Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-14. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Minimax Theorems in Graph Connectivity Augmentation

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August 24, 2001

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Abstract

This survey paper on polynomial-time solvable versions of graph connectivity augmentation problems, and on the corresponding minimax theorems, was written in August 2001 and has not been published earlier. Some of the references have been updated but the text is unchanged. For more recent developments the reader is referred to [29, 75].

1 Introduction

The problem of economically improving a network to meet given survivability conditions occurs in a number of areas. A straightforward problem of this type is concerned with creating more connections in a telephone or computer network so that it survives the failure of a given number of cables or terminals [43]. Similar problems arise in graph drawing [64], statics [3], and data security [65]. Graph theory provides a general framework to attack this type of questions. The general problem of this part of graph theory, called graph connectivity augmentation, is the following optimization problem.

General Problem: Let $G = (V, E)$ be a (directed) graph and let $r : V \times V \to Z_+$ be a function. Find a smallest (or cheapest) set F of new edges (arcs) such that $\lambda(u, v; G') \geq r(u, v)$ (or $\kappa(u, v; G') \geq r(u, v)$ holds for all $u, v \in V$ in $G' = (V, E \cup F)$.

Here $\lambda(u, v; H)$ ($\kappa(u, v; H)$) denotes the local edge-connectivity (local vertex-connectivity, respectively) in graph H, that is, the maximum number of pairwise edge-disjoint (vertex-disjoint) paths from u to v in H. The general problem includes several different subproblems, depending on whether G is undirected or not, and whether the requirements $r(u, v)$ are imposed for edge- or vertex-connectivity. When the goal is to find a cheapest set of new edges, this is meant with respect to a given weight function on the set of possible new edges.

This chapter is a survey on graph connectivity augmentation problems. We shall focus on the efficiently solvable variants of the problem, for which polynomial-time

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algorithms and minimax theorems are known. Thus, since most of the weigthed cases are NP-hard, we shall be interested in finding augmenting sets of minimum size. For a survey on NP-hard versions and approximation algorithms see Khuller [66]. The unweighted case alone is so rich of interesting results that we decided to focus on minimax theorems only and omit the details of the corresponding algorithms. For more on the algorithmic side see the recent survey of Nagamochi [72]. We shall also omit the detailed discussion of graph synthesis problems, which deal with the problem of constructing a graph with as few edges as possible from scratch, satisfying given connectvity requirements. These are special cases of the general problem with $E = \emptyset$.

Most of the theorems in this paper are less than ten years old, showing that this is a relatively new and active area of graph theory where substantial progress has been made in the last decade. The reader is referred to an earlier survey article of Frank [27] for more details on older results as well as for several other versions not treated in this chapter.

2 Edge-connectivity augmentation of graphs

The first papers on graph connectivity augmentation appeared in 1976, when Eswaran and Tarjan [20] and independently Plesnik [77] solved the 2-edge- (and 2-vertex-)connectivity augmentation problem. Although several graph synthesis problems (that is, augmentation problems where the starting graph has no edges) had been solved earlier (see e.g. Gomory and Hu [42], Frank and Chou [36]), these papers were the first to provide minimax theorems for arbitrary starting graphs.

Let $G = (V, E)$ be a graph which is not 2-edge-connected. A **2-component** G' of G is a maximal 2-edge-connected subgraph of G. A 2-component G' with $d_G(V(G')) = 1$ is a **leaf**. (We use $d_H(X)$ to denote the degree of a subset X of vertices in graph H.) Note that the 2-components are pairwise disjoint. Let $l(G)$ denote the number of leaves and let $c'(G)$ denote the number of those connected components of G which are 2-edge-connected. The leaves and these connected components are also pairwise disjoint. Since every new edge can increase the degree of at most two leaves or 2-edge-connected components (by one), it follows that at least $\lceil l(G)/2 \rceil + c'(G)$ new edges are needed to make G 2-edge-connected.

Theorem 2.1. *[20, 77] The minimum number of new edges which make $G = (V, E)$ 2-edge-connected equals $\lceil l(G)/2 \rceil + c'(G)$.*

By contracting each of the 2-components of G, we may assume that the starting graph G is a forest. It is also easy to reduce the problem to the case when G is a tree without vertices of degree 2 and with an even number of leaves. There exist a number of methods to find a smallest set F which makes such a tree 2-edge-connected. For example, connecting the end-vertices of a longest path and then contracting the resulting cycle reduces the number of leaves by two. Hence by repeating this procedure as long as possible, we obtain an (optimal) augmenting set of size $\lceil l(G)/2 \rceil$. Another solution is to number the leaves of G according to a DFS search and then connect "opposite" pairs of vertices.

Kajitani and Ueno [63] solved the k-edge-connectivity augmentation problem for every $k \geq 1$, provided the starting graph is a tree. For arbitrary starting graphs we can extend the lower bound used in Theorem 2.1 as follows. A **subpartition** of V is a family of pairwise disjoint non-empty subsets of V. Let $G = (V, E)$ be a graph and $k \geq 2$. Let

$$\alpha(G, k) = \max\{\sum_1^t (k - d(X)) : X_1, X_2, ..., X_t \text{ is a subpartition of } V\}. \tag{1}$$

and let $\gamma(G, k)$ be the size of a smallest augmenting set of G with respect to k-edge-connectivity. Every augmenting set must contain at least $k - d(X)$ edges entering X for every $X \subset V$ and every new edge can decrease this "deficiency" of at most two sets in any subpartition. Thus $\gamma(G, k) \geq \Phi(G, k)$, where $\Phi(G, k) = \lceil \alpha(G, k)/2 \rceil$. Theorem 2.1 implies that $\gamma(G, k) = \Phi(G, k)$ for $k = 2$. Watanabe and Nakamura [84] solved the k-edge-connectivity augmentation problem and proved that this minimax equality is valid for every $k \geq 2$.

Theorem 2.2. *[84] Let $G = (V, E)$ be a graph and let $k \geq 2$. Then $\gamma(G, k) = \Phi(G, k)$.*

Watanabe and Nakamura construct an increasing sequence of augmenting sets $F_1, F_2, ..., F_k$ such that for all $1 \leq i \leq k$ the set F_i is an optimal augmenting set of G with respect to i.

A different method was employed by Cai and Sun [11] (and suggested already by Plesnik [77] for $k = 2$). This method is based on edge splitting. Let $H = (V + s, E)$ be a graph with a designated vertex s. By **splitting off** a pair of edges su, sv we mean the operation that replaces su, sv by a new edge uv. The resulting graph is denoted by H_{uv}. A **complete splitting** at s is a sequence of splittings which isolates s. A complete splitting exists only if $d_H(s)$ is even. The splitting off method adds a new vertex s to the starting graph and constructs the augmenting set by splitting off edges from s. Frank [26] simplified and extended this method and showed that "generalized polymatroids" are in the background of this method. For more details on the use of polymatroids we refer to [26, 27, 34]. Here we describe Frank's algorithmic proof for Theorem 2.2 in terms of graphs.

Let $G = (V, E)$ be a graph. An **extension** $G' = (V + s, E')$ of G is obtained from G by adding a new vertex s and a set of new edges, such that every new edge is incident with s. An extension G' is said to be (k, s)**-edge-connected** if $\lambda(x, y; G') \geq k$ holds for every pair $x, y \in V$. G' is **minimally** (k, s)**-edge-connected** if $G' - e$ is no longer (k, s)-edge-connected for every edge e incident with s.

It follows from Menger's theorem that in a minimally (k, s)-edge-connected extension $G' = (V + s, E')$ every neighbour of s is contained by a set $X \subset V$ with $d_{G'}(X) = k$. Frank [26] proved that there exists a subpartition $\{X_1, ..., X_t\}$ of V consisting of sets of degree k which cover all neighbours of s in G', and hence $d_{G'}(s) = \sum_1^t (k - d_G(X_i))$ holds. Since $d_{G'}(s) \geq \sum_{X \in \mathcal{X}} (k - d_G(X))$ for any subpartition $\mathcal{X}$ of V, this implies that if G' is a minimally (k, s)-edge-connected extension of G then

$$d_{G'}(s) = \alpha(G, k). \tag{2}$$

This fact and the following result of Lovász [68] gives rise to a simple algorithmic proof of Theorem 2.2. Let $H = (V + s, E)$ be a (k, s)-edge-connected graph. We say that splitting off two edges us, sv is **k-admissible** if H_{uv} is also (k, s)-edge-connected. A **complete k-admissible splitting at** s is a sequence of k-admissible splittings which isolates s. Note that the graph on vertex set V, obtained from H by a complete k-admissible splitting, is k-edge-connected.

Theorem 2.3. *[68] Let $H = (V + s, E)$ be a (k, s)-edge-connected graph for some $k \geq 2$ and suppose that $d_H(s)$ is even. Then (a) for every edge su there exists an edge sv such that the pair su, sv is k-admissible; (b) there exists a complete k-admissible splitting at s in H.*

Note that (b) follows by $d_H(s)/2$ repeated applications of (a). We can summarize Frank's algorithm as follows.

Frank's algorithm

(PHASE 1) Given a graph $G = (V, E)$ and an integer $k \geq 2$, create a minimally (k, s)-edge-connected extension $G' = (V + s, E')$ of G.

(PHASE 2) If $d_{G'}(s)$ is odd, add a new edge sv for some arbitrary $v \in V$. Split off k-admissible pairs of edges incident to s in arbitrary order. When s becomes isolated, delete s.

It is clear that a (k, s)-edge-connected extension of G exists (say, by adding k parallel edges from s to each vertex of V will do). A minimally (k, s)-edge-connected extension $G' = (V + s, E')$ can be obtained by a greedy deletion procedure. In the second phase every edge incident to s can be split off by Theorem 2.3. The resulting graph is an optimal k-edge-connected augmentation of G since after the first phase $\gamma(G, k) \geq d_{G'}(s)/2$ by (2). This proves Theorem 2.2.

Frank [26] showed that, although finding an augmenting set with minimum total weigth is NP-hard, the k-edge-connectivity augmentation problem with vertex-induced edge weights can be solved in polynomial time. In this version we are given a weigth function $w : V \to Z_+$ on the vertices and the weight of an edge uv equals $w(u) + w(v)$. He also solved several degree constrained versions of the problem. For example, he proved the following characterisation.

Theorem 2.4. *[26] Let $G = (V, E)$ be a graph, $k \geq 2$ and integer, and let $f \leq g$ be two non-negative integer-valued functions on V. Then G can be made k-edge-connected by adding a set F of new edges so that $f(v) \leq d_F(v) \leq g(v)$ holds for every $v \in V$ if and only if $k - d(X) \leq g(X)$ for every $\emptyset \neq X \subset V$ and there is no partition $\mathcal{P} = \{X_0, X_1, ..., X_t\}$ of V, where only X_0 may be empty, and such that $f(X_0) = g(X_0)$, $g(X_i) = k - d(X_i)$ for $1 \leq i \leq t$, and $g(V)$ is odd.*

Naor, Gusfield and Martel [76] came up with yet another proof (and algorithm) for Theorem 2.2. Their algorithm increases the edge-connectivity one by one and is based on the concept of extreme sets from [84]. A set $X \subset V$ is **extreme** in a graph $G = (V, E)$ if $d(Y) > d(X)$ for every $Y \subset X$. Extreme sets form a laminar family (in fact, every extreme set is an edge-connectivity component of G, see [27]), and hence they can be represented by a tree.

It is easy to see that the sets X_i maximizing $\Phi(G,k)$ in Theorem 2.2 can be chosen to be extreme. The family of extreme sets suffices to compute $\gamma(G,k)$ for all $k \geq 2$ and, with some extra efforts, to find a smallest augmenting set as well. Benczúr and Karger [8] show how the extreme set tree can be used to find a minimally (k,s)-edge-connected extension G' of G and a complete k-admissible splitting in G'.

Naor, Gusfield and Martel use the Gomory-Hu tree of G to find the extreme sets. They also show how to employ the cactus representation of minimum edge cuts [18] to find a smallest set F which increases the edge-connectivity by one.

Let us return to the splitting off method, which enabled Frank [26] to solve the local edge-connectivity augmentation problem as well. A function $r : V \times V \to Z_+$ is called a **local requirement function** on V. Let r be a local requirement function on V, let $G = (V,E)$ be a graph and let $G' = (V+s, E')$ be an extension of G. We call G **r-edge-connected** if $\lambda(x,y;G) \geq r(x,y)$ for every $x,y \in V$. We say G' is **(r,s)-edge-connected** if $\lambda(x,y;G') \geq r(x,y)$ for every $x,y \in V$. Splitting off us, sv is **r-admissible** in G' if $\lambda(x,y;G'_{uv}) \geq r(x,y)$ for all $x,y \in V$. In the local edge-connectivity augmentation problem the goal is to find a smallest set F which makes the input graph $G = (V,E)$ r-edge-connected for a given local requirement function r.

Let function $R : 2^V \to Z_+$ be defined by letting $R(\emptyset) = R(V) = 0$ and

$$R(X) = \max\{r(x,y) : x \in X, y \in V - X\} \text{ for all } \emptyset \neq X \subset V. \tag{3}$$

By Menger's theorem an augmented graph G' is r-edge-connected if and only if $d_{G'}(X) \geq R(X)$ for every $X \subseteq V$. Let $q(X) = R(X) - d_G(X)$ for sets $\emptyset \neq X \subset V$ and let

$$\alpha(G,r) = \max\{\sum_{i=1}^{t} q(X_i) : X_1, X_2, \ldots, X_t \text{ is a subpartition of } V\} \tag{4}$$

An argument analogous to the uniform case shows that $\gamma(G,r) \geq \Phi(G,r)$, where $\gamma(G,r)$ is the size of a smallest augmenting set and $\Phi(G,r) = \lceil \alpha(G,r)/2 \rceil$. Theorem 2.2 claims that this lower bound is achievable if $r \equiv k \geq 2$. In the local version this does not necessarily hold. For example, consider a graph with four vertices and no edges and let $r \equiv 1$. The exceptional cases, however, can be dealt with as follows. Let $C \subset V$ be the vertex-set of a component of $G = (V,E)$. We say that C is a **marginal component** (with respect to r) if $q(C) \leq 1$ and $q(X) = 0$ for every proper subset of C. If G has a marginal component, we can reduce the problem as follows.

Theorem 2.5. *[26] Let $G = (V,E)$ be a graph and let C be a marginal component of G with respect to r, where r is a local requirement function on V. Let r' denote the restriction of r to $V - C$. Then $\gamma(G,r) = q(C) + \gamma(G - C, r')$.*

The proof of Theorem 2.5 shows how to find a smallest augmentation for G if a smallest augmenting set for $G - C$ (with respect to r') is available. Thus we can assume that G has no marginal components. In this case $\Phi(G, r)$ turns out to be sufficient to characterise $\gamma(G, r)$. The proof follows the steps of Frank's algorithm: first construct a minimally (r, s)-edge-connected extension $G' = (V + s, E')$ of G. It can be seen that this graph has $d_{G'}(s) = \alpha(G, r)$ [26]. Then, possibly after adding an edge sv to make $d_{G'}(s)$ even, perform a complete r-admissible splitting at s in G'. The set of split edges is a smallest augmenting set.

The fact that a complete r-admissible splitting exists in G' follows from Mader's [69] deep result, which entends Theorem 2.3 to local edge-connectivities.

Let $H = (V + s, E)$ be a graph. Let $r_\lambda(x, y) = \lambda(x, y; H)$ be a special requirement function, defined on pairs $x, y \in V$.

Theorem 2.6. *[69] Let $H = (V + s, E)$ be a graph with $d(s)$ is even and suppose that there is no cut-edge incident to s. Then there is a complete r_λ-admissible splitting at s.*

Observe that if G has no marginal components with respect to r then there is no cut-edge incident to s in G'. Thus, since we have $d_{G'}(s) = \alpha(G, r)$ and $\gamma(G, r) \geq \Phi(G, r)$, Theorem 2.6 completes the proof of Frank's theorem.

Theorem 2.7. *[26] Let $G = (V, E)$ be a graph and let r be a local requirement function on V. If G has no marginal components with respect to r then $\gamma(G, r) = \Phi(G, r)$.*

We say that an increasing sequence of local requirements $(r_1, r_2, ..., r_t)$ on V has the **successive augmentation property** if, for any starting graph $G = (V, E)$, there exists an increasing sequence $F_1 \subseteq F_2 \subseteq ... \subseteq F_t$ of sets of edges such that $G + F_i$ is an optimal augmentation of G with respect to r_i, for all $1 \leq i \leq t$. The proof of Theorem 2.2 by Watanabe and Nakamura [84] (and also Naor et al. [76]) implies that any increasing sequence of uniform requirements has the successive augmentation property in the edge-connectivity augmentation problem. By using an entirely different approach, Cheng and Jordán [13] generalised this to sequences with the following property:

$$r_{i+1}(u, v) - 1 = r_i(u, v) \geq 2, \text{ for all } u, v \in V, 1 \leq i \leq t - 1 \tag{5}$$

The proof is based on the fact that if $G' = (V + s, E')$ ($G'' = (V + s, E'')$) is a minimally r_{i-1}-edge-connected (minimally r_i-edge-connected, respectively) extension of G such that G' is a subgraph of G'', then any r_{i-1}-admissible splitting su, sv in G' is r_i-admissible in G''.

Theorem 2.8. *[13] Every increasing sequence $(r_1, r_2, ..., r_t)$ of local requirements satisfying (5) has the successive augmentation property in the edge-connectivity augmentation problem.*

A **mixed graph** $D = (V, E \cup A)$ has edges as well as arcs. Bang-Jensen, Frank, and Jackson [2] extended Theorem 2.6 and to mixed graphs and with the splitting off method, they generalised Theorem 2.7 to the case when the edge-connectivity of a mixed graph is to be increased by adding undirected edges. See also [27] for a list of theorems of this type.

3 Edge-connectivity of digraphs

Let $\gamma(D, k)$ denote the size of a smallest set F of new arcs which makes a given directed graph D k-edge-connected. The first result on digraph augmentation is due to Eswaran and Tarjan [20], who solved the strong connectivity augmentation problem and gave the following minimax formula for $\gamma(D, 1)$.

Let $D = (V, A)$ be a digraph. By contracting the strongly connected components of D we may assume that D is acyclic. A vertex v with $\rho(v) = 0$ ($\delta(v) = 0$) is a **source** (**sink**, respectively). For a set $X \subset V$ let X^+ and X^- denote the set of sources in X and the set of sinks in X, respectively. Clearly, to make D strongly connected we need at least $|V^+|$ ($|V^-|$) new arcs. For acyclic digraphs these bounds are sufficient to characterize $\gamma(D, 1)$.

Theorem 3.1. *[20] Let $D = (V, A)$ be an acyclic digraph. Then* $\gamma(D, 1) = \max\{|V^+|, |V^-|\}$.

It follows that for arbitrary starting graphs $\gamma(D, 1)$ equals the maximum number of pairwise disjoint sets $X_1, ..., X_t$ in D with $\rho(X_i) = 0$ for all $1 \leq i \leq t$ (or $\delta(X_i) = 0$ for all $1 \leq i \leq t$).

Kajitani and Ueno [63] solved the problem when D is a directed tree and $k \geq 1$ is arbitrary. The solution for arbitrary starting digraphs is due to Frank [26], who adapted the splitting off method to directed graphs and showed that, as in the undirected case, $\gamma(D, k)$ can be characterised by a subpartition-type lower bound. For a digraph $D = (V, A)$ let

$$\alpha_{in}(D, k) = \max\{\sum_1^t (k - \rho(X_i)) : X_1, ..., X_t \text{ is a subpartition of } V\},$$

$$\alpha_{out}(D, k) = \max\{\sum_1^t (k - \delta(X_i)) : X_1, ..., X_t \text{ is a subpartition of } V\}$$

Clearly, $\gamma(D, k) \geq \Phi(D, k)$, where $\Phi(D, k) = \max\{\alpha_{in}(D), \alpha_{out}(D)\}$. An **extension** $D' = (V + s, A')$ of a digraph $D = (V, A)$ is obtained from D by adding a new vertex s and a set of new arcs, such that each new arc leaves or enters s. A digraph $H = (V + s, A)$ with a designated vertex s is (k, s)**-edge-connected** if $\lambda(x, y; H) \geq k$ for every $x, y \in V$. H is **minimally** (k, s)**-edge-connected** if it is (k, s)-edge-connected but $H - e$ is no longer (k, s)-edge-connected for any arc e incident to s. Splitting off two arcs us, sv means replacing the arcs us, sv by a new arc uv. Splitting off two edges su, vs is k**-admissible** in a (k, s)-edge-connected digraph H if H_{uv} is also (k, s)-edge-connected.

Frank proved that if D' is a minimally (k, s)-edge-connected extension of D then we have $\max\{\rho_{D'}(s), \delta_{D'}(s)\} = \Phi(D, k)$. Suppose that D' has $\rho_{D'}(s) = \delta_{D'}(s) = \Phi(D, k)$ (this can be achieved by adding some extra edges incident to s). Then a complete k-admissible splitting, if exists, results in an (optimal) augmenting set on V of size $\Phi(D, k)$. The next theorem, due to Mader [70], guarantees that such a complete splitting always exists.

Theorem 3.2. *[70] Let $D = (V + s, A)$ be a (k, s)-edge-connected digraph with $\rho(s) = \delta(s)$. Then (a) for every arc su there exists an arc vs such that the pair su, vs is k-admissible, (b) there is a complete k-admissible splitting at s.*

This proves the minimax theorem for the directed k-edge-connectivity augmentation problem.

Theorem 3.3. *[26] Let $D = (V, A)$ be a directed graph and let $k \geq 1$. Then $\gamma(D, k) = \Phi(D, k)$.*

Frank [26] solved the weighted version with vertex-induced weight function as well as some degree constrained versions. Cheng and Jordán [13] proved that the successive augmentation property holds for any increasing sequence of uniform requirements in the directed edge-connectivity augmentation problem as well.

The local edge-connectivity augmentation problem in directed graphs is NP-hard, even if $r(u, v) \in \{0, 1\}$ for all $u, v \in V$ [26]. Bang-Jensen, Frank, and Jackson [2] generalised Theorem 3.3 to mixed graphs and special classes of local requirements. For instance, they showed that the local version is solvable for Eulerian digraphs (that is, for digraphs where $\rho(v) = \delta(v)$ for all $v \in V$). The proofs of these results rely on an edge splitting theorem, which is a common extension of Theorem 3.2 and a result of Frank [23] and Jackson [54] on splitting off edges in Eulerian digraphs preserving local edge-connectivities.

There is a different generalisation of Theorem 3.3, which is also tractable, although with entirely different techniques. Let $D = (V, A)$ be a digraph and let $S, T \subset V$ be two (possibly intersecting) non-empty subsets. We say that D is **k-edge-connected from S to T** if there are k edge-disjoint paths from every vertex of S to every vertex of T. This corresponds to k-edge-connectivity when $S = T = V$. We say that a family $\mathcal{X}$ of subsets of V is **(S, T)-independent** if for every pair $s \in S, t \in T$, the arc st enters at most one set in $\mathcal{X}$.

Theorem 3.4. *[30] A digraph $D = (V, A)$ can be made k-edge-connected from S to T by adding at most γ new arcs with tails in S and heads in T if and only if*

$$\sum_{j=1}^{t} (k - \varrho(X_j)) \leq \gamma$$

holds for every (S, T)-independent family $\mathcal{X} = \{X_1, X_2, ..., X_t\}$ of subsets of V with $T \cap X_j \neq \emptyset$ and $S - X_j \neq \emptyset$ for all $X_j \in \mathcal{X}$.

The proof of Theorem 3.4 is different from that of Theorem 3.3. It follows from a more general result (Theorem 7.10). No direct proof for Theorem 3.4 and no combinatorial algorithm for the k-ST-edge-connectivity augmentation problem is known, except when $k = 1$. For this special case Enni [19] gave a direct proof.

A different version of the mixed graph augmentation problem was investigated by Gusfield [44].

4 Constrained edge-connectivity augmentation problems

In each of the augmentation problems considered so far it was allowed to add (an arbitrary number of parallel copies of) any edge (or arc) connecting two vertices of the input graph. It is natural to consider (and in some cases the applications give rise to) variants where the set of new edges must meet certain additional constraints. In general, this variant is NP-hard. Frederickson and Jaja [37] proved that, given a tree $T = (V, E)$ and a set J of edges on V, it is NP-hard to find a smallest set $F \subseteq J$ for which $T' = (V, E \cup F)$ is 2-edge-connected. This problem remains NP-hard even if J is the edge-set of a cycle on the leaves of T [14]. For some types of constraints, however, the edge-connectivity augmentation problem remains tractable. This section contains extensions of some of the previous unconstrained theorems in this direction.

Motivated by a question in statics, Bang-Jensen, Gabow, Jordán, and Szigeti [3] solved the "partition-constrained" problem. Let $G = (V, E)$ be a graph an let $\mathcal{P} = \{P_1, P_2, ..., P_r\}$, $r \geq 2$, be a partition of V. In the partition-constrained k-edge-connectivity augmentation problem the goal is to find a smallest set F of new edges, such that every edge in F joins two distinct members of $\mathcal{P}$ and $G' = (V, E \cup F)$ is k-edge-connected. If G is a bipartite graph with bipartition $V = A \cup B$ and $\mathcal{P} = \{A, B\}$ then the problem is to augment a bipartite graph, preserving bipartiteness. By a theorem of Bolker and Crapo [10] this corresponds to increasing the rigidity of a square grid framework by adding a smallest set of extra diagonal rods. For the statics background see [3] and [78].

Let $\gamma(G, k, \mathcal{P})$ denote the size of a smallest augmenting set with respect to k and the given partition $\mathcal{P}$. Clearly, $\gamma(G, k, \mathcal{P}) \geq \gamma(G, k)$. The case $k = 1$ is easy, hence we assume $k \geq 2$. For $i = 1, 2, ..., r$ let

$$\beta_i = \max\{\sum_{Y \in \mathcal{Y}_j} (k - d(Y)) : \mathcal{Y}_j \text{ is a subpartition of } P_i\} \tag{6}$$

Since no new edge can join vertices in the same member P_i of $\mathcal{P}$, it follows that β_i is a lower bound for $\gamma(G, k, \mathcal{P})$ for all $1 \leq i \leq r$. By combining this bound and the lower bound of the unconstrained problem we obtain $\gamma(G, k, \mathcal{P}) \geq \Phi(G, k, \mathcal{P})$, where

$$\Phi(G, k, \mathcal{P}) = \max\{\lceil \alpha(G, k)/2 \rceil, \beta_1, \ldots, \beta_r\} \tag{7}$$

Simple examples show that $\gamma \geq \Phi + 1$ may hold. Consider a four-cycle C_4 and let $\mathcal{P}$ be the natural bipartition of C_4. Here we have $\Phi(C_4, 3, \mathcal{P}) = 2$ and $\gamma(C_4, 3, \mathcal{P}) = 3$. Now consider a six-cycle C_6 and let $\mathcal{P} = \{P_1, P_2, P_3\}$, where the members of $\mathcal{P}$ contain pairs of opposite vertices. For this graph and partition we have $\Phi(C_6, 3, \mathcal{P}) = 3$ and $\gamma(C_6, 3, \mathcal{P}) = 4$.

On the other hand, Bang-Jensen et al. [3] proved that $\gamma \leq \Phi + 1$ and characterised all graphs with $\gamma = \Phi + 1$. The proof employed the splitting off method. The first step was a complete solution of the corresponding constrained edge splitting problem. Let $H = (V + s, E)$ be a (k, s)-edge-connected graph and let $\mathcal{P} = \{P_1, P_2, ..., P_r\}$ be a partition of V. We say a splitting su, sv is **allowed** if it is k-admissible and respects the partition constraints, that is, u and v belong to distinct members of $\mathcal{P}$. If k is even, the following extension of Theorem 2.3(b) is not hard to prove.

Theorem 4.1. *[3] Let $H = (V+s, E)$ be a (k, s)-edge-connected graph, for some even integer k, let $\mathcal{P} = \{P_1, P_2, ..., P_r\}$ be a partition of V, and suppose that $d(s)$ is even. There exists a complete allowed splitting at s if and only if $d(s, P_i) \leq d(s)/2$ for all $1 \leq i \leq r$.*

For k odd, however, there exist more complicated "obstacles" that prevent a complete allowed splitting at s. Let S denote the set of neighbours of s.

A partition $A_1 \cup A_2 \cup B_1 \cup B_2$ of V is called a C_4**-obstacle** if it satisfies the following properties in H for some index i, $1 \leq i \leq r$:

(i) $d(A_1) = d(A_2) = d(B_1) = d(B_2) = k$;

(ii) $d(A_1, A_2) = d(B_1, B_2) = 0$;

(iii) $S \cap (A_1 \cup A_2) = S \cap P_i$ or $S \cap (B_1 \cup B_2) = S \cap P_i$;

(iv) $d(s, P_i) = d(s)/2$.

C_4 obstacles exist only for k odd. It is not difficult to see that if H contains a C_4-obstacle, then there exists no complete allowed splitting at s. A more special family of obstacles, called C_6**-obstacles**, can be defined when $r \geq 3$, k is odd, and $d(s) = 6$, see [3]. These two families suffice to characterise when there is no complete allowed splitting. Note that in the bipartition constrained case only C_4-obstacles may exist.

Theorem 4.2. *[3] Let $H = (V + s, E)$ be a (k, s)-edge-connected graph with $d(s)$ even and let $\mathcal{P} = \{P_1, P_2, ..., P_r\}$ be a partition of V. There exists a complete allowed splitting at vertex s in G if and only if*

(a) $d(s, P_i) \leq d(s)/2$ for $1 \leq i \leq r$,

(b) H contains no C_4- or C_6-obstacle.

Bang-Jensen et al. [3] show that there exists a (k, s)-edge-connected extension $G' = (V + s, E')$ of $G = (V, E)$ with $d(s) = 2\Phi(G, k, \mathcal{P})$ for which Theorem 4.2(a) holds. If G' satisfies Theorem 4.2(b), as well, a complete allowed splitting at s yields an optimal augmenting set (of size $\Phi(G, k, \mathcal{P})$). Since $\gamma \leq \Phi + 1$, it remains to characterise the exceptions, that is, those starting graphs G for which any extension G' with $d(s) = 2\Phi(G, k, \mathcal{P})$ contains an obstacle (and hence $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P}) + 1$ holds).

Let $G = (V, E)$ be a graph. A partition X_1, X_2, Y_1, Y_2 of V is a C_4**-configuration** if it satisfies the following properties in G:

(i) $d(A) < k$ for $A = X_1, X_2, Y_1, Y_2$;

(ii) $d(X_1, X_2) = d(Y_1, Y_2) = 0$;

(iii) There exist subpartitions $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}'_1, \mathcal{F}'_2$ of X_1, X_2, Y_1, Y_2 respectively, such that for A ranging over X_1, X_2, Y_1, Y_2 and $\mathcal{F}$ the corresponding subpartition of A, $k - d(A) = \sum_{U \in \mathcal{F}} (k - d(U))$. Furthermore for some $i \leq r$, P_i contains every set of either $\mathcal{F}_1 \cup \mathcal{F}_2$ or $\mathcal{F}'_1 \cup \mathcal{F}'_2$.

(iv) $(k - d(X_1)) + (k - d(X_2)) = (k - d(Y_1)) + (k - d(Y_2)) = \Phi(G, k, \mathcal{P})$.

As with C_4-obstacles, k must be odd in a C_4-configuration. A C_6**-configuration** is more specialized, since it only exists in graphs with $r \geq 3$ and $\Phi = 3$, see [3].

Theorem 4.3. *[3] Let $k \geq 2$ and let $G = (V, E)$ be a graph with a partition $\mathcal{P} = \{P_1, \dots, P_r\}$, $r \geq 2$ of V. Then $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P})$ unless G contains a C_4- or C_6-configuration, in which case $\gamma(G, k, \mathcal{P}) = \Phi(G, k, \mathcal{P}) + 1$.*

If each member of $\mathcal{P}$ is a single vertex then we are back at Theorem 2.2. The following special case solves the rigidity problem mentioned above. Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, let $\mathcal{P} = \{A, B\}$, and let

$$\beta'_1 = \sum_{v \in A} \max\{0, k - d(v)\},$$

$$\beta'_2 = \sum_{v \in B} \max\{0, k - d(v)\},$$

$$\Theta(G, k, \mathcal{P}) = \max\{\lceil \alpha(G, k, \mathcal{P})/2 \rceil, \beta'_1, \beta'_2\}.$$

Theorem 4.4. *[3] Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$ and let $\mathcal{P} = \{A, B\}$. Then $\gamma(G, k, \mathcal{P}) = \Theta(G, k, \mathcal{P})$ unless k is odd and G contains a C_4-configuration, in which case $\gamma(G, k, \mathcal{P}) = \Theta(G, k, \mathcal{P}) + 1$.*

As we shall see, the phenomenon of minimax theorems with exceptions will occur in other constrained augmentation (or edge splitting) problems as well.

The variant of the above problem, where the edges of the augmenting set F must lie within members of a given partition, is NP-hard [3]. The status of this variant is open if the number r of partition members is fixed, even if $r = 2$. The corresponding edge splitting problem, for $r = 2$, has been solved in [6].

A different application of Theorem 4.2 is concerned with permutation graphs. A **permutation graph** G^π of a graph G is obtained by taking two disjoint copies of G and adding a matching joining each vertex v in the first copy to $\pi(v)$ in the second copy, where π is a permutation of $V(G)$. Thus G has several permutation graphs. The edge-connectivity of any permutation graph of G is at most $\delta(G) + 1$, where $\delta(G)$ is the minimum degree of G. When does G have a k-edge-connected permutation graph for $k = \delta(G) + 1$?

Creating a permutation graph of G corresponds to performing a complete bipartition constrained splitting in G', where G' is obtained from $2G$ by adding a new vertex s and precisely one edge from s to each vertex of $2G$. (For some graph H we use $2H$ to denote the graph consisting of two disjoint copies of H.) If G is simple, it can be seen that G' is (k, s)-edge-connected. Thus Theorem 4.2 leads to the following characterisation, due to Goddard, Raines, and Slater [41].

Theorem 4.5. *[41] Let G be a simple graph without isolated vertices and let $k = \delta(G) + 1$. Then there is a k-edge-connected permutation graph of G unless $G = 2K_k$, and k is odd.*

The fact that constrained edge splitting results, such as Theorem 4.2, may lead to solutions of constrained augmentation problems, such as Theorem 4.3, motivated Jordán [61] to introduce a general method for solving constrained edge splitting problems. Let $G = (V + s, E)$ be a (k, s)-edge-connected graph ($k \geq 2$) and suppose that the goal is to decide if there is a complete k-admissible splitting at s, where the split edges satisfy some additional property Π. This property defines the **constraint graph** $D(G, s, \Pi)$ on $N(s)$, the neighbour-set of s. Two distinct vertices $u, v \in N(s)$ are adjacent in $D(G, s, \Pi)$ if and only if uv satisfies Π.

The **non-admissibility graph** $B(G, s)$ of G (with respect to s) is also defined on $N(s)$. Two distinct vertices $u, v \in N(s)$ are adjacent in $B(G, s)$ if and only if the pair su, sv is not k-admissible in G. Clearly, there is a k-admissible splitting satisfying Π if and only if $D(G, k, \Pi)$ is not a subgraph of $B(G, s)$. If, for some property Π, the constraint graph can never be a subgraph of the non-admissibility graph, it follows that there is always a complete k-admissible splitting satisfying Π. In some case this can be shown to hold by using the following general structural properties of non-admissibility graphs.

A complete (subgraph of a) graph will be called a **clique**. The union of two cliques with precisely one vertex in common is a **double clique**.

Theorem 4.6. *[61] Let $G = (V + s, E)$ be (k, s)-edge-connected and suppose that $d(s)$ is even and $|N(s)| \geq 2$. Then (a) if k is even then $B(G, s)$ is disconnected and each component of $B(G, s)$ is a clique, (b) if k is odd then each component of $B(G, s)$ is either a clique, or a double clique, or two disjoint cliques connected by a path, or a cycle of length at least four.*

Those graphs $G = (V + s, E)$ for which $B(G, s)$ is 2-edge-connected are of special interest. An example for this case is a wheel of size at least five, where s is the center, and $k = 3$. Here $B(G, s)$ is a cycle (corresponding to the rim of the wheel). This example can easily be extended to a family of "wheel-like" graphs, called **round**, see [61] for the definition. Round graphs exist for every odd $k \geq 3$ and their non-admissibility graphs are cycles.

Theorem 4.7. *[61] Let $G = (V + s, E)$ be (k, s)-edge-connected for some $k \geq 2$ and suppose that $d(s)$ is even and $|N(s)| \geq 2$. Then $B(G, s)$ is 2-edge-connected if and only if $B(G, s)$ is a cycle of length $d(s)$, k is odd, $d(s) \geq 4$, and G is round.*

We illustrate the non-admissibility graph method by the planarity-preserving k-edge-connectivity augmentation problem. In this problem we are given a planar graph $G = (V, E)$ and the goal is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and planar. The complexity of this problem is still open, even for $k = 2$. (Note that the corresponding problem for 2-vertex-connectivity is NP-hard, see [64].)

Consider the corresponding edge splitting problem. Let $H = (V + s, E)$ be a (k, s)-edge-connected planar graph $H = (V + s, E)$ with $d(s)$ even. Call a k-admissible splitting su, sv **non-crossing** if H_{uv} is also planar. It is easy to decide whether H has a non-crossing splitting at s, but how can we decide if there is a complete non-crossing splitting at s? Nagamochi and Eades [73] investigated this problem and pointed out that a complete non-crossing splitting does not always exist if $k \geq 5$ is odd.

Suppose now that k is even. Let us fix a planar embedding of H. This embedding determines a cyclic ordering $\mathcal{C}$ of the neighbours of s. Clearly, a k-admissible splitting su, sv for which u and v are consecutive in $\mathcal{C}$ is non-crossing (and a planar embedding of H_{uv} can be obtained without reembedding $H - \{su, sv\}$). We may assume $|N(s)| \geq 3$. Then the constraint graph $D(H, s, \Pi_{\mathcal{C}})$, defined by the property $\Pi_{\mathcal{C}}$ of being consecutive in $\mathcal{C}$, is a cycle on $N(s)$ corresponding to $\mathcal{C}$. Thus to see that a non-crossing splitting exists at s it is enough to prove that $(*)$ $D(H, s, \Pi_{\mathcal{C}})$ cannot be a subgraph of $B(G, s)$. This is straightforward from Theorem 4.6(a), since the constraint graph is connected.

Even though $N(s)$ as well as $B(H, s)$ and $D(H, s, \Pi_{\mathcal{C}})$ may change after a non-crossing splitting is performed, $(*)$ remains valid. Thus there exists a complete non-crossing splitting at s. Hence we have proved the first half of the following theorem of Nagamochi and Eades [73].

Theorem 4.8. *[73] Let $H = (V + s, E)$ be a planar graph with $d(s)$ even and suppose that H is (k, s)-edge-connected, where k is either even or $k = 3$. Then there exists a complete non-crossing splitting at s.*

The case when $k = 3$ can also be proved by this method, using the fact that $D(H, s, \Pi_{\mathcal{C}})$ is 2-edge-connected. By Theorem 4.6(b) and Theorem 4.7 this implies that if there is no consecutive spliting at s then H is round. Then a complete non-crossing splitting can be identified directly, using structural properties of round graphs, see [61]. It remains an open problem to characterise those (k, s)-edge-connected graphs $H = (V + s, E)$, for odd $k \geq 5$, which do not have complete non-crossing splittings at s.

If $G = (V, E)$ is outer-planar, any extension of G is planar. Thus, by using Theorem 4.8, the second phase of Frank's algorithm can be executed in such a way that the resulting graph is planar, provided k is even or $k = 3$. This proves the following result of Nagamochi and Eades [73]. For $k = 2$ it was proved earlier by Kant [64].

Theorem 4.9. *[73] Let $G = (V, E)$ be outer-planar and let k be even or $k = 3$. Then G can be made k-edge-connected and planar by adding $\Phi(G, k)$ edges.*

Non-admissibility graphs turn out to be useful in the following augmentation problem as well: given two graphs $G_1 = (V, E)$ and $G_2 = (V, I)$, and two integers $k, l \geq 2$, find a smallest set F of new edges for which $G_1' = (V, E \cup F)$ is k-edge-connected and $G_2' = (V, I \cup F)$ is l-edge-connected. We call this the simultaneous edge-connectivity augmentation problem.

As in the previous constrained problems, first we consider the corresponding edge splitting problem. Let $H_1 = (V+s, E)$ be (k, s)-edge-connected and let $H_2 = (V+s, I)$

be (l, s)-edge-connected, and suppose that the sets of edges incident to s are common in H_1 and H_2. A splitting su, sv is **legal** if it is k-admissible in H_1 and l-admissible in H_2. To show that a legal splitting exists we need to verify that the complement of the non-admissibility graph of H_2 is not a subgraph of the non-admissibility graph of H_1. This does not always hold but with Theorems 4.6 and 4.7 one can show the following.

Theorem 4.10. *[61]. Let $H_1 = (V + s, E)$ and $H_2 = (V + s, I)$ be (k, s)-edge-connected and (l, s)-edge-connected, respectively, for $k, l \geq 2$. Suppose that $d(s) = d_{H_1}(s) = d_{H_2}(s)$ is even. Then (a) if $d(s) \geq 6$ then there exists a legal splitting at s, (b) if k and l are both even then there exists a complete legal splitting at s.*

A subpartition-type lower bound for the size $\gamma(G_1, G_2, k, l)$ of a smallest simultaneous augmentating set is easy to find. Let

$$\alpha(G_1, G_2, k, l) = \max\{\sum_{1}^{r}(k - d_{G_1}(X_i)) + \sum_{r+1}^{t}(l - d_{G_2}(X_i)\}, \tag{8}$$

where the maximum is taken over all subpartitions $\{X_1, ..., X_t\}$ of V, $0 \leq r \leq t$. Clearly, $\gamma(G_1, G_2, k, l) \geq \Phi(G_1, G_2, k, l)$, where $\Phi(G_1, G_2, k, l) = \lceil \alpha(G_1, G_2, k, l)/2 \rceil$. With the help of generalised polymatroids, Jordán [61] proved that there exists a pair of extensions G_1' and G_2' for which the sets of edges incident to s are the same and $d(s) = \alpha(G_1, G_2, k, l)$. A more direct proof of this claim can be found in [74]. Using this fact and Theorem 4.10, the splitting off method yields the following near optimal solution for the simultaneous augmentation problem.

Theorem 4.11. *[61] Let $G_1 = (V, E)$ and $G_2 = (V, I)$ be two graphs and let $k, l \geq 2$ be two integers. Then $\Phi(G_1, G_2, k, l) \leq \gamma(G_1, G_2, k, l) \leq \Phi(G_1, G_2, k, l) + 1$. If k and l are both even then $\gamma(G_1, G_2, k, l) = \Phi(G_1, G_2, k, l)$ holds.*

The status of the simultaneous augmentation problem is still open in the case when k or l is odd. One approach for attacking this problem would be to complete Theorem 4.10 and characterise those pairs H_1, H_2 for which there is no complete legal splitting at s for a given pair k, l. This could lead to a minimax theorem with exceptions, as in the partition constrained problem. Note, however, that the simultaneous edge splitting problem, even if l is even, contains the partition-constrained edge splitting problem as a special case [61].

A recent result of Ishii and Nagamochi [50] solves a similar simultaneous augmentation problem where the goal is to make G_1 and G_2 k-edge-connected and 2-vertex-connected, respectively.

A natural constrained augmentation problem, which has been investigated by several authors, is the simplicity preserving k-edge-connectivity augmentation problem: given a simple graph $G = (V, E)$, find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and simple. Frank and Chou [36] solved this problem (even with local requirements) in the special case where the starting graph G has no edges. Some papers on arbitrary starting graphs G but with small target value

k followed. Let us denote the size of a smallest simplicity-preserving augmenting set F by $\gamma(G,k,S)$. Clearly, we have $\gamma(G,k,S) \geq \gamma(G,k)$. Following the algorithmic proof of Theorem 2.1, it can be checked that if G is simple, so is the augmented graph G'. This proves $\gamma(G,k,S) = \gamma(G,k)$ for $k = 2$. Watanabe and Yamakado [82] proved that $\gamma(G,k,S) = \gamma(G,k)$ holds for $k = 3$ as well. Taoka, Takafuji and Watanabe [80] pointed out that $\gamma(G,k,S) \geq \gamma(G,k)+1$ may hold if $k \geq 4$, even if the starting graph G is $(k-1)$-edge-connected. On the other hand, they showed that for $(k-1)$-edge-connected starting graphs one has $\gamma(G,k,S) \leq \gamma(G,k)+1$ for $k = 4, 5$. Moreover, in these special cases, we have $\gamma(G,k,S) = \gamma(G,k)$, provided $\gamma(G,k) \geq 4$. For general k, it was observed [76] that $\gamma(G,k,S) = \gamma(G,k)$ if G is $(k-1)$-edge-connected and the minimum degree of G is at least k.

Jordán [60] settled the complexity of the problem by proving that the simplicity-preserving k-edge-connectivity augmentation problem is NP-hard, even if the starting graph is $(k-1)$-edge-connected. For k fixed, however, the problem is solvable in polynomial time. This result of Bang-Jensen and Jordán [5] is based on the fact that if $\gamma(G,k)$ is large compared to k then $\gamma(G,k,S) = \gamma(G,k)$ holds. The proof of this equality employed the splitting off method.

Let $H = (V+s,E)$ be (k,s)-edge-connected and suppose that $H-s$ is simple. A k-admissible splitting su, sv is **feasible** if it does not create parallel edges, that is, if $H_{uv} - s$ is also simple.

Theorem 4.12. *[5] Let $G = (V,E)$ be a simple graph and let $G' = (V+s,E')$ be a minimally (k,s)-edge-connected extension of G with $d_{G'}(s)$ even. If $d_{G'}(s) \geq 3k^4$ then there exists a complete feasible splitting at s.*

Theorem 4.12 was obtained by extending the second phase of Frank's algorithm. Together with (2) this edge splitting result proves the above mentioned equality. The proofs are algorithmic and show that if $\gamma(G,k)$ is large then an optimal simplicity-preserving set can be found in polynomial time, even if k is part of the input.

Theorem 4.13. *[5] Let $G = (V,E)$ be a simple graph. If $\gamma(G,k) \geq 3k^4/2$ then $\gamma(G,k,S) = \gamma(G,k)$.*

The solution for graphs with small $\gamma(G,k)$ value uses two additional results [5]. First, for any graph G we have $\gamma(G,k,S) \leq \gamma(G,k)+2k^2$. Second, if $\gamma(G,k)$ is small, then there exists a small subset $T \subseteq V$, which can be found efficiently, and for which there exists an optimal augmenting set containing edges on T only. These facts make an exhaustive search on T possible and lead to an $O(n^4)$ algorithm for k fixed. Most of these results have been extended to the local version of the simplicity-preserving edge-connectivity augmentation problem [5].

The "opposite" of the simplicity-preserving augmentation problem is the "reinforcement" problem. In the latter problem we are given a connected graph $G = (V,E)$ and an integer $k \geq 2$, and the goal is to find a smallest set F of new edges for which $G' = (V, E \cup F)$ is k-edge-connected and every edge of F is parallel to some edge in E. This problem is also NP-hard [60].

In the rest of this section we consider constrained augmentation problems of directed graphs. The partition constrained k-edge-connectivity augmentation problem has been investigated for digraphs as well. Baglivo and Graver [1] showed that a 2-dimensional square grid framework with diagonal **cables** is rigid if and only if a natural bipartite digraph representation is strongly connected. Thus making a framework rigid by adding the smallest number of new cables corresponds to making a bipartite digraph strongly connected by adding a smallest set of new arcs that preserve bipartiteness. Gabow and Jordán [38] proved a minimax theorem for the more general problem, where an arbitrary digraph $D = (V, A)$ is to be made strongly connected by adding the smallest number $\gamma(D, \mathcal{P})$ of new arcs, such that each new arc joins vertices in distinct members of a given partition $\mathcal{P} = \{P_1, ..., P_r\}$ of V. As in the unconstrained strong connectivity augmentation problem (c.f. Theorem 3.1), we can reduce the problem to the case when D is acyclic by contracting each of its strong components C to a single vertex v_C and defining the corresponding partition $\mathcal{P}'$ accordingly: if C intersects at least two members of $\mathcal{P}$ then v_C becomes a singleton member of $\mathcal{P}'$, otherwise it keeps its "colour". Thus we may assume that $D = (V, A)$ is a dag (with $|V| \geq 2$ and $r \geq 2$). Recall the definition of X^+ and X^- from Section 3. Since new arcs must connect distinct members of $\mathcal{P}$, it is easy to see that the following value $\Phi(D, \mathcal{P})$ is a lower bound for $\gamma(D, \mathcal{P})$:

$$\Phi(D, \mathcal{P}) = \max\{|V^+|, |V^-|, |P_i^+| + |P_i^-| : 1 \leq i \leq r\} \tag{9}$$

There are digraphs for which the optimum is greater than this subpartition-type lower bound. Consider for example the bipartite digraph consisting of two independent arcs whose tails belong to different members of the (bi)partition. This graph has $\Phi = 2$ and $\gamma = 3$. The "exceptions", however, can be classified as follows. Let $R^-(X)$ denote the set of all sinks that can be reached from some source of X. Similarly, $R^+(X)$ denotes the set of all sources that can reach some sink of X.

We say a dag $D = (V, A)$ is a **blocker** if it is a 1-, 2-, 3-, or 4-blocker as defined by the conditions below. In these conditions B and W denote two distinct partition members and $\bar{B} = V - B$. A 1-**blocker** has

(i) $\Phi(D, \mathcal{P}) = |B^+| + |B^-|$;

(ii) $B^+, \bar{B}^+ \neq \emptyset$;

(iii) $R^-(B) \subseteq \bar{B}$.

The remaining blockers satisfy (i) $\Phi(D, \mathcal{P}) = |V^+| = |V^-|$. A 2-**blocker** has

(ii) $B^+, \bar{B}^+ \neq \emptyset$;

(iii) $R^-(\bar{B}) \subseteq B$.

A 3-**blocker** either has

(ii) $B^+, W^+ \neq \emptyset$;

(iii) $R^-(B) \subseteq W, R^-(W) \subseteq B, |V^+ - B - W| = 1$, or it has (ii) and (iii) with + and − interchanged. A 4-**blocker** has

(ii) $B^+, B^- \neq \emptyset$;

(iii) $R^-(B) \cup R^+(B) \subseteq W, |B^+| + |B^-| = \Phi(D, \mathcal{P}) - 1$.

It is not hard to show that any blocker D has $\gamma(D, \mathcal{P}) > \Phi(D, \mathcal{P})$. Gabow and Jordán [38] show that any dag satisfies $\gamma(D, \mathcal{P}) \leq \Phi(D, \mathcal{P}) + 1$ and every dag with $\gamma(D, \mathcal{P}) > \Phi(D, \mathcal{P})$ is a blocker.

Theorem 4.14. *[38] Let $D = (V, A)$ be a dag. Then $\gamma(D, \mathcal{P}) = \Phi(D, \mathcal{P})$ unless D is a blocker, in which case $\gamma(D, \mathcal{P}) = \Phi(D, \mathcal{P}) + 1$.*

The partition constrained k-edge-connectivity augmentation problem for directed graphs seems to be more complex than for undirected graphs. Even in the special case when $k = 1$ and $r = 2$, the directed problem has four exceptional families, while there is only one exceptional family for any $k \geq 2$ and $r = 2$ in the undirected case, see Theorem 4.3. Moreover, for any exceptional graph G we have $\gamma(G, \mathcal{P}) \leq 2k + 1$. In the directed case exceptional digraphs exist with arbitrary $\gamma(D, \mathcal{P})$ value.

Theorem 4.14 has been extended in two ways. Gabow and Jordán [39] solved the problem of optimally increasing the edge-connectivity of a $(k-1)$-edge-connected digraph by one, with bipartition constraints. Actually, they solved an even more general "abstract" problem, see the remark after Theorem 7.8. The (bi)partition-constrained k-edge-connectivity augmentation problem for arbitrary starting digraphs is still open. For this general problem Gabow and Jordán [40] developed a polynomial algorithm which finds a solution with at most k extra edges over the optimum and showed that the difference between a natural extension of the subpartition type lower bound (9) and the optimum is at most k. A key step in the proof is an extension of Theorem 3.2.

5 Vertex-connectivity of graphs

The vertex-connected versions of the augmentation problems are substantially more difficult than their edge-connected counterparts. This will be transparent by comparing the corresponding minimax theorems, the proof methods, as well as the hardness results and open questions. The following observation indicates that the k-vertex-connectivity augmentation problem, at least in the undirected case, has a different character. Suppose the goal is to make $G = (V, E)$ k-connected, optimally, where $k = |V| - 2$. Although this case may seem very special, it is in fact equivalent to the maximum matching problem. To see this observe that F is a good augmenting set if and only if the complement of $G + F$ consists of independent edges. Thus finding a smallest augmenting set for G corresponds to finding a maximum matching in its complement. The case $k = |V| - 3$, which is equivalent to finding a four-cycle free 2-matching of maximum size, is still open.

As in the edge-connected case, if k is small, the k-connectivity augmentation problem can be solved by considering the tree-like structure of the "k-connected components" of the graph. If $k = 2$, the familiar concept of 2-connected components, or "blocks", and the "block-cutvertex tree" helps. For simplicity, suppose $G = (V, E)$ is connected. Let $t(G)$ denote the number of end-blocks of G and let $b(G)$ denote the maximum number of components of $G - v$ over all vertices $v \in V$. Note that the end-blocks are pairwise disjoint. Since G' is 2-connected if and only if $t(G') = b(G') - 1 = 0$, and adding a new edge can decrease $t(G)$ by at most two and $b(G)$ by at most one, it follows that at least $\Psi(G) = \max\{\lceil t(G)/2 \rceil, b(G) - 1\}$ new edges are needed to make G 2-connected. Eswaran and Tarjan [20] and independently Plesnik [77] proved that this number can be achieved. (See also Hsu and Ramachandran [48].) Finding two end-blocks X, Y for which adding a new edge xy with $x \in X$ and $y \in Y$ decreases $\Psi(G)$ by one can be done, roughly speaking, by choosing the end-blocks corresponding to the end-vertices of a longest path in the block-cutvertex tree of G.

Theorem 5.1. *[20, 77] Let $G = (V, E)$ be a connected graph. Then G can be made 2-connected by adding* $\max\{\lceil t(G)/2 \rceil, b(G) - 1\}$ *edges.*

The lower bounds used in Theorem 5.1 can be extended to $k \geq 3$ and arbitrary starting graphs G as follows. A non-empty subset $X \subset V$ is a **fragment** if $V - X - N(X) \neq \emptyset$, where $N_G(X)$, or simply $N(X)$ if G is clear from the context, denotes the set of neighbours of vertex set X in G. It is easy to see that every set of new edges F which makes G k-connected must contain at least $k - |N(X)|$ edges from X to $V - X - N(X)$ for every fragment X. By summing up these "deficiencies" over pairwise disjoint fragments, we obtain a subpartition-type lower bound, similar to the one used in the corresponding edge-connectivity augmentation problem. Let

$$t(G, k) = \max\{\sum_{i=1}^{r}(k - |N(X_i)|) : X_1, ..., X_r \text{ are pairwise disjoint fragments in } V\} \tag{10}$$

Let $\gamma(G, k)$ denote the size of a smallest augmenting set of G with respect to k. Since an edge can decrease the 'deficiency' $k - |N(X_i)|$ of at most two sets X_i, we have $\gamma(G, k) \geq \lceil t(G, k)/2 \rceil$. For $K \subset V$ let $b(K, G)$ denote the number of components in $G - K$. Let

$$b(G, k) = \max\{b(K, G) : K \subset V, |K| = k - 1\} \tag{11}$$

Since $G' - K$ has to be connected in the augmented graph G', we have that $\gamma(G, k) \geq b(G, k) - 1$. Thus the following value $\Psi(G, k)$ is also a lower bound for $\gamma(G, k)$. Let

$$\Psi(G, k) = \max\{\lceil t(G, k)/2 \rceil, b(G, k) - 1\} \tag{12}$$

Theorem 5.1 implies that $\gamma(G, k) = \Psi(G, k)$ for $k = 2$. Watanabe and Nakamura [85] proved that this minimax equality is valid for $k = 3$, too. Hsu and Ramachandran [47] gave an alternative proof and a linear time algorithm, based on Tutte's decomposition theory of 2-connected graphs into 3-connected components. This method was further developed by Hsu [45], who solved the problem of making a 3-connected graph 4-connected by adding a smallest set of edges. His proof relies on the decomposition of 3-conneced graphs into 4-connected components.

This approach, however, which relies on the decomposition of a graph into its "k-connected components", is rather hopeless for $k \geq 5$. While k-edge-connected components have a nice structure, k-connected components are difficult to handle. Furthermore, the successive augmentation property does not hold for vertex-connectivity augmentation [13].

Although $\Psi(G,k)$ suffices to characterize $\gamma(G,k)$ for $k \leq 3$, there are examples showing that $\gamma(G,k)$ can be strictly larger than $\Psi(G,k)$. Consider for example the complete bipartite graph $K_{k-1,k-1}$ with target k. For $k \geq 4$ this graph has $\Psi = k-1$ and $\gamma = 2k-4$, showing that the gap can be as large as $k-3$. Jordán [58] showed that if the starting graph G is $(k-1)$-connected then this is the extremal case, that is, $\gamma(G,k) \leq \Psi(G,k) + k - 3$. He developed a polynomial time algorithm to find an augmenting set with at most $k-3$ surplus edges. This gap was later reduced to $(k-1)/2$ with the help of two additional lower bounds [59]. Cheriyan and Thurimella [15] gave a more efficient algorithm with the same approximation gap and showed how to compute $b(G,k)$ in polynomial time if G is $(k-1)$-connected. A near optimal solution can be found efficiently even if G is not $(k-1)$-connected. This was proved by Ishii and Nagamochi [49] and independently Jackson and Jordán [55]. The approximation gap in the latter paper is slightly smaller.

Theorem 5.2. *[55] Let $G = (V,E)$ be l-connected. Then $\gamma(G,k) \leq \Psi(G,k) + (k-l)k/2 + 4$.*

Jackson and Jordán [55] adapted the edge splitting method for vertex-connectivity. This method was subsequently employed to solve the problem optimally, in polynomial time, for k fixed.

Given an extension $G' = (V+s, E')$ of a graph $G = (V,E)$, define $\bar{d}(X) = |N_G(X)| + d'(s,X)$ for every $X \subseteq V$, where d' denotes the degree function in G'. We say that G' is (k,s)**-connected** if

$$\bar{d}(X) \geq k \text{ for every fragment } X \subset V, \tag{13}$$

and that it is **minimally** (k,s)**-connected** if the set of edges incident to s is inclusionwise minimal with respect to (13). The following result from [55] gives lower and upper bounds for $\gamma(G,k)$ in terms of $d'(s)$ in any minimally (k,s)-connected extension of G.

Theorem 5.3. *[55] Let $G' = (V+s, E')$ be a minimally (k,s)-connected extension of a graph G. Then $\lceil d'(s)/2 \rceil \leq \gamma(G,k) \leq d'(s) - 1$.*

Let $G' = (V+s, E')$ be a minimally (k,s)-connected extension of G. Splitting off su and sv in G' is k**-admissible** if G'_{uv} also satisfies (13). Notice that if G' has no edges incident to s then (13) is equivalent to the k-connectivity of G. Hence, as in the case of edge-connectivity, it would be desirable to know, when $d(s)$ is even, that there is a sequence of k-admissible splittings which isolates s. In this case, using the fact that $\gamma(G,k) \geq d'(s)/2$ by Theorem 5.3, the resulting graph on V would be an optimal augmentation of G with respect to k. This approach works for the k-edge-connectivity augmentation problem but does not always work in the vertex connectivity case. The reason is that complete k-admissible splittings do not necessarily exist. On the other hand, the splitting off results in [55, 56] are 'close enough' to yield an optimal algorithm for k-connectivity augmentation, which is polynomial for k fixed.

The obstacle for the existence of a k-admissible splitting can be described, provided $d'(s)$ is large enough compared to k. The proof of the following theorem is based on a new tripartite submodular inequality for $|N(X)|$, see [55].

Theorem 5.4. *[55] Let $G' = (V+s, E')$ be a minimally (k, s)-connected extension of $G = (V, E)$ and suppose that $d'(s) \geq k^2$. Then there is no k-admissible splitting at s in G' if and only if there is a set $K \subset V$ in G such that $|K| = k-1$ and $G - K$ has $d'(s)$ components $C_1, C_2, ..., C_{d'(s)}$ (and we have $d'(s, C_i) = 1$ for $1 \leq i \leq d'(s)$).*

Theorem 5.4 does not always hold if $d'(s)$ is small compared to k. To avoid this essential obstacle, Jackson and Jordán [56] introduced the following family of graphs. Let $G = (V, E)$ be a graph and k be an integer. Let X_1, X_2 be disjoint subsets of V. We say (X_1, X_2) is a k-**deficient pair** if $d(X_1, X_2) = 0$ and $|V-(X_1 \cup X_2)| \leq k-1$. We say two deficient pairs (X_1, X_2) and (Y_1, Y_2) are **independent** if for some $i \in \{1, 2\}$ we have either $X_i \subseteq V-(Y_1 \cup Y_2)$ or $Y_i \subseteq V-(X_1 \cup X_2)$, since in this case no edge can simultaneously connect X_1 to X_2 and Y_1 to Y_2. We say G is k-**independence free** if G does not have two independent k-deficient pairs. Note that if G is $(k-1)$-connected and (X_1, X_2) is a k-deficient pair then $V-(X_1 \cup X_2) = N(X_1) = N(X_2)$. For example (a) $(k-1)$-connected chordal graphs and graphs with minimum degree $2k-2$ are k-independence free; (b) all graphs are 1-independence free and all connected graphs are 2-independence free; (c) a graph with no edges and at least $k+1$ vertices is not k-independence free for any $k \geq 2$; (d) if G is k-independence free and H is obtained by adding edges to G then H is also k-independence free; (e) a k-independence free graph is l-independence free for all $l \leq k$. In general, a main difficulty in vertex-connectivity problems is that vertex cuts can cross each other in many different ways. In the case of an independence free graph G these difficulties can be overcome.

Theorem 5.5. *[56] If G is k-independence free then $\gamma(G, k) = \Psi(G, k)$.*

If G is not k-independence free but $t(G, k)$ is large, then the augmentation problem can be reduced to the independence free case by adding new edges. This crucial property is formulated by the next theorem.

Theorem 5.6. *[56] Let $G = (V, E)$ be $(k-1)$-connected and suppose that $t(G, k) \geq 8k^3 + 10k^2 - 43k + 22$. Then there exists a set of edges F for G such that $t(G+F, k) = t(G, k) - 2|F|$, $G+F$ is k-independence free, and $t(G+F, k) \geq 2k-1$.*

These results lead to the following theorem.

Theorem 5.7. *[56] Let G be $(k-1)$-connected. If $\gamma(G, k) \geq 8k^3 + 10k^2 - 43k + 21$ then*

$$\gamma(G, k) = \Psi(G, k).$$

The min-max equality in Theorem 5.7 is not valid if we remove the hypothesis that G is $(k-1)$-connected. To see this consider the graph G obtained from the complete bipartite graph $K_{m,k-2}$ by adding a new vertex x and joining x to j vertices in the m set of the $K_{m,k-2}$, where $j < k < m$. Then $b(G,k) = m$, $t(G,k) = 2m + k - 2j$ and $\gamma(G,k) = m - 1 + k - j$. However, by modifying the definition of $b(G,k)$ slightly, an analogous minimax theorem can be obtained for augmenting graphs of arbitrary connectivity. For a set $K \subset V$ with $|K| = k - 1$ let $\delta(K,k) = \max\{0, \max\{k - d(x) : x \in K\}\}$ and $b^*(K,G) = b(K,G) + \delta(K,k)$. Let $b^*(G,k) = \max\{b^*(K,G) : K \subset V, |K| = k - 1\}$. It is easy to see that $\gamma(G,k) \geq b^*(G,k) - 1$. Let

$$\Psi^*(G,k) = \max\{\lceil t(G,k)/2 \rceil, b^*(G,k) - 1\}$$

Theorem 5.8. *[56] Let $G = (V,E)$ be l-connected. If $\gamma(G,k) \geq 3(k-l+2)^3(k+1)^3$ then $\gamma(G,k) = \Psi^*(G,k)$.*

The lower bounds, in terms of k, are certainly not best possible in Theorems 5.6, 5.7, and 5.8. These bounds, however, depend only on k. This is the essential fact in the solution for k fixed. Note that by Theorem 5.3 one can efficiently decide whether $\gamma(G,k)$ (or $t(G,k)$) is large enough compared to k.

The proofs of these results are algorithmic and give rise to an algorithm which solves the k-vertex-connectivity augmentation problem optimally in polynomial time for k fixed. If $\gamma(G,k)$ is large, then the algorithm has polynomial running time even if k is not fixed. This phenomenon is similar to what we observed when we investigated the algorithm for the simplicity-preserving k-edge-connectivity augmentation problem.

One of the most exciting open questions of this area is the complexity of the k-vertex-connectivity augmentation problem, when k is part of the input. It remains open even if the starting graph is $(k-1)$-connected. The following conjecture[1] of Frank and Jordán [31] may give a good characterisation for this special case.

Let $G = (V,E)$ be $(k-1)$-connected. A **clump** of G is an ordered pair $C = (S, \mathcal{P})$ where $S \subset V$, $|S| = k-1$, and $\mathcal{P}$ is a partition of $V - S$ into non-empty subsets, called **pieces**, with the property that no edge of G joins two distinct pieces in C. (Note that a piece is not necessarily connected.) It can be seen that if $C = (S, \mathcal{P})$ is a clump of G then, in order to make G k-connected, we must add a set of at least $|\mathcal{P}| - 1$ edges between the pieces of C, where $|\mathcal{P}|$ is the number of pieces of $\mathcal{P}$. We shall say that a clump C **covers** a pair of vertices u, v of G if u and v belong to distinct pieces of C. A **bush** B of G is a set of clumps such that each pair of vertices of G is covered by at most two clumps in B. Thus, if B is a bush in G, then in order to make G k-connected, we must add a set of at least

$$def(B) = \lceil \frac{1}{2} \sum_{(S,P) \in B} (|\mathcal{P}| - 1) \rceil \tag{14}$$

edges between the pieces of the clumps in B. Two bushes B_1 and B_2 of G are **disjoint** if no pair of vertices of G is covered by clumps in both B_1 and B_2. Thus, if B_1 and

[1] The conjecture, and the polynomial-time solvability of the k-vertex-connectivity augmentation problem with $(k-1)$-connected input graphs were verified by L. Végh [81] in 2011.

B_2 are disjoint bushes, then the sets of edges which need to be added between the pieces of the clumps in B_1 and B_2 are disjoint. We can now state the conjecture.

Conjecture *Let G be a $(k-1)$-connected graph. Then the minimum number of edges which must be added to G to make it k-connected is equal to the maximum value of $\sum_{B \in D} def(B)$ taken over all sets of pairwise disjoint bushes D for G.*

The local connectivity augmentation problem, or LC for short, is NP-hard. This can be seen as follows. In the (decision version of) problem LC we are given a graph $G = (V, E)$, a positive integer $r(u, v)$ for each pair $u, v \in V$ and a positive integer L. The question is whether there is a set F of edges on V such that $|F| \leq L$ and $\kappa(u, v; G') \geq r(u, v)$ for each pair $u, v \in V$, where $G' = (V, E \cup F)$. In the proof we use the fact that problem LDC is NP-complete (a result from [26]). In problem LDC we are given a directed graph $D = (V, A)$, a set $T \subseteq V$ and a positive integer K and the question is whether there is set I of arcs with $|I| \leq K$ such that in the digraph $D' = (V, A \cup I)$ there exist a directed path from u to v for all $u, v \in T$.

Theorem 5.9. *Problem LC is NP-complete.*

Proof: It is clear that LC is in NP. Consider an instance $D = (V, A)$, $T \subset V$, and $K \in Z_+$ of LDC. Let $n = |V|$ and $V = \{v_1, v_2, ..., v_n\}$ and let us build an undirected graph $H = (Z \cup W, J)$ as follows. The vertex set is the union of two parts of size n each, denoted by $Z = \{z_1, ..., z_n\}$ and $W = \{w_1, ..., w_n\}$. Every edge connecting two vertices of Z or two vertices of W is in the edge set J of H as well as every edge of the form $z_i w_i$, $1 \leq i \leq n$. In addition, an edge $z_i w_j$ $(i \neq j)$ belongs to J if and only if $v_i v_j \in A$. It is easy to check that H is n-connected, and for every separating set S of size n in H we have $S \cap \{z_i, w_i\} = 1$, $1 \leq i \leq n$. Moreover, there exists a set S of size n separating z_i and w_j with $v_i, v_j \in T$, $i \neq j$, if and only if there is no directed path in D from v_i to v_j. Now let us define an instance of LC with the above graph $H = (Z \cup W, J)$, integer $L = K$ and with local requirements

$$r(z_i, w_j) = n + 1, \quad \text{if} \quad v_i, v_j \in T, \ i \neq j, \ \text{and} \ r(z_i, w_j) = 0 \ \text{ otherwise.} \tag{15}$$

The size of this instance is polynomial in the size of the given instance of the LDC problem. By the above properties of H it is easy to see that $H' = (Z \cup W, J \cup F)$ satisfies the local requirements of (15) if and only if for the corresponding set F' of edges on V there exists a directed path between each pair $u, v \in T$ in $D' = (V, A \cup F')$. This proves the theorem. •

The proof shows that it is already NP-hard to find a smallest augmenting set which increases the local vertex-connectivity up to k within a given subset of vertices of a $(k-1)$-connected graph. However, it may be possible to find efficient algorithms if k (or the maximum $r(u, v)$ value) is small (fixed). For instance, Watanabe et al. [83] gave a linear-time algorithm for optimally increasing the connectivity to two within a specified subset. Another interesting open problem is LC in the special case when the starting graph has no edges.

We close this section by a different generalisation of the connectivity augmentation problem. In some cases it is desirable to make the starting graph $G = (V, E)$ k-edge-connected as well as l-vertex-connected at the same time, by adding a new set of edges F. In this "multiple target" version l is typically small while k is arbitrary. We may always assume $k \leq l$. Hsu and Kao [46] solved a local version of this problem for $k = l = 2$. Ishii, Nagamochi, and Ibaraki [53, 51, 52] proved a number of results for $l \leq 3$ and k arbitrary, and presented near optimal polynomial-time algorithms when l as well as k is arbitrary.

A typical result is as follows. Let $k \geq 2$ and $l = 2$. By combining $\alpha(G, k)$ and $t(G, 2)$ define

$$\alpha'(G, k, 2) = \max\{\sum_{i=1}^{p}(k - d(X_i)) + \sum_{i=p+1}^{t}(2 - |N(X_i)|)\},$$

where the maximum is taken over all subpartitions $\{X_1, ..., X_p, X_{p+1}, ..., X_t\}$ of V for which X_i is a fragment for $p + 1 \leq i \leq t$. Clearly, $\lceil \alpha'(G, k, 2)/2 \rceil$ is a lower bound for this multiple target problem. By applying the splitting off method (and a new operation called "edge switching"), a common extension of Theorems 2.2 and 5.1 can be obtained.

Theorem 5.10. *[52] $G = (V, E)$ can be made k-edge-connected and 2-connected by adding γ new edges if and only if* $\max\{\lceil \alpha'(G, k, l)/2 \rceil, b(G, 2) - 1\} \leq \gamma$.

6 Vertex-connectivity of digraphs

From several aspects, the directed k-edge-connectivity augmentation problem is less tractable than its undirected version. This may suggest that the directed k-connectivity augmentation problem is harder than the (still unsolved) undirected problem. Indeed, after the basic result of Eswaran and Tarjan [20] on the case $k = 1$ (Theorem 3.1) only a few more results appeared for nearly twenty years. Masuzawa, Hagihara, Tokura [71] solved the special case when the starting digraph $D = (V, A)$ is an arborescence (that is, a directed tree with a root vertex r such that there is a directed path from r to every $v \in V$). Let $\gamma(D, k)$ denote the size of a smallest augmenting set with respect to the target vertex-connectivity k.

Theorem 6.1. *[71] Let $B = (V, A)$ be an arborescence. Then $\gamma(B, k) = \sum_{v \in V} \max\{0, k - \delta(v)\}$.*

For arbitrary starting digraphs $D = (V, A)$ there is a natural subpartition-type lower bound for $\gamma(D, k)$, similar to $t(G, k)$. We say $X \subset V$ is an **in-fragment** if $V - X - N^-(X) \neq \emptyset$. If $V - X - N^+(X) \neq \emptyset$ then X is called an **out-fragment**. Let

$$t_{in}(D, k) = \max\{\sum_{i=1}^{r}(k - |N^-(X)|) : X_1, ..., X_r \text{ are pairwise disjoint in-fragments in } V\},$$

$$t_{out}(D,k) = \max\{\sum_{i=1}^{r}(k-|N^+(X)|) : X_1,...,X_r \text{ are pairwise disjoint out-fragments in } V\}$$

and let

$$\Psi(D,k) = \max\{t_{in}(D,k), t_{out}(D,k)\}.$$

It can bee seen that $\gamma(D,k) \geq \Psi(D,k)$ holds. Theorem 3.1 shows $\gamma(D,1) = \Psi(D,1)$ and Theorem 6.1 implies that $\gamma(B,k) = \Psi(B,k)$ for every arborescence B and every $k \geq 1$. However, Jordán [57] pointed out that $\gamma(D,k) \geq \Psi(D,k) + k - 1$ may hold for every $k \geq 2$, even if D is $(k-1)$-connected. On the other hand, for $(k-1)$-connected starting digraphs the gap cannot be larger than $k-1$ [57].

A stronger lower bound can be obtained by considering deficient pairs of subsets of V rather than deficient in- or out-fragments. We say that an ordered pair (X,Y), $\emptyset \neq X, Y \subset V$, $X \cap Y = \emptyset$ is a **one-way pair** in a digraph $D = (V,A)$ if there is no arc in D with tail in X and head in Y. We call X and Y the **tail** and the **head** of the pair, respectively. The **deficiency** of a one-way pair-with respect to k-connectivity is $def_k(X,Y) = \max\{0, k - |V - (X \cup Y)|\}$. Two pairs are **independent** if their tails or their heads are disjoint. For a family $\mathcal{F}$ of pairwise independent one-way pairs we define $def_k(\mathcal{F}) = \sum_{(X,Y)\in\mathcal{F}} def_k(X,Y)$. By Menger's theorem every augmenting set F with respect to k must contain at least $def_k(X,Y)$ arcs from X to Y for every one-way pair (X,Y). Moreover, these arcs are distinct for independent one-way pairs. This proves $\gamma(D,k) \geq def_k(\mathcal{F})$ for families $\mathcal{F}$ of pairwise independent one-way pairs. Frank and Jordán [30] solved the k-connectivity augmentation problem for digraphs by showing that this lower bound can be attained.

Theorem 6.2. *[30] A digraph $D = (V,A)$ can be made k-connected by adding at most γ new arcs if and only if*

$$def_k(\mathcal{F}) \leq \gamma \qquad (16)$$

holds for all families $\mathcal{F}$ of pairwise independent one-way pairs.

This result is a corollary of a more general theorem on "bi-supermodular functions", see Theorem 7.10. It would be interesting to find a direct proof which could lead to a combinatorial polynomial algorithm.

The minimax formula of Theorem 6.2 was later refined by Frank and Jordán [32]. Among others, they showed that if $def_k(\mathcal{F}) \geq 2k^2 - 1$, then the tails or the heads of the pairs in $\mathcal{F}$ are pairwise disjoint. This implies that if $\gamma(D,k) \geq 2k^2 - 1$ then the simpler lower bound $\Psi(D,k)$ suffices. With the help of this refined version, one can deduce Theorem 6.1 from Theorem 6.2 as well. A related conjecture of Frank [27] claims that $\gamma(D,k) = \Psi(D,k)$ for every acyclic starting digraph D.

If D is strongly connected and $k = 2$ then a direct proof and a simplified minimax theorem was given in [25] by applying the splitting off method. In a strongly connected digraph $D = (V,A)$ there are two types of deficient sets with respect to $k = 2$: in-fragments X with $|N^-(X)| = 1$ (called **in-tight**) and out-fragments X with $|N^+(X)| = 1$ (called **out-tight**).

Theorem 6.3. *[25] Let $D = (V,A)$ be strongly connected. If $t_{in}(D,2) = t_{out}(D,2) = 2$, then D can be made 2-connected by adding two arcs unless $(*)$ there exist three* *in-tight (or three out-tight) sets B_1, B_2, B_3, such that*

$$B_1 \cap B_2 \neq \emptyset, |B_3| = 1, \quad \textit{and } V - (B_1 \cup B_2) = B_3.$$

If $(*)$ holds, then D can be made 2-connected by adding 3 arcs.

Theorem 6.4. *[25] Let $D = (V, A)$ be strongly connected. Then $\gamma(D, 2) = \Psi(D, 2)$ holds, unless $t_{in}(D) = t_{out}(D) = 2$ and $(*)$ holds, in which case $\gamma(D, 2) = 3$.*

7 Hypergraph augmentation and coverings of set functions

Connectivity augmentation is about adding new edges to a graph or digraph so that it becomes sufficiently highly connected. Applying Menger's theorem, every augmentation problem has an equivalent formulation where the goal is to add new edges so that each "cut" receives at least as many new edges as its deficiency with respect to the given target. Cut typically means a subset of vertices, but it may also be a pair or collection of subsets of the vertex set. The deficiency function, say $k - \rho(X)$ or $R(X) - d(X)$, is determined by the input graph. This leads to a more abstract point of view: given a function p on subsets of a ground-set V, find a smallest "cover" of p, that is, a smallest set of edges F such that at least $p(X)$ edges enter every subset $X \subset V$. Deficiency functions related to connectivity problems have certain "supermodular" properties. This motivates the study of minimum covers of functions of this type.

This is not just for the sake of proving more general minimax theorems. In some cases (e.g. in the directed k-connectivity augmentation problem) the only known way to the solution is via an abstract result. In other cases (e.g. in the k-edge-connectivity augmentation problem) generalizations lead to simpler proofs, algorithms, and extensions (to local requirements or vertex-induced cost functions) by showing the background of the problem.

An intermediate step towards an abstract formulation is to consider hypergraphs. A **hypergraph** is a pair $\mathcal{G} = (V, E)$ where V is a finite set (the set of **vertices** of $\mathcal{G}$) and E is a finite collection of **hyperedges**. Each hyperedge e is a set $Z \subseteq V$ with $|Z| \geq 2$. The **size** of e is $|Z|$. Thus (loopless) graphs correspond to hypergraphs with edges of size two only. A hyperedge of size two is called a **graph edge**. Let $d_{\mathcal{G}}(X)$ denote the number of hyperedges intersecting both X and $V - X$. A hypergraph is k**-edge-connected** if $d_{\mathcal{G}}(X) \geq k$ for all $\emptyset \neq X \subset V$. A **component** of $\mathcal{G}$ is a maximal connected subhypergraph of $\mathcal{G}$. Let $w(\mathcal{G})$ denote the number of components of $\mathcal{G}$.

One possible way to generalize the k-edge-connectivity augmentation problem is to search for a smallest set of graph edges whose addition makes a given hypergraph k-edge-connected. Cheng [12] was the first to prove a result in this direction. He determined the minimum number of new graph edges needed to make a $(k-1)$-edge-connected hypergraph $\mathcal{G}$ k-edge-connected, by invoking deep structural results of Cunningham [17] on decompositions of submodular functions. His result was soon extended to arbitrary hypergraphs by Bang-Jensen and Jackson [4]. They employed and extended the splitting off method to hypergraphs.

A hypergraph $\mathcal{H} = (V + s, E)$ is (k, s)**-edge-connected** if s is incident to graph edges only and $d_{\mathcal{H}}(X) \geq k$ for all $\emptyset \neq X \subset V$. Splitting off two edges su, sv is k-admissible if $\mathcal{H}_{uv}$ is also (k, s)-edge-connected. The extension of Theorem 2.3 to hypergraphs is as follows.

Theorem 7.1. *[4] Let $\mathcal{H} = (V + s, E)$ be a (k, s)-edge-connected hypergraph with $d_{\mathcal{H}}(s) = 2m$. Then exactly one of the following statements holds.*
(i) there is a complete k-admissible splitting at s, or
(ii) there exists $A \subset E$ with $|A| = k - 1$ and $w(\mathcal{H} - s - A) \geq m + 2$.

Let $\mathcal{G} = (V, E)$ be a hypergraph. Let

$$\alpha(\mathcal{G}, k) = \max\{\sum_{i=1}^{t}(k - d(X_i)) : X_1, ..., X_t \text{ is a subpartition of } V\} \tag{17}$$

$$c(\mathcal{G}, k) = \max\{w(\mathcal{G} - A) : A \subset E, |A| = k - 1\}. \tag{18}$$

As in the case of graphs, $\lceil \alpha(\mathcal{G}, k)/2 \rceil$ is a lower bound for the size of a smallest augmentation. Another lower bound is $c(\mathcal{G}, k) - 1$. Note that if $\mathcal{G}$ is a graph, $k \geq 2$, and $\mathcal{G} - A$ has $c(\mathcal{G}, k)$ components $C_1, C_2, ..., C_c$ for some $A \subset E$ with $|A| = k - 1$ then $\sum_1^c (k - d(C_i)) = kc - \sum_1^c d(C_i) \geq kc - 2(k-1) = 2(c-1) + (k-2)(c-2) \geq 2(c-1)$, and hence $\alpha(\mathcal{G}, k)/2 \geq c(\mathcal{G}, k) - 1$.

Theorem 7.2. *[4] The minimum number of new graph edges which makes a hypergraph $\mathcal{G}$ k-edge-connected equals*

$$\max\{\lceil \alpha(\mathcal{G}, k)/2 \rceil, c(\mathcal{G}, k) - 1\}. \tag{19}$$

The proof in [4] is simpler than Cheng's proof and gives Cheng's result in the special case when $\mathcal{G}$ is $(k-1)$-edge-connected. In a recent work Cosh [16] solved the bipartition constrained version of Theorem 7.2 by using the non-admissibility graph method. Note that the successive augmentation property does not hold for hypergraphs [13]. Cosh, Jackson, and Király [67] proved that the local version of the hypergraph edge-connectivity augmentation problem (with graph edges) is NP-hard, even if the starting hypergraph is connected and the maximum requirement is two.

Benczúr and Frank [7] solved an abstract generalisation of Theorem 7.2. Let V be a finite set and let $p : 2^V \to Z$ be a function with $p(\emptyset) = p(V) = 0$. p is **symmetric** if $p(X) = p(V - X)$ holds for every $X \subseteq V$. We say that p is **crossing supermodular** if it satisfies the following inequality for each pair of crossing sets $X, Y \subset V$:

$$p(X) + p(Y) \leq p(X \cup Y) + p(X \cap Y) \tag{20}$$

We say that a set of edges F on V **covers** p if $d_F(X) \geq p(X)$ for all $X \subset V$.

Now suppose p is a symmetric crossing supermodular function on V. What is the minimum size $\gamma(p)$ of a cover of p? A subpartition-type lower bound is the following. Let

$$\alpha(p) = \max\{\sum_{i=1}^{t} p(X_i) : \{X_1, ..., X_t\} \text{ is a subpartition of } V\}$$

Since an edge can cover at most two sets of a subpartition, we have $\gamma(p) \geq \lceil \alpha(p)/2 \rceil$. This lower bound may be strictly less than $\gamma(p)$. To see this consider a ground set V with 4 elements and let $p \equiv 1$. Here $\alpha(p) = 4$ but, since every cover forms a connected graph on V, we have $\gamma(p) \geq 3$. This example leads to the following notions. We call a partition $\mathcal{Q} = \{Y_1, \ldots, Y_r\}$ of V with $r \geq 4$ p-**full** if

$$p(\cup_{Y \in \mathcal{Q}'} Y) \geq 1 \text{ for every non-empty subfamily } \mathcal{Q}' \subseteq \mathcal{Q}. \tag{21}$$

The maximum size of a p-full partition is called the **dimension** of p and is denoted by $\dim(p)$. If there is no p-full partition, then $\dim(p) = 0$. Since every cover induces a connected graph on the members of a p-full partition, we have $\gamma(p) \geq \dim(p) - 1$. Thus the minimum size of a cover is at least $\Phi(p) = \max\{\lceil \alpha(p)/2 \rceil, \dim(p) - 1\}$.

Theorem 7.3. *[7] Let $p : 2^V \to Z$ be a symmetric crossing supermodular function. Then $\gamma(p) = \Phi(p)$.*

The proof of Theorem 7.3 yields a polynomial time algorithm to find a smallest cover, provided a polynomial time submodular function minimization oracle is available. The deficiency function of a (hyper)graph is symmetric and supermodular: Theorems 2.2 and 7.2 follow by taking $p(X) = k - d_{\mathcal{G}}(X)$ for all $X \subset V$ and $p(\emptyset) = p(V) = 0$, where $\mathcal{G}$ is the starting (hyper)graph, see [7]. The special case of Theorem 7.3 where $p(X) \in \{0, 1\}$ for all $X \subset V$ follows also from a result of Fleiner and Jordán [22].

One may also want to augment a hypergraph by adding hyperedges. The minimum number of new hyperedges which make a given hypergraph $\mathcal{G} = (V, E)$ k-edge-connected is easy to determine: add l copies of the hyperedge containing all vertices of V, where $l = \max\{k - d_{\mathcal{G}}(X) : \emptyset \neq X \subset V\}$. So it is natural to either set an upper bound on the size of the new edges ([22] contains partial results in this direction) or to make the "cost" of a new hyperedge depend on its size. Szigeti [79] solved (an abstract generalisation of) the latter version.

Let V be a ground set and let $p : 2^V \to Z$ with $p(\emptyset) = p(V) = 0$. p is **skew-supermodular** if for each pair $X, Y \subset V$ we have

$$p(X) + p(Y) \leq \max\{p(X \cap Y) + p(X \cup Y), p(X - Y) + p(Y - X)\}.$$

For example, symmetric crossing supermodular functions are skew-supermodular. Also, $q(X)$ (defined as the deficiency of X with respect to local edge-connectivity requirements in a (hyper)graph $\mathcal{G}$, see Section 2) is skew-supermodular [26]. The minimum total size of a cover, consisting of hyperedges, can be characterised by a subpartition-type lower bound.

Theorem 7.4. *[79] Let $p : 2^V \to Z$ be a symmetric skew-supermodular function. Then*

$$\min\{\sum_{e \in F} |e| : F \textit{ is a cover of } p\} = \max\{\sum_{1}^{t} p(X_i) : X_1, ..., X_r \textit{ is a subpartition of } V\}. \tag{22}$$

Theorem 7.4 gives the minimum total size of hyperedges which make a hypergraph $\mathcal{G}$ r-edge-connected for given local edge-connectivity requirements $r(u,v)$, by taking $p(X) = q(X) = R(X) - d_{\mathcal{G}}(X)$. Another corollary of Theorem 7.4 is a solution for the local edge-connectivity augmentation problem for hypergraphs, where graph edges of fractional capacities can also be added. If p is an even valued skew-supermodular function, the minimum size of a cover consisting of graph edges can also be determined.

We have seen that edge splitting results are important ingredients in solutions of connectivity augmentation problems and hence generalisations of some augmentation problems lead to extensions of edge splitting theorems. This works the other way round as well. Consider the following operation. Let $G = (V + s, E)$ be a graph with a designated vertex s. A **degree specification** for s is a sequence $\mathcal{S} = (d_1, ..., d_p)$ of positive integers with $\sum_{j=1}^{p} d_j = d(s)$. An **$\mathcal{S}$-detachment** of s in G is obtained by replacing s by p vertices $s_1, ..., s_p$ and replacing every edge su by an edge $s_i u$ for some $1 \le i \le p$ so that $d(s_i) = d_i$ holds in the new graph for $1 \le i \le p$. If $d_i = 2$ for all $1 \le i \le p$ then an $\mathcal{S}$-detachment corresponds to a complete splitting in a natural way. Given a requirement function $r : V \times V \to Z_+$, an $\mathcal{S}$-detachment is called **r-admissible** if the detached graph G' satisfies $\lambda(x, y; G') \ge r(x, y)$ for every pair $x, y \in V$.

Extending an earlier theorem of Fleiner [21] on the case when $r \equiv k$ for some $k \ge 2$, Jordán and Szigeti [62] gave a necessary and sufficient condition for the existence of an r-admissible $\mathcal{S}$-detachment. We call r **proper** if $r(x, y) \le \lambda(x, y; G)$ for every pair $x, y \in V$.

Theorem 7.5. *[62] Let r be a requirement function for $G = (V + s, E)$ and suppose that G is 2-edge-connected and $r(u, v) \ge 2$ for each pair $u, v \in V$. Let $\mathcal{S} = (d_1, ..., d_p)$ be a degree specification for s with $d_i \ge 2$, $i = 1, ..., p$. Then there exists an r-admissible $\mathcal{S}$-detachment of s if and only if r is proper and*

$$\lambda(u, v; G - s) \ge r(u, v) - \sum_{i=1}^{p} \lfloor d_i/2 \rfloor \tag{23}$$

holds for every pair $u, v \in V$.

Theorem 7.5 implies Theorem 2.6 by letting $r \equiv r_\lambda$ and $d_i \equiv 2$. Theorem 7.5 gives the following extension of Theorem 2.7. By **attaching** a star of degree d to a graph $G = (V, E)$ we mean the addition of new vertex s and d edges from s to vertices in V. Let $G = (V, E)$ be a graph and suppose that we are given local requirements $r(u, v)$ for each pair $u, v \in V$ as well as a set of integers $d_1, ..., d_p$ $(d_j \ge 2)$. Can we make G r-edge-connected by attaching p stars with degrees $d_1, d_2, ..., d_p$? The first phase of Frank's algorithm combined with Theorem 7.5 gives a necessary and sufficient condition [62]. Recall the definition of $q(X)$ from Section 2. For simplicity, we can assume that G has no marginal components with respect to r (otherwise a reduction similar to Theorem 2.5 works).

Theorem 7.6. *[62] Let a graph $G = (V, E)$ and local edge-connectivity requirements $r(u, v)$, $u, v \in V$ be given such that G has no marginal components with respect to*

r. Then G can be made r-edge-connected by attaching p stars with degrees $d_1, ..., d_p$ ($d_j \geq 2$, $1 \leq j \leq p$) if and only if

$$\sum_{i=1}^{t} q(X_i) \leq \sum_{j=1}^{p} d_j \tag{24}$$

holds for every subpartition $\{X_1, ..., X_t\}$ of V and $\lambda(u, v; G) \geq r(u, v) - \sum_{j=1}^{p} \lfloor d_j/2 \rfloor$ for every pair $u, v \in V$.

With the help of Theorem 7.6 it is easy to deduce a minimax formula for the following optimization problem: given G, r, and an integer $w \geq 2$, determine the minimum number γ for which G can be made r-edge-connected by attaching γ stars of degree w. If $w = 2$ then we are back at Theorem 2.7.

In the rest of this section we consider extensions of some of the previous results on augmentations of directed graphs to directed hypergraphs and to directed covers of certain set functions.

A **directed hypergraph** (or **dypergraph**, for short) is a pair $\mathcal{D} = (V, A)$, where V is a finite set (the set of **vertices** of D) and A is a finite collection of **hyperarcs**. Each hyperarc e is a set $Z \subseteq V$, $|Z| \geq 2$, with a specified **head vertex** $v \in Z$. We also use (Z, v) to denote a hyperarc on set Z and with head v. The **size** of e is $|Z|$. Thus a directed graph (without loops) is a dypergraph with hyperarcs of size two only. We say that a hyperarc (Z, v) **enters** a set $X \subset V$ if $v \in X$ and $Z - X \neq \emptyset$. Let $\rho(X)$ denote the number of hyperarcs entering X. A dypergraph $\mathcal{D} = (V, A)$ is k**-edge-connected** if $\rho(X) \geq k$ for every $\emptyset \neq X \subset V$. In a recent paper Berg, Jackson, and Jordán [9] extended Theorem 3.2 to dypergraphs and, among others, proved the following extension of Theorem 3.3.

Theorem 7.7. *[9] Let $\mathcal{D} = (V, A)$ be a dypergraph. Then D can be made k-edge-connected by adding γ new hyperarcs of size at most t if and only if*

$$\gamma \geq \sum_{1}^{r} (k - \rho(X_i)) \tag{25}$$

and

$$(t-1)\gamma \geq \sum_{1}^{r} (k - \delta(X_i)) \tag{26}$$

hold for every subpartition $\{X_1, X_2, ..., X_r\}$ of V.

Directed covers of set functions have also been investigated. Frank [27] proved the directed version of Theorem 7.3. We say that a set of arcs F on ground set V **covers** a function $p : 2^V \to Z$ if $\rho_F(X) \geq p(X)$ for all $X \subset V$.

Theorem 7.8. *[27] Let $p : 2^V \to Z$ be a crossing supermodular function. Then p can be covered by γ arcs if and only if*

$$\gamma \geq \sum_{1}^{t} p(X_i) \tag{27}$$

and

$$\gamma \geq \sum_{1}^{t} p(V - X_i) \tag{28}$$

hold for every subpartition $\{X_1, X_2, ..., X_t\}$ *of* V.

If $p(X) \in \{0,1\}$ for all $X \subset V$, the problem corresponds to covering a "crossing family" of subsets of V by a smallest set of arcs. Gabow and Jordán [39] solved the bipartition-constrained version of this special case, extending Theorem 4.14. In this abstract formulation the minimax theorem has six exceptional families.

Since $p(X) = k - \rho(X)$ is crossing supermodular, Theorem 7.8 implies Theorem 3.3. The following generalisation of the directed k-edge-connectivity augmentation problem can also be solved by Theorem 7.8. Let $D = (V, A)$ be a directed graph with a specified root vertex $r \in V$ and let $k \geq l \geq 0$ be integers. D is called (k,l)**-edge-connected (from** r**)** if $\lambda(r, v; D) \geq k$ and $\lambda(v, r; D) \geq l$ for every vertex $v \in V - r$. Clearly, D is k-edge-connected if and only if D is (k,k)-edge-connected. The extension, due to Frank [24], is as follows. Let $p_{kl}(X) = \max\{k - \rho(X), 0\}$ for sets $\emptyset \neq X \subseteq V - r$ and let $p_{kl}(X) = \max\{l - \rho(X), 0\}$ for sets $X \subset V$ with $r \in X$.

Theorem 7.9. *[24] Let* $D = (V, A)$ *be a digraph and let* $r \in V$. D *can be made* (k,l)*-edge-connected from* r *by adding* γ *new arcs if and only if*

$$\gamma \geq \sum_{1}^{t} p_{kl}(X_i) \tag{29}$$

and

$$\gamma \geq \sum_{1}^{t} p_{kl}(V - X_i) \tag{30}$$

hold for every partition $\{X_1, X_2, ..., X_t\}$ *of* V.

The next abstract result concerning digraphs, due to Frank and Jordán [30], has a number of corollaries. Among others, it leads to the solution of the k-connectivity augmentation problem for directed graphs.

Let V be a ground set and let $p(X, Y)$ be an integer-valued function defined on ordered pairs of disjoint subsets $X, Y \subset V$. We call p **crossing bi-supermodular** if

$$p(X,Y) + p(X',Y') \leq p(X \cap X', Y \cup Y') + p(X \cup X', Y \cap Y')$$

holds whenever $X \cap X', Y \cap Y' \neq \emptyset$. A set of arcs F **covers** p if there are at least $p(X,Y)$ arcs in F with tail in X and head in Y for every pair $X, Y \subset V$, $X \cap Y = \emptyset$. Two pairs $(X,Y), (X',Y')$ are **independent** if $X \cap X'$ or $Y \cap Y'$ is empty.

Theorem 7.10. *[30] Let* p *be an integer-valued crossing bi-supermodular function on* V. *Then* p *can be covered by* γ *arcs if and only if* $\sum_{(X,Y)\in\mathcal{F}} p(X,Y) \leq \gamma$ *holds for every family* $\mathcal{F}$ *of pairwise independent pairs.*

Let $D = (V, A)$ be a digraph. By taking $p(X, Y) = k - |(V - (X \cup Y)|$ for one-way pairs (X, Y) we can deduce Theorem 6.2. Furthermore, Theorem 7.10 implies Theorems 3.2, 3.3 and 3.4 as well as Edmonds' matroid partition theorem, a theorem of Győri on covering a rectilinear polygon with rectangles, and a theorem of Frank on $K_{t,t}$-free t-mathcings in bipartite graphs, see [28, 30] for more details.

On the other hand, the proof of Theorem 7.10 is not algorithmic. For the previously unsolved corollaries, such as the directed k-ST-edge-connectivity and k-vertex-connectivity augmentation problems, it does imply a polynomial-time algorithm, via the ellipsoid method. However, a constructive proof and a combinatorial algorithm would be desirable at least for these two special cases. (Actually, the directed k-ST-edge-connectivity augmentation problem contains the directed k-vertex-connectivity augmentation problem as a special case [32].) Frank and Végh [35] developed such an algorithm for the case when $p(X, Y) \in \{0, 1\}$ for all pairs (X, Y).

The idea of abstract formulations may also lead to graph augmentation problems with somewhat different but still connectivity related objectives. A recent result of Frank and Király [33] solves the problem of optimally augmenting a graph G by adding a set F of edges so that $G + F$ has k edge-disjoint spanning trees.

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