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# Stable multicommodity flows

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**Abstract**

We extend the stable flow model of Fleiner to multicommodity flows. In addition to the preference lists of agents on trading partners for each commodity, every trading pair has a preference list on the commodities that the seller can sell to the buyer. A blocking path (with respect to a certain commodity) may include saturated arcs, provided that a positive amount of less preferred commodity is traded on the arc. We prove that a stable multicommodity flow always exists.

## 1 Introduction

Just as network flows generalize bipartite matchings, the stable marriage problem can be generalized to stable flows. An acyclic network model was presented by Ostrovsky [4], while Fleiner [3] introduced a stable flow model where the network is not necessarily acyclic. The aim of this paper is to extend the model of Fleiner to multicommodity flows, but first we briefly describe his model and results.

An instance of the stable flow problem consists of a network on digraph $D = (V, A)$ with $s, t \in V$ and capacities $c \in \mathbb{R}_+^A$, and additionally linear orders $<_v$ for each node $v \in V$ on the arc set incident to $v$. We assume that $s$ has no incoming arcs and $t$ has no outgoing arcs. The network along with the set of these preference orders is called a *network with preferences*. We note that outgoing arcs are never compared to incoming arcs, so the information that we really need is a linear order on the set of outgoing arcs $\delta^{out}(v)$ and one on the set of incoming arcs $\delta^{in}(v)$.

We refer to directed paths simply as *paths* in this paper. A *rooted cycle* is a directed cycle in which one node is designated as the root — it can be regarded as a path which ends at its starting node. Let $f$ be a flow of network $(D, s, t, c)$. A path or rooted cycle $P = (v_1, a_1, v_2, a_2, \dots, a_{k-1}, v_k)$ is said to *block* $f$ if the following hold:

(i) $v_i \neq s, t$ if $i \in \{2, 3, \dots, k-1\}$,

(ii) each arc $a_i$ is unsaturated in $f$,

(iii) $v_1 = s$ or there is an arc $a' = v_1 u$ for which $f(a') > 0$ and $a' <_{v_1} a_1$,

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(iv) $v_k = t$ or there is an arc $a'' = wv_k$ for which $f(a'') > 0$ and $a'' <_{v_k} a_{k-1}$.

A flow is called *stable* if there is no path or rooted cycle blocking it.

The problem can be motivated by a network trading model: the nodes are traders that can buy and sell amounts of a certain product along the arcs of the digraph, and have preferences with whom they would like to trade (an arc that is bigger in the linear order is more preferred). A blocking path represents a possible chain of transactions with which the starting and ending trader would be happier than with some transaction they make in $f$. In this interpretation, nodes $s$ and $t$ represent the producers and the consumers.

Fleiner [3] proved the following result, by reducing the stable flow problem to the stable allocation problem of Baïou and Balinski [1]. A different proof based on the Gale-Shapley algorithm, as well as an extension to flows over time, was given by Cseh, Matuschke and Skutella [2].

**Theorem 1.1** (Fleiner [3])**.** *Every network with preferences has a stable flow. If the capacities are integral, then there is an integral stable flow.*

## 1.1 Stable multicommodity flows

In this paper we present a way to include multiple commodities in the above network trading model. In the multicommodity setting, every arc has individual capacities for specific commodities as well as a cumulative capacity, and each trader has a preference order for each commodity on the possible buyers and sellers, so the preferred trading partners may be different for different commodities. In addition, for each trading pair (i.e. arc in $D$) there is a preference order on the commodities, which expresses that the pair is more inclined to trade certain goods than others.

A blocking path in our model represents a chain of transactions for a specific commodity, with which the starting and ending trader would be more pleased than before, and each intermediate transaction either corresponds to an arc with free capacity or it can replace the trade of a less preferred commodity.

The formal definitions are as follows. The network consists of a digraph $D = (V, A)$ with capacities $c \in \mathbb{R}^A_+$. There are $\ell$ commodities, and each has its own capacity bound $c^j \in \mathbb{R}^A_+$ ($j \in [\ell]$). Each commodity has a *source* $s^j \in V$ and a *sink* $t^j \in V$. Sources and sinks of different commodities may coincide, but we assume that $c^j$ is 0 on every arc entering $s^j$ and on every arc leaving $t^j$.

For each commodity $j \in [\ell]$ there are linear orders $<^j_v$ for each node $v \in V \setminus \{s^j, t^j\}$ on the arc set incident to $v$. In addition, for each arc $a \in A$ there is a linear order $<_a$ on the set of commodities. The network, together with these preference orders, is called a *multicommodity network with preferences*.

We say that $f = (f^1, \dots, f^\ell)$ is a multicommodity flow of the network if $f^j : A \to \mathbb{R}_+$ satisfies Kirchhoff's law at every node except at $s^j$ and $t^j$, $f^j(a) \le c^j_a$ for every $a \in A$ and $j \in [\ell]$, and $\sum_{j=1}^{\ell} f^j(a) \le c_a$ for every $a \in A$.

A path or rooted cycle $P = (v_1, a_1, v_2, a_2, \dots, a_{k-1}, v_k)$ is said to be *blocking with respect to commodity* $j$ if the following hold:

(i) $v_i \neq s^j, t^j$ if $i \in \{2, 3, \ldots, k-1\}$,

(ii) each arc $a_i$ is unsaturated by $f^j$, that is, $f^j(a_i) < c^j_{a_i}$,

(iii) $v_1 = s^j$ or there is an arc $a' = v_1 u$ for which $f^j(a') > 0$ and $a' <^j_{v_1} a_1$,

(iv) $v_k = t^j$ or there is an arc $a'' = w v_k$ for which $f^j(a'') > 0$ and $a'' <^j_{v_k} a_{k-1}$,

(v) if an arc $a_i$ of $P$ is saturated by $f$, i.e. $\sum_{j=1}^{\ell} f^j(a_i) = c_{a_i}$, then there is a commodity $j'$ such that $f^{j'}(a_i) > 0$ and $j' <_{a_i} j$.

The last condition expresses that a chain of transactions can be blocking even if some of them have to replace transactions of other commodities, provided that the latter are less preferred commodities for the trading pairs. A multicommodity flow is called *stable* if there is no path or cycle blocking it.

Before proving the main result on the existence of stable multicommodity flows, we describe a version of Sperner's Lemma that serves as the main tool in our proof.

## 1.2 A polyhedral version of Sperner's Lemma

The following polyhedral version of Sperner's Lemma was introduced in [5], and it was used in [7] to give an alternative proof of Theorem 1.1. An *extreme direction* of a polyhedron is an extreme ray of its characteristic cone. For a colouring of the facets of a pointed polyhedron $P$, a vertex of $P$ is *multicoloured* if it lies on facets of every colour.

**Theorem 1.2** ([5]). *Let $P \subseteq \mathbb{R}^n$ be an $n$-dimensional pointed polyhedron whose characteristic cone is generated by $n$ linearly independent vectors. If the facets of the polyhedron are coloured with $n$ colours such that facets containing the $i$-th extreme direction do not get colour $i$, then there is a multicoloured vertex.*

We note that there is no known polynomial algorithm to find a multicoloured vertex; in fact, it is shown in [6] that this problem is PPAD-complete.

# 2 Existence of a stable multicommodity flow

**Theorem 2.1.** *In every multicommodity network with preferences there exists a stable multicommodity flow.*

*Proof.* Let us consider a multicommodity network with preferences, as defined in subsection 1.1. Let $\mathcal{P}$ denote the set of all paths and rooted cycles, and for an arc $a \in A$ let $\mathcal{P}_a$ be the set of paths and rooted cycles that contain $a$. Furthermore, for a node $v \in V$, let $\mathcal{P}^{\text{out}}_v$ and $\mathcal{P}^{\text{in}}_v$ denote the set of paths and rooted cycles that start and end at $v$, respectively.

Let us introduce the variables $x^j_P$ ($j \in [\ell], P \in \mathcal{P}$) and $y^j_a$ ($j \in [\ell], a \in A$). Given a set of paths $\mathcal{P}' \subseteq \mathcal{P}$, we use the notation $x^j(\mathcal{P}')$ for $\sum_{P \in \mathcal{P}'} x^j_P$.

These variables do not have a clear meaning in terms of the problem that we want to solve; in order to help understanding their role, we describe the overall structure of the proof. We will define a full-dimensional polyhedron $\Pi$ in the space of these variables that satisfies the conditions of Theorem 1.2, and has the property that we remain in $\Pi$ if we decrease a coordinate $x^j_P$ or if we increase it to 0, and we remain in $\Pi$ if we increase a coordinate $y^j_a$ or if we decrease it to $c^j_a$. In place of Kirchhoff's law, we use the inequalities that at every node in $V \setminus \{s^j, t^j\}$ the incoming $x^j$ is at most the outgoing $y^j$, and the outgoing $x^j$ is at most the incoming $y^j$. The face

$$\{(x, y) \in \Pi : x^j_P = 0 \text{ if } P \text{ is not a single arc, } x^j_a = y^j_a \text{ for every } a \in A \text{ and } j \in [\ell]\}$$

corresponds to the set of multicommodity flows. We will prove the following:

- We can define a suitable colouring of the facets of $\Pi$ such that any multicoloured vertex of $\Pi$ is on the face corresponding to multicommodity flows,
- We show that a multicoloured vertex corresponds to a stable multicommodity flow. This is where the variables $x^j_P$ (where $P$ has length at least 2) play a role: in some sense, they correspond to possibilities of changing a feasible solution along a blocking path.

Let us turn to the details of the proof. We consider the polyhedron $\Pi$ described by the following inequalities:

$$y^j_a \geq 0 \qquad \forall a \in A \ \forall j \in [\ell], \tag{1}$$

$$x^j(\mathcal{P}') - y^j_a \leq 0 \qquad \forall \mathcal{P}' \subseteq \mathcal{P}_a \ \forall a \in A \ \forall j \in [\ell], \tag{2}$$

$$x^j(\mathcal{P}') \leq c^j_a \qquad \forall \mathcal{P}' \subseteq \mathcal{P}_a \ \forall a \in A \ \forall j \in [\ell], \tag{3}$$

$$x^j(\mathcal{P}') - y^j(\delta') \leq c^j(\delta^{in}(v) \setminus \delta') \qquad \forall \emptyset \neq \mathcal{P}' \subseteq \mathcal{P}^{\text{out}}_v,\ \delta' \subseteq \delta^{in}(v),\ v \in V \setminus \{s^j, t^j\},\ j \in [\ell], \tag{4}$$

$$x^j(\mathcal{P}') - y^j(\delta') \leq c^j(\delta^{out}(v) \setminus \delta') \qquad \forall \emptyset \neq \mathcal{P}' \subseteq \mathcal{P}^{\text{in}}_v,\ \delta' \subseteq \delta^{out}(v),\ v \in V \setminus \{s^j, t^j\},\ j \in [\ell], \tag{5}$$

$$\sum_{j=1}^{\ell} x^j(\mathcal{P}'_j) \leq c_a \qquad \forall \mathcal{P}'_j \subseteq \mathcal{P}_a \ (j \in [\ell]) \ \forall a \in A. \tag{6}$$

First let us determine the set of extreme directions of $\Pi$. Clearly $-x^j_P$ for $P \in \mathcal{P}, j \in [\ell]$ and $y^j_a$ for $a \in A, j \in [\ell]$ give infinite directions. Since $x^j_P$ is bounded from above and $y^j_a$ from below, there is no infinite direction which is not in the cone of the above. So the number of extreme directions equals the dimension.

Now let us assign colours (that is, variables, since each variable corresponds to an extreme direction of $\Pi$) to each inequality:

- to an inequality of type (1) or type (2) we assign $y^j_a$,
- to an inequality of type (3) we assign $x^j_P$ for a longest possible path $P \in \mathcal{P}'$,

- to an inequality of type (4) we assign $x_P^j$ for a path $P \in \mathcal{P}'$ in which the outgoing arc from $v$ is smallest possible in the order $<_v^j$ from $\mathcal{P}'$, and among these, we choose $P$ to be one of the longest paths,
- to an inequality of type (5) we assign $x_P^j$ for a path $P \in \mathcal{P}'$ in which the incoming arc to $v$ is smallest possible in the order $<_v^j$ from $\mathcal{P}'$, and among these, we choose $P$ to be one of the longest paths,
- to an inequality of type (6) we assign $x_P^j$ where $j$ is the commodity which is smallest in the order $<_a$ among those with nonempty $\mathcal{P}'_j$, and from $\mathcal{P}'_j$ we choose $P$ to be one of the longest paths.

Since the assigned colour of each inequality is a coordinate with nonzero coefficient, the colouring fulfils the criteria of Theorem 1.2. Thus there exists a multicoloured vertex $(\overline{x}, \overline{y})$ of $\Pi$.

**Claim 2.2.** *$\overline{x} \geq \mathbf{0}$, and $\overline{y}^j \leq c^j$ for every $j \in [\ell]$.*

*Proof.* Suppose that $\overline{x}_P^j$ is negative for some $P \in \mathcal{P}$. Then by increasing $\overline{x}_P^j$ to zero we get a vector that is still in $\Pi$, because every inequality where $x_P^j$ has positive coefficient is also present with the coefficient changed to zero (except for the inequalities where $\mathcal{P}' = \{P\}$, but changing the coefficient of $x_p^j$ in those to zero is satisfied too since $y_a^j$ and $c_a^j$ are nonnegative). On the other hand we know that $-x_P^j$ is an infinite direction, so $(\overline{x}, \overline{y})$ could not have been a vertex. Thus $\overline{x}_P^j$ is nonnegative for every $P \in \mathcal{P}$ and $j \in [\ell]$. Similarly we get that $y_a^j \leq c_a^j$ for every $a \in A$ and $j \in [\ell]$. ♢

**Claim 2.3.** *$\overline{x}^j(\mathcal{P}_a) = \overline{y}_a^j$ for every arc $a \in A$ and $j \in [\ell]$.*

*Proof.* Since $(\overline{x}, \overline{y})$ is multicoloured, there is a tight inequality which has colour $y_a^j$. If this inequality is of type (1), then using Claim 2.2 we have $0 \leq \overline{x}^j(\mathcal{P}_a) \leq y_a^j = 0$, thus equality holds. If the tight inequality is of type (2) for some $\mathcal{P}' \subseteq \mathcal{P}_a$, then by Claim 2.2, $\overline{x}^j(\mathcal{P}_a) \geq \overline{x}^j(\mathcal{P}') = y_a^j \geq \overline{x}^j(\mathcal{P}_a)$, so equality holds again. ♢

**Claim 2.4.** *For every $j \in [\ell]$, $\overline{x}_P^j = 0$ for every path or rooted cycle $P$ that has at least 2 arcs.*

*Proof.* Suppose that $\overline{x}_P^j > 0$ for a path or rooted cycle $P = (v_1, a_1, \dots, v_k)$, where $k \geq 3$. Let $Q$ be the one-arc path $(v_1, a_1, v_2)$ and let $R$ be the path $(v_2, a_2, \dots v_k)$.

Consider the inequality that $(\overline{x}, \overline{y})$ satisfies with equality and has colour $x_Q^j$. It can not be of type (3), since because of $P$, we would not have chosen $x_Q^j$ as colour. It also can not be of type (4) or type (6) for the same reason. Thus it is of type (5). This means that there exists some $\mathcal{P}' \subseteq \mathcal{P}_{v_2}^{\text{in}}$ and $\delta' \subseteq \delta^{out}(v_2)$ for which $\overline{x}^j(\mathcal{P}') - \overline{y}^j(\delta') = c^j(\delta^{out}(v_2) \setminus \delta')$. Using Claim 2.2 this holds also for the whole sets $\mathcal{P}_{v_2}^{\text{in}}$ and $\delta^{out}(v_2)$, that is, $\overline{x}^j(\mathcal{P}_{v_2}^{\text{in}}) = \overline{y}^j(\delta^{out}(v_2))$.

Using $\overline{x}_P^j > 0$, and considering the sum of inequalities of type (2) for the arcs in $\delta^{in}(v_2)$, we get

$$\overline{y}^j(\delta^{out}(v_2)) = \overline{x}^j(\mathcal{P}_{v_2}^{\text{in}}) < \overline{x}^j(\bigcup_{a \in \delta^{in}(v_2)} \mathcal{P}_a) \leq \overline{y}^j(\delta^{in}(v_2)).$$

By the same argument for the subpath $R$, we get

$$\overline{y}^j(\delta^{in}(v_2)) = \overline{x}^j(\mathcal{P}^{\text{out}}_{v_2}) < \overline{y}^j(\delta^{out}(v_2)),$$

which is a contradiction. ◊

We obtained that $\overline{x}$ is positive only on the arcs (that is, on paths of length 1), and by Claim 2.3, $\overline{x}^j_a = \overline{y}^j_a$ for every arc $a$ and every $j \in [\ell]$. By inequalities (4) and (5), we have

$$\overline{x}^j(\delta^{out}(v)) \le \overline{y}^j(\delta^{in}(v)) = \overline{x}^j(\delta^{in}(v)) \le \overline{y}^j(\delta^{out}(v)) = \overline{x}^j(\delta^{out}(v))$$

for every $v \in V \setminus \{s^j, t^j\}$, so $\overline{y}^j$ is a flow. By inequality (3), it satisfies the capacity constraints for commodity $j$, and by inequality (6), $\overline{y}$ satisfies the cumulative capacity constraints, so it is a feasible multicommodity flow. We are done by proving our last claim.

**Claim 2.5.** *No path or rooted cycle blocks $\overline{y}$ with respect to any commodity.*

*Proof.* Let $P = (v_1, a_1, \dots, v_k)$ be an arbitrary path or rooted cycle and $j$ an arbitrary commodity. Since $(\overline{x}, \overline{y})$ is a multicoloured vertex, there is a tight inequality of colour $x^j_P$. If it is of the form (3), then the arc $a$ is saturated for commodity $j$, so $P$ does not block $\overline{y}$ with respect to commodity $j$.

If it is of the form (4) for node $v_1$, $\mathcal{P}' \subseteq \mathcal{P}^{\text{out}}_{v_1}$ and $\delta' \subseteq \delta^{in}(v_1)$, then $a' \notin \mathcal{P}'$ whenever $a' \in \delta^{out}(v_1)$ and $a' <^j_{v_1} a_1$. Thus $\overline{x}^j_{a'} = 0$, because if $\overline{x}^j_{a'}$ would be positive, then adding $x^j_{a'}$ to this tight inequality would not hold for $(\overline{x}, \overline{y})$, although it is also an inequality of the system. This implies that $P$ does not block the flow $\overline{y}$ with respect to commodity $j$.

The case when the tight inequality of colour $x^j_P$ is of type (5) is analogous. Finally, if the inequality of colour $x^j_P$ is of type (6), then the arc $a$ is saturated, and furthermore $a \notin \mathcal{P}'_{j'}$ whenever $j' <_a j$ since $x^j_P$ is selected as colour. This implies that $\overline{x}^{j'}_a = 0$ whenever $j' <_a j$: if $\overline{x}^{j'}_a > 0$ would hold, then by adding $a$ to $\mathcal{P}'_{j'}$ we would obtain an inequality that is violated by $(\overline{x}, \overline{y})$. Again, this means that $P$ is not blocking for commodity $j$. ◊

We obtained that there is no blocking path or rooted cycle, so $\overline{y}$ is a stable multicommodity flow. □

We remark that even if the capacities are integer, there may be no integer stable multicommodity flow, and furthermore the denominator cannot be bounded. As an example, consider the network consisting of nodes $v_1, \dots, v_n$, and arcs $v_i v_{i+1}$ ($i \in [n]$), where we use the notation $v_{n+1} = v_1$. There are $n$ commodities; the sources and sinks are defined by $s^j = v_{j+1}$, $t^j = v_j$, and the capacity $c^j$ is 1 on all arcs except for $v_j v_{j+1}$ where it is 0. This means that commodity $j$ has a unique path from $s^j$ to $t^j$, namely $(v_{j+1}, v_{j+2}, \dots, v_1, \dots, v_j)$, so the preference orders at the nodes do not play any role. The cumulative capacity $c$ is 1 everywhere. The preference order at arc $a = v_i v_{i+1}$ is defined by $i - 1 >_a \cdots >_a 1 >_a n >_a \cdots >_a i + 1$ (commodity

$i$ does not appear in the ordering because it has capacity 0 on this arc). We claim that the only stable multicommodity flow is obtained by setting $f^j(a) = \frac{1}{n-1}$ on every arc $a$ except for $v_j v_{j+1}$. Indeed, the all-zero flow is not stable, and if $f$ is a nonzero feasible multicommodity flow different from the above, then there must be an index $j$ such that commodity $j$ has positive flow and the arc $v_j v_{j+1}$ is unsaturated. It can be checked that the path $(v_{j+2}, v_{j+3}, \ldots, v_1, \ldots, v_{j+1})$ is blocking with respect to commodity $j + 1$.


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