\mathcal{S}$ by $\tilde{m}(\mathcal{S}v) = m(v)$ for each vertex orbit $\mathcal{S}v$.

Let $(H/\mathcal{S}, \phi)$ be the quotient $\mathcal{S}$-gain graph of $H$. Recall that each (oriented) edge orbit $\mathcal{S}e$ connecting $\mathcal{S}u$ and $\mathcal{S}v$ with gain $h_e$ can be written by $\mathcal{S}e = \{\{gu, gh_ev\} : g \in \mathcal{S}\}$. The system (7) is hence written by

$$\langle m(gu) - m(gh_ev), p(gu) - p(gh_ev) \rangle = 0 \qquad \text{for all } \{gu, gh_ev\} \in \mathcal{S}e \qquad (12)$$

over all edge orbits $\mathcal{S}e \in E(H)/\mathcal{S}$. Recall that the transpose of $g$ is $g^{-1}$ for any $g \in \mathcal{O}(\mathbb{R}^d)$. By (10) and (11),

$$\begin{aligned}\langle m(gu) - m(gh_e v), p(gu) - p(gh_e v)\rangle &= \langle m(u) - h_e m(v), p(u) - h_e p(v)\rangle \\ &= \langle m(u), p(u) - h_e p(v)\rangle + \langle m(v), p(v) - h_e^{-1} p(u)\rangle \\ &= \langle \tilde{m}(\mathcal{S}u), \tilde{p}(\mathcal{S}u) - h_e \tilde{p}(\mathcal{S}v)\rangle + \langle \tilde{m}(\mathcal{S}v), \tilde{p}(\mathcal{S}v) - h_e^{-1}\tilde{p}(\mathcal{S}u)\rangle.\end{aligned}$$

Therefore, for $\tilde{p} : V(H)/\mathcal{S} \to \mathbb{R}^d$, a mapping $\tilde{m} : H/\mathcal{S} \to \mathbb{R}^d$ is an infinitesimal motion of $(H/\mathcal{S}, \tilde{p})$ if and only if

$$\langle \tilde{m}(\mathcal{S}u), \tilde{p}(\mathcal{S}u) - h_e \tilde{p}(\mathcal{S}v)\rangle + \langle \tilde{m}(\mathcal{S}v), \tilde{p}(\mathcal{S}v) - h_e^{-1}\tilde{p}(\mathcal{S}u)\rangle = 0 \tag{13}$$

for every oriented edge orbit $\mathcal{S}e$ with $\phi(\mathcal{S}e) = h_e$. By regarding (13) as a system of linear equations of $\tilde{m}$, the corresponding $|E(H)/\mathcal{S}| \times d|V(H)/\mathcal{S}|$-matrix is called the *orbit rigidity matrix.*

In general, for an $\mathcal{S}$-gain graph $(G, \phi)$ and $\tilde{p} : V \to \mathbb{R}^d$, we shall define the *orbit rigidity matrix* as an $|E(G)| \times d|V(G)|$-matrix, in which each row corresponds to an edge, each vertex is associated with a $d$-tuple of columns, and the row corresponding to $e = (u, v) \in E(G)$ is written by

$$0 \dots 0 \quad \overbrace{\tilde{p}(u) - \phi(e)\tilde{p}(v)}^{u} \quad 0 \dots 0 \quad \overbrace{\tilde{p}(v) - \phi(e)^{-1}\tilde{p}(u)}^{v} \quad 0 \dots 0$$

if $e$ is not a loop, and by

$$0 \dots 0 \quad \overbrace{(2I_d - \phi(e) - \phi(e)^{-1})\tilde{p}(v)}^{v} \quad 0 \dots 0$$

if $e$ is a loop. The orbit rigidity matrix of $(G, \phi, \tilde{p})$ is denoted by $O(G, \phi, \tilde{p})$. ¿From the above discussion, it follows that the dimension of the $\mathcal{S}$-symmetric infinitesimal motions can be computed from the rank of the orbit rigidity matrix of the corresponding quotient gain graph, which is formally stated as follows:

**Theorem 5.1** (Schulze and Whiteley [27]). *Let $(H, p)$ be an $(\mathcal{S}, \rho)$-symmetric framework with a free action $\rho$. Then,*

$$\dim L_{\mathcal{S}}(H, p) = d|V(H)/\mathcal{S}| - \operatorname{rank} O(H/\mathcal{S}, \phi, \tilde{p}),$$

*where $(H/\mathcal{S}, \phi)$ is the quotient $\mathcal{S}$-gain graph and $\tilde{p}$ is a joint-configuration of vertex orbits corresponding to $p$.*

## 5.5 Necessary condition for symmetric infinitesimal rigidity

Combining some observations given in §2, we can show a necessary condition for the row independence of orbit rigidity matrices.

**Lemma 5.2.** *Let $(G, \phi)$ be an $\mathcal{S}$-gain graph with underlying graph $G = (V, E)$, and let $p : V \to \mathbb{R}^d$. If $O(G, \phi, p)$ is row independent, then*

$$|F| \leq \sum_{F_i \in C(F)} \{d|V(F_i)| - \dim \mathrm{iso}_{\langle F_i \rangle_{\phi,w}}(p(F_i))\}$$

*for all $F \subseteq E$ and $w \in V(F_i)$, where $p(F_i) = \{gp(v) \colon v \in V(F_i), g \in \mathcal{S}\}$.*

*Proof.* Let $R_F$ be the linear space spanned by the row vectors associated with $F$ in $O(G, \phi, p)$. Observe that each non-zero entry of the row vector associated with $e \in F$ is in the columns associated with $V(F)$. This means that $R_F$ is the direct sum of $R_{F'}$ for $F' \in C(F)$, and hence it suffices to check the statement for a connected $F$ with $V(F) = V$.

Clearly, $\dim R_F \leq d|V|$. Since $|F| \leq \dim R_F$, we now show that $\dim R_F^\perp \geq \dim \mathrm{iso}_{\langle F \rangle_{\phi,w}}(p(F))$, where $R_F^\perp$ denotes the orthogonal complement of $R_F$.

To see this we first check that a switching operation does not change the rank of the orbit rigidity matrix. Let $\phi'$ be the gain function obtained from $\phi$ by a switching operation at $v_0$ with $g_0 \in \mathcal{S}$. We define $p' : V \to \mathbb{R}^d$ by

$$p'(u) = \begin{cases} p(u) & \text{if } u \neq v_0 \\ g_0 p(u) & \text{if } u = v_0. \end{cases} \tag{14}$$

Note that $p'(F) = \{gp'(v) \colon v \in V, g \in \mathcal{S}\} = p(F)$. We now show

$$\mathrm{rank}\, O(G, \phi, p) = \mathrm{rank}\, O(G, \phi', p'). \tag{15}$$

Let us consider a non-loop edge $e = (u, v_0)$ oriented to $v_0$ in $G$. The row corresponding to $e$ in $O(G, \phi', p')$ is written by

| | | | | |
|---|---|---|---|---|
| | $u$ | | $v_0$ | |
| $0 \ldots 0$ | $p'(u) - \phi'(e)p'(v_0)$ | $0 \ldots 0$ | $p'(v_0) - \phi'(e)^{-1}p'(u)$ | $0 \ldots 0$ |

By (1), we have $\phi'(e) = \phi(e)g_0^{-1}$. Thus, by using (14), the row of $e$ becomes

| | | | | |
|---|---|---|---|---|
| | $u$ | | $v_0$ | |
| $0 \ldots 0$ | $p(u) - \phi(e)p(v_0)$ | $0 \ldots 0$ | $g_0(p(v_0) - \phi(e)^{-1}p(u))$ | $0 \ldots 0$ |

Similarly, for a non-loop edge $e = (v_0, u)$ oriented from $v_0$ in $G$, the row of $e$ becomes exactly the same form as above. By using the same calculation, for a loop $e$ incident to $v_0$ in $G$, the row of $e$ in $O(G', \phi', p')$ can be written as

| | | |
|---|---|---|
| | $v_0$ | |
| $0 \ldots 0$ | $g_0(2I_d - \phi(e) - \phi(e)^{-1})p(v_0)$ | $0 \ldots 0$ |

By performing column operations within the $d$ columns associated with $v_0$, these are converted to

| | | | | |
|---|---|---|---|---|
| | $u$ | | $v_0$ | |
| $0 \ldots 0$ | $p(u) - \phi(e)p(v_0)$ | $0 \ldots 0$ | $p(v_0) - \phi(e)^{-1}p(u)$ | $0 \ldots 0$ |

and

$$\begin{array}{c|c|c} & v_0 & \\ \hline 0\dots0 & (2I_d - \phi(e) - \phi(e)^{-1})p(v_0) & 0\dots0 \end{array}$$

respectively, which implies that $\operatorname{rank} O(G,\phi,p) = \operatorname{rank} O(G,\phi',p')$. Therefore, the row independence of the orbit rigidity matrix is invariant under switching operations. Moreover, since $p(F) = p'(F)$, $\dim \mathrm{iso}_{\langle F\rangle_{\phi,w}}(p(F)) = \dim \mathrm{iso}_{\langle F\rangle_{\phi',w}}(p'(F))$. So it suffices to prove the statement for $O(G,\phi',p')$.

Let $T$ be a spanning tree of $G$. Since we can freely apply switching operations, we may assume that $\phi(e) = \mathrm{id}$ for all $e \in T$. Then, by Lemma 2.4, $\langle F\rangle_{\phi,w} = \langle \phi(e) \colon e \in F \setminus T\rangle$ for a vertex $w \in V(F)$.

Let us take any $m \in \mathrm{iso}_{\langle F\rangle_{\phi,w}}(p(F))$ and let $\tilde{m} : V \to \mathbb{R}^d$ be defined by $\tilde{m}(v) = m(p(v))$ for $v \in V$. We show that $\tilde{m}$ is in the orthogonal complement of $R_F$. To check it, let us consider any edge $e = (u,v) \in F$ with gain $h = \phi(e)$. Since $m \in \mathrm{iso}(p(F))$, we have

$$\langle p(u) - hp(v), m(p(u)) - m(hp(v))\rangle = 0.$$

Since $m$ is $\langle F\rangle_{\phi,w}$-symmetric, we also have $m(hp(v)) = hm(p(v))$. Therefore, we obtain

$$0 = \langle p(u) - hp(v), m(p(u)) - m(hp(v))\rangle = \langle p(u) - hp(v), \tilde{m}(u) - h\tilde{m}(v)\rangle,$$

implying that $\tilde{m}$ is in the orthogonal complement of $R_F$. Consequently, $\dim R_F^{\perp} \geq \dim \mathrm{iso}_{\langle F\rangle_{\phi,w}}(p(F))$, and hence $|F| \leq \dim R_F \leq d|V| - \dim \mathrm{iso}_{\langle F\rangle_{\phi,w}}(p(F))$. □

This, together with Theorem 5.1, directly implies a necessary condition for symmetric frameworks to be symmetry-forced infinitesimally rigid.

Recall that $\mathcal{S}$ is a finite family of orthogonal matrices. Let $\mathbb{Q}_{\mathcal{S}}$ be the field generated by $\mathbb{Q}$ and the entries of all the matrices in $\mathcal{S}$. Since $\mathcal{S}$ is finite, almost all numbers in $\mathbb{R}$ are transcendental over $\mathbb{Q}_{\mathcal{S}}$. For a given gain graph $(G,\phi)$, a mapping $\tilde{p} : V(G) \to \mathbb{R}^d$ is called *$\mathcal{S}$-generic* if the set of coordinates of $\tilde{p}(v)$ for all $v \in V(G)$ is algebraically independent over $\mathbb{Q}_{\mathcal{S}}$. Similarly, for a given $(\mathcal{S},\rho)$-symmetric graph $H$, an $(\mathcal{S},\rho)$-symmetric joint-configuration $p : V(H) \to \mathbb{R}^d$ is called *$\mathcal{S}$-generic* if the corresponding joint-configuration $\tilde{p}$ of the vertex orbits is $\mathcal{S}$-generic. An $\mathcal{S}$-symmetric framework is called *generic* if the joint configuration is $\mathcal{S}$-generic.

In §6 and §8 we will check that the condition of Lemma 5.2 is indeed sufficient for generic symmetric frameworks in the plane with cyclic groups and dihedral groups with odd order, respectively.

# 6 Combinatorial Characterization of Generic Rigidity with Cyclic Symmetry

In this section we shall prove a combinatorial characterization of the infinitesimal rigidity of $\mathcal{S}$-generic symmetric frameworks for cyclic point groups in the plane.

Let $(H, p)$ be an $(\mathcal{S}, \rho)$-symmetric framework with a point group $\mathcal{S} \in \mathcal{O}(\mathbb{R}^2)$, and suppose that $p$ is $\mathcal{S}$-generic. We only focus on the case when $\rho$ is a free action. As in §5.4, we fix a representative vertex $v$ from each vertex orbit, which determines a one-to-one correspondence between $p$ and the joint-configuration $\tilde{p}$ of vertex orbits by $\tilde{p}(\mathcal{S}v) = p(v)$. We prove that the row matroid of the orbit rigidity matrix $O(H/\mathcal{S}, \phi, \tilde{p})$ is equal to the $(2,3)$-g-matroid $\mathcal{M}(g_{2,3})$ of $(H/\mathcal{S}, \phi)$. We shall make use of extensions of the quotient gain graphs to "extend" frameworks by keeping the rigidity. The following lemma is a key observation, which is an extension of the one given in [30, 32] for proving Laman's theorem. The lemma is not limited to cyclic groups.

**Lemma 6.1.** *Let $(G, \phi)$ be an $\mathcal{S}$-gain graph for a point group $\mathcal{S} \subset \mathcal{O}(\mathbb{R}^2)$. Let $(G', \phi')$ be an $\mathcal{S}$-gain graph obtained from $(G, \phi)$ by a 0-extension, 1-extension, or loop-1-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

*Proof.* If there is a $p$ such that $O(G, \phi, p)$ is row independent, then $O(G, \phi, q)$ is row independent for all $\mathcal{S}$-generic $q$. Hence, we may assume that $p$ is $\mathcal{S}$-generic.

(Case 1) Suppose that $G'$ is obtained from $G$ by a 0-extension, by adding a vertex $v$ and non-loop edges $e_1$ and $e_2$ incident to $v$. Let $u_i$ be the other endvertex of $e_i$ and let $g_i = \phi(e_i)$ for $i = 1, 2$.

By the definition of 0-extensions, $g_1 \neq g_2$ if $u_1 = u_2$. Therefore, as $p$ is generic, $g_1 p(u_1) \neq g_2 p(u_2)$. Let us take $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for all $w \in V(G)$ and $p'(v)$ is a point not on the line through $g_1 p(u_1)$ and $g_2 p(u_2)$. Then $O(G', \phi', p')$ can be described as follows, by decomposing it into four blocks:

| | $v$ | $V(G)$ |
|---|---|---|
| $e_1$ | $p'(v) - g_1 p(u_1)$ | $*$ |
| $e_2$ | $p'(v) - g_2 p(u_2)$ | $*$ |
| $E(G)$ | $0$ | $O(G, \phi, p)$ |

where the right-bottom block, corresponding to $E(G)$ and $V(G)$, is equal to $O(G, \phi, p)$. Since $O(G, \phi, p)$ is row independent, it suffices to show that the top-left block is row independent. Since $p'(v)$ does not lie on the line through $g_1 p(u_1)$ and $g_2 p(u_2)$, the top-left block is indeed row independent.

(Case 2) Suppose that $G'$ is obtained from $G$ by a loop-1-extension, by adding a vertex $v$ with a non-loop edge $e$ and a loop $l$ incident to $v$. Let $u$ be the other endvertex of $e$ and let $g = \phi(e)$ and $h = \phi(l)$. Without loss of generality, we may assume that $e$ is outgoing from $v$.

By the definition of loop-1-extensions, $h$ is not equal to the $2 \times 2$-identity matrix $I_2$, and hence $2I_2 - h - h^{-1}$ is nonzero. Therefore, there is a point $q \in \mathbb{R}^2$ such that $\{(2I_2 - h - h^{-1})q, q - gp(u)\}$ is linearly independent (c.f. Lemma 8.3). We define $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for all $w \in V(G)$ and $p'(v) = q$. Then,

$O(G', \phi', p')$ can be described as follows:

| | $v$ | $V(G)$ |
|---|---|---|
| $e$ | $q - gp(u)$ | $*$ |
| $l$ | $(2I_2 - h - h^{-1})q$ | $0$ |
| $E(G)$ | $0$ | $O(G, \phi, p)$ |

Since the top-left block and the bottom-right block are both row independent, $O(G', \phi', p')$ is row independent.

(Case 3) Suppose that $G'$ is obtained from $G$ by a 1-extension, by removing an existing edge $e$ and adding a new vertex $v$ with three new non-loop edges $e_1, e_2, e_3$ incident to $v$. We may assume that $e_i$ is outgoing from $v$. Let $u_i$ be the other endvertex of $e_i$, and let $g_i = \phi'(e_i)$ and $p_i = p(u_i)$ for $i = 1, 2, 3$. By the definition of 1-extension, we have $\phi(e) = g_1^{-1} g_2$.

**Claim 6.2.** *The three points $g_i p_i$ $(i = 1, 2, 3)$ do not lie on a line.*

*Proof.* Since $p$ is $\mathcal{S}$-generic, $u_1 = u_2 = u_3$ should hold if they lie on a line. Then $p_1 = p_2 = p_3$. By the definition of 1-extensions, $g_i \neq g_j$ if $u_i = u_j$. This implies that $g_1 p_1, g_2 p_2, g_3 p_3$ are three distinct points on a circle. Thus, they do not lie on a line. □

We take $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for all $w \in V(G)$, and $p'(v)$ is a point on the line through $g_1 p_1$ and $g_2 p_2$ but neither $g_1 p_1$ nor $g_2 p_2$. $O(G', \phi', p')$ is described as follows: if $u_1 \neq u_2$

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_3$ | $p'(v) - g_3 p_3$ | $*$ | $*$ | $*$ |
| $e_1$ | $p'(v) - g_1 p_1$ | $p_1 - g_1^{-1} p'(v)$ | $0$ | $0$ |
| $e_2$ | $p'(v) - g_2 p_2$ | $0$ | $p_2 - g_2^{-1} p'(v)$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | | |

where the right-bottom block $O(G - e, \phi, p)$ denotes the orbit rigidity matrix obtained from $O(G, \phi, p)$ by removing the row of $e$, whereas, if $u_1 = u_2$,

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_3$ | $p'(v) - g_3 p_3$ | $*$ | $*$ |
| $e_1$ | $p'(v) - g_1 p_1$ | $p_1 - g_1^{-1} p'(v)$ | $0$ |
| $e_2$ | $p'(v) - g_2 p_1$ | $p_1 - g_2^{-1} p'(v)$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

We consider the case when $u_1 \neq u_2$. (The case when $u_1 = u_2$ is similar.) Since $p'(v)$ lies on the line through $g_1 p_1$ and $g_2 p_2$, $p'(v) - g_i p(u_i)$ is a scalar multiple of $g_1 p_1 - g_2 p_2$ for $i = 1, 2$. Hence, by multiplying the rows of $e_1$ and $e_2$ by an appropriate scalar, $O(G', \phi', p')$ becomes

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_3$ | $p'(v) - g_3 p_3$ | $*$ | $*$ | $*$ |
| $e_1$ | $g_1 p_1 - g_2 p_2$ | $-g_1^{-1}(g_1 p_1 - g_2 p_2)$ | $0$ | $0$ |
| $e_2$ | $g_1 p_1 - g_2 p_2$ | $0$ | $-g_2^{-1}(g_1 p_1 - g_2 p_2)$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | | |

Subtracting the row of $e_1$ from that of $e_2$, we finally get

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_3$ | $p'(v) - g_3p_3$ | $*$ | $*$ | $*$ |
| $e_1$ | $g_1p_1 - g_2p_2$ | $-g_1^{-1}(g_1p_1 - g_2p_2)$ | $0$ | $0$ |
| $e_2$ | $0$ | $p_1 - g_1^{-1}g_2p_2$ | $p_2 - g_2^{-1}g_1p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | | |

Since $\phi(e) = g_1^{-1}g_2$, the row of $e_2$ is equal to the row of $e$ in $O(G, \phi, p)$. This means that the right-bottom block together with the row of $e_2$ forms $O(G, \phi, p)$, which is row independent. Thus, the matrix is row independent if and only if the top-left block is row independent. Since $g_ip_i$ $(i = 1, 2, 3)$ are not on a line by Claim 6.2, the line through $p'(v)$ and $g_3p_3$ is not parallel to the line through $g_1p_1$ and $g_2p_2$. This implies that the top-left block is row independent, and consequently $O(G', \phi', p')$ is row independent. □

**Theorem 6.3.** *Let $\mathcal{C} \subset \mathcal{O}(\mathbb{R}^2)$ be a cyclic point group, that is, either a group of k-fold rotations or a group of a reflection, and let $(H, p)$ be a generic $(\mathcal{C}, \rho)$-symmetric framework in the plane with a free action $\rho$. Then $(H, p)$ is symmetry-forced infinitesimally rigid if and only if the quotient $\mathcal{C}$-gain graph contains a spanning maximum $(2, 3)$-g-tight subgraph.*

*Proof.* By Theorem 5.1 it suffices to show that for the quotient $\mathcal{C}$-gain graph $(H/\mathcal{C}, \phi)$ and any $\mathcal{C}$-generic $\tilde{p} : V(H/\mathcal{C}) \to \mathbb{R}^2$, $O(H/\mathcal{C}, \phi, \tilde{p})$ is row independent if and only if $(H/\mathcal{C}, \phi)$ is $(2, 3)$-g-sparse. Let us simply denote $G = H/\mathcal{C}$.

("If part") It suffices to consider the case when $G$ is maximum $(2, 3)$-g-tight. The proof is done by induction on $|V(G)|$. For $|V(G)| = 1$, $G$ consists of single vertex with an unbalanced loop. Then $O(G, \phi, \tilde{p})$ consists of a nonzero row, which implies that $O(G, \phi, \tilde{p})$ is row-independent.

For $|V(G)| > 1$, by Theorem 4.4, $G$ can be built up from a $\mathcal{C}$-gain graph with one vertex and an unbalanced loop with a sequence of 0-extensions, 1-extensions, and loop-1-extensions. Thus, there is a maximum $(2, 3)$-g-tight graph $(G', \phi')$ from which $(G, \phi)$ is constructed by a 0-extension, 1-extension, or loop-1-extension. By induction, there is a $p'$ such that $O(G, \phi', p')$ is row independent. Thus, Lemma 6.1 implies that there is a $p$ such that $O(G, \phi, p)$ is row independent, which in turn implies that $O(G, \phi, q)$ is row independent for all $\mathcal{C}$-generic $q$.

("Only-if part") The necessity is based on Lemma 5.2. Suppose that $O(G, \phi, \tilde{p})$ is row independent. Recall that we have seen the exact value of $\dim \mathrm{iso}_{\mathcal{C}}(P)$ for $\mathcal{C} \subset \mathcal{O}(\mathbb{R}^2)$ and a $\mathcal{C}$-symmetric point set $P \subseteq \mathbb{R}^2$ in Example 5.2. Since $\tilde{p}$ is $\mathcal{C}$-generic, we have

$$\mathrm{iso}_{\langle F\rangle_v}(\tilde{p}(F)) = \begin{cases} 3 & \text{(if } F \text{ is balanced)} \\ 1 & \text{(otherwise)} \end{cases}$$

for all connected $F \subseteq E(G)$ and $v \in V(F)$, where $\tilde{p}(F) = \{g\tilde{p}(v) \colon v \in V(F), g \in \mathcal{C}\}$. Therefore, by Lemma 5.2, we have

$$|F| \leq \sum_{F' \in C(F)} \{2|V(F')| - \mathrm{iso}_{\langle F'\rangle_v}(\tilde{p}(F'))\} \leq 2|V(F)| - \begin{cases} 3 & \text{(if } F \text{ is balanced)} \\ 1 & \text{(otherwise)} \end{cases}$$

for all $F \subseteq E(G)$. Therefore, $(G, \phi)$ is $(2,3)$-g-sparse. □

*Remark* 6.1. We have seen in Lemma 6.1 that the 0-extension, 1-extension, and loop-1-extension operations all preserve the row independence of the orbit rigidity matrix. In the covering graph these operations can be seen as graph operations that preserve the underlying symmetry. Some of them can be recognized as performing so-called Henneberg operations [30, 32] simultaneously. See Figure 7. In case of 3-fold rotation symmetry, these operations are considered by Schulze [26].

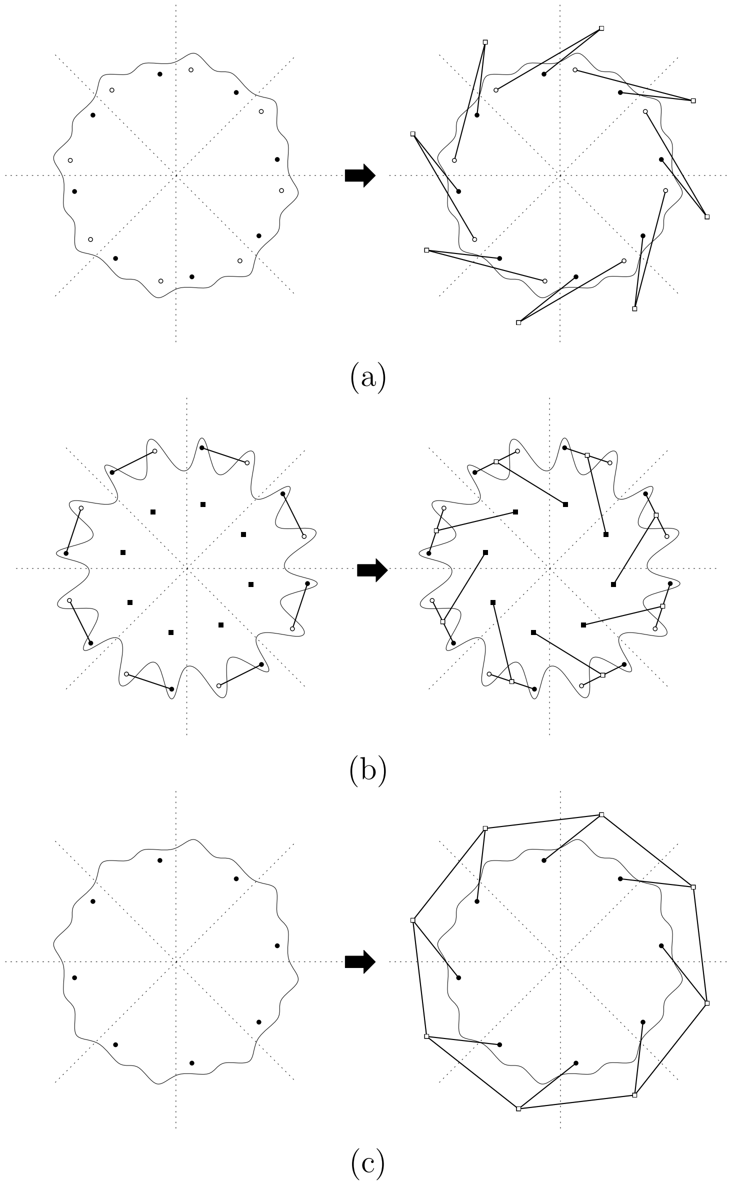


Figure 7: (a) 0-extension, (b) 1-extension, (c) loop-1-extension in the covering graph.

# 7 Constructive Characterization of Maximum $\mathcal{D}$-tight Graphs

In the previous sections we gave a constructive characterization of $(2,3)$-g-sparse graphs and their realizations as symmetry-forced rigid frameworks in the plane with cyclic point group symmetry. We next move to non-cyclic point groups, that is, dihedral groups that we denote by $\mathcal{D}_k$ (or simply by $\mathcal{D}$). The corresponding matroid, that we construct in the next subsection, is slightly different from the $(2,3)$-g-count matroid, as we need to take into account the fact that the underlying group is not cyclic.

## 7.1 $\mathcal{D}$-sparsity

Let $(G,\phi)$ be a $\mathcal{D}$-gain graph with underlying graph $G=(V,E)$. We define a function $f_{\mathcal{D}}$ on $E$ by

$$f_{\mathcal{D}}(X) = 2|V(X)| - 3 + \beta(X) \qquad (X \subseteq E)$$

where

$$\beta(X) = \begin{cases} 0 & \text{(if } X \text{ is balanced)} \\ 2 & \text{(if } X \text{ is unbalanced and cyclic)} \\ 3 & \text{(otherwise)}, \end{cases}$$

and define a class of sparse graphs determined by $f_{\mathcal{D}}$ as follows.

**Definition 7.1.** Let $(G,\phi)$ be a $\mathcal{D}$-gain graph. An edge set $X \subseteq E(G)$ is called *$\mathcal{D}$-sparse* if $|F| \leq f_{\mathcal{D}}(F)$ for any nonempty $F \subseteq X$, and it is called *$\mathcal{D}$-tight* if it is $\mathcal{D}$-sparse with $|X| = f_{\mathcal{D}}(X)$.

$(G,\phi)$ is said to be *$\mathcal{D}$-sparse* if so is $E(G)$, and it is called *maximum $\mathcal{D}$-tight* if it is $\mathcal{D}$-sparse with $|E(G)| = 2|V(G)|$.

By a simple degree of freedom counting argument based on Example 5.2 and Lemma 5.2, it is not difficult to see that the $\mathcal{D}$-sparsity is a necessary condition for orbit rigidity matrices to be row independent in case of dihedral symmetry. (A formal proof will be given in Lemma 8.1.) To prove the sufficiency, the first question is whether $\mathcal{D}$-sparsity defines a collection of independent sets of a matroid. This will be proved in this subsection.

We will use the following technical lemmas on properties of $\mathcal{D}$-tight sets.

**Lemma 7.1.** *Let $(G,\phi)$ be a $\mathcal{D}$-sparse graph with $G=(V,E)$ and $F \subseteq E$ be a $\mathcal{D}$-tight set. Then, the following holds.*

**(i)** *If $F$ is cyclic, then $F$ is connected.*

**(ii)** *If $F$ is balanced with $|F|>1$, then $F$ has neither parallel edges nor loops and is 2-connected and essentially 3-edge-connected.*

*Proof.* Since $G$ is $\mathcal{D}$-sparse and $\beta$ is monotone nondecreasing, we have $|F| \leq \sum_{F' \in C(F)} f_{\mathcal{D}}(F') \leq 2|V(F)| - (3 - \beta(F))c$, where $c$ denotes the number of connected components in $F$. Hence, if $F$ is not connected and $\beta(F) < 3$, then $|F| < 2|V(F)| - 3 + \beta(F)$, implying that $F$ is not $\mathcal{D}$-tight. Therefore if $\beta(F) < 3$ then $X$ is connected.

Suppose further that $F$ is balanced. Then we have $\beta(X) = 0$ for any $X \subseteq F$. This means that $|X| \leq f_{2,3}(X)$ for any nonempty $X \subseteq F$, and $|F| = f_{\mathcal{D}}(F) = 2|V(F)| - 3 = f_{2,3}(F)$. In other words, $F$ is independent in the generic 2-rigidity matroid $\mathcal{M}(f_{2,3})$ of $G[F]$. It is known that, in the generic 2-rigidity matroid, an independent set $F$ with $|F| = f_{2,3}(F)$ and $|F| > 1$ has neither parallel edges nor a loop and is 2-connected and essentially 3-edge-connected (see e.g. [12]). □

**Lemma 7.2.** *Let $(G, \phi)$ be a $\mathcal{D}$-sparse graph with $G = (V, E)$. Let $X, Y \subseteq E$ be $\mathcal{D}$-tight edge sets with $X \cap Y \neq \emptyset$. Then $X \cup Y$ is $\mathcal{D}$-tight.*

*Proof.* Without loss of generality, assume $\beta(X) \geq \beta(Y)$.

Let $d = 2|V(X \cup Y)| - |X \cup Y|$. Note that $X \cup Y$ is $\mathcal{D}$-tight if one of the following holds: (i) $d = 0$, (ii) $d \leq 1$ and $X \cup Y$ is cyclic, or (iii) $d \leq 3$ and $X \cup Y$ is balanced.

Let $c_0$ be the number of isolated vertices in the graph $(V(X) \cap V(Y), X \cap Y)$ and $c_1$ be the number of connected components in $X \cap Y$. We have $|X| = 2|V(X)| - 3 + \beta(X)$ and $|Y| = 2|V(Y)| - 3 + \beta(Y)$. We also have

$$\begin{aligned} |X \cap Y| &\leq \sum_{F \in C(X \cap Y)} f_{\mathcal{D}}(F) = 2|V(X \cap Y)| - 3c_1 + \sum_{F \in C(X \cap Y)} \beta(F) \\ &= 2|V(X) \cap V(Y)| - 2c_0 - 3c_1 + \sum_{F \in C(X \cap Y)} \beta(F) \\ &\leq 2|V(X) \cap V(Y)| - 2c_0 - 3c_1 + \beta(Y)c_1 \end{aligned} \tag{16}$$

since $\beta$ is monotone non-decreasing. Therefore,

$$\begin{aligned} d &= 2|V(X \cup Y)| - |X \cup Y| = 2|V(X \cup Y)| - (|X| + |Y| - |X \cap Y|) \\ &\leq 6 - \beta(X) - \beta(Y) - 2c_0 - 3c_1 + \beta(Y)c_1 \\ &\leq 3 - \beta(X) - 2c_0 - (3 - \beta(Y))(c_1 - 1). \end{aligned} \tag{17}$$

Note that $c_1 \geq 1$ by $X \cap Y \neq \emptyset$ and hence $(3 - \beta(Y))(c_1 - 1) \geq 0$.

If $\beta(X) = 3$, then (17) implies that $d = 0$ and hence $X \cup Y$ is $\mathcal{D}$-tight.

Therefore we assume $\beta(X) < 3$. Then $X$ and $Y$ are connected by Lemma 7.1. We split the proof into two cases depending on the value of $\beta(X)$.

(Case 1) If $\beta(X) = 2$, then (17) implies that $d \leq 1$. Since $d = 0$ implies the $\mathcal{D}$-tightness of $X \cup Y$, let us assume $d = 1$ and prove that $X \cup Y$ is cyclic. If $d = 1$, then the inequalities of (16) and (17) hold with equalities, and in particular $c_0 = 0$, $c_1 = 1$ and

$$|X \cap Y| = 2|V(X \cap Y)| - 3 + \beta(Y). \tag{18}$$

By $c_0 = 0$ and $c_1 = 1$, the number of connected components in the graph $(V(X) \cap V(Y), X \cap Y)$ is one. If $\beta(Y) = 2$, then $X \cap Y$ is unbalanced cyclic by (18) and hence

Lemma 2.5(3) implies that $X \cup Y$ is cyclic. If $\beta(Y) = 0$, then $Y$ is balanced and, again, Lemma 2.5(2) implies that $X \cup Y$ is cyclic. Thus $X \cup Y$ is $\mathcal{D}$-tight.

(Case 2) If $\beta(X) = 0$, then $\beta(Y) = 0$ and we have $d \leq 6 - 2c_0 - 3c_1$ by (17). By $c_1 \geq 1$, we have three possible pairs $(c_0, c_1) = (0,1), (1,1), (0,2)$. If $(c_0, c_1) = (0,1)$, then $d \leq 3$ and Lemma 2.5 implies that $X \cup Y$ is balanced. Thus, $X \cup Y$ is a balanced $\mathcal{D}$-tight set. If $(c_0, c_1) = (1,1)$ or $(c_0, c_1) = (0,2)$, then $d \leq 1$ and Lemma 2.6 implies that $X \cup Y$ is cyclic. Thus, $X \cup Y$ is a cyclic $\mathcal{D}$-tight set.

This completes the proof. □

**Lemma 7.3.** *Let $(G, \phi)$ be a $\mathcal{D}$-gain graph with $G = (V, E)$ and $X$ and $Y$ be $\mathcal{D}$-tight sets with $X \subseteq Y \subseteq E$. For $e \in E \setminus Y$, if $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X+e)$, then $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y+e)$.*

*Proof.* Since $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X+e)$, the endvertices of $e$ are contained in $V(X)$, implying $V(Y+e) = V(Y)$. If $X$ or $Y$ is not cyclic, then we have $\beta(Y) = \beta(Y+e) = 3$, meaning that $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y+e)$.

We hence assume that $X$ and $Y$ are cyclic, and they are connected by Lemma 7.1. Take a spanning tree $T$ in $G[Y]$ such that $X \cap T$ is a spanning tree of $G[X]$. By Proposition 2.3, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(f) = \mathsf{id}$ for $f \in T$. By Lemma 2.4, there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi'(f) \in \mathcal{C}$ for every $f \in Y$, where $\mathcal{C}$ is the identity group if $Y$ is balanced. Since $f_{\mathcal{D}}(X) = f_{\mathcal{D}}(X+e)$ and $X \subseteq Y$, we have $\phi'(e) \in \bar{\mathcal{C}}$, and hence $f_{\mathcal{D}}(Y) = f_{\mathcal{D}}(Y+e)$ holds. □

We are ready to prove that the family of $\mathcal{D}$-sparse edge subsets is a family of independent sets of a matroid on ground-set $E$. We shall also characterize the rank function of this matroid.

**Theorem 7.4.** *Let $(G, \phi)$ be a $\mathcal{D}$-gain graph with $G = (V, E)$ and $\mathcal{I}$ be the family of all $\mathcal{D}$-sparse edge subsets in $E$. Then $\mathcal{M}_{\mathcal{D}}(G, \phi) = (E, \mathcal{I})$ is a matroid on ground-set $E$. The rank of a set $E' \subseteq E$ in $\mathcal{M}_{\mathcal{D}}(G, \phi)$ is equal to*

$$\min \left\{ \sum_{i=1}^{t} f_{\mathcal{D}}(E'_i) \colon \{E'_1, \dots, E'_t\} \text{ is a partition of } E' \right\}.$$

*Proof.* For a partition $\mathcal{P} = \{E'_1, \dots, E'_t\}$ of $E' \subseteq E$, we denote $\mathrm{val}(\mathcal{P}) = \sum_{i=1}^{t} f_{\mathcal{D}}(E'_t)$. We shall check the following independence axiom of matroids: (I1) $\emptyset \in \mathcal{I}$; (I2) for any $X, Y \subseteq E$ with $X \subseteq Y$, $Y \in \mathcal{I}$ implies $X \in \mathcal{I}$; (I3) for any $E' \subseteq E$, maximal subsets of $E'$ belonging to $\mathcal{I}$ have the same cardinality.

$\mathcal{I}$ obviously satisfies (I1) and (I2). To see (I3), let $E' \subseteq E$ and let $F \subseteq E'$ be a maximal subset of $E'$ in $\mathcal{I}$. Since $F \in \mathcal{I}$ we have $|F| \leq \mathrm{val}(\mathcal{P})$ for all partitions $\mathcal{P}$ of $E'$. We shall prove that there is a partition $\mathcal{P}$ of $E'$ with $|F| = \mathrm{val}(\mathcal{P})$, from which (I3) follows.

Let $J = (V, F)$ denote the subgraph with the edge set $F$. Consider the family $\{F_1, F_2, \dots, F_t\}$ of all maximal $\mathcal{D}$-tight sets in $J$. Since each edge $f \in F$ forms a $\mathcal{D}$-tight set, $\cup_{i=1}^{t} F_i = F$ holds. Since $F_i \cap F_j = \emptyset$ for every pair $1 \leq i < j \leq t$ by Lemma 7.2 and the maximality, $\mathcal{P}_F = \{F_1, F_2, \dots, F_t\}$ is a partition of $F$ and $|F| = \mathrm{val}(\mathcal{P}_F)$ follows.

Based on $\mathcal{P}_F$, we construct a partition $\mathcal{P}$ of $E'$ with $\text{val}(\mathcal{P}) = \text{val}(\mathcal{P}_F) = |F|$. Consider an edge $(u, v) = e \in E' - F$. Since $F$ is a maximal subset of $E'$ in $\mathcal{I}$ we have $F + e \notin \mathcal{I}$. Hence there must be a tight set $X_e$ in $J$ with $u, v \in V(X_e)$ and $X_e + e \notin \mathcal{I}$. $X_e \subseteq F_i$ for some $1 \le i \le t$. Choose such an $F_i$ for every $e \in E' - F$ and define $E_i = F_i \cup \{e : F_i \text{ was chosen for } e\}$. Clearly $\mathcal{P} = \{E_1, E_2, \ldots, E_t\}$ is a partition of $E'$. By Lemma 7.3, $f_{\mathcal{D}}(F_i) = f_{\mathcal{D}}(E_i)$ for every $1 \le i \le t$ and hence $\text{val}(\mathcal{P}) = \text{val}(\mathcal{P}_F) = |F|$. □

The matroid which was introduced and denoted by $\mathcal{M}_{\mathcal{D}}(G, \phi)$ in Theorem 7.4 is called the *$\mathcal{D}$-sparsity matroid* of $(G, \phi)$.

## 7.2 Constructive characterization of maximum $\mathcal{D}$-tight graphs

We now present a constructive characterization of maximum $\mathcal{D}$-tight graphs. Notice that the average vertex degree in a maximum $\mathcal{D}$-tight graph $(G, \phi)$ is four, which means that $G$ has a vertex of degree at most 3 if and only if $G$ is not 4-regular. Thus we shall take a special care of 4-regular $\mathcal{D}$-sparse graphs.

### 7.2.1 0-extension, 1-extension, and loop-1-extension

Before looking at 4-regular graphs and vertices of degree four, we consider the 0-extension, 1-extension, and loop-1-extension operations. Recall that the corresponding inverse operations are called reductions. A reduction is *admissible* if the resulting graph is $\mathcal{D}$-sparse.

**Lemma 7.5.** *Let $(G, \phi)$ be a $\mathcal{D}$-sparse graph with $G = (V, E)$. Applying a 0-extension, 1-extension or loop-1-extension to $G$ results in a $\mathcal{D}$-sparse graph with $|V| + 1$ vertices and $|E| + 2$ edges.*

*Conversely, for any vertex $v$ of degree 2 or 3, the 0-reduction, loop-1-reduction, or some of the 1-reductions at $v$ is admissible if $|V| \ge 2$.*

*Proof.* The proof of the first claim is exactly the same as the proof of Lemma 4.1. (Indeed, we just need to change $2\alpha_{\mathcal{S}}(F)$ with $\beta(F)$ in the proof of Lemma 4.1.)

To see that some reduction is admissible at a vertex $v$ of degree three, we just need to observe that each circuit of $\mathcal{M}(g_{2,3})$ appearing in the proof of Claim 4.3 is also a circuit in $\mathcal{M}_{\mathcal{D}}(G, \phi)$. We can thus apply exactly the same proof as in Lemma 4.2 to conclude that some reduction is admissible at $v$. □

### 7.2.2 2-extension and loop-2-extension

Besides 0-extensions, 1-extensions and loop-1-extensions, we shall introduce *2-extensions* and *loop-2-extensions* for constructing 4-regular $\mathcal{D}$-sparse graphs.

In a *2-extension*, we take two existing edges $e = (v_1, v_2)$ and $f = (v_3, v_4)$ and pinch them by inserting a new vertex $v$. More precisely, a 2-extension removes $e$ and $f$, inserts a new vertex $v$ with four new edges, $e_i$ from $v_i$ to $v$ for each $i = 1, \ldots, 4$. The

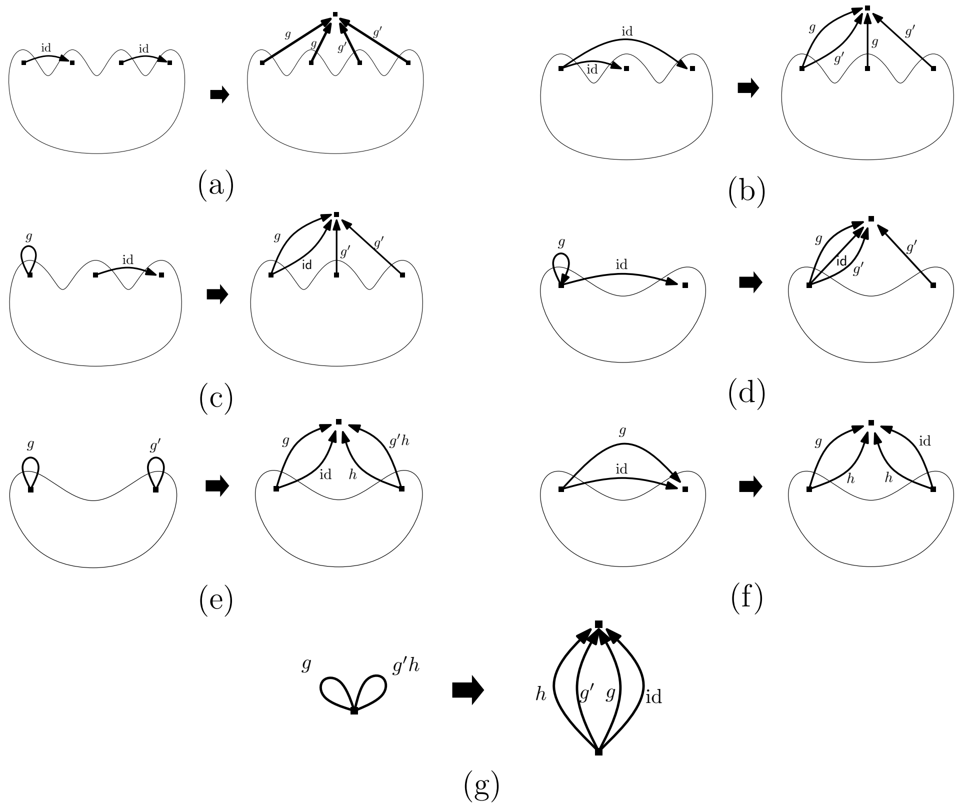



Figure 8: 2-extensions

gain function $\phi$ is extended on $E \cup \{e_1, \ldots, e_4\}$ so that $\phi(e_1) \cdot \phi(e_2)^{-1} = \phi(e)$, $\phi(e_3) \cdot \phi(e_4)^{-1} = \phi(f)$ and it is locally $\mathcal{D}$-sparse, i.e., $\{e_1, \ldots, e_4\}$ is $\mathcal{D}$-sparse. Depending on the multiplicity of the $v_i$'s we have seven cases as shown in Figure 8.

In a *loop-2-extension*, we remove an existing edge $e = (v_1, v_2)$, insert a new vertex $v$, a new loop $l$ at $v$ and two new edges, $e_i$ from $v_i$ to $v$ for each $i = 1, 2$. $\phi$ is extended on $E \cup \{e_1, e_2, l\}$ so that $\phi(e_1) \cdot \phi(e_2)^{-1} = \phi(e)$, $\phi(l) \neq \text{id}$ and it is locally $\mathcal{D}$-sparse. Depending on whether $e$ is a loop or not, we have two cases as shown in Figure 9.

The following lemma shows that these operations preserve $\mathcal{D}$-sparsity.

**Lemma 7.6.** *Let $(G, \phi)$ be a $\mathcal{D}$-sparse graph. Then, any $\mathcal{D}$-gain graph $(G', \phi')$ obtained from $G$ by a 2-extension or a loop-2-extension is $\mathcal{D}$-sparse.*

*Proof.* Suppose that $(G', \phi')$ is obtained by a 2-extension. Let us denote the removed edges by $e$ and $f$ and the new edges by $e_1, \ldots, e_4$ as above. Suppose that there is $F \subseteq E(G')$ that violates the $\mathcal{D}$-sparsity condition. Let $F' = F \setminus \{e_1, \ldots, e_4\}$. Since $\{e_1, \ldots, e_4\}$ satisfies the $\mathcal{D}$-sparsity condition, $F' \neq \emptyset$. Let us add $e$ to $F'$ if $\{e_1, e_2\} \subseteq F$ and add $f$ to $F'$ if $\{e_3, e_4\} \subseteq F$. Observe that $|F'| \geq |F| - 2$, $|V(F)| \geq |V(F')| + 1$ and $\beta(F) \geq \beta(F')$. Since $|F| > f_{\mathcal{D}}(F)$, we obtain $|F'| \geq |F| - 2 > f_{\mathcal{D}}(F) - 2 = 2|V(F)| - 3 + \beta(F) - 2 \geq 2|V(F')| - 3 + \beta(F') = f_{\mathcal{D}}(F')$. This contradicts the $\mathcal{D}$-sparsity of $G$ since $\emptyset \neq F' \subseteq E(G)$. Therefore $(G', \phi')$ is $\mathcal{D}$-sparse.

In the same manner, it can be easily checked that a loop-2-extension also preserves $\mathcal{D}$-sparsity. □

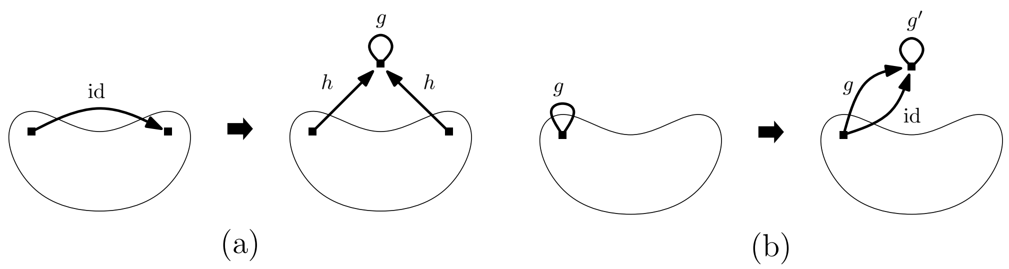



Figure 9: Loop-2-extensions.

We shall define the inverse moves of these operations. Recall that, for a vertex $v$ and two incoming non-loop edges $e_1 = (u, v)$ and $e_2 = (w, v)$, we denote by $e_1 \cdot e_2^{-1}$ a new edge from $u$ to $w$ with gain $\phi(e_1) \cdot \phi(e_2)^{-1}$.

Let $v$ be a vertex of degree four, not incident to a loop, and $e_i = (v_i, v)$ for $i = 1, \ldots, 4$ be the edges incident to $v$, assuming that all of them are oriented to $v$. The *2-reduction* (at $v$) deletes $v$ and adds one of $\{e_1 \cdot e_2^{-1}, e_3 \cdot e_4^{-1}\}$, $\{e_1 \cdot e_3^{-1}, e_2 \cdot e_4^{-1}\}$ and $\{e_1 \cdot e_4^{-1}, e_2 \cdot e_3^{-1}\}$. We sometimes refer to a specific one: the *2-reduction at $v$ through $(e_i, e_j)$ and $(e_k, e_l)$* deletes $v$ and adds $\{e_i \cdot e_j^{-1}, e_k \cdot e_l^{-1}\}$.

Let $v$ be a vertex of degree four, incident to a loop $l$, and $e_i = (v_i, v)$ for $i = 1, 2$ be the non-loop edges incident to $v$, assuming that all of them are oriented to $v$. The *loop-2-reduction* (at $v$) deletes $v$ and adds $e_1 \cdot e_2^{-1}$.

A 2-reduction or loop-2-reduction is said to be *admissible* if the resulting graph is $\mathcal{D}$-sparse.

### 7.2.3 Base graphs

Our main theorem asserts that these operations are sufficient to construct all 4-regular $\mathcal{D}$-sparse graphs from certain classes of $\mathcal{D}$-sparse graphs. Here, the classes can be categorized into three groups: the first group includes special small graphs as in the conventional constructive characterizations, the second group is a class of graphs, which are obtained from cycles by duplicating each edge, and the third one consists of *near-cyclic 4-regular graphs*.

The first group consists of three types of special $\mathcal{D}$-tight graphs, called *trivial graphs*, *fancy triangles*, and *fancy hats*. A *trivial graph* is a $\mathcal{D}$-sparse graph with a single vertex and with two loops as shown in Figure 10(a). The gain function is assigned so that the gains of two loops generate a non-cyclic group.

A *fancy triangle* is a $\mathcal{D}$-gain graph whose underlying graph is obtained from a triangle by adding a loop to each vertex, as shown in Figure 10(b). The gain function is assigned so that it is $\mathcal{D}$-sparse and the triangle is balanced.

A *hat* is a graph obtained from $K_{2,3}$ by adding an edge to the class of cardinality two, and the *fancy hat* is a $\mathcal{D}$-gain graph obtained from the hat by adding a loop to each degree two vertex, as shown in Figure 10(c). The gain function is assigned so that it is $\mathcal{D}$-sparse and the hat is balanced.

The second group consists of $\mathcal{D}$-sparse graphs whose underlying graphs are double cycles, where, for $n \geq 2$, the *double cycle* $C_n^2$ is defined as the graph obtained from the

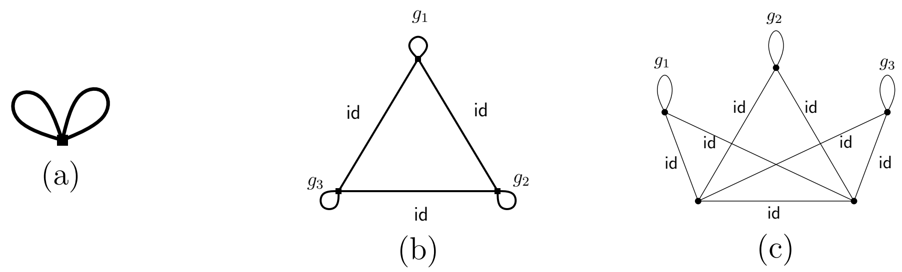



Figure 10: Special graphs: (a) a trivial graph, (b) a fancy triangle, and (c) a fancy hat.

cycle on $n$ vertices by replacing each edge by two parallel edges as shown in Figure 11. As we will see later, key properties of this group depend on whether the order of the underlying dihedral group $\mathcal{D}$ is odd or even.

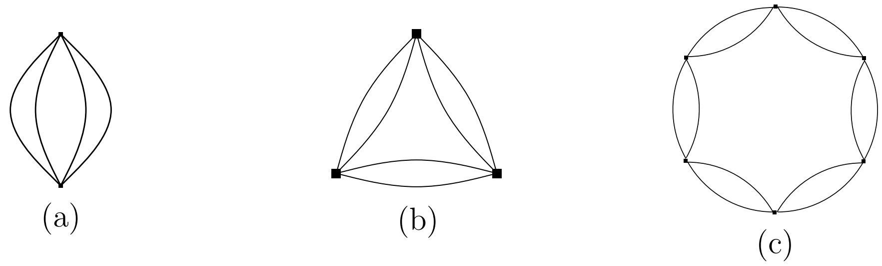



Figure 11: Double cycles: (a)$C_2^2$, (b)$C_3^2$, (c)$C_6^2$.

The third group consists of *near-cyclic* graphs, which, intuitively speaking, are the $\mathcal{D}$-tight graphs closest to $(2,3)$-g-tight graphs. By definition, any $(2,3)$-g-tight graph is also $\mathcal{D}$-sparse. Hence, if we add a new edge with an appropriate gain to a maximum $(2,3)$-g-tight graph, we can obtain a maximum $\mathcal{D}$-tight graph. The following lemma indicates one of the easiest situations in which such an operation works.

**Lemma 7.7.** *Let $(G,\phi)$ be a $(2,3)$-g-sparse graph with $G=(V,E)$, and suppose that there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi(e)\in\mathcal{C}$ for all $e\in E$. If we add a new edge $e$ having a gain in $\mathcal{D}\setminus\bar{\mathcal{C}}$, then $(G+e,\phi)$ is $\mathcal{D}$-sparse.*

*Proof.* Suppose that $(G+e,\phi)$ is not $\mathcal{D}$-sparse. Then there is a subset $F\subseteq E$ such that $|F+e|>f_{\mathcal{D}}(F+e)$. Since $F$ is cyclic, either (i) $\beta(F)=0$ or (ii) $\beta(F)=2$.

By Lemma 7.1(i), $F$ is connected, and clearly the endvertices of $e$ are contained in $V(F)$. Moreover, every cycle in $F+e$ that passes through $e$ has a gain not contained in $\bar{\mathcal{C}}$, as the gain of $e$ is not in $\bar{\mathcal{C}}$. Thus, $F+e$ contains a cycle whose gain is not contained in $\bar{\mathcal{C}}$. This means that (i) $\beta(F)=0$ implies $\beta(F+e)=1$, and (ii) $\beta(F)=2$ implies $\beta(F+e)=3$. In each case, we obtain $\beta(F+e)-\beta(F)\geq 1$. Therefore, $|F|=|F+e|-1>f_{\mathcal{D}}(F+e)-1\geq f_{\mathcal{D}}(F)$, and this contradicts the $\mathcal{D}$-sparsity of $(G,\phi)$ as $\emptyset\neq F\subseteq E$. □

Motivated by this fact we say that a $\mathcal{D}$-sparse graph is *near-cyclic* if removing an edge results in a cyclic graph.

### 7.2.4 Constructive characterizations

We are ready to state our constructive characterization of 4-regular $\mathcal{D}$-sparse graphs. We say that a 4-regular $\mathcal{D}$-sparse graph is a *base graph* if it is a trivial graph, a fancy triangle, a fancy hat, or a near-cyclic graph.

**Theorem 7.8.** *Let $(G, \phi)$ be a $\mathcal{D}$-gain graph. Then, $(G, \phi)$ is 4-regular and $\mathcal{D}$-sparse if and only if it can be built up from a disjoint union of base graphs and $\mathcal{D}$-sparse double cycles by a sequence of 2-extension and loop-2-extension operations.*

We have proved that these operations preserve $\mathcal{D}$-sparsity in Lemma 7.6. The proof of the converse direction will be given in §9, where we will show that a 2-reduction or a loop-2-reduction is admissible at some vertex if the graph is neither a base graph nor a double cycle.

Combining Theorem 7.8 and Lemma 7.5, we obtain the following:

**Theorem 7.9.** *Let $(G, \phi)$ be a $\mathcal{D}$-gain graph. Then, $(G, \phi)$ is maximum $\mathcal{D}$-tight if and only if it can be built up from a disjoint union of base graphs and $\mathcal{D}$-sparse double cycles by a sequence of 0-extension, 1-extension, loop-1-extension, 2-extension and loop-2-extension operations.*

The theorems can be strengthened if the order $k$ of $\mathcal{D}$ is odd, in which case every $\mathcal{D}$-sparse double cycle can be reduced to a trivial graph. To see this, let us prove the following technical lemma.

**Lemma 7.10.** *Let $\mathcal{D}_k$ be a dihedral group of odd order. Let $g_1, g_2, g_3, g_4$ be elements of $\mathcal{D}_k$ such that*

- *$g_1, g_2$ and $g_3$ are distinct non-identity elements,*
- *$\{g_1, g_2, g_3\}$ generates a non-cyclic group, and*
- *$g_4 = \mathsf{id}$.*

*Then, at least one of $\{g_1 g_2^{-1}, g_3 g_4^{-1}\}$, $\{g_1 g_3^{-1}, g_2 g_4^{-1}\}$ and $\{g_1 g_4^{-1}, g_2 g_3^{-1}\}$ generates a non-cyclic group.*

*Proof.* Since $\{g_1, g_2, g_3\}$ generates a non-cyclic group, we may assume that $g_1$ is a reflection $r$ along a line. Suppose that $\{g_1, g_2 g_3^{-1}\}$ is cyclic. Then, $g_2 g_3^{-1} = \mathsf{id}$ or $g_2 g_3^{-1} = r$. Since $g_2 \neq g_3$, we have $g_2 = r g_3$.

If $g_3$ is also a reflection $r'$, which is different from $r$, then $\{g_1 g_2^{-1}, g_3\} = \{r r' r^{-1}, r'\}$. Clearly $r r' r^{-1} \neq id$. If $r r' r^{-1} = r'$ or equivalently $(r r')^2 = id$ then $k$ has to be even which is a contradiction. Thus $\{g_1 g_2^{-1}, g_3\}$ generates a non-cyclic group.

If $g_3$ denotes a rotation $C$, then $\{g_1 g_3^{-1}, g_2\} = \{r C^{-1}, r C\}$. Since $r C^{-1}$ and $r C$ are non-identity and reflections, if they generate a cyclic group, then $r C^{-1} = r C$, implying $C^2 = \mathsf{id}$. This contradicts the parity of $k$. □

**Lemma 7.11.** *Let $\mathcal{D}_k$ be a dihedral group of odd order $k \geq 1$, and $(G, \phi)$ be a $\mathcal{D}_k$-sparse double cycle $C_n^2$ with $n \geq 2$. Then, a 2-reduction is admissible at some vertex.*

Figure 12: $G'$.

*Proof.* Let $v$ be a vertex, and we denote the edges incident to $v$ by $e_i$ for $i = 1, \ldots, 4$. Without loss of generality, we assume that all of $e_i$ are oriented to $v$, $e_1$ and $e_2$ are parallel, and $e_3$ and $e_4$ are parallel.

We first perform the 2-reduction at $v$ through $(e_1, e_2)$ and $(e_3, e_4)$. Then, the resulting graph $(G', \phi')$ is, as shown in Figure 12, a path of parallel edges with loops at its endvertices. Using the fact that each 2-cycle is unbalanced in $G$, it is easy to check that $|F| \leq 2|V(F)| - 3$ for any balanced $F \subseteq E(G')$ and $|F| \leq 2|V(F)| - 1$ for any proper subset $F \subset E(G')$. Thus, $(G', \phi')$ is $\mathcal{D}_k$-sparse if $E(G')$ is not cyclic.

Suppose that $E(G')$ is cyclic. Then, by Lemma 2.4, we may assume that there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}_k$ such that all gains of $E(G')$ are contained in $\mathcal{C}$. Let $a = \phi(e_1 \cdot e_2^{-1})$ and $a' = \phi(e_3 \cdot e_4^{-1})$. Since any 2-cycle of $G$ is unbalanced, $a$ and $a'$ are non-identity. Moreover, $\phi(e_1) \cdot \phi(e_2)^{-1} = a \in \mathcal{C}$ and $\phi(e_3) \cdot \phi(e_4)^{-1} = a' \in \mathcal{C}$. Hence, by using some elements $b_1, b_2 \in \mathcal{D}_k$, we can express $\phi(e_i)$ by

$$\phi(e_1) = ab_1, \quad \phi(e_2) = b_1, \quad \phi(e_3) = a'b_2, \quad \phi(e_4) = b_2.$$

Let us perform the switching operation at $v$ with $b_2$. Then we have

$$\phi(e_1) = ab, \quad \phi(e_2) = b, \quad \phi(e_3) = a', \quad \phi(e_4) = \mathsf{id}, \tag{19}$$

where $b = b_1 b_2^{-1}$. Notice that $\phi(e) \in \mathcal{C}$ for all $e \in E(G) \setminus \{e_1, e_2\}$. Since $(G, \phi)$ is maximum $\mathcal{D}_k$-tight, we must have $b \notin \bar{\mathcal{C}}$.

We now consider the remaining two possible 2-reductions at $v$. In each reduction, the resulting underlying graph is $C^2_{n-1}$, and it can be easily checked that the 2-reduction is admissible if one of the resulting $\mathcal{D}_k$-gain graphs $(G_1, \phi_1)$ and $(G_2, \phi_2)$ is not cyclic.

To see that $(G_1, \phi_1)$ or $(G_2, \phi_2)$ is not cyclic, let $g_i = \phi(e_i)$ for $i = 1, \ldots, 4$. Observe that $\{g_1, \ldots, g_4\}$ satisfies the condition of Lemma 7.10. Since $\{g_1 \cdot g_2^{-1}, g_3 \cdot g_4^{-1}\}$ generates a cyclic group, this implies, by Lemma 7.10, that $\{g_1 \cdot g_3^{-1}, g_2 \cdot g_4^{-1}\}$ or $\{g_1 \cdot g_4^{-1}, g_2 \cdot g_3^{-1}\}$ is not cyclic, implying that $(G_1, \phi_1)$ or $(G_2, \phi_2)$ is not cyclic. □

Combining Theorem 7.9 and Lemma 7.11, we obtain the following constructive characterization.

**Theorem 7.12.** *Let $\mathcal{D}_k$ be a dihedral group of odd order $k$. Then a $\mathcal{D}_k$-gain graph $(G, \phi)$ is maximum $\mathcal{D}_k$-tight if and only if it can be built up from a disjoint union of base graphs by a sequence of 0-extension, 1-extension, loop-1-extension, 2-extension and loop-2-extension operations.*

Lemma 7.11 does not hold for dihedral groups of even order. See Figure 13 for examples. In the next section we will see how the combinatorial properties given in the preceeding two lemmas lead to substantial differences between the rigidity properties of frameworks with odd or even order dihedral symmetry.

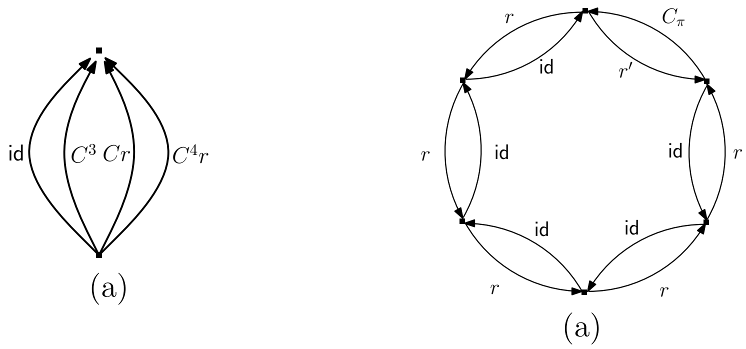



Figure 13: Double cycles without admissible 2-reductions. (a) a $\mathcal{D}_6$-sparse $C_2^2$, where $C$ denotes a 6-fold rotation and $r$ denotes a reflection. (b) a $\mathcal{D}_2$-sparse $C_6^2$, where $C_\pi$ denotes a 2-fold rotation and $r$ and $r'$ denote distinct reflections.

# 8 Combinatorial Characterization of Generic Rigidity with Dihedral Symmetry

In this section we discuss our combinatorial characterization of symmetry-forced infinitesimal rigidity with dihedral symmetry. We begin with a necessary condition based on Lemma 5.2.

**Lemma 8.1.** *Let $\mathcal{D}_k$ be a dihedral group of order $k \geq 2$, and $(H, p)$ be a generic $(\mathcal{D}_k, \rho)$-symmetric framework with a free action $\rho$. If $(H, p)$ is symmetry-forced infinitesimally rigid, then the quotient gain graph contains a spanning maximum $\mathcal{D}_k$-tight subgraph.*

*Proof.* Let $(H/\mathcal{D}_k, \phi)$ be the quotient gain graph of $H$ and $\tilde{p}$ be a joint configuration of the vertex orbits $V(H/\mathcal{D}_k)$ corresponding to $p$. By Theorem 5.1, it suffices to prove that if $O(H/\mathcal{D}_k, \phi, \tilde{p})$ is row independent, then $(H/\mathcal{D}_k, \phi)$ is $\mathcal{D}_k$-sparse.

Since $\tilde{p}$ is generic, according to the exact value given in Example 5.2, we have

$$\mathrm{iso}_{\langle F\rangle_{\phi,u}}(\tilde{p}(F)) = \begin{cases} 3 & \text{(if } F \text{ is balanced)} \\ 1 & \text{(if } F \text{ is unbalanced and cyclic)} \\ 0 & \text{(otherwise)} \end{cases}$$

for any connected $F \subseteq E(H/\mathcal{D}_k)$ and $u \in V(F)$, where $\tilde{p}(F) = \{g\tilde{p}(v) \colon v \in V(F), g \in \mathcal{D}_k\}$. By this and Lemma 5.2, we have that $|F| \leq f_{\mathcal{D}_k}(F)$ for any $F \subseteq E(H/\mathcal{D}_k)$. In other words, $H/\mathcal{D}_k$ is $\mathcal{D}_k$-sparse. $\square$

In Section 8.1 we shall prove that $\mathcal{D}_k$-sparsity is also sufficient for row independence when $k \geq 3$ is odd. On the other hand, in Section 8.2 we give a family of examples showing that this implication does not always hold when $k$ is even.

## 8.1 Combinatorial characterization of symmetry-forced rigidity with odd order dihedral symmetry

Our goal of this subsection is to prove the following characterization of symmetry-forced infinitesimal rigidity.

**Theorem 8.2.** *Let $\mathcal{D}_k$ be a dihedral group of odd order $k \geq 3$, and $(H, p)$ be a generic $(\mathcal{D}_k, \rho)$-symmetric framework with a free action $\rho$. Then $(H, p)$ is symmetry-forced infinitesimally rigid if and only if the quotient gain graph contains a spanning maximum $\mathcal{D}_k$-tight subgraph.*

Necessity follows from Lemma 8.1. Therefore, by Theorem 5.1, it suffices to prove that, for a maximum $\mathcal{D}_k$-tight graph $(G, \phi)$, there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent. The proof of this claim is based on the constructive characterization of maximum $\mathcal{D}_k$-tight graphs formulated in Section 7.

By Theorem 7.12, $(G, \phi)$ can be constructed from a disjoint union of base graphs by 0-extension, 1-extension, loop-1-extension, 2-extension, and loop-2-extension operations. Therefore, what we have to prove is that (i) the orbit rigidity matrix of each base graph is row independent and (ii) each extension preserves the row independence of the orbit rigidity matrix by extending $p$ appropriately. (i) will be solved in Lemma 8.4 whereas (ii) will be solved in Lemmas 8.5 and 8.7. Note that there is no parity condition in these lemmas.

In the rest of this section, we identify $\mathcal{D}_k$ with the symmetry group of a regular $k$-gon, which consists of $k$-fold rotations around the origin and reflections along (fixed) lines. For a line $L$ through the origin, we denote by $L^\perp$ the orthogonal complement of $L$, that is, the line orthogonal to $L$ and through the origin. We first note an elementary fact from geometry.

**Lemma 8.3.** *Let $g \in \mathcal{O}(\mathbb{R}^2)$.*

- *If $g$ is the reflection along a line $L$, then $(I_2 - g)p \in L^\perp \setminus \{0\}$ for any $p \in \mathbb{R}^2 \setminus L$.*
- *If $g$ is a rotation, then $(2I_2 - g - g^{-1})p \in \mathrm{span}\{p\} \setminus \{0\}$ for any $p \in \mathbb{R}^2 \setminus \{0\}$.*

**Lemma 8.4.** *Let $(G, \phi)$ be a base graph. Then, there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent.*

*Proof.* (Case 1) Suppose that $(G, \phi)$ is a trivial graph. Let $v$ be the vertex. Take $p : V(G) \to \mathbb{R}^2$ such that $p(v)$ does not lie on reflection lines $L$ in $\mathcal{D}_k$ and their orthogonal complements $L^\perp$. Then, $O(G, \phi, p)$ consists of two row vectors, which are linearly independent by Lemma 8.3.

(Case 2) Suppose that $(G, \phi)$ is a fancy triangle. Let $V(G) = \{v_1, v_2, v_3\}$, and let $l_i$ be the loop attached to $v_i$. Also, we denote $g_i = \phi(l_i)$ and $p_i = p(v_i)$ for $i = 1, 2, 3$. Recall that the triangle of $G$ is balanced by definition, and hence we may assume that $\phi(e) = \mathsf{id}$ for all non-loop edges $e$. Since $(G, \phi)$ is not cyclic, there is a loop whose gain is a reflection. Hence, without loss of generality, we may assume that $g_1$ is the reflection along the vertical line $L$. Also, since $(G, \phi)$ is not cyclic, we may assume that $g_2 \neq g_1$.

We take $p : V(G) \to \mathbb{R}^2$ such that

- $p_1$ is any point not on $L \cup L^{\perp}$,
- $p_2$ is any point on the horizontally right side of $p_1$ such that the line through $p_1$ and $p_2$ is parallel to $L^{\perp}$ with $p_2 \notin L$, and
- $p_3$ is any point such that (i) $p_3$ is not on the line through $p_2$ and the origin and (ii) the line through $p_2$ and $p_3$ is neither parallel nor orthogonal to any of the reflection lines of $\mathcal{D}_k$. See Figure 14.

We show that $O(G, \phi, p)$ is row independent.

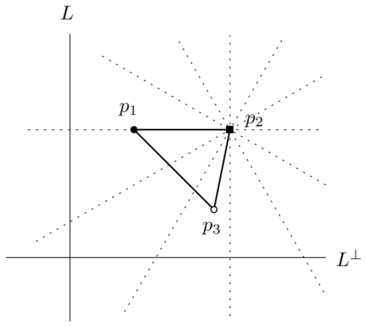



Figure 14: Proof of Lemma 8.4

Let us consider any infinitesimal motion $m : V(G) \to \mathbb{R}^2$. Since the triangle of $G$ is balanced, $m$ should be a trivial infinitesimal motion of the framework $(G', p)$ for the graph $G'$ obtained from $G$ by removing the loops. By Lemma 8.3, $m(v_1) \in L$ holds because of the row associated with $l_1$. Similarly, since $g_1 \neq g_2$, Lemma 8.3 implies that $m(v_2) \notin L$. However, since $p_1 - p_2 \in L^{\perp}$, if we consider the equation corresponding to the edge $(v_1, v_2)$, we have

$$0 = \langle m(v_1) - m(v_2), p_1 - p_2 \rangle = -\langle m(v_2), p_1 - p_2 \rangle,$$

implying $m(v_2) = 0$. (Hence, $m$ is an infinitesimal rotation of $(G', p)$ around $p_2$.) This implies that $m(v_3)$ is orthogonal to $p_2 - p_3$. However, because of the row associated with $l_3$, $m(v_3)$ cannot be a nonzero vector orthogonal to $p_2 - p_3$. In other words, $m(v_3) = 0$ and hence $m(v_1) = 0$. Since any infinitesimal motion is zero, we conclude that $O(G, \phi, p)$ is row independent.

(Case 3) Suppose that $(G, \phi)$ is a fancy hat. The proof is exactly the same as Case 2. Indeed, we just need to replace a balanced triangle with a hat, which is balanced by definition and also admits only a trivial infinitesimal motion when it is realized in a generic position.

(Case 4) Suppose that $(G, \phi)$ is near-cyclic. Then there is an edge $e$ such that $G - e$ is cyclic. Let $\mathcal{C} = \langle E - e \rangle_v$ for a vertex $v \in V(G)$, and denote $g_e = \phi(e)$. We may assume that the labels of the edges in $E - e$ are all contained in $\mathcal{C}$. Then $g_e \notin \bar{\mathcal{C}}$.

By Theorem 6.3, $O(G - e, \phi, p)$ is row independent for any $\mathcal{D}_k$-generic joint configuration $p : V(G) \to \mathbb{R}^2$, and the kernel space of $O(G - e, \phi, p)$ is one-dimensional.

Let $m : i \in V(G) \mapsto m_i \in \mathbb{R}^2$ be a nonzero infinitesimal motion. Also, denote $p_i = p(i)$ for $i \in V(G)$. Then either (i) $\mathcal{C}$ is the group of the reflection along a line $L$, in which case there is a $t \in L$ such that $m_i = t$ for all $i \in V(G)$, or (ii) $\mathcal{C}$ is a group of rotations, in which case $m_i = C_{\pi/2}p_i$ for $i \in V(G)$. We show that $m$ does not satisfy the equation associated with $e = (i, j)$:

$$\langle p_i - g_e p_j, m_i - g_e m_j \rangle = 0. \tag{20}$$

First suppose that $\mathcal{C}$ is the group of the reflection along a line $L$. Then (20) implies

$$0 = \langle p_i - g_e p_j, t - g_e t \rangle = \langle (I_2 - g_e^{-1})p_i + (I_2 - g_e)p_j, t \rangle.$$

Thus $(I_2 - g_e^{-1})p_i + (I_2 - g_e)p_j \in L^\perp$. As $p$ is generic, the only possible situation is that $p_i = p_j$ and $g_e$ is the reflection along $L$ by Lemma 8.3. This however implies that $G$ is cyclic, a contradiction. Thus, $m$ does not satisfy (20).

Next suppose that $\mathcal{C}$ is a group of rotations. If $e$ is a loop (and hence $p_i = p_j$), then the left side of (20) becomes

$$\langle (I_2 - g_e)p_i, (I_2 - g_e)m_i \rangle = \langle (I_2 - g_e)p_i, (I_2 - g_e)C_{\pi/2}p_i \rangle.$$

Note that $g_e$ is a reflection by $g_e \notin \bar{\mathcal{C}}$, and thus this inner product is nonzero by Lemma 8.3. If $e$ is not a loop, by $\langle p_i - g_e p_j, C_{\pi/2}(p_i - g_e p_j) \rangle = 0$, (20) becomes

$$\begin{aligned} 0 = \langle p_i - g_e p_j, m_i - g_e m_j \rangle &= \langle p_i - g_e p_j, C_{\pi/2}p_i - g_e C_{\pi/2}p_j \rangle \\ &= \langle p_i - g_e p_j, (C_{\pi/2}g_e - g_e C_{\pi/2})p_j \rangle. \end{aligned}$$

Since $p$ is generic and $p_i \neq p_j$, we have $C_{\pi/2}g_e = g_e C_{\pi/2}$. Since $g_e$ is a reflection, basic properties of the dihedral groups imply that $g_e C_{\pi/2} = C_{\pi/2}^{-1} g_e$. These equalities imply $C_{\pi/2} = C_{\pi/2}^{-1}$, a contradiction. □

The next two lemmas show that loop-2-extensions and 2-extensions preserve the independence of rigidity matrices.

**Lemma 8.5.** *Let $(G, \phi)$ be a maximum $\mathcal{D}_k$-tight graph with $k \geq 2$ and $(G', \phi')$ a maximum $\mathcal{D}_k$-tight graph obtained from $(G, \phi)$ by a loop-2-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

*Proof.* We may assume that $p$ is $\mathcal{D}_k$-generic. Suppose that $G'$ is obtained from $G$ by a loop-2-extension, by removing an existing edge $e$, adding a new vertex $v$ with new non-loop edges $e_1$ and $e_2$ and a new loop $l$ incident to $v$. (See Figure 9.) We may assume that $e_1$ and $e_2$ are outgoing from $v$. Let $u_i$ be the other endvertex of $e_i$ and let $g_i = \phi'(e_i)$ for $i = 1, 2$. By the definition of loop-2-extension, $\phi(e) = g_1^{-1}g_2$. Also, denote $h = \phi'(l)$.

Let $p_i = p(u_i)$ for $i = 1, 2$. Note that $g_1 p_1 \neq g_2 p_2$, as $G'$ is $\mathcal{D}_k$-sparse and $p$ is $\mathcal{D}_k$-generic. Let $L$ be the line through $g_1 p_1$ and $g_2 p_2$. We take a point $q \in L \setminus \{g_1 p_1, g_2 p_2\}$, and define $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for $w \in V(G)$ and $p'(v) = q$. $O(G', \phi', p')$ is then described as follows: if $u_1 \neq u_2$

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $l$ | $(2I_2 - h - h^{-1})q$ | $0$ | $0$ | $0$ |
| $e_1$ | $q - g_1 p_1$ | $p_1 - g_1^{-1}q$ | $0$ | $0$ |
| $e_2$ | $q - g_2 p_2$ | $0$ | $p_2 - g_2^{-1}q$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | | |

whereas, if $u_1 = u_2$ (and hence $p_1 = p_2$),

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $(2I_2 - h - h^{-1})q$ | $0$ | $0$ |
| $e_1$ | $q - g_1p_1$ | $p_1 - g_1^{-1}q$ | $0$ |
| $e_2$ | $q - g_2p_1$ | $p_1 - g_2^{-1}q$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

Since $q \in L \setminus \{g_1p_1, g_2p_2\}$, $q - g_ip_i$ is a scalar multiple of $g_1p_1 - g_2p_2$ for $i = 1, 2$. Hence, as in the proof of Lemma 6.1, by multiplying the rows of $e_1$ and $e_2$ by appropriate scalars and then subtracting the row of $e_1$ from that of $e_2$, $O(G', \phi', p')$ becomes one of the following matrices,

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $l$ | $(2I_2 - h - h^{-1})q$ | $0$ | $0$ | $0$ |
| $e_1$ | $g_1p_1 - g_2p_2$ | $-g_1^{-1}(g_1p_1 - g_2p_2)$ | $0$ | $0$ |
| $e_2$ | $0$ | $p_1 - g_1^{-1}g_2p_2$ | $p_2 - g_2^{-1}g_1p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | | |

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $(2I_2 - h - h^{-1})q$ | $0$ | $0$ |
| $e_1$ | $g_1p_1 - g_2p_1$ | $-g_1^{-1}(g_1p_1 - g_2p_1)$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - g_1^{-1}g_2 - g_2^{-1}g_1)p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

depending on whether $u_1 \neq u_2$ or $u_1 = u_2$. The right-bottom block together with the row of $e_2$ forms $O(G, \phi, p)$, which is row independent. Hence, $O(G', \phi', p')$ is row independent if and only if $\{(2I_2 - h - h^{-1})q, g_1p_1 - g_2p_2\}$ is linearly independent. We have the following sufficient condition for linear independence.

**Claim 8.6.** *If there is no point $q \in L \setminus \{g_1p_1, g_2p_2\}$ such that $\{(2I_2 - h - h^{-1})q, g_1p_1 - g_2p_2\}$ is linearly independent, then either*

**(1)** *$u_1 = u_2$ and $h$ is the reflection along the line orthogonal to $L$ with $h = g_2g_1^{-1}$, or*

**(2)** *$u_1 = u_2$, $h$ is a rotation, and $g_2g_1^{-1}$ is the 2-fold rotation.*

*Proof.* We split the proof into two cases.

Suppose that $h$ is the reflection along some line $R$ through the origin. By Lemma 8.3, $(2I_2 - h - h^{-1})q$ is orthogonal to $R$. This means that, if $\{(2I_2 - h - h^{-1})q, g_1p_1 - g_2p_2\}$ is dependent, $L$ is orthogonal to $R$. Since $p$ is $\mathcal{D}_k$-generic, $L$ cannot be orthogonal to reflection line $R$ if $p_1 \neq p_2$. Thus, $p_1 = p_2$ (and hence $u_1 = u_2$), and $h = g_2g_1^{-1}$ as $p$ is $\mathcal{D}_k$-generic.

Suppose that $h$ is a rotation. By Lemma 8.3, $(2I_2 - h - h^{-1})q$ is a scalar multiple of $q$. Hence, if $\{(2I_2 - h - h^{-1})q, g_1p_1 - g_2p_2\}$ is dependent for any $q \in L \setminus \{g_1p_1, g_2p_2\}$, $L$ passes through the origin. Since $p$ is $\mathcal{D}_k$-generic, $L$ passes through the origin if and only if $p_1 = p_2$ (and hence $u_1 = u_2$) and $g_2p_1$ is the antipodal point of $g_1p_1$. Observe that $g_2p_1$ is the antipodal point of $g_1p_1$ if and only if $g_2g_1^{-1}$ is the 2-fold rotation as $p$ is $\mathcal{D}_k$-generic. $\square$

By Claim 8.6 we may focus on cases (1) and (2) of Claim 8.6.

Case (1). Suppose that $u_1 = u_2$ and $h$ is the reflection along $R$ with $h = g_2 g_1^{-1}$, where $R$ is the line orthogonal to $L$ and through the origin. Note that $\phi(e) = g_1^{-1} g_2$ is a reflection since $g_2 g_1^{-1}$ is a reflection. (A conjugate of a reflection is also a reflection.)

We take a point $x \in \mathbb{R}^2 \setminus (L \cup R)$ and redefine $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for $w \in V(G)$ and $p'(v) = x$. Then, the orbit rigidity matrix becomes

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $(I_2 - h)x$ | $0$ | $0$ |
| $e_1$ | $x - g_1 p_1$ | $p_1 - g_1^{-1}x$ | $0$ |
| $e_2$ | $x - g_2 p_1$ | $p_1 - g_2^{-1}x$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

By subtracting the row of $e_1$ from that of $e_2$, it changes to

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $(I_2 - h)x$ | $0$ | $0$ |
| $e_1$ | $x - g_1 p_1$ | $-g_1^{-1}(x - g_1 p_1)$ | $0$ |
| $e_2$ | $g_1 p_1 - g_2 p_1$ | $g_1^{-1}x - g_2^{-1}x$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

By Lemma 8.3, $(I_2 - h)x$ is orthogonal to $R$. Since $R$ is orthogonal to $L$, we deduce that $(I_2 - h)x$ is a scalar multiple of $g_1 p_1 - g_2 p_1$. Thus, by subtracting a scalar multiple of the first row, the row of $e_2$ is changed to the following form:

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_2$ | $0$ | $g_1^{-1}x - g_2^{-1}x$ | $0$ |

Moreover, $g_1^{-1}x - g_2^{-1}x = (I_2 - g_2^{-1} g_1) g_1^{-1} x = (I_2 - \phi(e)) g_1^{-1} x$, which is a scalar multiple of $(2I_2 - \phi(e) - \phi(e)^{-1})p_1$ by Lemma 8.3 (using the fact that $\phi(e)$ is a reflection). Thus, by multiplying the row of $e_2$ by another scalar, the matrix is changed to

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $(I_2 - h)x$ | $0$ | $0$ |
| $e_1$ | $x - g_1 p_1$ | $-g_1^{-1}(x - g_1 p_1)$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - \phi(e) - \phi(e)^{-1})p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

where the right-bottom block together with the row of $e_2$ forms $O(G, \phi, p)$, which is row independent, and the left-top block is also row independent as $(I_2 - h)x \in R^{\perp}$ and $x - g_1 p_1 \notin R^{\perp}$ by $x \notin L$ and $g_1 p_1 \in L$. Thus, $O(G', \phi', p')$ is row independent.

Case (2). Suppose that $u_1 = u_2$, $h$ is a rotation, and $g_2 g_1^{-1}$ is the 2-fold rotation. We redefine $p' : V(G') \to \mathbb{R}^2$ such that $p'(w) = p(w)$ for $w \in V(G)$ and $p'(v) = 0$. Let us consider the rank of $O(G', \phi', p')$. Since $p'(v) = 0$, the row of $l$ is a zero vector in $O(G', \phi', q')$. We hence imaginary put $d \in \mathbb{R}^2$ in place of $(2I - h - h^{-1})p'(v)$, where $d$ is a vector linearly independent from $(g_2 - g_1)p_1$. The resulting matrix, denoted by

$\bar{O}(G', \phi', p')$, can be written in the following way:

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $d$ | $0$ | $0$ |
| $e_1$ | $-g_1 p_1$ | $p_1$ | $0$ |
| $e_2$ | $-g_2 p_1$ | $p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

We first compute the rank of $\bar{O}(G', \phi', p')$. To do this first we recall that $g_2 g_1^{-1}$ is the 2-fold rotation. This means that $L$ contains the origin, and $g_1 p_1 + g_2 p_1 = 0$. Also, since $\phi(e) = g_1^{-1} g_2$ is a rotation, $p_1$ is proportional to $(2I_2 - \phi(e) - \phi(e)^{-1})p_1$ by Lemma 8.3. Therefore, by appropriate row operations, $\bar{O}(G', \phi', p')$ will look like this:

| | $v$ | $u_1$ | |
|---|---|---|---|
| $l$ | $d$ | $0$ | $0$ |
| $e_1$ | $(g_2 - g_1)p_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - \phi(e) - \phi(e)^{-1})p_1$ | $0$ |
| $E(G) - e$ | $0$ | $O(G - e, \phi, p)$ | |

where the right-bottom block together with the row of $e_2$ forms $O(G, \phi, p)$, which is row independent, and the left-top block is also row independent by the choice of $d$. Thus, $\bar{O}(G', \phi', p')$ is row independent.

To avoid the situation where $p'(v) = 0$, we continuously perturb $p'(v)$ in the direction of $d$. To see the perturbation more precisely, for each $t \in \mathbb{R}$, let us define $p'_t : V \cup \{v\} \to \mathbb{R}^2$ by $p'_t(v) = td$ and $p'_t(u) = p'(u)$ for $u \in V$. Then, observe that for all $t \in \mathbb{R} \setminus \{0\}$ the row of $l$ in $O(G', \phi', p'_t)$ is a nonzero scalar multiple of that of $l$ in $\bar{O}(G', \phi', p')$ by Lemma 8.3. Therefore, rank $O(G', \phi', p'_t) =$ rank $\bar{O}(G', \phi', p'_t)$ for all $t \in \mathbb{R} \setminus \{0\}$. Since $\bar{O}(G', \phi', p'_0) = \bar{O}(G', \phi', p')$ and the latter matrix is row independent, it follows that $\bar{O}(G', \phi', p'_t)$ is row independent for almost all $t$. This in turn implies that $O(G', \phi', p'_t)$ is row independent for almost all $t \in \mathbb{R} \setminus \{0\}$.

This complete the proof of the lemma. □

**Lemma 8.7.** *Let $(G, \phi)$ be a maximum $\mathcal{D}_k$-tight graph with $k \geq 2$ and $(G', \phi')$ a maximum $\mathcal{D}_k$-tight graph obtained from $(G, \phi)$ by a 2-extension. If there is a mapping $p : V(G) \to \mathbb{R}^2$ such that $O(G, \phi, p)$ is row independent, then there is a mapping $p' : V(G') \to \mathbb{R}^2$ such that $O(G', \phi', p')$ is row independent.*

*Proof.* We may assume that $p$ is $\mathcal{D}_k$-generic. Suppose that $G'$ is obtained from $G$ by a 2-extension, by removing two existing edges $e$ and $f$ and adding a new vertex $v$ with new non-loop edges $e_1, e_2, e_3, e_4$ incident to $v$. (See Figure 8.) We may assume that $e_i$ is outgoing from $v$, and $e = e_1^{-1} \cdot e_2$ and $f = e_3^{-1} \cdot e_4$. Let $u_i$ be the other endvertex of $e_i$ and let $g_i = \phi'(e_i)$. We then have $\phi(e) = g_1^{-1} g_2$ and $\phi(f) = g_3^{-1} g_4$.

Let $p_i = p(u_i)$ for $i = 1, \ldots, 4$, $L$ be the line through $g_1 p_1$ and $g_2 p_2$, and $L'$ be the line through $g_3 p_3$ and $g_4 p_4$. We have the following elementary geometric observation.

**Claim 8.8.** *(i) No three points among $\{g_i p_i : i = 1, \ldots, 4\}$ are colinear.*

*(ii) If $L$ and $L'$ are parallel, then the following holds:*

- *$L \neq L'$,*
- *$u_1 = u_2$ and $u_3 = u_4$, and*
- *$g_2 g_1^{-1}$ is the reflection along $L^\perp$ with $g_2 g_1^{-1} = g_4 g_3^{-1}$.*

*Proof.* The first claim follows from the proof of Claim 6.2.

For the second claim, suppose that $L$ and $L'$ are parallel. Without loss of generality, we have the following four cases: (i) $p_1 \notin \{p_2, p_3, p_4\}$, (ii) $p_1 = p_2 = p_3 = p_4$, (iii) $p_1 = p_2 \neq p_3 = p_4$, and (iv) $p_1 = p_3 \neq p_2 = p_4$.

In case (i), the $\mathcal{D}_k$-genericity of $p$ implies that $g_1 p_1$ has no relation to the other three points, and hence $L$ and $L'$ intersect at a point.

In case (ii), $g_1 p_1, \ldots, g_4 p_4$ lie on a circle $C$. Moreover, since $u_1 = u_2 = u_3 = u_4$, $e$ and $f$ are loops attached to a vertex (i.e., the 2-extension is type (g) of Figure 8). This implies that the group generated by $\{g_1^{-1} g_2, g_3^{-1} g_4\}$ is not cyclic by the $\mathcal{D}_k$-sparsity of $G$.

Now, $L$ is the line through $g_1 p_1$ and $g_2 p_1$ while $L'$ is the line through $g_3 p_1$ and $g_4 p_1$. We have two subcases depending on whether $g_2 g_1^{-1}$ is a reflection or a rotation.

(ii-1) If $g_2 g_1^{-1}$ is a reflection, then it is the reflection along the bisector $L^\perp$ of $g_1 p_1$ and $g_2 p_1$. If $L$ and $L'$ are parallel, then this reflection also sends $g_3 p_3$ to $g_4 p_4$. This means that $g_2 g_1^{-1}$ is the reflection along $L^\perp$ with $g_2 g_1^{-1} = g_4 g_3^{-1}$, which implies the statement.

(ii-2) If $g_2 g_1^{-1}$ is a rotation, $g_1 p_1$ and $g_2 p_1$ are vertices of a regular $k$-gon inscribing $C$. Since $p$ is generic, if $L'$ is parallel to $L$, $g_3 p_1$ and $g_4 p_1$ are also vertices of this regular $k$-gon, and hence $g_4 g_3^{-1}$ is also a rotation. Since a conjugate of a rotation is also a rotation, we deduce that $g_1^{-1} g_2$ and $g_3^{-1} g_4$ are rotations as well. This however contradicts the fact that $\langle g_1^{-1} g_2, g_3^{-1} g_4 \rangle$ is not cyclic.

In case (iii), $L$ is the line through $g_1 p_1$ and $g_2 p_1$ while $L'$ is the line through $g_3 p_3$ and $g_4 p_3$. Observe that, if $g_2 g_1^{-1}$ is a rotation, the line $L$ can have any slope, by moving $p_1$. Therefore, if $L$ and $L'$ are parallel for generic $p$, then $g_2 g_1^{-1}$ and $g_4 g_3^{-1}$ are both reflections. When $g_2 g_1^{-1}$ is a reflection, it is the reflection along the bisector $L^\perp$ of $g_1 p_1$ and $g_2 p_1$. As $g_4 g_3^{-1}$ is a reflection and $L$ and $L'$ are parallel, $g_4 g_3^{-1}$ is the reflection along $L^\perp$, which implies the statement.

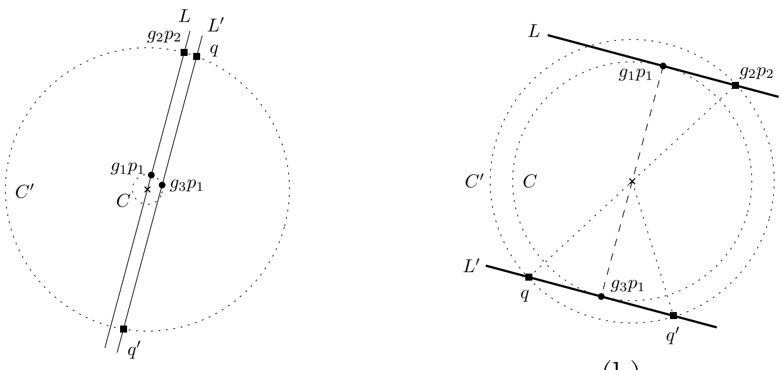
In case (iv), $L$ is the line through $g_1 p_1$ and $g_2 p_2$ and $L'$ is the line through $g_3 p_1$ and $g_4 p_2$. Also, $u_1 = u_3 \neq u_2 = u_4$ implies that $\{e, f\}$ forms a 2-cycle in $G'$ (i.e, the 2-extension is type (f) in Figure 8). Hence, $\phi(e) \neq \phi(f)$, and equivalently, $g_1^{-1} g_2 \neq g_3^{-1} g_4$. This implies

$$g_1 g_3^{-1} \neq g_2 g_4^{-1}. \tag{21}$$

We prove that $L$ and $L'$ cannot be parallel if $p$ is generic.

Let $C$ be the circle whose center is the origin and which passes through $g_1 p_1$ (and hence through $g_3 p_1$). We split the proof into two cases depending on whether $g_3 g_1^{-1}$ is the 2-fold rotation $C_\pi$ or not.

(iv-1) Suppose that $g_3 g_1^{-1} \neq C_\pi$. Let $C'$ be a circle whose center is the origin and the diameter is much larger than that of $C$. We shall relocate $g_2 p_2$ on $C'$ such that $g_2 p_2$ is on the line through $g_1 p_1$ and the origin as shown in Figure 15(a). Then, if $L$




(a) (b)

Figure 15: Proof of case (iv) in Claim 8.8.

and $L'$ are parallel, we have only two possible locations $q$ and $q'$ for $g_4p_2$ (as shown in Figure 15(a)). Since the diameter of $C'$ can be arbitrarily large, $\mathcal{D}_k$ has no element that sends $g_2p_2$ to $q$ or $q'$. In other words, if $p$ is generic, $L$ and $L'$ are not parallel.

(iv-2) Suppose that $g_3g_1^{-1} = C_\pi$. Then $g_3p_1$ is the antipodal point of $g_1p_1$ in $C$ as shown in Figure 15(b). Let $C'$ be a circle whose center is the origin and the diameter is slightly larger than that of $C$. We shall relocate $g_2p_2$ on $C'$ such that $L$ is the tangent of $C$ at $g_1p_1$ (see Figure 15(b)). Then, we have only two possible locations $q$ and $q'$ for $g_4p_2$ as $L$ and $L'$ are parallel and $g_4p_2$ is on $C'$, where $q$ is the antipodal point of $g_2p_2$ with respect to the origin and $q'$ is the reflection of $q_2p_2$ along the line parallel to $L$ and through the origin. When $p$ is generic, $L$ is not parallel to any reflection lines in $\mathcal{D}_k$, implying $g_4p_2 \neq q'$. Hence, $g_4p_2 = q$. This means that $g_4g_2^{-1}$ is also the 2-fold rotation $C_\pi$.

Recall that $C_\pi$ is in the center of $\mathcal{O}(\mathbb{R}^2)$, i.e., $gC_\pi = C_\pi g$ for any $g \in \mathcal{O}(\mathbb{R}^2)$. Thus, by $g_3g_1^{-1} = C_\pi$, we have $g_1^{-1}g_3 = g_1^{-1}C_\pi g_1 = C_\pi$. Symmetrically, by $g_4g_2^{-1} = C_\pi$, we have $g_2^{-1}g_4 = C_\pi$. This however implies that $g_1^{-1}g_3 = g_2^{-1}g_4$, which contradicts (21). □

Following the statement of Claim 8.8, we shall split the proof into two cases.

(Case 1) Suppose that $L$ and $L'$ are not parallel. Let $q$ be the intersection of $L$ and $L'$. By Claim 8.8(i), we have $q \neq g_ip_i$. We define $p' : V(G') \to \mathbb{R}^2$ by $p'(w) = p(w)$ for $w \in V(G)$ and $p'(v) = q$ for the added vertex $v$. Then, $O(G', \phi', p')$ can be written as follows:

| | $v$ | $V(G)$ |
|---|---|---|
| $e_1$ | $q - g_1p_1$ | $*$ |
| $e_2$ | $q - g_2p_2$ | $*$ |
| $e_3$ | $q - g_3p_3$ | $*$ |
| $e_4$ | $q - g_4p_4$ | $*$ |
| $E(G) - e - f$ | $0$ | $O(G - e - f, \phi, p)$ |

where $O(G - e - f, \phi, p)$ is the matrix obtained from $O(G, \phi, p)$ by removing the rows of $e$ and $f$. Consider the rows associated with $e_1$ and $e_2$. Since $q$ is on $L$, $q - g_ip_i$ is a scalar multiple of $g_1p_1 - g_2p_2$, and hence these two rows can be transformed to the following form by row operations: if $u_1 \neq u_2$

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_1$ | $g_1p_1 - g_2p_2$ | $-p_1 + g_1^{-1}g_2p_2$ | $0$ | $0$ |
| $e_2$ | $0$ | $p_1 - g_1^{-1}g_2p_2$ | $p_2 - g_2^{-1}g_1p_1$ | $0$ |

and, if $u_1 = u_2$,

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_1$ | $g_1p_1 - g_2p_2$ | $-p_1 + g_1^{-1}g_2p_2$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - g_1^{-1}g_2 - g_2^{-1}g_1)p_1$ | $0$ |

Notice that, in each case, the row of $e_2$ is converted to that of $e$ in $O(G, \phi, p)$. In a symmetric manner, the rows of $e_3$ and $e_4$ can be converted to the above form, simply by replacing 1 and 2 with 3 and 4, respectively. Thus, $O(G', \phi', p')$ is converted to

| | $v$ | |
|---|---|---|
| $e_1$ | $g_1p_1 - g_2p_2$ | $*$ |
| $e_3$ | $g_3p_3 - g_4p_4$ | $*$ |
| $E(G)$ | $0$ | $O(G, \phi, p)$ |

The right-bottom block $O(G, \phi, p)$ is row independent while the left-top block is also row independent since $L$ and $L'$ are not parallel. In other words, $O(G', \phi', p')$ is row independent.

(Case 2) Suppose that $L$ and $L'$ are parallel. By Claim 8.8, $L \neq L'$, $p_1 = p_2$, $p_3 = p_4$, and $g_1^{-1}g_2$ and $g_3^{-1}g_4$ are reflections. Let $q$ be any point on $L$ with $q \neq g_1p_1$ and $q \neq g_2p_1$. We define $p' : V(G') \to \mathbb{R}^2$ by $p'(w) = p(w)$ for $w \in V(G)$ and $p'(v) = q$ for the new vertex $v$. Then, the orbit rigidity matrix is described as follows:

| | $v$ | $u_1$ | $u_3$ | $V(G)$ |
|---|---|---|---|---|
| $e_1$ | $q - g_1p_1$ | $p_1 - g_1^{-1}q$ | $0$ | $0$ |
| $e_2$ | $q - g_2p_1$ | $p_1 - g_2^{-1}q$ | $0$ | $0$ |
| $e_3$ | $q - g_3p_3$ | $0$ | $p_3 - g_3^{-1}q$ | $0$ |
| $e_4$ | $q - g_4p_3$ | $0$ | $p_3 - g_4^{-1}q$ | $0$ |
| $E(G) - e - f$ | $0$ | $O(G - e - f, \phi, p)$ | | |

Since $q$ is on the line $L$, $q - g_ip_i$ is a scalar multiple of $(g_1 - g_2)p_1$ for $i = 1, 2$. Hence, the rows of $e_1$ and $e_2$ can be converted to

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_1$ | $(g_1 - g_2)p_1$ | $-g_1^{-1}(g_1 - g_2)p_1$ | $0$ |
| $e_2$ | $(g_1 - g_2)p_1$ | $-g_2^{-1}(g_1 - g_2)p_1$ | $0$ |

and then to

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_1$ | $(g_1 - g_2)p_1$ | $-(I_2 - g_1^{-1}g_2)p_1$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - g_1^{-1}g_2 - g_2^{-1}g_1)p_1$ | $0$ |

Since $g_1^{-1}g_2$ is a reflection, we have $g_1^{-1}g_2 = g_2^{-1}g_1$. Hence, by adding the half of the second row to the first row, we obtain

| | $v$ | $u_1$ | |
|---|---|---|---|
| $e_1$ | $(g_1 - g_2)p_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - g_1^{-1}g_2 - g_2^{-1}g_1)p_1$ | $0$ |

Next we consider the rows of $e_3$ and $e_4$. By subtracting the row of $e_3$ from that of $e_4$, we obtain

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_4$ | $(g_3 - g_4)p_3$ | $0$ | $(g_3^{-1} - g_4^{-1})q$ | $0$ |

Since $L$ and $L'$ are parallel, $\{(g_1 - g_2)p_1, (g_3 - g_4)p_3\}$ is linearly dependent. Thus, by subtracting the row of $e_1$ from that of $e_4$, we have

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_4$ | $0$ | $0$ | $(g_3^{-1} - g_4^{-1})q$ | $0$ |

Moreover, since $g_4^{-1}g_3$ is a reflection, Lemma 8.3 implies that $(I_2 - g_4^{-1}g_3)g_3^{-1}q$ is a scalar multiple of $(I_2 - g_4^{-1}g_3)p_3$, and hence $(g_3^{-1} - g_4^{-1})q$ is a scalar multiple of $(I_2 - g_4^{-1}g_3)p_3$. Therefore, by using $g_3^{-1}g_4 = g_4^{-1}g_3$, the row of $e_4$ can be converted by a scalar multiplication to

| | $v$ | $u_1$ | $u_2$ | |
|---|---|---|---|---|
| $e_4$ | $0$ | $0$ | $(2I_2 - g_3^{-1}g_4 - g_4^{-1}g_3)p_3$ | $0$ |

In total, $O(G', \phi', p')$ is changed to the following form by row-operations:

| | $v$ | $u_1$ | $u_3$ | $V(G)$ |
|---|---|---|---|---|
| $e_1$ | $(g_1 - g_2)p_1$ | $0$ | $0$ | $0$ |
| $e_3$ | $q - g_3p_3$ | $p_3 - g_3^{-1}q$ | $0$ | $0$ |
| $e_2$ | $0$ | $(2I_2 - g_1^{-1}g_2 - g_2^{-1}g_1)p_1$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $(2I_2 - g_3^{-1}g_4 - g_4^{-1}g_3)p_3$ | $0$ |
| $E(G) - e - f$ | $0$ | $O(G - e - f, \phi, p)$ | | |

The right-bottom block together with the rows of $e_2$ and $e_4$ forms $O(G, \phi, p)$, which is row independent. Also, since $q$ is on $L$, but not on $L'$, $\{(g_1 - g_2)p_1, q - g_3p_3\}$ is linearly independent. Therefore, $O(G', \phi', p)$ is row independent. □

Combining Theorem 7.12, Lemma 6.1, Lemma 8.1, Lemma 8.4, Lemma 8.5, and Lemma 8.7, we can now complete the proof of Theorem 8.2.

## 8.2 Symmetric infinitesimal motions with even order dihedral symmetry

Notice that all the lemmas given in the last subsection are independent of the parity of the order $k$. Therefore, we obtain the following statement even for a dihedral group $\mathcal{D}_k$ of even order $k$: for a generic $(\mathcal{D}_k, \rho)$-symmetric framework $(H, p)$ with even order $k$ and a free action $\rho$, $(H, p)$ is symmetry-forced infinitesimally rigid if the quotient gain graph can be constructed from a disjoint union of base graphs by 0-extensions, 1-extensions, loop-1-extensions, 2-extensions and loop-2-extensions. However, as we have seen in Figure 13, there are infinitely many gain graphs that cannot be constructed from base graphs. By Theorem 7.9, minimal examples are $\mathcal{D}_k$-sparse double cycles $C_n^2$. Below, we show that some of them indeed have symmetric infinitesimal motions.

For $C_n^2$, the vertex set is denoted by $\{1, \dots, n\}$ and the edges of the 2-cycle between $i$ and $i+1$ (mod $n$) are denoted by $e_{i,1}$ and $e_{i,2}$ for $i = 1, \dots, n$.

**Theorem 8.9.** *Let $\mathcal{D}_2$ be the dihedral group of order 2, which consists of the identity $I_2$, the 2-fold rotation $C_\pi$, and two reflections $r$ and $r'$. Let $(G, \phi)$ be a $\mathcal{D}_2$-sparse $C_n^2$ such that*

- $\phi(e_{i,1}) = \mathsf{id}$ *and* $\phi(e_{i,2}) = r'$ *for* $i = 1, \dots, n-1$;
- $\phi(e_{n,1}) = C_\pi$ *and* $\phi(e_{n,2}) = r$.

*Then, for any $\mathcal{D}_2$-generic $p : V(G) \to \mathbb{R}^2$, $\operatorname{rank} O(G, \phi, p) = 2n$ if and only if $n$ is odd.*

*Proof.* Let $p : i \in V(G) \mapsto (x_i, y_i) \in \mathbb{R}^2$ be a $\mathcal{D}_2$-generic mapping. Then $C_\pi p(i) = (-x_i, -y_i)$, $rp(i) = (-x_i, y_i)$, $r'p(i) = (x_i, -y_i)$. The rows of $O(G, \phi, p)$ are as follows,

| | | $i$ | $i+1$ | | | $i$ | $i+1$ | |
|---|---|---|---|---|---|---|---|---|
| $e_{i,1}$ | 0 | $x_i - x_{i+1}$ | $x_{i+1} - x_i$ | 0 | 0 | $y_i - y_{i+1}$ | $y_{i+1} - y_i$ | 0 |
| $e_{i,2}$ | 0 | $x_i - x_{i+1}$ | $x_{i+1} - x_i$ | 0 | 0 | $y_i + y_{i+1}$ | $y_{i+1} + y_i$ | 0 |

and

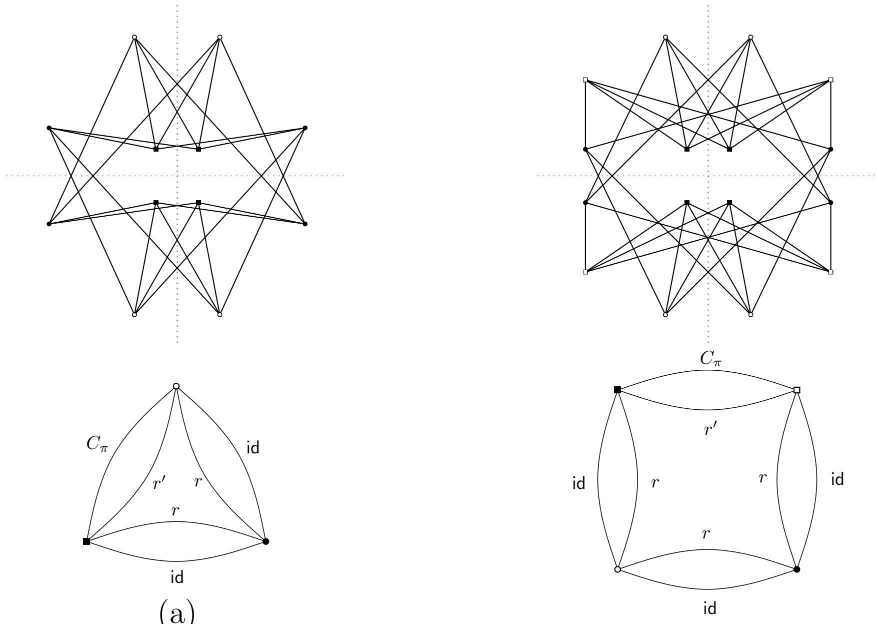
| | | $n$ | $1$ | | | $n$ | $1$ | |
|---|---|---|---|---|---|---|---|---|
| $e_{n,1}$ | 0 | $x_n + x_1$ | $x_1 + x_n$ | 0 | 0 | $y_n + y_1$ | $y_1 + y_n$ | 0 |
| $e_{n,2}$ | 0 | $x_n + x_1$ | $x_1 + x_n$ | 0 | 0 | $y_n - y_1$ | $y_1 - y_n$ | 0 |

where the left and the right half sides correspond to $x$- and $y$-coordinates, respectively. For each $i$, we subtract the first row from the second row and then multiply the first row by an appropriate scalar. We then have, for each $i = 1, \dots, n-1$,

| | | $i$ | $i+1$ | | | $i$ | $i+1$ | |
|---|---|---|---|---|---|---|---|---|
| $e_{i,1}$ | 0 | 1 | $-1$ | 0 | 0 | $*$ | $*$ | 0 |
| $e_{i,2}$ | 0 | 0 | 0 | 0 | 0 | $y_{i+1}$ | $y_i$ | 0 |

and

| | | $n$ | $1$ | | | $n$ | $1$ | |
|---|---|---|---|---|---|---|---|---|
| $e_{n,1}$ | 0 | 1 | 1 | 0 | 0 | $*$ | $*$ | 0 |
| $e_{n,2}$ | 0 | 0 | 0 | 0 | 0 | $y_1$ | $y_n$ | 0 |




(a) (b)

Figure 16: Examples of symmetric frameworks given in Theorem 8.9. (b) has a symmetric infinitesimal motion, but (a) does not.

In other words, $O(G, \phi, p)$ is converted to the following form,

$$\left(\begin{array}{cccccc|cccccc} 1 & -1 & & & & & & & & & & \\ & 1 & -1 & & & & & & & * & & \\ & & \ddots & \ddots & & & & & & & & \\ & & & 1 & -1 & & & & & & & \\ 1 & & & & 1 & & & & & & & \\ \hline & & & & & & y_2 & y_1 & & & & \\ & & & & & & & y_3 & y_2 & & & \\ & & 0 & & & & & & \ddots & \ddots & & \\ & & & & & & & & & y_n & y_{n-1} & \\ & & & & & & y_n & & & & y_1 & \end{array}\right)$$

The determinant of this matrix is $2(1-(-1)^{n-1})\prod_{i=1}^{n} y_i$, which is equal to zero if and only if $n$ is even. □

See Figure 16 for examples of frameworks given in Theorem 8.9. For $n = 2$, the covering graph is $K_{4,4}$ and the corresponding framework is known as Bottema's mechanism (see [27, Section 7.2.1]).

# 9 Proof of Theorem 7.8

In this section we prove Theorem 7.8. For simplicity, a $\mathcal{D}$-gain graph satisfying the conditions of Theorem 7.8 is called *essential*, i.e., $\mathcal{D}$-sparse, 4-regular, not a base graph, and not a double cycle. Lemma 7.6 shows that 2-extensions and loop-2-extensions preserve $\mathcal{D}$-sparsity, and hence what we have to prove is the following theorem.

**Theorem 9.1.** *Any essential graph $(G, \phi)$ has a vertex at which a 2-reduction or a loop-2-reduction is admissible.*

For simplicity, in the subsequent discussion we omit gain functions $\phi$ when referring to gain graphs if it is clear from the context. Also an edge $(u, v)$ from $u$ to $v$ is simply denoted by $uv$, and a $\mathcal{D}$-tight set is called a tight set.

The proof of Theorem 9.1 consists of four parts. In §9.1, we shall prove useful lemmas for subsequent discussion. In §9.2, we prove Theorem 9.1 for the following graphs,

- graphs consisting of only special vertices (Lemma 9.5), where a vertex is called special if it is incident with a loop or two parallel classes of edges;
- graphs that are not 2-connected (Lemma 9.6),
- "almost" near-cyclic graphs (Lemma 9.8), defined below,
- graphs that are not essentially 4-edge-connected (Lemma 9.9),
- graphs having a vertex $v$ with $|N(v)| = 2$.

In §9.3 we discuss graphs not belonging to the above classes. In §9.4 we put everything together to complete the proof of Theorem 9.1.

## 9.1 Preliminary facts

The following fundamental properties of 4-regular graphs will be frequently used.

- A 4-regular graph is Eulerian. Hence, a 4-regular connected graph is 2-edge-connected.
- Let $G = (V, E)$ be a graph with maximum degree at most 4. Then, for any $X \subseteq V$, $i_G(X) \leq 2|X| - \lfloor d_G(X)/2 \rfloor$, where $i_G(X)$ denotes the number of edges induced by $X$. In particular, if $G$ is 4-regular, $i_G(X) = 2|X| - d_G(X)/2$.

The next lemma asserts that if the maximum degree is at most 4, then $\mathcal{D}$-sparsity is equivalent to the following simpler properties:

**(C1)** $|F| \leq 2|V(F)| - 3$ for every nonempty balanced set $F \subseteq E$;

**(C2)** $G$ is not cyclic for some $v \in V$.

**Lemma 9.2.** *Let $G = (V, E)$ be a $\mathcal{D}$-gain graph with maximum vertex degree at most 4. If $G$ is connected, then $G$ is $\mathcal{D}$-sparse if and only if*

**(i)** *$G$ is not 4-regular and condition (C1) is satisfied, or*

**(ii)** *$G$ is 4-regular and conditions (C1) and (C2) are satisfied.*

*If $G$ is not connected, $G$ is $\mathcal{D}$-sparse if and only if each connected component is $\mathcal{D}$-sparse.*

*Proof.* If the maximum degree is at most 4, $|F| \leq 2|V(F)|$ for any $F \subseteq E$. In particular, if $G$ is connected, we have $|F| \leq i_G(V(F)) \leq 2|V(F)| - \lfloor d_G(V(F))/2 \rfloor \leq 2|V(F)| - 1$ for any $F \subseteq E$ with $V(F) \neq V$. Therefore, $|F| \geq 2|V(F)|$ holds if and only if $G$ is 4-regular and $F = E$. □

Thus, to prove Theorem 9.1, we shall investigate whether (C1) and (C2) are satisfied after the reductions. The next lemma will be used when (C2) is not satisfied. We say that $(G, \phi)$ is *almost near-cyclic* if there are two incident edges $e$ and $f$ such that $G - e - f$ is cyclic.

**Lemma 9.3.** *Let $(G, \phi)$ be a connected 4-regular $\mathcal{D}$-sparse graph with $G = (V, E)$ and $v$ be a vertex in $G$ that is not incident to a loop. Let $e_1, e_2, e_3, e_4$ be the edges incoming to $v$, and suppose that $G - v + e_1 \cdot e_2^{-1} + e_3 \cdot e_4^{-1}$ is connected and cyclic. Then, there is an equivalent gain function $\phi'$ to $\phi$ and a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that*

- *$\phi'(e) \in \mathcal{C}$ for every $e \in E \setminus \{e_3, e_4\}$, and*
- *$\phi'(e_3) \notin \bar{\mathcal{C}}$ and $\phi'(e_4) \notin \bar{\mathcal{C}}$.*

*In particular, $G$ is almost near-cyclic.*

*Proof.* Let $G' = G - v + e_1 \cdot e_2^{-1} + e_3 \cdot e_4^{-1}$. Since $G'$ is connected and cyclic, by Lemma 2.4, there are an equivalent gain function $\phi'$ to $\phi$ and a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi'(e) \in \mathcal{C}$ for all $e \in E(G')$. Let $a = \phi'(e_1 \cdot e_2^{-1}) \in \mathcal{C}$ and $a' = \phi'(e_3 \cdot e_4^{-1}) \in \mathcal{C}$. Then, by using some elements $b_1, b_2 \in \mathcal{D}$, we can express $\phi'(e_i)$ by

$$\phi'(e_1) = ab_1, \quad \phi'(e_2) = b_1, \quad \phi'(e_3) = a'b_2, \quad \phi'(e_4) = b_2.$$

We further perform the switching operation at $v$ with $b_1$. We consequently have an equivalent gain function $\phi'$ to $\phi$ such that

$$\phi'(e_1) = a, \quad \phi'(e_2) = \mathsf{id}, \quad \phi'(e_3) = a'b, \quad \phi'(e_4) = b,$$

where $b = b_2 b_1^{-1}$. Notice that $\phi'(e) \in \mathcal{C}$ for all $e \in E \setminus \{e_3, e_4\}$. Since $G$ is not cyclic, we must have $b \notin \bar{\mathcal{C}}$, implying that $\phi'(e_3) \notin \bar{\mathcal{C}}$ and $\phi'(e_4) \notin \bar{\mathcal{C}}$. □

The following technical lemma is one of the key observations. A vertex in a 4-regular graph is called *special* if it is incident with a loop or two parallel classes of edges with $|N(v)| = 2$.

**Lemma 9.4.** *Let $(G, \phi)$ be a connected 4-regular $\mathcal{D}$-sparse graph with $G = (V, E)$, $v$ be a vertex in $G$ that is not special, and $e_1, e_2, e_3, e_4$ be the edges incoming to $v$. If $G - e_3 - e_4$ or $G - v + e_1 \cdot e_2^{-1} + e_3 \cdot e_4^{-1}$ is connected and cyclic, then at least one of the following holds*

**(a)** *$G$ is near-cyclic.*

**(b)** *$G - v + e_1 \cdot e_3^{-1} + e_2 \cdot e_4^{-1}$ is $\mathcal{D}$-sparse.*

**(c)** *$v$ is a cut-vertex in $G$ and $G - v + e_1 \cdot e_3^{-1} + e_2 \cdot e_4^{-1}$ is connected.*

*Proof.* For simplicity, we denote $e_{i,j} = e_i \cdot e_j^{-1}$ for $i, j \in \{1, 2, 3, 4\}$. We assume that (a) does not occur and show that (b) or (c) holds.

We claim that there are an equivalent gain function $\phi'$ to $\phi$ and a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi'(e) \in \mathcal{C}$ holds for $e \in E \setminus \{e_3, e_4\}$ and $\phi'(e_3) \notin \bar{\mathcal{C}}$ and $\phi'(e_4) \notin \bar{\mathcal{C}}$.

To see this, first observe that if $G - v + e_1 \cdot e_2^{-1} + e_3 \cdot e_4^{-1}$ is connected and cyclic, then Lemma 9.3 implies the claim. On the other hand, if $G - e_3 - e_4$ is connected and cyclic, then by Lemma 2.4, there is an equivalent $\phi'$ to $\phi$ and a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that $\phi'(e) \in \mathcal{C}$ for $e \in E \setminus \{e_3, e_4\}$. Since $G$ is neither cyclic nor near-cyclic, we have $\phi'(e_3) \notin \bar{\mathcal{C}}$, and $\phi'(e_4) \notin \bar{\mathcal{C}}$.

Note that $\phi'(e_{1,3}) \notin \bar{\mathcal{C}}$ and $\phi'(e_{2,4}) \notin \bar{\mathcal{C}}$.

Let us consider $G - v$. Since $G - v$ is cyclic with $|E(G-v)| = 2|V(G-v)| - 2$, $G - v$ is $(2, 3)$-g-sparse. Applying Lemma 7.7 with $\phi'(e_{1,3}) \notin \bar{\mathcal{C}}$, we deduce that $G - v + e_{1,3}$ is $\mathcal{D}$-sparse. Let $G' = G - v + e_{1,3} + e_{2,4}$. We now show that, if $G'$ is not $\mathcal{D}$-sparse (i.e., (b) does not hold), then (c) holds. To see this, let us assume that $G'$ is not $\mathcal{D}$-sparse. By Lemma 9.2, $G'$ (or a connected component of $G'$) violates (C1) or (C2).

Case 1: If (C1) is violated, then $G - v + e_{1,3}$ contains a balanced tight set $F$ such that $V(F)$ contains the endvertices of $e_{2,4}$ and $F + e_{2,4}$ is balanced. Let $s$ and $t$ be the endvertices of $e_{2,4}$, which are possibly the same vertex. By Lemma 7.1, if $|F| > 1$, $F$ contains a path from $s$ to $t$ that does not pass through $e_{1,3}$. Recall that the gain of each edge in this path is included in $\mathcal{C}$, and the concatenation of the path and $e_{2,4}$ forms an unbalanced closed walk in $F + e_{2,4}$, contradicting that $F + e_{2,4}$ is balanced. Therefore, $|F| = 1$ holds; in particular, since $s, t \in V(F)$ and $F + e_{2,4}$ is balanced, it follows that $F = \{e_{1,3}\}$ and $\{e_{1,3}, e_{2,4}\}$ forms a balanced 2-cycle in $G'$. This implies that $v$ is special in $G$, contradicting the assumption of the lemma.

Case 2: We next consider the case when (C2) is violated in $G'$. Suppose that $v$ is not a cut-vertex. Note that, since $|E(G - v)| = 2|V(G - v)| - 2$, $G - v$ contains an unbalanced cycle $C$, whose gain is included in $\mathcal{C}$. Let $s$ and $t$ be the endvertices of $e_{2,4}$, which are possibly the same vertex. Since $G - v$ is connected, there is a path $P$ from $s$ to a vertex in $V(C)$. We consider a closed walk $W_1$ that first passes through $P$ starting at $s$, then goes around $C$, and comes back to $s$ through $P^{-1}$. We then have $\phi'(W_1) \in \mathcal{C}$. Also, since $G - v$ is connected, $G - v$ has a path $P'$ connecting $s$ and $t$. The concatenation of $P'$ with $e_{2,4}$ forms a closed walk $W_2$ starting at $s$ with $\phi(W_2) \notin \bar{\mathcal{C}}$. Thus, $\{\phi'(W_1), \phi'(W_2)\}$ generates a non-cyclic group. Hence, $G'$ satisfies (C2), a contradiction. Thus, $v$ is a cut-vertex in $G$.

Suppose that $G'$ is not connected. Then, by the 4-regularity of $G$, $G'$ consists of two connected components, denoted $G'_1$ and $G'_2$ with $e_{1,3} \in E(G'_1)$ and $e_{2,4} \in E(G'_2)$. We have already seen that $G - v + e_{1,3}$ is $\mathcal{D}$-sparse, and hence its subgraph $G'_1$ is $\mathcal{D}$-sparse. However, since $G'_1$ is 4-regular, $G'_1$ is indeed maximum $\mathcal{D}$-tight. By the symmetry between $e_{1,3}$ and $e_{2,4}$, $G'_2$ is also maximum $\mathcal{D}$-tight, and thus $G'$ is maximum $\mathcal{D}$-tight, a contradiction. Thus (c) must hold. □

## 9.2 Special cases

Recall that a vertex is special if it is incident with a loop or two parallel classes of edges. A graph which consists of only special vertices is called a *special graph*. Special graphs are classified into the following three classes $C_n^2$, $C_n^\circ$ and $P_n^2$ for $n \geq 2$ (Figure 17): As defined in § 7.2, $C_n^2$ is the graph obtained from the cycle of $n$ vertices by replacing each edge by two parallel copies; $C_n^\circ$ is the cycle of $n$ vertices, each of which is incident to a loop; $P_n^2$ is the graph obtained from a path of $n$ vertices by replacing each edge by two parallel copies and adding one loop to each endvertex of the path.

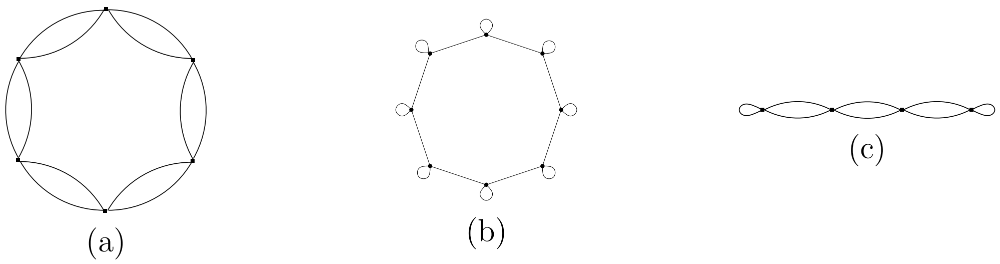



Figure 17: Special graphs: (a)$C_6^2$, (b)$C_8^\circ$, (c)$P_4^2$.

**Lemma 9.5.** *Let $(G, \phi)$ be an essential $\mathcal{D}$-gain graph whose underlying graph $G = (V, E)$ is special. Then there is a vertex at which a 2-reduction or a loop-2-reduction is admissible.*

*Proof.* Since $(G, \phi)$ is essential, the underlying graph is either $P_n^2$ or $C_n^\circ$.

Suppose that the underlying graph is $P_n^2$. We perform the loop-2-reduction at a vertex incident to a loop $l$. The resulting graph is $P_{n-1}^2$ and clearly it satisfies (C1). If it does not satisfy (C2), then the resulting graph is cyclic and there is a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$ such that the gain of every cycle in $G$ except for the loop $l$ is in $\mathcal{C}$. This in turn implies that $G - l$ is cyclic, contradicting the assumption that $G$ is essential.

Suppose that the underlying graph is $C_n^\circ$. We may assume $n \geq 3$ since $C_2^\circ = P_2^2$. We perform the 2-reduction at a vertex incident to a loop $l$. The resulting $\mathcal{D}$-gain graph, denoted $G'$, has the underlying graph $C_{n-1}^\circ$.

If $G'$ does not satisfy (C2), then the gain of each cycle in $G$ except for the loop $l$ is included in a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$, which again contradicts the fact that $G$ is essential.

It can be easily observed that $G'$ satisfies (C1) if $n > 3$. For $n = 3$, (C1) is violated if the 2-cycle of $G'$ is balanced, but in such a case the triangle in the original graph $G$ is balanced, and $G$ turns out to be a fancy triangle, contradicting the fact that $G$ is essential. □

The next lemma solves the case when the graph can be disconnected by removing one vertex.

**Lemma 9.6.** *Let $G = (V, E)$ be a connected essential $\mathcal{D}$-gain graph with $|V| \geq 2$. Suppose that $G$ is not 2-connected. Then a 2-reduction is admissible at some vertex.*

*Proof.* By Lemma 9.5, we may assume that $G$ is not equal to $P^2_{|V|}$. Then $G$ has a cut-vertex $v$ which is not special. We show that a 2-reduction at $v$ is admissible. Note that $G - v$ consists of two connected components by the 4-regularity of $G$. Let $e_1, e_2, e_3, e_4$ be the edges incident to $v$, all of them are directed to $v$. ¿From the 2-edge-connectivity of $G$, we can assume, without loss of generality, that the endvertices of $e_1$ and $e_3$ are included in a connected component of $G - v$ while those of $e_2$ and $e_4$ are included in the other component.

Consider the 2-reduction at $v$ through $(e_1, e_2)$ and $(e_3, e_4)$. Let $G'$ be the resulting graph. Note that $G'$ is connected. Let us check that $G'$ satisfies (C1). To see this, recall that any balanced tight set consisting of more than one edge is 2-connected by Lemma 7.1. Note also that $e_3 \cdot e_4^{-1}$ is not parallel to $e_1 \cdot e_2^{-1}$ as $v$ is not special. Since the endvertices of $e_3 \cdot e_4^{-1}$ belong to different connected components in $G - v$ and $e_1 \cdot e_2^{-1}$ is the bridge in $G - v + e_1 \cdot e_2^{-1}$, $G - v + e_1 \cdot e_2^{-1}$ has no balanced tight set $F$ such that $V(F)$ contains both endvertices of $e_3 \cdot e_4^{-1}$. This implies that $G'$ satisfies (C1).

Therefore, if $G'$ satisfies (C2), then $G'$ is $\mathcal{D}$-sparse by Lemma 9.2, and a 2-reduction is admissible at $v$. Suppose that $G'$ does not satisfy (C2). Then, $G'$ is connected and cyclic. To apply Lemma 9.4, we next consider the 2-reduction at $v$ through $(e_1, e_3)$ and $(e_2, e_4)$. The resulting graph, denoted by $G''$, is disconnected. Lemma 9.4 thus implies that $G''$ is $\mathcal{D}$-sparse. □

Thus, in the subsequent discussion, we may focus on 2-connected graphs. The next lemma solves the case when $G$ has a special vertex not incident to a loop.

**Lemma 9.7.** *Let $G = (V, E)$ be a 2-connected essential $\mathcal{D}$-gain graph. Suppose that $G$ has a special vertex not incident to a loop. Then, $G$ has a vertex at which a 2-reduction is admissible.*

*Proof.* Let $w$ be a special vertex not incident to a loop. By definition of special vertices, $|N(w)| = 2$ and $w$ is incident to two parallel classes of edges. Since $G \neq C^2_n$, $G$ contains two adjacent vertices $u$ and $v$ such that $v$ is not special and $u$ is special not incident to a loop (where $u$ is possibly equal to $w$). Depending on the size of $N(\{u, v\})$, we have two possible cases as shown in Figure 18.


Figure 18: Proof of Lemma 9.7.

Let us denote the edges incident to $u$ by $e_1, e_2, e_3, e_4$, where $e_1$ and $e_2$ are linking from $v$ to $u$ and $e_3$ and $e_4$ are linking from a vertex in $V \setminus \{u, v\}$ to $u$. We perform the 2-reduction at $u$ through $(e_1, e_2)$ and $(e_3, e_4)$. Since both new edges are unbalanced loops and adding unbalanced loops does not violate (C1), the resulting graph $G'$

satisfies (C1). Therefore, if the 2-reduction is not admissible at $u$, then $G'$ does not satisfy (C2), and hence $G - e_1 - e_2$ is cyclic by Lemma 9.3.

Let $a, b, c \in V$ such that $N(v) = \{u, a, b\}$ and $N(u) = \{v, c\}$. Since $|N(u)| = 2$ with $v \in N(u)$, without loss of generality we may assume $a \notin N(u)$ (where $b = c$ possibly holds). Recall that $G - e_1 - e_2$ is connected and cyclic, and hence we can apply Lemma 9.4 to deduce that the 2-reduction at $v$ through $(bv, e_1)$ and $(av, e_2)$ is admissible. Indeed, since $G$ is not near-cyclic and $v$ is neither a cut-vertex nor a special vertex, Lemma 9.4 implies that this 2-reduction at $v$ is admissible. □

The next lemma solves the case when $G$ is almost near-cyclic.

**Lemma 9.8.** *Let $G = (V, E)$ be a 2-connected essential $\mathcal{D}$-gain graph with at least two vertices. Suppose that $G$ is almost near-cyclic. Then a 2-reduction or a loop-2-reduction is admissible at some vertex in $G$.*

*Proof.* Since $G$ is almost near-cyclic, there are two edges $e_1$ and $e_2$ for which $e_1$ and $e_2$ are incident to a vertex $v$ and $G - e_1 - e_2$ is cyclic.

Suppose that $v$ is not special. Then, since $v$ is not a cut-vertex, a 2-reduction is admissible at $v$ by Lemma 9.4. Therefore, let us consider the case when $v$ is special. If $v$ is not incident to a loop, then Lemma 9.7 directly implies the claim. We can thus assume that $v$ is incident to a loop.

Suppose that both $e_1$ and $e_2$ are non-loop edges. By Lemma 2.4, we may assume that the label of each edge in $G - e_1 - e_2$ is contained in a cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$. By further performing a switching operation at $v$ with $\phi(e_1)$, $\phi$ is converted such that $\phi(e_1) = \mathsf{id}$ and $\phi(e) \in \mathcal{C}$ for all edges $e$ not incident to $v$. This implies that if we remove $e_2$ and the loop incident to $v$ from $G$, the resulting graph is cyclic. In other words, it suffices to consider the case when $e_1$ or $e_2$ is a loop.

We hence assume that $e_1$ is the loop incident to $v$. Let $e_3$ be the remaining non-loop edge incident to $v$, where $\phi(e_3) \in \mathcal{C}$. Observe that the gain of the non-loop edge $e_2$ is not included in $\bar{\mathcal{C}}$, since otherwise $G - e_1$ becomes cyclic, contradicting the assumption that $G$ is essential. Therefore, $\phi(e_2 \cdot e_3^{-1}) \notin \bar{\mathcal{C}}$, and the loop-2-reduction at $v$ adds the edge $e_2 \cdot e_3^{-1}$ to the cyclic $(2, 3)$-g-sparse graph $G - v$. By Lemma 7.7, the resulting gain graph is $\mathcal{D}$-sparse. □

By using Lemma 9.8, we can now prove an important consequence for graphs that are not essentially 4-edge-connected.

**Lemma 9.9.** *Let $G = (V, E)$ be a 2-connected essential $\mathcal{D}$-gain graph with $|V| = n \geq 4$. Suppose that $G$ is not essentially 4-edge-connected. Then, $G$ has a vertex at which a 2-reduction or a loop-2-reduction is admissible.*

*Proof.* Since $G$ is 2-edge-connected and is not essentially 4-edge-connected, there exists a subset $X$ of $V$ for which $|X| > 1$, $|V \setminus X| > 1$ and $d_G(X) = 2$. Since $G$ is not $C_n^{\circ}$, we can suppose that $B(X)$ contains a vertex $v$ not incident to a loop, where $B(X)$ denotes a set of vertices of $X$ adjacent to some vertices of $V \setminus X$. By the 2-connectivity, $v$ is not a cut-vertex. Hence, denoting the four edges incident to $v$ by

$e_1, ..., e_4$, we may assume that $e_1, e_2, e_3$ are included in the subgraph induced by $X$ while $e_4$ is not.

Note that $v$ is a vertex of degree 3 in $G - e_4$, and hence, by Lemma 7.5, a 1-reduction at $v$ is admissible in $G - e_4$. Without loss of generality, we may assume that $G - v + e_1 \cdot e_2^{-1}$ (obtained by a 1-reduction at $v$ in $G - e_4$) is $\mathcal{D}$-sparse.

We now consider adding $e_3 \cdot e_4^{-1}$ to $G - v + e_1 \cdot e_2^{-1}$ to complete the 2-reduction at $v$. Let $G' = G - v + e_1 \cdot e_2^{-1} + e_3 \cdot e_4^{-1}$, and suppose that $G'$ does not satisfy (C1). Since any balanced tight set $F$ is 2-edge-connected if $|F| > 1$, there is no balanced tight set $F$ for which $V(F)$ contains both endvertices of $e_3 \cdot e_4^{-1}$ unless $|F| = 1$. If $G - v + e_1 \cdot e_2^{-1}$ has a balanced set $F$ such that $|F| = 1$ and $V(F)$ contains both endvertices of $e_3 \cdot e_4^{-1}$, then the edge in $F$, denoted by $f$, is incident to $e_3$ and $e_4$ and connects between $X$ and $V \setminus X$. However, since $d_G(X) = 2$, $|X| > 1$ and $|V \setminus X| > 1$, the vertex incident to $e_4$ and $f$ turns out to be a cut-vertex of $G$, contradicting the 2-connectivity of $G$. Thus, $G'$ satisfies (C1).

If $G'$ does not satisfy (C2), it is cyclic. By Lemma 9.3, $G$ is almost near-cyclic, and we can apply Lemma 9.8 to conclude that a 2-reduction or a loop-2-reduction is admissible at some vertex $v$. □

The final special case is when $G$ has a vertex $v$ with $|N(v)| = 2$.

**Lemma 9.10.** *Let $G = (V, E)$ be a 2-connected essential $\mathcal{D}$-gain graph. Suppose that $G$ has a vertex $v$ with $|N(v)| = 2$ that is not incident to a loop. Then, there is a vertex at which a 2-reduction is admissible.*

*Proof.* If $v$ is special, Lemma 9.7 implies the claim.

If $v$ is not special, then there are three parallel edges between $v$ and a neighbor of $v$. By the 4-regularity, if $|V| \geq 4$, $G$ is not essentially-4-edge-connected, and thus Lemma 9.9 implies the statement.

If $|V| = 3$, $G$ is equal to the graph (shown in Figure 20) of three vertices $V = \{u, v, w\}$, three parallel edges $e_1, e_2, e_3$ between $u$ and $v$, a loop $l$ attached to $w$, and two remaining edges $uw$ and $vw$, denoted by $f_1$ and $f_2$, respectively. We may assume $\phi(f_1) = \phi(f_2) = \mathsf{id}$. Let $\mathcal{C}$ be the subgroup generated by $\phi(l)$. Since $G$ is not cyclic, there is an unbalanced cycle whose gain is not included in $\bar{\mathcal{C}}$.

If a triangle, say $e_1 f_1 f_2$ has a gain not included in $\bar{\mathcal{C}}$, then the 2-reduction at $u$ through $(e_1, f_1)$ and $(e_2, e_3)$ results in a $\mathcal{D}$-sparse $P_2^2$. Otherwise, removing $e_2$ and $e_3$ results in a cyclic graph. Then $G$ is almost near-cyclic, and Lemma 9.8 implies the statement. □

## 9.3 The remaining cases

In a graph $G$, the *star* of a vertex $v$ means the subgraph of $G$ whose vertex set is $N(v) \cup \{v\}$ and the edge set is the set of edges incident to $v$. A *hat subgraph* is a balanced subgraph whose underlying graph is a hat. See Figure 19 for an example. The following claim, together with the previous lemmas, will complete the proof of Theorem 9.1.

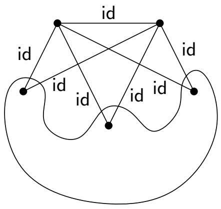



Figure 19: A hat subgraph.

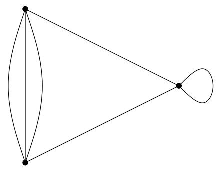


Figure 20: The special graph given in the proof of Lemma 9.10.

**Theorem 9.11.** *Let $G = (V, E)$ be 2-connected, essentially 4-edge-connected, and essential graph with $|V| \geq 3$. Suppose also that $G$ is not almost near-cyclic. Then, for every vertex $v \in V$ that is not incident to a loop with $|N(v)| \geq 3$, either a 2-reduction at $v$ is admissible or the star of $v$ is contained in a hat subgraph.*

In §9.3.1, we focus on the case of $|N(v)| = 4$. Lemma 9.12 says that if the 2-reduction is not admissible then $G$ has an *obstacle* around $v$. We will investigate intersection properties of obstacles. The corresponding results for the case of $|N(v)| = 3$ will be given in §9.3.2. In §9.3.3, we prove Theorem 9.11 based on the intersection properties of obstacles.

In the rest of this section, $\mathrm{cl}_{\mathcal{D}}$ denotes the closure operator of the underlying matroid $\mathcal{M}_{\mathcal{D}}(G, \phi)$.

### 9.3.1 Obstacles around a vertex $v$ with $|N(v)| = 4$

Throughout §9.3.1, $(G, \phi)$ denotes a $\mathcal{D}$-gain graph satisfying the assumptions of Theorem 9.11, $v$ denotes a vertex with $|N(v)| = 4$, $N(v) = \{a, b, c, d\}$, and $E_v$ denotes the set of edges incident to $v$.

An edge subset $F$ is called *sub-tight* if $|F| = 2|V(F)| - 4$ and $F$ is balanced. We first make a simple observation which describes the situation where 2-reductions are not admissible.

**Lemma 9.12.** *Suppose that the 2-reduction through $(av, vb)$ and $(cv, vd)$ is not admissible. Then there is an edge subset $F \subseteq E \setminus E_v$ satisfying one of the following properties:*

*(i) $F$ is balanced tight with $a, b \in V(F)$ and $av \cdot vb \in \mathrm{cl}_{\mathcal{D}}(F)$;*

*(ii) $F$ is balanced tight with $c, d \in V(F)$ and $cv \cdot vd \in \mathrm{cl}_{\mathcal{D}}(F)$;*

*(iii) $F$ is sub-tight with $a, b, c, d \in V(F)$, $F + av \cdot vb$ is balanced tight, and $cv \cdot vd \in \mathrm{cl}_{\mathcal{D}}(F + av \cdot vb)$.*

*Proof.* Let us first consider the graph $G' = G - v + av \cdot vb$. If $G'$ is not $\mathcal{D}$-sparse, then, by Lemma 9.2, $E \setminus E_v$ has a balanced tight set $F$ with $a, b \in V(F)$ and $av \cdot vb \in \mathrm{cl}_{\mathcal{D}}(F)$, which satisfies property (i).

Hence, let us assume that $G'$ is $\mathcal{D}$-sparse. If $G' + cv \cdot vd$ is cyclic, Lemma 9.3 implies that $G$ is almost near-cyclic, contradicting the assumption that $G$ is not almost near-cyclic. Therefore, $G' + cv \cdot vd$ satisfies (C2). By Lemma 9.2, there exists a balanced tight set $F' \subseteq E \setminus E_v \cup \{av \cdot vb\}$ with $c, d \in V(F')$ and $cv \cdot vd \in \mathrm{cl}_{\mathcal{D}}(F')$. Depending on whether $av \cdot vb \in F'$ or not, we find a desired subset of the statement; if $av \cdot vb \notin F'$ then $F'$ is the one satisfying property (ii); otherwise $F' - av \cdot vb$ satisfies property (iii). (We remark that, in the latter case, $V(F' - av \cdot vb)$ contains $a, b, c, d$ since $F'$ is 2-edge-connected.) □

Since the first and the second cases of the statement of Lemma 9.12 are symmetric, we basically have two types of *obstacles*: for a vertex $v$ and $N(v) = \{a, b, c, d\}$, $F \subseteq E \setminus E_v$ is called an *obstacle of type 1* (for the 2-reduction through $(av, vb)$ and $(cv, vd)$) if $F$ satisfies (i) or (ii) of Lemma 9.12; $F$ is called an *obstacle of type 2* if $F$ satisfies (iii).

As noted above, we have three possible ways for a 2-reduction at $v$, through $(av, vb)$ and $(cv, vd)$, through $(av, vc)$ and $(bv, vd)$, and through $(av, vd)$ and $(bv, vc)$. By Lemma 9.12, if none of them are admissible, $E \setminus E_v$ contains three corresponding obstacles $X, Y, Z$. We now investigate properties of these obstacles.

We begin with a property of type 2 obstacles.

**Lemma 9.13.** *Suppose that $X$ is an obstacle of type 2 for the 2-reduction through $(av, vb)$ and $(cv, vd)$. Then, the following holds for $X$:*

- $|X \cup E_v| = 2|V(X \cup E_v)| - 2$;
- *There is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in X \cup \{va, vb\}$, and $\phi'(vc) = \phi'(vd) \neq \mathsf{id}$;*
- *$X \cup E_v$ is cyclic.*

*Proof.* By definition, $|X| = 2|V(X)| - 4$, and hence $|X \cup E_v| = 2|V(X \cup E_v)| - 2$ by $N(v) \subseteq V(X)$.

Since $cv \cdot vd \in \mathrm{cl}_{\mathcal{D}}(X + av \cdot vb)$ and $X + av \cdot vb$ is balanced, $X + av \cdot vb + cv \cdot vd$ is also balanced. Hence, by Lemma 2.4, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in X$ and $\phi'(av \cdot vb) = \phi'(cv \cdot vd) = \mathsf{id}$. We thus have $\phi'(av) = \phi'(bv) = g$ and $\phi'(cv) = \phi'(dv) = g'$ for some $g, g' \in \mathcal{D}$. By performing a switching operation at $v$ with $g$ if necessary, we may assume that $\phi'(av) = \phi'(bv) = \mathsf{id}$ and $\phi'(cv) = \phi'(dv) = g'g^{-1}$. If $g'g^{-1} = \mathsf{id}$, $X \cup E_v$ becomes a balanced set with $|X \cup E_v| > 2|V(X \cup E_v)| - 3$, contradicting the $\mathcal{D}$-sparsity of $G$. Thus, $\phi'(cv) = \phi'(dv) \neq \mathsf{id}$, and $X \cup E_v$ is cyclic. □

In the same manner we also have the following technical lemma.

**Lemma 9.14.** *Let $X$ and $Y$ be obstacles for the 2-reduction through $(av, vb)$ and $(cv, vd)$ and through $(av, vc)$ and $(bv, vd)$, respectively. Suppose that $X$ is type 2 and $X \cup Y$ is cyclic. Then, $X \cup Y \cup E_v$ is cyclic.*

*Proof.* Since $X$ is balanced and $X \cup Y$ is cyclic, for some cyclic subgroup $\mathcal{C}$ of $\mathcal{D}$, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for every $e \in X$ and $\phi'(e) \in \mathcal{C}$ for every $e \in Y$ by Lemma 2.4. Moreover, since $X + av \cdot vb$ and $X + av \cdot vb + cv \cdot vd$ are balanced, we have $\phi'(av \cdot vb) = \phi'(cv \cdot vd) = \mathsf{id}$. As in the previous proof, by applying a switching operation at $v$, we may assume that $\phi'(va) = \phi'(vb) = \mathsf{id}$ and $\phi'(vc) = \phi'(vd)$.

By the definition of the obstacles (whether type 1 or type 2), $Y+$ $Y + av \cdot vc$ or $Y + bv \cdot vd$ is connected and balanced. Hence $\phi'(av \cdot vc) \in \bar{\mathcal{C}}$ or $\phi'(bv \cdot vd) \in \bar{\mathcal{C}}$, which implies $\phi'(vc) = \phi'(vd) \in \bar{\mathcal{C}}$. Thus, every label of $X \cup Y \cup E_v$ is included in $\bar{\mathcal{C}}$. □

The following lemmas describe different relations among obstacles.

**Lemma 9.15.** *Let $X$ and $Y$ be obstacles for the 2-reduction through $(av, vb)$ and $(cv, vd)$ and through $(av, vc)$ and $(bv, vd)$, respectively. If $X \cap Y \neq \emptyset$, then $X \cup Y$ is not a balanced set.*

*Proof.* Suppose for a contradiction that $X \cup Y$ is a balanced set with $X \cap Y \neq \emptyset$.

(Case 1) If both $X$ and $Y$ are of type 1, $X \cup Y$ is tight by Lemma 7.2 and hence $|X \cup Y| = 2|V(X \cup Y)| - 3$. Without loss of generality, we may assume that $a, b, c \in V(X \cup Y)$, $av \cdot vb \in \mathrm{cl}_{\mathcal{D}}(X)$ and $av \cdot vc \in \mathrm{cl}_{\mathcal{D}}(Y)$. Since $X \cup Y$ is balanced, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in X \cup Y$. Moreover, since $av \cdot vb \in \mathrm{cl}_{\mathcal{D}}(X)$ and $av \cdot vc \in \mathrm{cl}_{\mathcal{D}}(Y)$, we have $\phi'(av) = \phi'(bv) = \phi'(cv)$. This implies that $X \cup Y \cup \{av, bv, cv\}$ is a balanced set. However, since $|X \cup Y \cup \{av, bv, cv\}| > 2|V(X \cup Y \cup \{av, bv, cv\})| - 3$, the existence of such a balanced set contradicts the $\mathcal{D}$-sparsity of $G$.

(Case 2) Let us consider the case when $X$ is type 2. By definition of obstacles (whether type 1 or type2), $Y + av \cdot vc$ or $Y + bv \cdot vd$ is balanced and 2-edge-connected. Without loss of generality, we assume that $Y + av \cdot vc$ is balanced and 2-edge-connected. By Lemma 9.13, there exists an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in X \cup \{va, vb\}$ and $\phi'(vc) = \phi'(vd) \neq \mathsf{id}$. Moreover, since $X \cup Y$ is balanced, we may assume that $\phi'(e) = \mathsf{id}$ for $e \in Y$. Since $\phi'(av \cdot vc) \neq \mathsf{id}$ but $\phi'(e) = \mathsf{id}$ for $e \in Y$, $Y + av \cdot vc$ is unbalanced, a contradiction. □

**Lemma 9.16.** *Let $X$ and $Y$ be obstacles for the 2-reductions through $(av, vb)$ and $(cv, vd)$ and through $(av, vc)$ and $(bv, vd)$, respectively. If $|X| > 1$ and $|Y| > 1$, then $X \cap Y \neq \emptyset$.*

*Proof.* Without loss of generality, we assume $a \in V(X) \cap V(Y)$. Recall that each balanced tight set is 2-connected if the size is more than one. By the 4-regularity of $G$, each vertex of $N(v)$ has degree three in $G - v$. Hence, if $X$ and $Y$ are type 1 with $|X| > 1$ and $|Y| > 1$, then $X \cap Y$ contains an edge incident to $a$.

If $X$ is type 2, then $X + av \cdot vb$ is balanced tight with $a, b, c, d \in V(X + av \cdot vb)$ by definition. Hence, if $Y$ is type 1, then $X \cap Y$ contains an edge incident to $c$ or $d$.

If both $X$ and $Y$ are type 2, then $X \cap Y$ contains an edge incident to $d$. □

**Lemma 9.17.** *Let $X$, $Y$, $Z$ be obstacles for the 2-reductions through $(av, vb)$ and $(cv, vd)$, through $(av, vc)$ and $(bv, vd)$, and through $(av, vd)$ and $(bv, vc)$, respectively. If there is no hat subgraph containing the star of $v$, then $X \cap Y \neq \emptyset$, $Y \cap Z \neq \emptyset$ or $Z \cap X \neq \emptyset$ holds.*

*Proof.* Note that a type 2 obstacle consists of more than one edge. If two of $X, Y$ and $Z$ are not singleton sets, then the lemma follows from Lemma 9.16. Hence we may assume that $|Y| = |Z| = 1$, and denote $Y = \{e_y\}$ and $Z = \{e_z\}$. Clearly, $e_y \neq e_z$.

(Case 1) Let us first consider the case when $X$ is also a singleton set. Let $X = \{e_x\}$. Depending on the relative position of $e_x, e_y$ and $e_z$, we have two situations: (I) $e_x, e_y$ and $e_z$ share a vertex or (II) $e_x, e_y$ and $e_z$ form a triangle.

In case (I), the star of $v$ is included in a hat subgraph. Indeed, if denoting without loss of generality $e_x = ab$, $e_y = ac$, and $e_z = ad$, $\{e_x, e_y, e_z, va, vb, vc, vd\}$ forms a hat if it is balanced. Since $X, Y$ and $Z$ are obstacles, we have $\phi(e_x) = \phi(av \cdot vb)$, $\phi(e_y) = \phi(av \cdot vc)$ and $\phi(e_z) = \phi(av \cdot vd)$, and hence this subgraph is indeed balanced.

In case (II), without loss of generality, we assume $e_x = ab, e_y = bc$ and $e_z = ca$. Then $\{e_x, e_y, e_z, va, vb, vc\}$ forms $K_4$. Since $\phi(e_x) = \phi(av \cdot vb)$, $\phi(e_y) = \phi(bv \cdot vc)$ and $\phi(e_z) = \phi(cv \cdot va)$, this $K_4$ does not have any unbalanced cycle. Therefore, Case (II) cannot happen because of the $\mathcal{D}$-sparsity of $G$, as a balanced $K_4$ is not $\mathcal{D}$-sparse.

(Case 2) Next, we consider the case when $|X| > 1$. We further split the proof into two subcases depending on whether $X$ is type 1 or type 2.

If $X$ is type 2, then $|X \cup E_v| = 2|V(X \cup E_v)| - 2$ by Lemma 9.13. Also, by Lemma 9.13, there exists an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in X \cup \{va, vb\}$ and $\phi'(vc) = \phi'(vd) \neq \mathsf{id}$. Denote $\phi'(vc)$ by $g$. Since $Y$ and $Z$ are obstacles, we have $\phi'(e_y) = \phi'(e_z) = g$, which in particular implies $e_y, e_z \notin X$. By $N(v) \subseteq V(X)$ and $e_y \neq e_z$, $|X \cup Y \cup Z \cup E_v| = 2|V(X \cup Y \cup Z \cup E_v)|$, which in turn implies $E = X \cup Y \cup Z \cup E_v$. Notice that the label of each edge in $X \cup Y \cup Z \cup E_v$ is either the identity or $g$. In other words, $X \cup Y \cup Z \cup E_v$ is cyclic, contradicting the $\mathcal{D}$-sparsity of $G$.

The remaining case is when $X$ is type 1. Without loss of generality we assume $a, b \in V(X)$. By $|X| > 1$ and Lemma 7.1, $d_X(a) \geq 2$ and $d_X(b) \geq 2$. Since $e_y$ is either $ac$ or $bd$ and $e_z$ is either $ad$ or $bc$, it suffices to consider the following two cases by symmetry: (i)$(e_y, e_z) = (ac, ad)$, and (ii)$(e_y, e_z) = (ac, bc)$.

In subcase (i), $X \cap Y$ or $X \cap Z$ contains an edge incident to $a$ as $d_X(a) \geq 2$ and $d_{G-v}(a) = 3$.

In subcase (ii), notice that, $\{av, bv, cv, e_y, e_z, av \cdot vb\}$ is a circuit of the underlying $\mathcal{D}$-sparsity matroid since it forms a balanced $K_4$. By $av \cdot vb \in \mathrm{cl}_{\mathcal{D}}(X)$, we have $cv \in \mathrm{cl}_{\mathcal{D}}(X + av + bv + e_y + e_z) \subseteq \mathrm{cl}_{\mathcal{D}}(E - cv)$, contradicting the independence of $E$. Therefore, this case does not occur and the proof is complete. □

### 9.3.2 Obstacles around a vertex $v$ with $|N(v)| = 3$

In this subsection we shall investigate *obstacles* for a 2-reduction at a vertex $v$ with $|N(v)| = 3$. Most of the arguments are similar to the previous subsection. Throughout §9.3.2, $(G, \phi)$ denotes a $\mathcal{D}$-gain graph satisfying the assumptions of Theorem 9.11, $v$ denotes a vertex with $|N(v)| = 3$, $N(v) = \{a, b, c\}$, and $E_v$ denotes the set of edges incident to $v$. Without loss of generality, we assume that there are parallel edges $e_1$ and $e_2$ between $v$ and $a$, and we denote $E_v = \{e_1, e_2, vb, vc\}$.

We again have three possible ways for a 2-reduction at $v$. In each case, there exists an obstacle if the operation is not admissible. The proof of the following claim is identical to that of Lemma 9.12 and hence is omitted.

**Lemma 9.18.** *Suppose that the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$ is not admissible. Then there is an edge subset $F \subseteq E \setminus E_v$ satisfying one of the following properties:*

*(i) $F$ is balanced tight with $a, b \in V(F)$ and $e_1 \cdot vb \in \mathrm{cl}_{\mathcal{D}}(F)$;*

*(ii) $F$ is balanced tight with $a, c \in V(F)$ and $e_2 \cdot vc \in \mathrm{cl}_{\mathcal{D}}(F)$;*

*(iii) $F$ is sub-tight with $a, b, c \in V(F)$, $F + e_1 \cdot vb$ is balanced tight, and $e_2 \cdot vc \in \mathrm{cl}_{\mathcal{D}}(F + e_1 \cdot vb)$.*

For the 2-reduction through $(e_1, e_2)$ and $(bv, vc)$, we encounter an even simpler situation.

**Lemma 9.19.** *Suppose that the 2-reduction through $(e_1, e_2)$ and $(bv, vc)$ is not admissible. Then there is a balanced tight set $F \subseteq E \setminus E_v$ with $b, c \in V(F)$ and $bv \cdot vc \in \mathrm{cl}_{\mathcal{D}}(F)$.*

*Proof.* Note that $e_1 \cdot e_2^{-1}$ is a loop. $G - v + e_1 \cdot e_2^{-1}$ is $\mathcal{D}$-sparse by Lemma 9.2 since adding an unbalanced loop does not affect (C1). Note that $G - v + e_1 \cdot e_2^{-1} + bv \cdot vc$ is connected. If $G - v + e_1 \cdot e_2^{-1} + bv \cdot vc$ does not satisfy (C2), then Lemma 9.3 implies that $G$ is almost near-cyclic, which contradicts our assumption on $G$. If $G - v + e_1 \cdot e_2^{-1} + bv \cdot vc$ does not satisfy (C1), then $G - v + e_1 \cdot e_2^{-1}$ contains a balanced tight set $F$ with $b, c \in V(F)$ and $bv \cdot vc \in \mathrm{cl}_{\mathcal{D}}(F)$. Since a balanced tight set does not contain a loop by Lemma 7.1, we have $F \subseteq E \setminus E_v$. □

According to Lemmas 9.18 and 9.19, we can define *the type of an obstacle* as in the previous subsection. Lemma 9.19 also says that we only encounter type 1 obstacles for the 2-reduction through $(e_1, e_2)$ and $(bv, vc)$. The next two lemmas are counterparts of Lemmas 9.14 and 9.15, respectively, with identical proofs, which are omitted.

**Lemma 9.20.** *Let $X$ and $Y$ be obstacles for distinct 2-reductions at $v$. If $X$ is type 2 and $X \cup Y$ is cyclic, then $X \cup Y \cup E_v$ is cyclic.*

**Lemma 9.21.** *Let $X$ and $Y$ be obstacles for distinct 2-reductions at $v$. Then, if $X \cap Y \neq \emptyset$, then $X \cup Y$ is balanced.*

To prove the counterpart of Lemma 9.17, we need the following two additional lemmas.

**Lemma 9.22.** *Suppose that $Z$ is an obstacle of type 1 for the 2-reduction through $(e_1, e_2)$ and $(bv, vc)$. Then, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in Z \cup \{vb, vc\}$.*

*Proof.* $Z + bv \cdot vc$ is balanced. Hence, by Lemma 2.4, there is an equivalent gain function $\phi'$ to $\phi$ such that $\phi'(e) = \mathsf{id}$ for $e \in Z + bv \cdot vc$. By performing a switching operation at $v$ with $\phi'(bv)$ if necessary, we may assume that $\phi'(bv) = \phi'(vc) = \mathsf{id}$. □

**Lemma 9.23.** *Let $X$ be an obstacle of type 2 for the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$. Suppose further that there is no obstacle of type 1 for the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$. Then $d_X(a) + d_X(b) + d_X(c) \geq 5$ holds.*

*Proof.* Let $X' = X + e_1 \cdot vb$. By definition, $X'$ is balanced tight with $a, b, c \in V(X')$ and $|X'| > 1$. Such a balanced tight set is 2-connected and essentially 3-edge-connected by Lemma 7.1. We thus have $d_{X'}(u) \geq 2$ for $u \in \{a, b, c\}$.

Suppose that $d_{X'}(a) = d_{X'}(b) = 2$. Since $X'$ is essentially 3-edge-connected and $e_1 \cdot vb$ is incident to $a$ and $b$, $X'$ must be a triangle on $a, b, c$. This means that $X$ contains an edge linking from $a$ to $c$, denoted by $e'$. Recall that $X' + e_2 \cdot vc$ is balanced by definition of type 2 obstacles. However, since $e'$ and $e_2 \cdot vc$ are parallel, for $X' + e_2 \cdot vc$ to be balanced, $\{e', e_2 \cdot vc\}$ has to be a balanced 2-cycle, that is, $\{e'\}$ is a type 1 obstacle for the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$, contradicting the assumption of the lemma.

Therefore, $d_{X'}(a) \geq 3$ or $d_{X'}(b) \geq 3$, implying $d_{X'}(a) + d_{X'}(b) + d_{X'}(c) \geq 7$. Since $X' = X + e_1 \cdot vb$, we obtain $d_X(a) + d_X(b) + d_X(c) \geq 5$. □

**Lemma 9.24.** *Let $X$, $Y$, $Z$ be obstacles for the 2-reductions through $(e_1, vb)$ and $(e_2, vc)$, through $(e_1, vc)$ and $(e_2, vb)$, and through $(e_1, e_2)$ and $(bv, vc)$, respectively. Then, $X \cap Y \neq \emptyset$, $Y \cap Z \neq \emptyset$, or $Z \cap X \neq \emptyset$ holds.*

*Proof.* We split the proof into two cases depending on whether a type 1 obstacle exists for the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$.

(Case 1) Suppose that there is no type 1 obstacle for the 2-reduction through $(e_1, vb)$ and $(e_2, vc)$. Then, $X$ is type 2. By Lemma 9.23, $d_X(a) + d_X(b) + d_X(c) \geq 5$ holds. If $d_X(a) \geq 2$, then $X \cap Y$ contains an edge incident to $a$ since $d_{G-v}(a) = 2$ and $d_Y(a) \geq 1$. If $d_X(a) = 1$, then we have $d_X(b) \geq 2$ and $d_X(c) \geq 2$. Since $d_{G-v}(b) = d_{G-v}(c) = 3$, $|Z| = 1$ holds if $X \cap Z = \emptyset$. However, in this case, we have $d_{X \cup Z}(b) = d_{X \cup Z}(c) = 3$, and thus $X \cap Y$ or $Y \cap Z$ contains an edge incident to $b$ or $c$.

In a symmetric manner, we are done in the case when a type 1 obstacle does not exist for the 2-reduction through $(e_1, vc)$ and $(e_2, vb)$.

(Case 2) We now consider the case when both $X$ and $Y$ are type 1. If $|X| > 1$ or $|Y| > 1$, then $X$ or $Y$ is 2-connected, and hence $X \cap Y$ contains an edge incident to $a$ as $d_{G-v}(a) = 2$. We thus assume $|X| = |Y| = 1$ and $X \neq Y$. Let us denote $X = \{e_x\}$ and $Y = \{e_y\}$. Without loss of generality, we assume that $e_x$ connects from $a$ to $b$. Also, by Lemma 9.22, we may assume $\phi(e) = \mathsf{id}$ for $e \in Z \cup \{vb, vc\}$. Since $e_1 \cdot vb \in \mathrm{cl}_{\mathcal{D}}(X)$, we have $\phi(e_x) = \phi(e_1 \cdot vb) = \phi(e_1)$. The proof is completed by a further case analysis: (i) $e_y$ connects from $a$ to $c$ or (ii) $e_y$ connects from $a$ to $b$ (see Figure 21).

In case (i), we have $e_1 \cdot vc \in \mathrm{cl}_{\mathcal{D}}(Y)$ by definition. Therefore, $\phi(e_y) = \phi(e_1 \cdot vc) = \phi(e_1)$. Notice that $\{e_1, vb, vc, e_x, e_y, bv \cdot vc\}$ forms a $K_4$ without unbalanced cycles by $\phi(e_y) = \phi(e_1) = \phi(e_x)$. Moreover, since $bv \cdot vc \in \mathrm{cl}_{\mathcal{D}}(Z)$, we obtain $e_1 \in \mathrm{cl}_{\mathcal{D}}(\{vb, vc, e_x, e_y, bv \cdot vc\}) \subseteq \mathrm{cl}_{\mathcal{D}}(E - e_1)$. This contradicts the independence of $E$ in the underlying $\mathcal{D}$-sparsity matroid.

Let us consider case (ii). If $|Z| > 1$, then $X \cap Z$ or $Y \cap Z$ contains an edge incident to $b$ as $Z$ is type 1 and $d_Z(b) \geq 2$. Suppose that $|Z| = 1$, $X \cap Y = \emptyset$, $X \cap Z = \emptyset$ and

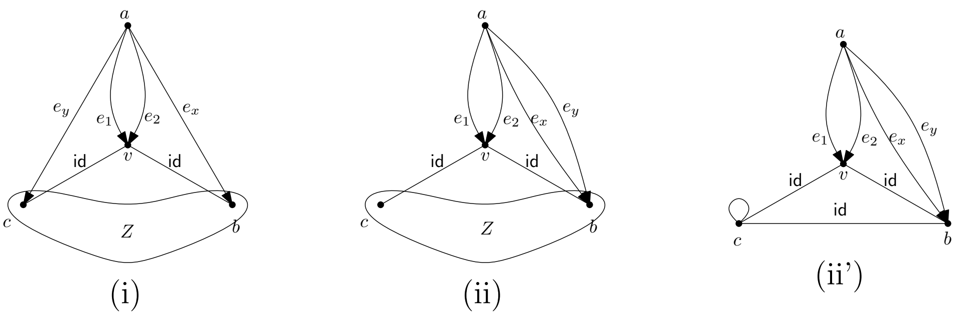



Figure 21

$Y \cap Z = \emptyset$. Then $X \cup Y \cup Z \cup E_v$ induces a subgraph in which $v, a$ and $b$ have degree four. So, if $|V| > 4$, then $c$ becomes a cut-vertex, contradicting the 2-connectivity of $G$. On the other hand, if $|V| = 4$, then $G$ becomes the graph shown in Figure 21(ii'). In this case removing $e_2$ and $e_y$ results in a cyclic graph (where any cycle except the loop is balanced by $\phi(e_1) = \phi(e_x)$). This means that $G$ is almost near-cyclic, a contradiction. □

### 9.3.3 Proof of Theorem 9.11

*Proof of Theorem 9.11.* Suppose that no 2-reduction is admissible at $v$. Then we have three obstacles $X$, $Y$ and $Z$ for the three possible 2-reductions at $v$. Suppose further that the star of $v$ is not contained in a hat subgraph. Then, by Lemma 9.17 and Lemma 9.24, we may assume without loss of generality that $X \cap Y \neq \emptyset$ holds.

If $|X \cup Y| \geq 2|V(X \cup Y)| - 1$, then $V(X \cup Y) \cup \{v\} = V$ must hold since $G$ is essentially 4-edge-connected. We then have $|X \cup Y \cup E_v| \geq 2|V| + 1$, contradicting the $\mathcal{D}$-sparsity of $G$.

Therefore we have

$$|X \cup Y| \leq 2|V(X \cup Y)| - 2. \tag{22}$$

To derive a contradiction, we next show that the number of connected components in $(V(X) \cap V(Y), X \cap Y)$ is equal to two. To see this, let $c_0$ be the number of trivial connected components (i.e., singleton vertex components) in $(V(X) \cap V(Y), X \cap Y)$ while let $c_1$ be the number of nontrivial connected components in it. Then,

$$|X| + |Y| \geq 2|V(X)| - 4 + 2|V(Y)| - 4 = 2|V(X \cup Y)| + 2|V(X \cap Y)| + 2c_0 - 8, \tag{23}$$

$$|X \cap Y| \leq 2|V(X \cap Y)| - 3c_1, \tag{24}$$

where the last inequality comes from $|F| \leq 2|V(F)| - 3$ for any non-empty $F \subseteq X \cap Y$. ¿From (22)(23)(24), we obtain $2c_0 + 3c_1 \leq 6$. On the other hand by $X \cap Y \neq \emptyset$ we also have $c_1 \geq 1$. Hence we get $c_1 + c_2 \leq 2$, and the number of connected components in the graph $(V(X) \cap V(Y), X \cap Y)$ is at most two.

If the number of connected components in $(V(X) \cap V(Y), X \cap Y)$ is one, then, since $X$ and $Y$ are connected and balanced, Lemma 2.5(1) implies that $X \cup Y$ is balanced, which contradicts Lemmas 9.15 and 9.21.

Thus the number of connected components in $(V(X) \cap V(Y), X \cap Y)$ is two. Then $2c_0 + 3c_1 \geq 5$. Hence by (23) and (24) we have

$$|X \cup Y| \geq 2|V(X \cup Y)| - 3. \tag{25}$$

Also by Lemma 2.6 $X \cup Y$ is cyclic. This implies that $X \cup Y$ is not tight, as $X \cup Y$ cannot be cyclic tight by (22).

If both $X$ and $Y$ are type 1, then $X \cup Y$ is tight by Lemma 7.2, which does not happen. Hence $X$ or $Y$ is type 2, and Lemmas 9.14 and 9.20 imply that $X \cup Y \cup E_v$ is also cyclic. Also by (25) and $N(v) \subseteq X \cup Y$ (as $X$ or $Y$ is type 2) we obtain $|X \cup Y \cup E_v| \geq 2|V(X \cup Y \cup E_v)| - 1$. Thus, due to the essential 4-edge-connecitivity of $G$, $|V(X \cup Y \cup E_v)| \geq |V| - 1$ must hold.

If $V(X \cup Y \cup E_v) = V$, then $|X \cup Y \cup E_v| = |E| - 1$, and hence $G$ is near cyclic, as $X \cup Y \cup E_v$ is cyclic. On the other hand, if $V(X \cup Y \cup E_v) = V - u$ for some $u \in V$, then $u$ is incident to a loop and two non-loop edges by the 4-regularity. Observe that removing this loop and one of the two non-loop edges results in a cyclic graph. This means that $G$ is almost near-cyclic.

In both cases $G$ turns out to be almost near-cyclic, which contradicts the assumption on $G$. This completes the proof. □

## 9.4 Proof of the Main Theorem

We are now ready to prove Theorem 9.1, which also completes the proof of Theorem 7.8.

*Proof of Theorem 9.1.* By Lemmas 9.5, 9.6, 9.8 and 9.9, we may assume that $G$ is 2-connected, essentially 4-edge-connected, not special, and not almost near-cyclic. Also, by Lemma 9.10, we may assume that every vertex $v$ with $N(v) = 2$ is incident to a loop.

Since $G$ is not special, $G$ has a vertex $v$ that is not incident to a loop. Then $|N(v)| \geq 3$. By Theorem 9.11, either the 2-reduction at $v$ is admissible or the star of $v$ is contained in a hat subgraph $H$. Suppose the latter holds. We denote the vertices of $H$ by $a_1, a_2, b_1, b_2, b_3$, and assume that $a_1$ and $a_2$ have degree four in $H$ (and hence $a_1$ or $a_2$ is $v$). Since $H$ is balanced, we may assume that all labels in $H$ are identity. Moreover, since $G$ is not a fancy hat, we may assume that $b_1$ is not incident to a loop.

We prove that some 2-reduction at $b_1$ is admissible. Suppose that no 2-reduction is admissible at $b_1$. Then, by Theorem 9.11, the star of $b_1$ is contained in a hat subgraph $H'$. Note that $H'$ is different from $H$.

We claim that $H'$ contains a triangle on $b_1, a_i, b_j$ for some $i \in \{1,2\}$ and $j \in \{2,3\}$. To see this first suppose that $a_1a_2 \notin E(H')$. Then, since each vertex has degree at least 2 in $H'$, we have $a_1b_2 \in E(H')$ or $a_1b_3 \in E(H')$ by $N_G(a_1) = \{a_2, b_1, b_2, b_3\}$ and $a_1a_2 \notin E(H')$. This also implies $b_1b_2 \in E(H')$ or $b_1b_3 \in E(H')$, respectively, as $b_1$ is incident to all the vertices of $H'$. Thus $H'$ has a triangle on $b_1, a_1, b_j$ for some $j \in \{2,3\}$.

If $a_1a_2 \in E(H')$, then $H'$ contains a triangle on $b_1, a_1, a_2$. In a hat subgraph, two vertices of each triangle have degree four, which implies $N(a_i) \subseteq V(H')$ for some

$i \in \{1, 2\}$. Therefore, $a_ib_2 \in E(H')$ and $b_1b_2 \in E(H')$, and hence $b_1b_2a_i$ forms a triangle.

Consequently, without loss of generality, we may assume that $H'$ contains a triangle on $b_1, b_2, a_1$. Recall that a hat subgraph is balanced. Since $\phi(a_1b_1) = \phi(a_1b_2) = \mathsf{id}$, we obtain $\phi(b_1b_2) = \mathsf{id}$ as $H'$ contains a triangle on $a_1, b_1, b_2$. Observe then that $\{a_1, a_2, b_1, b_2\}$ induces a $K_4$ in which the label of each edge is identity. This contradicts the $\mathcal{D}$-sparsity of $G$. Consequently, the 2-reduction at $b_1$ is admissible. □

# 10 Concluding Remarks

The main results of this paper (Theorems 6.3 and 8.2) give rise to efficient algorithms for testing generic symmetric rigidity with cyclic or odd-order dihedral symmetry. This can be done by computing the rank of the quotient graphs in the corresponding matroids $\mathcal{M}(g_{2,3})$ or $\mathcal{M}_{\mathcal{D}}(G, \phi)$.

Here we briefly describe the main algorithmic ideas and show that testing independece in these matroids can be done in polynomial time. We omit the proofs and a detailed and improved running time analysis.

Let $(G, \phi)$ be a gain graph with $G = (V, E)$. First consider $\mathcal{M}(g_{2,3})$, in which $E$ is independent if and only if (i) $G$ is $(2, 1)$-sparse and (ii) every nonempty balanced subset $F \subseteq E$ is $(2, 3)$-sparse, c.f. Lemma 3.1. There exist efficient algorithms for testing $(k, l)$-sparsity for any pair of integers $k, l$, see e.g. [2, 19], so checking (i) is easy. Observe that $G$ satisfies (ii) if and only if every minimally non-$(2, 3)$-sparse graph (also called a $(2, 3)$-circuit or an $M$-circuit) is unbalanced. Suppose that $G$ satisfies (i) and consider one of its $M$-components, i.e. a subgraph $H$ of $G$ induced by a connected component of the $(2, 3)$-sparsity matroid of $G$ (see [2, 12] for more details on $M$-components). Each $(2, 3)$-circuit is a subgraph of some $M$-component, so we may deal with them separately. The key observation is that within $H$ the complements of the $(2, 3)$-circuits are pairwise edge-disjoint. Since the $M$-components are pairwise edge-disjoint, this shows that the number of $(2, 3)$-circuits in $G$ is $O(n)$ and they can easily be enumerated. Then it remains to test whether each of these circuits is unbalanced, which can be done by switching and using Lemma 2.4. (Similar arguments are given in [1].)

Next consider $\mathcal{M}_{\mathcal{D}}(G, \phi)$. the odd-order dihedral case, in which $E$ is independent if and only if (i) $G$ is $(2, 0)$-sparse and (ii) every cyclic subset $F \subseteq E$ is $(2, 1)$-sparse, and (iii) every balanced subset $F \subseteq E$ is $(2, 3)$-sparse. As above, testing $(2, 0)$-sparsity is easy. We can again observe that $G$ satisfies (ii) if and only if every minimally non-$(2, 1)$-sparse graph (a $(2, 1)$-circuit) is non-cyclic. Suppose that $G$ satisfies (i). Then it is easy to see that these circuits are edge-disjoint, which shows that we have $O(n)$ circuits to check. As above, they can easily be enumerated, and we can use switching and Lemma 2.4 to see whether they are all non-cyclic. So suppose $G$ satisfies (ii) as well. As above, it remains to check whether every $(2, 3)$-circuit is unbalanced. Let $H$ be an $M$-component of $G$. It is not hard to see that $H - e$ is $(2, 1)$-sparse for all $e \in E(H)$. Thus, by using the arguments above, it follows that we have $O(n^2)$ circuits to enumerate and test, which can also be done efficiently by the same techniques.

## Acknowledgments

This work was supported by the Grant-in-Aid for Scientific Research from Japan Society for Promotion of Science, the MTA-ELTE Egerváry Research Group on Combinatorial Optimization, the Hungarian Scientific Research Fund grant no. K81472, and the National Development Agency of Hungary, grant no. CK 80124, based on a source from the Research and Technology Innovation Fund.

## References

[1] M. Berardi, B. Heeringa, J. Malestein, and L. Theran, Rigid components in fixed-lattice and cone frameworks, Proc. CCCG 2011, Toronto ON, August 2011.

[2] A.R. Berg and T. Jordán, Algorithms for graph rigidity and scene analysis, Proc. 11th Annual European Symposium on Algorithms (ESA) 2003, (G. Di Battista, U. Zwick, eds) Springer Lecture Notes in Computer Science 2832, pp. 78-89, 2003.

[3] N. Biggs, *Algebraic Graph Theory.* Cambridge University Press, 2nd edition, 1994.

[4] C. Borcea and I. Streinu, Periodic frameworks and flexibility. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 466(2121):2633–2649, 2010.

[5] C. Borcea and I. Streinu, Minimally rigid periodic graphs. *Bulletin of the London Mathematical Society*, 43(6):1093–1103, 2011.

[6] R. Connelly, P. Fowler, S. Guest, B. Schulze, and W. Whiteley, When is a symmetric pin-jointed framework isostatic? *International Journal of Solids and Structures*, 46(3-4):762–773, 2009.

[7] J. Edmonds, Submodular functions, matroids, and certain polyhedra. In R. Guy, H. Hanani, N. Sauer, and J. Schönheim, editors, *Combinatorial Structures and Their Applications*, pages 69–87, 1970.

[8] P. Fowler and S. Guest, A symmetry extension of Maxwell's rule for rigidity of frames. *International Journal of Solids and Structures*, 37(12):1793–1804, 2000.

[9] A. Frank, *Connections in Combinatorial Optimization.* Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2011.

[10] J. Graver, B. Servatius, and H. Servatius, *Combinatorial Rigidity*, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[11] J. L. Gross and T. W. Tucker, *Topological Graph Theory.* Dover, 1987.

[12] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs, *J. Combinatorial Theory Ser B*, Vol. 94, 1-29, 2005.

[13] J. Malestein and L. Theran, Generic combinatorial rigidity of periodic frameworks. *Advances in Math.*, to appear.

[14] J. Malestein and L. Theran, Generic rigidity of frameworks with orientation-preserving crystallographic symmetry, manuscript, *arXiv:1108.2518*, 2011.

[15] J. Malestein and L. Theran, Generic rigidity of reflection frameworks, manuscript, arXiv:1203.2276, 2012.

[16] Y. Kanno, M. Ohsaki, K. Murota, and N. Katoh, Group symmetry in interior-point methods for semidefinite program, *Optimization and Engineering*, 2, 293-320, 2001.

[17] A. Nixon and E. Ross Periodic Rigidity on a Variable Torus Using Inductive Constructions, manuscript, arXiv:1204.1349, 2012.

[18] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[19] A. Lee and I. Streinu, Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 308(8):1425–1437, 2008.

[20] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* 3 (1982), no. 1, 91–98.

[21] J. Owen and S. Power, Frameworks, symmetry and rigidity. *arXiv:0812.3785*, 2008.

[22] E. Ross, *Geometric and combinatorial rigidity of periodic frameworks as graphs on the torus.* PhD thesis, York University, Toronto, May 2011.

[23] A. Schrijver, *Combinatorial optimization: polyhedra and efficiency.* Springer, 2003.

[24] B. Schulze, *Combinatorial and geometric rigidity with symmetric constraints.* Ph. thesis, York University, 2009.

[25] B. Schulze, Symmetric versions of Laman's theorem. *Discrete & Computational Geometry*, 44(4):946–972, 2010.

[26] B. Schulze, Symmetry as a sufficient condition for a finite flex. *SIAM J. Discrete Math.*, 24(4):1291–1312, 2010.

[27] B. Schulze and W. Whiteley, The orbit rigidity matrix of a symmetric framework. *Discrete Comput. Geom.*, 46(3):561–598, 2011.

[28] T. Tarnai, Simultaneous static and kinematic indeterminacy of space trusses with cyclic symmetry, *Int. J. Solids Structures* Vol. 16, pp. 347-359, 1980.

[29] S. Tanigawa, Matroids of gain graphs in discrete applied geometry, manuscript, arXiv:1207.3601v1, 2012.

[30] T. S. Tay and W. Whiteley, Generating isostatic graphs. *Structural Topology*, 11:21–68, 1985.

[31] F. Wegner, Rigid-unit modes in tetrahedral crystals, *Journal of Physics: Condensed Matter*, Volume 19, Issue 40, (2007).

[32] W. Whiteley, Some matroids from discrete applied geometry. Matroid theory (Seattle, WA, 1995), 171–311, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.

[33] W. Whiteley, Rigidity and scene analysis, in: Handbook of Discrete and Computational Geometry (J. E. Goodman and J. O'Rourke, eds.), CRC Press, Second Edition, pp. 1327-1354, 2004.

[34] W. Whiteley, Counting out to the flexibility of molecules. *Physical Biology*, 2:S116-S126, 2005.

[35] T. Zaslavsky, Biased graphs. I. bias, balance, and gains. *Journal of Combinatorial Theory, Series B*, 47(1):32–52, 1989.

[36] T. Zaslavsky, Biased graphs. II. the three matroids. *Journal of Combinatorial Theory, Series B*, 51(1):46–72, 1991.