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Abstract

We show that certain polyhedral versions of Sperner's Lemma, where the colouring is given explicitly as part of the input, are PPAD-complete. The proofs are based on two recent results on the complexity of computational problems in game theory: the PPAD-completeness of 2-player Nash, proved by Chen and Deng, and of Scarf's Lemma, proved by Kintali. We show how colourings of polyhedra provide a link between these two results.

1 Introduction

1.1 Polyhedral versions of Sperner's Lemma

Sperner's Lemma on the existence of a multicoloured triangle in a suitable colouring of a triangulation has many versions and generalizations. We consider a variant of the standard n-dimensional Sperner Lemma, formulated in terms of colourings of n-dimensional polytopes, see for example [9]. Given a colouring of the vertices of a polytope by n colours, a facet is called *multicoloured* if it contains vertices of each colour.

Theorem 1.1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P by n colours such that F_0 is multicoloured. Then there is another multicoloured facet.*

This theorem leads naturally to a computational problem where the task is to find a multicoloured facet different from F_0.

Polytopal Sperner

Input: vectors $v^i \in \mathbb{Q}^n$ $(i = 1, \dots, m)$ whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a multicoloured simplex facet F_0 of P.

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Output: n affine independent vectors $v^{i_1}, \ldots, v^{i_n}$ with different colours which lie on a facet of P different from F_0.

Obviously, the polar version of the theorem is also true, and the resulting computational problem is equivalent. A vertex of a full-dimensional polyhedron is called *simple* if it is on exactly n facets.

Polar polytopal Sperner

Input: a matrix $A \in \mathbb{Q}^{n \times n}$ and a vector $b \in \mathbb{Q}^n$, such that the polyhedron $P = \{x : Ax \leq b\}$ is bounded and full-dimensional; a colouring of the facets by n colours; a multicoloured simple vertex v_0 of P.

Output: a multicoloured vertex of P different from v_0.

A related but slightly different version of Sperner's Lemma was introduced by the authors in [9]. Recall that the extreme directions of a polyhedron are the extreme rays of its characteristic cone.

Theorem 1.2. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets containing the i-th extreme direction do not get colour i, then there is a multicoloured vertex.*

This version is practical since it provides a way to give short and transparent proofs of several known combinatorial and game theoretic results, see [8, 9, 10]. The corresponding computational problem is the following.

Extreme direction Sperner

Input: matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^m$ such that $P = \{x : Ax \leq b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets containing the i-th extreme direction do not get colour i.

Output: a multicoloured vertex of P.

In this note we show, using recent developments on the computational complexity of problems in game theory, that the following two natural special cases of this problem are already PPAD-complete.

0-1 extreme direction Sperner

Input: matrix $A \in \{0,1\}^{m \times n}$ with no all-0 column; a colouring of the facets of $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i.

Output: a multicoloured vertex of P.

Extreme direction Sperner with $2n$ facets

Input: a matrix $A \in \mathbb{Q}_+^{n \times n}$; a colouring of the facets of the polyhedron $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i and every colour appears exactly twice.

Output: a multicoloured vertex of P.

We also show that extreme direction Sperner provides a link between the complexity of Scarf's Lemma and that of finding Nash equilibria in 2-player games.

1.2 The class PPAD and PPAD-completeness

The complexity class PPAD is defined as the set of total search problems which are Karp-reducible to its prototypical problem, end of the line, which is the following.

End of the line

Input: a directed graph on $\{0,1\}^n$ given implicitly by an algorithm (described for example as a Turing machine) with running time polynomial in n. It is required that in the graph, every vertex has at most one out-neighbour and at most one in-neighbour, and $\mathbf{0}$ has no in-neighbour, but it has an out-neighbour. The input of the algorithm describing the graph is a vertex, that is, an n-bit binary string, and its output is the out-neighbour and the in-neighbour of the vertex.

Output: any vertex in $\{0,1\}^n \setminus \{\mathbf{0}\}$ that has degree 1 (where the degree is the in-degree plus the out-degree).

A problem in PPAD is called *PPAD-complete* if every other problem in PPAD is Karp-reducible to it. The class PPAD was introduced by Papadimitriou [11], who proved among other results that a computational version of 3D Sperner's Lemma is PPAD-complete. Later Chen and Deng [2] proved that the 2 dimensional problem is also PPAD-complete. The input of these computational versions is the description of a polynomial algorithm that computes a legal colouring, while the number of vertices to be coloured is exponential in the input size. This is conceptually different from the computational problems that we consider, where the input explicitly contains the vertices or facets of a polyhedron and their colouring. In these problems the difficulty lies not in the large number of vertices but in that the structure is encoded as a polyhedron. We note that in fixed dimension they are solvable in polynomial time since then the number of facets is polynomial in the number of vertices.

For a long time it had been open to find natural PPAD-complete problems that do not have a description of a Turing machine in their input. In 2006, Daskalakis, Goldberg and Papadimitriou [1] proved that approximating Nash-equilibria in 4-player games is PPAD-hard. Building on their work, Chen and Deng [4] managed to prove the same for 2-player Nash-equilibria, which is considered a breakthrough result in the area. In another line of research, Kintali [7] proved that the computational version of

Scarf's Lemma (Theorem 4.3) is PPAD-complete, along with other related problems, see [6].

In this paper we give natural PPAD-complete polyhedral problems that do not have descriptions of algorithms in their input. We show that these problems are related both to the Nash-equilibrium results and to Scarf's lemma, thus providing a link between these two game-theoretic computational problems.

First, we show that our problems belong to the class PPAD. Then in Section 3 we use the results of Kintali [7] to show that 0-1 EXTREME DIRECTION SPERNER is PPAD-complete even in the case when each row of A contains at most three 1s. In contrast, the problem is solvable in polynomial time if each row contains at most two 1s. Finally, in Section 4 we prove using the result of Chen and Deng [4] that EXTREME DIRECTION SPERNER WITH $2n$ FACETS is PPAD-complete. We also show that this problem is in fact a special case of SCARF, thus providing an alternative proof of its PPAD-completeness.

2 Membership in PPAD

Proposition 2.1. POLYTOPAL SPERNER *is in PPAD.*

Proof. We reduce it to the problem END OF THE LINE. We can compute in polynomial time a perturbation of the vertices in the input such that every facet of the convex hull of the perturbed vertices is a simplex, and every facet (as a vertex set) is a subset of an original facet. Assume that the set of colours is $[n]$. We define a digraph whose nodes are the facets that contain all colours in $[n-1]$ (formally, we may associate a node to each n-tuple of vertices, all other nodes being isolated). Each $(n-2)$-dimensional face with all colours in $[n-1]$ is in exactly two facets. We can say that one of them is on the left side of the face and the other is on the right side, with respect to a fixed orientation: we compute the sign of the two determinants of the vectors going from a fixed inner point of P to the $n-1$ vertices of the $(n-2)$-dimensional face (in the order according to the colours) and the n-th vertex of the two facets; the facet whose determinant is positive is on the left side, the other is on the right. For each such $(n-2)$-dimensional face, we introduce an arc from the node corresponding to the facet on the left side to the node corresponding to the facet on the right side.

The obtained digraph has in-degree and out-degree at most 1 in every node, and the neighbours of a node can be computed in polynomial time. A node has degree 1 if and only if the corresponding facet is multicoloured. We may assume without loss of generality that the node corresponding to F_0 is a source, so the solution of END OF THE LINE for this digraph corresponds to finding a multicoloured facet different from F_0. □

Proposition 2.2. EXTREME DIRECTION SPERNER *is in PPAD.*

Proof. We prove that EXTREME DIRECTION SPERNER is Karp-reducible to POLYTOPAL SPERNER. Suppose that matrix A and vector b are an instance of EXTREME DIRECTION SPERNER and let $P = \{x : Ax \le b\}$. We can translate P so that it contains the origin in its interior. In this case its polar P^{Δ} is a polytope whose vertices can be obtained easily from A and b. The colouring of P defines a colouring of the vertices of P^Δ except for the origin which corresponds to the infinite facet of P. Let us cut off the origin with a hyperplane H – such a hyperplane can be computed in polynomial time. This way, since the origin is a simple vertex of P, we introduce exactly n new vertices and a simplex facet. The i-th new vertex lies on the facets that correspond to all but the i-th extreme direction of P; let the colour of it be i. We obtained a colouring of $P^\Delta \cap H^+$ (where H^+ is the halfspace bounded by H not containing the origin) which satisfies the criteria, so it is an instance of POLYTOPAL SPERNER. A multicoloured facet of $P^\Delta \cap H^+$ which is different from $P^\Delta \cap H$, corresponds to a multicoloured vertex of P. □

3 PPAD-completeness of 0-1 EXTREME DIRECTION SPERNER

Theorem 3.1. 0-1 EXTREME DIRECTION SPERNER *is PPAD-complete, even when every row of A contains at most three 1s.*

Proof. The proof is similar to the proof of PPAD-completeness of SCARF by Kintali [7], and builds on his result that the problem 3-STRONG KERNEL defined below is PPAD-complete. A digraph $D = (V, E)$ is called *clique-acyclic* if there is no directed cycle in a clique, whose arcs do not appear reversed. Equivalently, for each clique K, there is a node $v \in K$ whose closed out-neighbourhood contains K (the node itself is included in the closed out-neighbourhod). A *strong fractional kernel* of D is a vector $x : V \to \mathbb{R}_+$ such that $x(K) \leq 1$ for every clique K, and for each node v there is at least one clique K in the closed out-neighbourhood of v such that $x(K) = 1$.

3-STRONG KERNEL

Input: A clique-acyclic digraph D with maximum clique size at most 3.

Output: A strong fractional kernel of D.

To reduce 3-STRONG KERNEL on digraph $D = (V, E)$ to 0-1 EXTREME DIRECTION SPERNER, we assume that $V = [n]$, and consider the polyhedron

$$P = \{x \in \mathbb{R}^n : x(K) \leq 1 \text{ for every clique } K \text{ of } D\}.$$

Since every clique has size at most 3, the number of cliques is polynomial in n. The extreme directions of P are $-e_j$ ($j \in [n]$). As a set of colours, we use $[n]$. Let the colour of the facet $x(K) = 1$ be a node of K whose closed out-neighbourhood contains K. This colouring satisfies the criterion in Theorem 1.2, so we have a valid input for 0-1 EXTREME DIRECTION SPERNER, and furthermore every row of the describing system contains at most three 1s. Let x^* be a mulitcoloured vertex. For each node v, there is a clique K such that the facet $x(K) = 1$ contains x^* and has colour v, hence K is in the closed out-neighbourhood of v. This means that x^* is a strong fractional kernel. □

Corollary 3.2. POLYTOPAL SPERNER *and* POLAR POLYTOPAL SPERNER *are PPAD-complete.*

Proof. This follows directly from Proposition 2.1, Proposition 2.2 and Theorem 3.1. □

Next we show that three 1s in a row is best possible.

Proposition 3.3. 0-1 EXTREME DIRECTION SPERNER *can be solved in polynomial time if every row of A contains at most two 1s.*

Proof. Let A be the matrix in the input. We may assume that every row of A contains exactly two 1s. Consider the graph on node set $[n]$ whose edge-node incidence matrix is A. The colouring of the facets corresponding to the rows of A determines an orientation of this graph: let the head of each edge be the colour of the corresponding facet. Let D denote the resulting directed graph. The goal is to find a vertex of the polyhedron $P = \{x \in \mathbb{R}^n : \ Ax \leq \mathbf{1}, \ x \leq \mathbf{1}\}$ for which for every $i \in [n]$ we have $x_i = 1$ or there is an arc ji of D which is saturated, that is, $x_j + x_i = 1$.

If each node has an incoming arc, then the vector $z = \frac{1}{2}\mathbf{1}$ is a multicoloured element of P, because $Az = \mathbf{1}$, that is, all arcs are saturated. Thus we can find a vertex x^* of P for which $Ax^* = \mathbf{1}$, which is therefore multicoloured.

If there is a source node i of D, then x_i has to be 1, because the only inequality with colour i is $x_i \leq 1$. If j is an out-neighbour of i, then to make arc ij saturated, let x_j be 0. This guarantees that x lies on facets of every colour in the closed outneighbourhood of i. We can delete the closed outneighbourhood of i and repeat the above, until we get a graph with no source node, and set the remaining variables as we did in the case where each node had an incoming arc. □

The above proof works only when the right side of every inequality is 1. If we remove this restriction, then we obtain an interesting problem on vertex covers of graphs. For an undirected graph $G = ([n], E)$ and a vector $w \in \mathbb{N}^E$, a vector $x \in \mathbb{N}^n$ is called a *w-cover* if $x_i + x_j \geq w_{ij}$ for every $ij \in E$.

Theorem 3.4. *Let $D = ([n], E)$ be a directed graph and let $w \in \mathbb{N}^E$. Then there is a $2w$-cover x of the underlying undirected graph of D such that for every node i with $x_i > 0$ there is an arc ji with $x_j + x_i = 2w_{ji}$.*

Proof. Let A be the edge-node incidence matrix of the underlying undirected graph and consider the polyhedron

$$P = \{x \in \mathbb{R}^n : \ Ax \geq 2w, \ x \geq \mathbf{0}\}.$$

Let us colour an inequality corresponding to an arc ji with colour i and an inequality $x_j \geq 0$ with colour j. Using Theorem 1.2 there is a multicoloured vertex x^* of P. By a result of Gallai [5], P is an integer polyhedron. Therefore x^* is the characteristic vector of a $2w$-cover which by multicolouredness has the desired properties. □

Question. *Can we find the $2w$-cover guaranteed by the theorem in polynomial time?*

The special case when the in-degree of every node is 1 can be solved in polynomial time.

Proposition 3.5. *The $2w$-cover in Theorem 3.4 can be found in polynomial time in the case when each node has in-degree 1 in D.*

Proof. Let us first assume that D is a directed cycle, and the nodes are indexed according to the cyclic order. We can check in polynomial time if there is a nonnegative $2w$-cover x where every arc is tight, that is, $x_i + x_{i+1} = 2w_{i,i+1}$ for every $i \in [n]$. If there is no such $2w$-cover, then there must be a node i where $x_i = 0$. We claim that if $x_i = 0$, then this uniquely determines the next node j in the order where $x_j = 0$. Suppose for convenience that $x_0 = 0$. Then x_1 has to be $2w_{0,1}$. Thus x_2 has to be the minimum of 0 and $2w_{1,2} - 2w_{0,1}$, and so forth, x_i has to be $w_{i-1,i} - w_{i-2,i-1} + - \cdots \pm w_{0,1}$, so far as these values are positive. If we reach a node i where this value is negative or 0, we have to set x_i to 0, and then repeating the above we get the values of the forthcoming nodes. If we determined all the x_i values, then the edge $n1$ is either covered, in which case we are done, or not. Since Theorem 3.4 guarantees a solution, thus by trying all possible starting points, we will find a solution.

In the general case each component of D contains one directed cycle and some arborescences rooted on nodes of the cycle. First we solve the problem restricted to the cycle, then we can traverse the arborescences starting from the root; the values are uniquely determined and we get a solution. □

We note that the prescription of in-degree 1 means that the corresponding polyhedron has $2n$ facets and each colour appears exactly twice. This leads us to the topic of the next section.

4 PPAD-completeness of EXTREME DIRECTION SPERNER WITH $2n$ FACETS

It is a well-known result in game theory that finding a symmetric Nash equilibrium in a symmetric finite 2-player game is as hard as finding a Nash equilibrium in a not necessarily symmetric 2-player game. A nice property of symmetric games is that symmetric Nash equilibria can be characterized as vertices of a polyhedron having a certain complementarity property. Thus the search problem for symmetric finite 2-player games can be described as follows.
SYMMETRIC 2-NASH

Input: a matrix $A \in \mathbb{Q}_+^{n\times n}$, such that the polyhedron $P = \{x : Ax \le \mathbf{1}, x \ge 0\}$ is bounded and full-dimensional.

Output: a nonzero vertex v of P with the property that $a_i v = 1$ whenever $v_i > 0$, where a_i is the i-th row of A.

The results of Chen and Deng [4] imply the following.

Theorem 4.1 ([4]). SYMMETRIC 2-NASH *is PPAD-complete.*

We can observe that SYMMETRIC 2-NASH is a special case of POLAR POLYTOPAL SPERNER: if we colour the facet corresponding to the i-th row of A by colour i, and also the facet $x_i \geq 0$ by colour i, we can set v_0 to be $\mathbf{0}$ since it is a simple and multicoloured vertex, and a nonzero multicoloured vertex v clearly satisfies that $a_i v = 1$ whenever $v_i > 0$.

In the following, we prove that SYMMETRIC 2-NASH is Karp-reducible to a special case of EXTREME DIRECTION SPERNER WITH $2n$ FACETS, which turns out to be a special case of SCARF.

Theorem 4.2. EXTREME DIRECTION SPERNER WITH $2n$ FACETS *is PPAD-complete.*

Proof. We reduce SYMMETRIC 2-NASH to it. Let $A \in \mathbb{Q}_+^{n \times n}$ define an instance of SYMMETRIC 2-NASH, and let $P = \{x : Ax \leq \mathbf{1}, x \geq 0\}$. The vertex $\mathbf{0}$ is simple, so it has n neighbouring vertices $v^1, \dots, v^n$, which furthermore have the form $v^i = \lambda_i e_i$ for some $\lambda_i > 0$. These vertices can be computed in polynomial time, and we can check if one of them satisfies the conditions. We can assume that none of them does, that is, $a_i v^i < 1$ $(i \in [n])$.

Let $P_0 = \{x : Ax \leq \mathbf{1}, x \geq 0, \sum_{j=1}^n \frac{1}{\lambda_j} x_j \geq 1\}$, which is the convex hull of the vertices of P except for the origin. The new facet F_0 contains the vertices $v_1, \dots, v_n$.

We can translate P_0 so that it contains the origin in its interior. In this case its polar P_0^Δ is a polytope; its vertices can be computed. Let w_0 be the vertex of P_0^Δ corresponding to F_0, and let $F_1, \dots, F_n$ denote the facets of P_0^Δ corresponding to the vertices $v^1, \dots, v^n$.

Let $w^1, \dots, w^n$ be the vertices of P_0^Δ adjacent to w^0, indexed such that w_i is not on facet F_i. We can apply an affine transformation that takes w^0 to $\mathbf{0}$ and w^i to e_i $(i \in [n])$; let Q be the resulting polytope. The polar Q^Δ is a polyhedron of the form $\{x : Ax \leq \mathbf{1}, x \leq 1\}$.

Let us take the colouring of P that we used for POLAR POLYTOPAL SPERNER: the facet corresponding to the i-th row of A and the facet $x_i \geq 0$ has colour i. This colouring induces a colouring of the facets of Q^Δ according to the two polarities. Clearly every colour appears exactly twice. We claim that the facets with extreme direction $-e_i$ do not get colour i. A facet of Q^Δ with extreme direction $-e_i$ corresponds to a vertex of P_0^Δ on facet F_i, which in turn corresponds to a facet of P containing vertex v^i. Since $v_i^i > 0$ and $a_i v^i < 1$, no facet containing v^i has colour i.

Suppose that we can find a multicoloured vertex v of Q^Δ. By a similar argument as above, this corresponds to a multicoloured vertex of P, which completes the proof. □

We close this section by showing a relation between EXTREME DIRECTION SPERNER WITH $2n$ FACETS and Scarf's Lemma [12]. In Scarf's Lemma we consider a bounded polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$, where A is an $m \times n$ nonnegative matrix (with non-zero columns) and $b \in \mathbb{R}^m$ is a positive vector. In addition, for every row a_i of A $(i \in [m])$, a total order $<_i$ of $\text{supp}(a_i)$ is given. If $j \in \text{supp}(a_i)$ and $K \subseteq \text{supp}(a_i)$, we use the notation $j \leq_i K$ as an abbreviation for "$j \leq_i k$ for every $k \in K$".

A vertex x^* of P *dominates* column j if there is a row i where $a_i x^* = b_i$ and $j \leq_i \text{supp}(x^*) \cap \text{supp}(a_i)$ (this implies that $j \in \text{supp}(a_i)$). A vertex x^* of P is *maximal* if by increasing any coordinate of x^* we leave P (or formally, $(\{x^*\} + \mathbb{R}^n_+) \cap P = \{x^*\}$).

Theorem 4.3 (Scarf's Lemma [12])**.** *Let $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ and let $<_i$ be a total order on supp(a_i) ($i \in [m]$), where a_i is the i-th row of A. Then P has a maximal vertex that dominates every column.*

It was shown by Kintali [7] that the following computational version of Scarf's Lemma is PPAD-complete.

SCARF

Input: a matrix $A \in \mathbb{Q}^{m \times n}_+$ and a vector $b \in \mathbb{Q}^m_+$; a total order $<_i$ on $\text{supp}(a_i)$ for every $i \in [m]$.

Output: a maximal vertex of $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ that dominates every column.

Proposition 4.4. EXTREME DIRECTION SPERNER WITH $2n$ FACETS *is a special case of* SCARF.

Proof. Let us consider an instance (A, c) of EXTREME DIRECTION SPERNER WITH $2n$ FACETS, where c_i is the colour of the facet determined by the i-th row. We can assume without loss of generality that all vertices of $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ are strictly positive, since we can get an equivalent problem of the same form by scaling from center $\mathbf{1}$. We can transform this into an instance $(A', b', <)$ of SCARF by setting $A' = \binom{A}{I}$, $b' = \mathbf{1}$, and defining $<_i$ to be an arbitrary total order on $\text{supp}(a_i)$ whose smallest element is c_i (the order of the other elements does not matter). Let $P' = \{x \in \mathbb{R}^n : A'x \leq b',\ x \geq 0\}$.

If v is a multicoloured vertex of P, then for every $j \in [n]$, either $v_j = 1$, or there is an index i such that $a_i v = 1$, $j \in \text{supp}(a_i)$ and $c_i = j$. This means that v dominates every column according to $<$. It is also a maximal vertex of P' since it is a vertex of P.

It is easy to check (using that every vertex of P is strictly positive) that the reverse also holds: any dominating maximal vertex of P' is a multicoloured vertex of P. $\square$

An interesting observation is that we obtain a special case of SCARF where the total orders $<_i$ do not play any role apart from designating a single element from $\text{supp}(a_i)$. Therefore the hardness of SCARF is not due to the extra structure given by total orders (compared to colouring).

It is natural to ask whether the problem remains PPAD-complete if we restrict it to 0-1 matrices. We pose this as an open question.

Question. *Is* 0-1 EXTREME DIRECTION SPERNER *PPAD-complete in the special case when A is an $n \times n$ matrix and every colour appears exactly twice?*

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