Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2012-10. Published by the Egerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres. ISSN 1587–4451.

On universally rigid frameworks on the line

Tibor Jordán and Viet-Hang Nguyen

July 14, 2012

On universally rigid frameworks on the line

Tibor Jordán* and Viet-Hang Nguyen**

Abstract

We give a complete characterization of universally rigid one-dimensional bar-and-joint frameworks in general position with a complete bipartite underlying graph. We also discuss several open questions concerning generically universally rigid graphs and the universal rigidity of general frameworks on the line.

1 Introduction

A d-dimensional (bar-and-joint) *framework* is a pair (G,p), where $G=(V,E)$ is a graph and p is a *configuration* of the vertices, that is, a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are *equivalent* if $||p(u)-p(v)||=||q(u)-q(v)||$ holds for all pairs u,v with $uv\in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $||p(u)-p(v)||=||q(u)-q(v)||$ holds for all pairs u,v with $u,v\in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$.

Let (G,p) be a d-dimensional framework for some $d\geq 1$. We say that (G,p) is *rigid* in $\mathbb{R}^d$ if there is a neighborhood U_p in the space of configurations in $\mathbb{R}^d$ such that if a d-dimensional framework (G,q) is equivalent to (G,p) and $q\in U_p$, then q is congruent to p. The framework (G,p) is called *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework (G,q) which is equivalent to (G,p) is congruent to (G,p). We obtain an even stronger property by extending this condition to equivalent realizations in any dimension: we say that (G,p) is *universally rigid* if it is a unique realization of G, up to congruence, with the given edge lengths, in all dimensions $\mathbb{R}^{d'}$, $d'\geq 1$.

It seems to be a hard problem to decide if a given framework is rigid, globally rigid, or universally rigid. Indeed, Abbott [1] verified that recognizing rigid frameworks in the plane is NP-hard and Saxe [16] proved that it is NP-hard to decide if even a 1-dimensional framework is globally rigid. The complexity of the corresponding decision problem for universal rigidity seems to be open, even for $d=1$.

These problems become more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A

*Department of Operations Research, Eötvös University, Pázmány sétány 1/C, 1117 Budapest, Hungary. e-mail: jordan@cs.elte.hu

**Laboratoire G-SCOP, Grenoble UJF, 46 avenue Félix Viallet, 38000 Grenoble, France. e-mail: Viet-Hang.Nguyen@grenoble-inp.fr

framework (G, p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. It is well-known [5] that rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. Global rigidity is also a generic property in $\mathbb{R}^d$, for all d [9, 13]. This property does not hold for universal rigidity, even if $d = 1$, which follows by considering different generic realizations of a four-cycle on the line.

A graph G is called *generically rigid* (resp. *generically globally rigid*, *generically universally rigid*) in $\mathbb{R}^d$ if every d-dimensional generic framework (G, p) is rigid (resp. globally rigid, universally rigid). We shall also use the shorter versions d-GR, d-GGR, and d-GUR, respectively, for these families of graphs. d-GR and d-GGR graphs are well-characterized for $d \leq 2$. It remains an open problem to extend these results to higher dimensions or to characterize d-GUR graphs for any $d \geq 1$. We refer the reader to [17] for more details on the theory of rigid graphs and frameworks.

Let (G, p) be a framework in $\mathbb{R}^d$ with $G = (V, E)$. An *equilibrium stress* (or *stress*, for short) on (G, p) is an assignment of scalars ω_{ij} to the edges $v_i v_j$ such that for each $v_i \in V$ we have

$$\sum_{j | v_i v_j \in E} \omega_{ij}(p(v_i) - p(v_j)) = \mathbf{0}$$

Given a stress, there is an associated $|V| \times |V|$ symmetric matrix Ω, the *stress matrix* such that for $i \neq j$, the i, j entry of Ω is $-\omega_{ij}$, and the diagonal entries for i, i are $\sum_{j \neq i} \omega_{ij}$. Here we follow the convention that an equilibrium stress can be extended to non-adjacent pairs v_i, v_j by putting $w_{ij} = 0$. Note that all row and column sums are now zero. It is easy to see that the rank of Ω is at most $|V| - d - 1$. We say that Ω is of *full rank* if its rank is equal to $|V| - d - 1$.

Connelly [8] and Gortler and Thurston [12] show that a generic framework (G, p) in $\mathbb{R}^d$ on at least $d + 2$ vertices is universally rigid if and only if it has a positive semi-definite (PSD) stress matrix of full rank. The 'if' direction also holds for frameworks in general position by a theorem of Alfakih and Ye [4].

2 Complete bipartite graphs

In this section we give a complete characterization of the universally rigid one-dimensional realizations of complete bipartite graphs. As a corollary we shall deduce that no bipartite graph (other than $K_{1,1}$) is 1-GUR.

We will need the following result.

Theorem 2.1 (Alfakih [3]). *Let (G, p) be a framework in general position. Then (G, p) has a non-zero PSD stress matrix Ω if and only if (G, p) has no equivalent realization in $\mathbb{R}^{|V(G)|-1}$.*

Let (G, p) be a framework on the line with $G = (V, E)$. A pair of vertices $u, v \in V$ is called *universally linked* in (G, p) if $||q(u) - q(v)|| = ||p(u) - p(v)||$ holds for all frameworks (G, q) which are equivalent to (G, p) (in all dimensions). Let C be a cycle of G passing through $v_1, \ldots, v_k$ with $E(C) = \{v_1 v_2, \ldots, v_{k-1} v_k, v_k v_1\}$. If $p(v_1) <$

$p(v_2) < \cdots < p(v_k)$ then C is called a *stretched cycle* in (G, p). If C is a stretched cycle in (G, p) then it is not difficult to see that every pair of vertices of C is universally linked in (G, p).

Theorem 2.2. *Let G be a complete bipartite graph on at least three vertices with bipartition $X = \{x_1, \ldots, x_m\}$, $Y = \{y_1, \ldots, y_n\}$ and p a realization of G on the line.*

1. *If $p(x_1) < \cdots < p(x_m) < p(y_1) < \cdots < p(y_n)$, then (G, p) is not universally rigid.*

2. *If $p(x_1) < \cdots < p(x_k) < p(y_1) < \cdots < p(y_n) < p(x_{k+1}) < \cdots < p(x_m)$ (or symmetrically $p(y_1) < \cdots < p(y_k) < p(x_1) < \cdots < p(x_m) < p(y_{k+1}) < \cdots < p(y_n)$), then (G, p) is not universally rigid.*

3. *If both of the two conditions above do not hold, then (G, p) is universally rigid.*

Proof.
1. Suppose that $p(x_1) < \cdots < p(x_m) < p(y_1) < \cdots < p(y_n)$ holds and consider a PSD stress matrix Ω of (G, p). We shall prove that Ω is the zero matrix.

Let $r_{ij} = p(y_j) - p(x_i) > 0$ denote the distance between x_i and y_j in (G, p), and w_{ij} the stress on the edge $x_i y_j$, for $1 \leq i \leq m$ and $1 \leq j \leq n$. The equilibrium condition at vertices in X gives

$$\sum_j r_{ij} w_{ij} = 0, \quad \text{for every } i = 1, \ldots, m. \tag{1}$$

Let $s_j = p(y_j) - p(y_1)$ be the distance between y_1 and y_j. Then we have $r_{ij} - r_{i1} = s_j$, for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The entries on the diagonal of Ω are $\sum_{j=1}^n w_{ij}$, for $i = 1, \ldots, m$, and $\sum_{i=1}^m w_{ij}$, for $j = 1, \ldots, n$. Since Ω is PSD, these entries are all non-negative. Therefore,

$$\sum_{j=1}^n r_{i1} w_{ij} \geq 0, \quad \text{for } i = 1, \ldots, m.$$

Using (1), we have

$$0 \leq \sum_j r_{i1} w_{ij} = \sum_j r_{i1} w_{ij} - \sum_j r_{ij} w_{ij} = \sum_j (r_{i1} - r_{ij}) w_{ij} = -\sum_{j>1} s_j w_{ij}.$$

Therefore, $\sum_{j>1} s_j w_{ij} \leq 0$, for $j = 1, \ldots, n$. Then, since $s_j > 0$ for $j = 2, \ldots, n$,

$$0 \leq \sum_{j>1} s_j \sum_i w_{ij} = \sum_i \sum_{j>1} s_j w_{ij} \leq 0$$

which implies that equality holds everywhere. Thus, all entries on the diagonal of Ω are 0's with possibly an exception of the entry corresponding to (x_1, x_1). However, by using the symmetry of the graph, we can deduce that this entry must also be 0.

Therefore, the sum of all eigenvalues of Ω is 0. Hence Ω is the zero matrix. Theorem 2.1 now implies that (G, p) is not universally rigid, in fact, it has an equivalent realization in dimension $m + n - 1$.

2. Suppose that $p(x_1) < \cdots < p(x_k) < p(y_1) < \cdots < p(y_n) < p(x_{k+1}) < \cdots < p(x_m)$ holds and consider a PSD stress matrix Ω of (G, p). We shall prove that Ω is the zero matrix.

Let r_{ij} be the distance between x_i and y_j in this realization and w_{ij} the stress on the edge x_iy_j. Let

$$q_j = \begin{cases} r_{ij} - r_{i1}, & \text{for } i \leq k \\ r_{i1} - r_{ij}, & \text{for } i \geq k+1 \end{cases}$$

and

$$t_i = \begin{cases} r_{k+1,j} + r_{ij}, & \text{for } i \leq k \\ r_{ij} - r_{k+1,j}, & \text{for } i \geq k+1 \end{cases}$$

Then $q_j \geq 0$, $t_i \geq 0$ for every i, j and $q_j > 0$ if $j \neq 1$ and $t_i > 0$ if $i \neq k+1$.

Let $A_i = r_{i1} \sum_j w_{ij}$. Since $\sum_j r_{ij} w_{ij} = 0$ for every i by the equilibrium condition at vertices in X, we have

$$\begin{aligned} A_i &= r_{i1} \sum_j w_{ij} - \sum_j r_{ij} w_{ij} \\ &= \sum_j (r_{i1} - r_{ij}) w_{ij} \\ &= \begin{cases} -\sum_j q_j w_{ij}, & \text{for } i \leq k \\ \sum_j q_j w_{ij}, & \text{for } i \geq k+1 \end{cases} \end{aligned}$$

Let $B_j = r_{k+1,j} \sum_i w_{ij}$. Since $\sum_{i \leq k} r_{ij} w_{ij} - \sum_{i \geq k+1} r_{ij} w_{ij} = 0$ for every $j = 1, \ldots, n$ by the equilibrium condition at vertices in Y, we have

$$\begin{aligned} B_j &= \sum_{i \leq k} (r_{k+1,j} + r_{ij}) w_{ij} + \sum_{i \geq k+1} (r_{k+1,j} - r_{ij}) w_{ij} \\ &= \sum_{i \leq k} t_i w_{ij} - \sum_{i \geq k+1} t_i w_{ij}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_i t_i A_i &= \sum_{i \leq k} t_i A_i + \sum_{i \geq k+1} t_i A_i \\ &= \sum_{i \leq k} t_i (-\sum_j q_j w_{ij}) + \sum_{i \geq k+1} t_i (\sum_j q_j w_{ij}) \\ &= -\sum_j q_j \sum_{i \leq k} t_i w_{ij} + \sum_j q_j \sum_{i \geq k+1} t_i w_{ij} \\ &= -\sum_j q_j B_j. \end{aligned}$$

Since Ω is PSD and $r_{ij} > 0$, $A_i, B_j \geq 0$ hold for every i, j. Hence $0 \leq \sum_i t_i A_i = -\sum_j q_j B_j \leq 0$ holds. Therefore, equality must occur everywhere, which means that $A_i = 0$ for $i \neq k+1$ and $B_j = 0$ for $j \neq 1$. Thus every entry on the diagonal of Ω with possible exceptions of the entries corresponding to (x_{k+1}, x_{k+1}) and (y_1, y_1) must be zero. However, by using the symmetry of the graph, we can deduce that these entries must also be zero, so every entry on the diagonal of the PSD matrix Ω is zero. Therefore, Ω is the zero matrix. Theorem 2.1 now implies that (G, p) is not universally rigid, in fact, it has an equivalent realization in dimension $m + n - 1$.

3. If both conditions in 1 and 2 do not hold, then there exist, say x_1, x_2, y_1, y_2, such that $p(x_1) < p(y_1) < p(x_2) < p(y_2)$. Then x_1, y_1, x_2, y_2 form a stretched cycle in (G, p) and hence x_1, x_2 and y_1, y_2 are universally linked in (G, p). This implies that the pairwise distances among these four vertices are the same in all realizations of G equivalent to (G, p) and hence (G, p) is universally rigid if and only if (G', p) is universally rigid, where $G' = G + x_1x_2 + y_1y_2$. It remains to observe that (G', p) can be obtained from a framework on a complete graph on four vertices by iteratively attaching vertices of degree two (and adding edges). These operations are known to preserve universal rigidity on the line. Therefore (G', p) and hence (G, p) are universally rigid, as required. □

Theorem 2.2 implies that a realization of a complete bipartite graph (on at least three vertices) on the line is universally rigid if and only if it contains a stretched cycle. It also implies the following observation of Connelly[1].

Corollary 2.3 (Connelly [10]). *The only generically universally rigid bipartite graph in $\mathbb{R}^1$ is the single edge $K_{1,1}$.*

We also have some further questions and remarks on the relation of PSD stress matrices and equivalent realizations.

Question 2.4. *Is it true that a framework (G, p) in general position has a PSD stress matrix Ω of rank at least k if and only if (G, p) has no equivalent realization in $\mathbb{R}^{|V(G)|-i}$ for all $1 \leq i \leq k$.*

The "only if" part follows from the following result.

Theorem 2.5 (Alfakih [3]). *Let (G, p) be a framework and Ω a PSD stress matrix of (G, p). Then Ω is a stress matrix for every framework (G, q) equivalent to (G, p).*

[1]Connelly's argument is as follows: map the bipartite graph $G = K_{m,n}$ onto the unit interval on the line. This framework has a realization as a subframework of a unit-length simplex (S, p) in R^d, where $d = m + n - 1$. Then perturb the realization on the line to a generic one and follow it with a modified realization of the simplex in R^d. The inverse function theorem can be used to verify the construction. (In detail, consider the rigidity map f_G on the d-dimensional realizations of G which assigns the edge lengths to the realizations. Since the simplex is minimally infinitesimally rigid in $\mathbb{R}^d$, p is a regular point of f_S. By the inverse function theorem, we can choose an open neibourhood U_p of p and an open neighbourhood W of $f_S(p)$ such that f_S maps U_p diffeomorphically onto W. Thus there is a realization of S for which the edge lengths of the complete bipartite subframework are consistent with the edge lengths of the perturbed one-dimensional framework.)

In fact, suppose that Ω is a PSD stress matrix of (G, p) of rank at least k, and (G, q) a framework equivalent to (G, p). Then Ω is a stress matrix for (G, q). If d is the dimension of (G, q) then $\operatorname{rank}\Omega \leq |V(G)| - d - 1$. Therefore, $d \leq |V(G)| - k - 1$.

Alfakih also conjectured that if a general framework (G, p) is universally rigid in $\mathbb{R}^d$ then it has a PSD stress matrix of rank $|V(G)| - d - 1$. Note that an affirmative answer to Question 2.4 would imply the truth of this conjecture.

We close this section with another question, motivated by Theorem 2.2.

Question 2.6. *Is it true that the universal rigidity of a general position framework (G, p) in $\mathbb{R}^1$ depends only on the ordering of vertices on the line (and not on the coordinates)?*

3 Generic universal rigidity on the line

In this section we consider generic frameworks in $\mathbb{R}^1$ and list a few questions and observations concerning the family of 1-GUR graphs. As we noted earlier, the complexity of recognizing these graphs is still an open question.

First we recall a conjectured inductive construction of 1-GUR graphs.

Conjecture 3.1. *[10] A graph G on at least three vertices is* 1*-GUR if and only if G can be obtained from K_3 by the following operations:*
(i) add an edge,
(ii) choose two graph G_1, G_2 built by these operations, choose two sets U_1, U_2 of each with $|U_1| = |U_2| \geq 2$, delete all edges joining vertices of U_1 in G_1, then glue the two graphs together along the vertices in U_1 and U_2.

The "if" direction of Conjecture 3.1 follows from a recent result of Ratmanski [15]. Note that the graphs built up from a triangle by operations (i) and (ii) must contain a triangle. Thus finding triangle-free 1-GUR graphs would be interesting, c.f. Section 4. Furthermore, Conjecture 3.1, if true, does not seem to provide a good characterization of 1-GUR graphs since it is not clear how to test whether G can be constructed from a triangle by the above operations.

This leads us to minimally 1-GUR graphs, for which the deletion of any edge makes them not 1-GUR. These graphs may be sparse and may have small vertex separations, along which they may be decomposed by the inverse operation of (ii).

Question 3.2. *Let $G = (V, E)$ be a minimally* 1*-GUR graph. Can we prove a (linear) upper bound on $|E|$ as a function of $|V|$?*

We remark here that there is no constant k for which the k-vertex-connectivity of G would imply that G is 1-GUR, and there exist dense not 1-GUR graphs, for example, the complete bipartite graphs (c.f. Corollary 2.3). However, the end of the proof of Theorem 2.2 shows that by adding an edge to a complete bipartite graph we obtain a 1-GUR graph which contains a sparse 1-GUR spanning subgraph.

Let $G = (V, E)$ be a graph. A pair (G_1, G_2), where G_1, G_2 are subgraphs of G, is called a k-separator of G if $V(G_1) \cup V(G_2) = V$, $E(G_1) \cup E(G_2) = E$, and $|V(G_1) \cup$

$V(G_2)| = k$ hold. For a subset $X \subseteq V$ let $G + K(X)$ denote the supergraph of G obtained by adding all edges connecting pairs of vertices of X (which are non-adjacent in G). The first observation about separations is as follows.

Lemma 3.3. *Let G be a 1-GUR graph and let (G_1, G_2) a k-separator of G with $X = V(G_1) \cap V(G_2)$. Then $G_i + K(X)$ is 1-GUR for $i = 1, 2$.*

Proof. Suppose that $\overline{G}_1 = G_1 + K(X)$ is not 1-GUR. Then there exists a generic realization $(\overline{G}_1, p_1)$ of $\overline{G}_1$ in $\mathbb{R}^1$ which is not UR and hence there exists a realization $(\overline{G}_1, p_1')$ equivalent but non congruent to $(\overline{G}_1, p_1)$. We can assume that $p_1'(v) = p_1(v)$ for every v in X. Extend p_1 to a generic realization p of G in $\mathbb{R}^1$. Let

$$p'(v) = \begin{cases} p_1'(v), & v \in V(G_1) \\ p(v), & v \in V(G_2) \end{cases}$$

Then (G, p') is equivalent but not congruent to (G, p), which means that G is not 1-GUR, a contradiction. □

Lemma 3.3 implies that we can cut a 1-GUR graph along a separating vertex pair u, v into two smaller 1-GUR graphs if we add the edge uv to both pieces[2]. What if we are not allowed to add the edge? In this context the following statement may help.

A pair of vertices u, v in graph G is called *generically universally linked* if the distance between u and v is the same in every pair of equivalent generic realizations of G.

Conjecture 3.4. *Suppose that u, v is not generically universally linked in G on the line, for some pair $u, v \in V$. Then there exist generic 1-dimensional realizations $(G, p), (G, q)$ of G with the property that there exist a realization (G, p') equivalent to (G, p) and a realization (G, q') equivalent to (G, q), such that $\|p'(u) - p'(v)\| > \|p(u) - p(v)\|$ and $\|q'(u) - q'(v)\| < \|q(u) - q(v)\|$.*

Note that a pair of vertices is generically globally linked in $\mathbb{R}^1$ if and only if there exist two vertex-disjoint paths from u to v. From this it is easy to see that the "globally linked" version of Conjecture 3.4 is true.

The truth of this conjecture would imply:

Conjecture 3.5. *Let G be 1-GUR and let (G_1, G_2) be a 2-separation in G with $V(G_1) \cap V(G_2) = \{x, y\}$. Then G_1 or G_2 is 1-GUR.*

Proof. (assuming the truth of Conjecture 3.4) Suppose, for a contradiction, that G_1 and G_2 are not 1-GUR. We may assume that x, y is not generically universally linked in G_1 and G_2. Thus there is a generic realization (G_1, p) in $\mathbb{R}^1$ and an equivalent realization (G_1, q) such that the distance between $p(x)$ and $p(y)$ is, say, stricly smaller than the distance between $q(x)$ and $q(y)$. By assuming the truth of Conjecture 3.4 we can find a generic realization (G_2, p') in $\mathbb{R}^1$ and an equivalent realization (G_2, q') such

[2]It is easy to see that every 1-GUR graph (in fact, every 1-GGR graph) is 2-connected. Thus we may begin the study of small separators with the 2-separations.

that the distance between $p'(x)$ and $p'(y)$ is, say, stricly smaller than the distance between $q'(x)$ and $q'(y)$. By using a result of Alfakih [2] we can show that every realization close enough to (G_2, p') has this property. Therefore, by carefully choosing the generic realization (G_2, p') and rescaling, if necessary, we may assume that $||p(x) - p(y)|| = ||p'(x) - p'(y)||$. Now we can use a result of Bezdek and Connelly [3] to obtain a pair of realizations (G_1, r) and (G_2, r') for which $||r(x) - r(y)|| = ||r'(x) - r'(y)|| > ||p(x) - p(y)||$ and such that (G_1, r) is equivalent to (G_1, p) and (G_2, r') is equivalent to (G_2, p'). By glueing together (G_1, p) and (G_2, p') as well as (G_1, r) and (G_2, r') along the pair x, y we obtain two equivalent but not congruent realizations of G, where the former realization is generic. This contradicts the fact that G is 1-GUR. □

Conjecture 3.5 would imply Conjecture 3.1 by induction in the case when there is a 2-separation.

We close this section with the following question.

Question 3.6. *Let $G = (V, E)$ be 1-GUR. Does this imply that*
(a) $|E| \geq 2|V| - 3$ holds?
(b) G is 2-GR?

Note that the truth of Conjecture 3.1 would imply an affirmative answer to (b), and hence also to (a), since both operations preserve generic rigidity in $\mathbb{R}^2$.

4 Cover graphs and universal rigidity

Since it is probably difficult to characterize 1-GUR graphs, special families of 1-GUR (or not 1-GUR) graphs may be of interest. In this context we offer the study of the following family of graphs as a candidate for being not 1-GUR.

Let $G = (V, E)$ be a graph and let $\vec{G}$ be an acyclic orientation of G. An edge e of G is *dependent* if the reversal of e in $\vec{G}$ creates a directed cycle. An orientation without dependent edges is called *strongly acyclic*. We say that G is a *cover graph* if G has a strongly acyclic orientation. (It is known that G is a cover graph if and only if it is the Hasse diagram of some partially ordered set on V.) Note that complete bipartite graphs are cover graphs: orient all edges from one colour class to the other. Also note that cover graphs are triangle-free. We should also remark that it is NP-hard to test whether a given graph is a cover graph [7, 14].

Question 4.1. *Is it true that no cover graph is 1-GUR (except $K_{1,1}$)?*

It is also known that triangle-free planar graphs (and more generally, triangle-free 3-colorable graphs) are cover graphs [11]. (Recall that by a theorem of Grötzsch, every triangle-free planar graph is 3-colorable.) These special cases would also be interesting:

[3]Bezdek and Connelly [6] proved that if $p = (p_1, p_2, ..., p_n)$ and $q = (q_1, q_2, ..., q_n)$ are two configurations in $\mathbb{R}^d$ then there is a continuous motion $p(t)$ in $\mathbb{R}^{2d}$, that is analytic in t, such that $p(0) = p$, $p(1) = q$ and for $0 \leq t \leq 1$, $\|p_i(t) - p_j(t)\|$ is monotone for all $1 \leq i < j \leq n$.

Question 4.2. *Is it true that no triangle-free planar graph (or even triangle-free* 3*-colorable graph) is* 1*-GUR (except* $K_{1,1}$*)?*

We may also ask whether all non-cover graphs are 1-GUR. An interesting graph to analyse is the Grötzsch graph, which is triangle-free and 4-chromatic, see Figure 1. This graph is not a cover graph [11]. Is it 1-GUR? Since it is triangle-free, an affirmative answer to this question would disprove Conjecture 3.1.

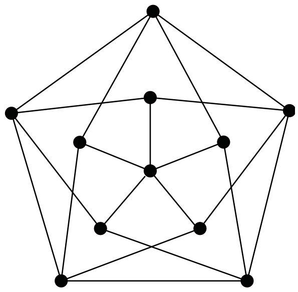

Figure 1: The Grötzsch graph.

5 Further observations on cover graphs

This section contains some further questions and observations about cover graphs, loosely related to (universal) rigidity of graphs. Let $G = (V, E)$ be a graph. We say that G is $(2, 4)$-*sparse* if for all subsets $X \subseteq V$ with $|X| \geq 3$ the subgraph induced by X has at most $2|X| - 4$ edges. For example, triangle-free planar graphs are $(2, 4)$-sparse. Perhaps the following larger family also consists of cover graphs.

Question 5.1. *Is every* $(2, 4)$*-sparse graph a cover graph?*

A $(2, 4)$-sparse graph is clearly triangle-free. It is also independent in the 2-dimensional generic rigidity matroid by Laman's theorem. This leads us to a further extension:

Question 5.2. *Is every triangle free graph which is independent in the two-dimensional generic rigidity matroid a cover graph?*

One proof method for a positive result here would use the following well-known Henneberg operations (typically used in rigidity problems in two dimensions). Let $G = (V, E)$ be a graph. The 0-*extension* operation adds a new vertex v to G and two new edges vx, vy connecting v to existing vertices. The 1-*extension* operation deletes an edge xy of G, adds a new vertex v, and three new edges vx, vy, vz, for some vertex $z \neq x, y$. The next lemmas on the construction of cover graphs show that this approach may be useful.

Lemma 5.3. *Let* G *be a triangle-free graph obtained from a graph* H *by a* 0*-extension operation. Then* G *is a cover graph if and only if* H *is a cover graph.*

Proof. Since H is a subgraph of G, necessity is obvious. To see the other direction consider a strongly acyclic orientation $\vec{H}$ of H. Suppose that $G = H + vx + vy$. Since $\vec{H}$ is acyclic, we cannot have an (x, y)-directed path and a (y, x)-directed path in $\vec{H}$ simultaneously. Thus we have three cases to consider.

Case 1: There is an (x, y)-directed path in $\vec{H}$. Then we orient vx from x to v and vy from v to y.

Case 2: There is an (y, x)-directed path in $\vec{H}$. Then we orient vx from v to x and vy from y to v.

Case 3: There is neither (x, y)-directed path nor (y, x)-directed path in $\vec{H}$. Then we orient vx from v to x and vy from v to y. □

Lemma 5.4. *Let G be a triangle-free graph obtained from a cover graph H by a 1-extension operation. Then G is also a cover graph.*

Proof. Consider a strongly acyclic orientation $\vec{H}$ of H. Suppose that $G = H - xy + vx + vy + vz$. We orient the edges vx, vy, vz as follows.

Case 1: There is an (x, z)-directed path in $\vec{H} - xy$. Then there is no (z, y)-directed path in $\vec{H} - xy$.

Case 1.1: There is a (y, z)-directed path in $\vec{H} - xy$. We orient vx from x to v, vy from y to v and vz from v to z. (Figure 2)

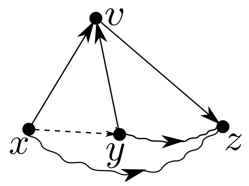

Figure 2: Case 1.1

Case 1.2: There is no (y, z)-directed path in $\vec{H} - xy$. We orient vx from x to v, vy from v to y and vz from v to z. (Figure 3)

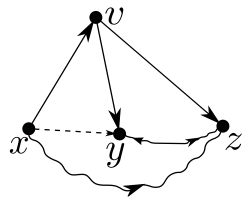

Figure 3: Case 1.2

Case 2: There is an (z, x)-directed path in $\vec{H} - xy$. Then every (y, z)-path in $\vec{H} - xy$ has at least two backward edges. We orient vx from v to x, vy from v to y and vz from z to v. (Figure 4)

Case 3: There is neither (x, z)-directed path nor (z, x)-directed path in $\vec{H} - xy$.

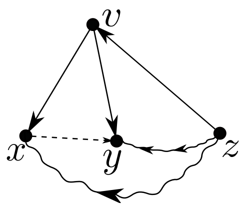

Figure 4: Case 2

Case 3.1: There is a (y, z)-directed path in $\vec{H} - xy$. Then every (x, z)-path in $\vec{H} - xy$ has at least two forward edges. We orient vx from x to v, vy from y to v and vz from v to z. (Figure 5)

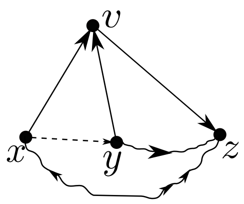

Figure 5: Case 3.1

Case 3.2: There is a (z, y)-directed path in $\vec{H} - xy$. We orient vx from x to v, vy from v to y and vz from z to v. (Figure 6)

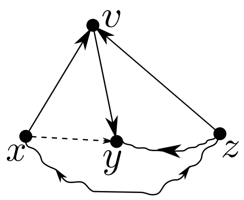

Figure 6: Case 3.2

Case 3.3: There is neither (y, z)-directed path nor (z, y)-directed path in $\vec{H} - xy$. We orient vx from x to v, vy from y to v and vz from z to v. (Figure 7) $\square$

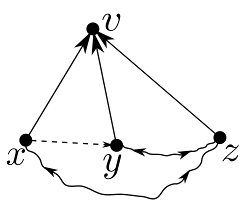

Figure 7: Case 3.3

Acknowledgements

This work was supported by the Hungarian Scientific Research Fund, grant nos. NK 72845 and K81472, the EU-ESF grant no. TÁMOP 4.2.1./B-09/KMR-2010-0003, the MTA-ELTE Egerváry Research Group on Combinatorial Optimization and the TEO-MATRO grant ANR-10-BLAN 0207. We thank Bill Jackson, Viktória Kaszanitzky, and Ferenc Péterfalvi for helpful discussions.

References

[1] T.G. Abbott, Generalizations of Kempe's universality theorem, Master's thesis, Massachusetts Institute of Technology, Dept. of Electrical Engineering and Computer Science, 2008.

[2] A.Y. Alfakih, Graph rigidity via Euclidean distance matrices, *Linear Algebra and Its Applications* 310, 149-165, 2000.

[3] A.Y. Alfakih, On bar frameworks, stress matrices and semidefinite programming, *Math. Program., Ser. B* (2011) 129:113-128.

[4] A.Y. Alfakih and Y. Ye, On affine motions of bar frameworks in general position, Arxiv preprint arXiv:1009.3318, 2010 - arxiv.org.

[5] L. Asimow and B. Roth, Rigidity of graphs II, *J. Math. Anal. Appl.* 68 (1979) 171-190.

[6] K. Bezdek and R. Connelly, Pushing disks apart - the Kneser-Poulsen conjecture in the plane, J. reine angew. Math. 553 (2002) 221-236.

[7] G. Brightwell, On the complexity of diagram testing, *Order* 10 (1993), 297-303.

[8] R. Connelly, Stress and stability, manuscript, 2001.

[9] R. Connelly, Generic global rigidity, *Discrete Comp. Geometry* **33** (2005), pp 549-563.

[10] R. Connelly (ed.), Conjectures and questions on global rigidity, Open problems of a mini-workshop at Cornell University, February 2011.

[11] D.C. Fisher, K. Fraughnaugh, L. Langley, and D.B. West, The number of dependent arcs in an acyclic orientation, *J. Combin. Theory, Ser. B* 71, 73-78 (1997).

[12] S. Gortler and D. Thurston, Characterizing the universal rigidity of generic frameworks, Arxiv preprint arXiv:1001.0172, 2009.

[13] S. Gortler, A. Healy, and D. Thurston, Characterizing generic global rigidity, *American Journal of Mathematics*, Volume 132, Number 4, August 2010, pp. 897-939.

[14] J. Nesetril and V. Rödl, More on complexity of diagrams, *Comment. Math. Univ. Carolinae* 2 (1995), 269-278.

[15] K. Ratmanski, Universally rigid framework attachments, Arxiv preprint arXiv:1011.4094, 2010 - arxiv.org.

[16] J.B. Saxe, Embeddability of weighted graphs in k-space is strongly NP-hard. Technical report, Computer Science Department, Carnegie Mellon University, 1979.

[17] W. Whiteley, Some matroids from discrete applied geometry. Matroid theory (Seattle, WA, 1995), 171–311, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.