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The triangle-free 2-matching polytope of subcubic graphs

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Abstract: The problem of determining the maximum size of a C_k-free 2-matching (that is, a 2-matching not containing cycles of length k) is a much studied question of matching theory. Cornuéjols and Pulleyblank showed that deciding the existence of a C_k-free 2-factor is NP-complete for $k \geq 5$, while Hartvigsen gave an algorithm for the triangle-free case ($k = 3$). The existence of a C_4-free 2-matching is still open.

The description of the C_k-free 2-matching polytope is also of interest. Király showed that finding a maximum weight square-free 2-factor is NP-complete even in bipartite graphs with $0 - 1$ weights, hence we should not expect a nice polyhedral description for $k \geq 4$. However, imposing the condition that the graph has maximum degree 3, these problems become considerably easier. The polyhedral description of the triangle-free 2-factor and 2-matching polytopes of subcubic simple graphs was given by Hartvigsen and Li. In this paper, we give slight generalizations of their nice results by using a shrinking method inspired by Edmonds' maximum matching algorithm.

Considering the general case, a new class of valid inequalities for the triangle-free 2-matching polytope is introduced. With the help of these inequalities, we propose a conjecture for the polyhedral description of the triangle-free 2-matching polytope of simple graphs.

1 Introduction

A cornerstone of matching theory is Edmonds' [7] description of the perfect matching polytope, the convex hull of incidence vectors of perfect matchings of a graph $G = (V, E)$.

Theorem 1.1. *The perfect matching polytope is determined by*

$$\begin{aligned} &(i)\ x_e \geq 0 && (e \in E),\\ &(ii)\ x(\delta(v)) = 1 && (v \in V),\\ &(iii)\ x(\delta(K)) \geq 1 && (K \subseteq V, |K| \textit{ odd}). \end{aligned} \qquad (P_1)$$

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Here $\delta(K)$ denotes the set of edges having exactly one end in K. Observe that the incidence vector of a perfect matching satisfies all these conditions. The theorem yields that the set of vertices of the above polytope is identical to the set of incidence vectors of perfect matchings.

A natural generalization of perfect matchings are b-factors, with 1-factors being perfect matchings. Given a graph $G = (V, E)$ and a degree prescription $b : V \to \mathbb{Z}_+$ on the nodes, a b**-factor** is a subset $M \subseteq E$ of edges such that $d_M(v)$, the number of edges in M incident to v, equals $b(v)$ for each $v \in V$. This is often called **simple** b-factor in the literature, since multiple copies of the same edge are not allowed. If not stated otherwise, all b-factors considered will be simple throughout the paper. We call $K \subseteq V, F \subseteq \delta(K)$ a **pair** if F does not contain loops (by notation, this only means restriction in case of $|K| = 1$). The pair is **odd** if $b(K) + |F|$ is odd. The b**-factor polytope** is the convex hull of the incidence vectors of b-matchings of G. In the same paper [7], Edmonds gave the following characterization of the b-factor polytope.

Theorem 1.2. *The b-factor polytope is determined by*

$$\begin{aligned} (i)\ & 0 \le x_e \le 1 && (e \in E), \\ (ii)\ & x(\dot\delta(v)) = b(v) && (v \in V), \\ (iii)\ & x(\delta(K) \setminus F) - x(F) \ge 1 - |F| && ((K, F)\ \textit{odd}). \end{aligned} \qquad (P_2)$$

Note that $b(K) = \sum_{v \in K} b(v)$, while $\dot\delta(v)$ denotes the family of edges incident to $v \in V$, that is, any loop at v occurs twice in $\dot\delta(v)$. The set of loops at $v \in V$ is denoted by $l(v)$.

A closely related concept is b**-matching**, where instead of $d_F(v) = b(v)$, only $d_F(v) \le b(v)$ is required. A polyhedral description of b-matchings can easily be derived form Theorem 1.2.

Theorem 1.3. *The b-matching polytope is determined by*

$$\begin{aligned} (i)\ & 0 \le x_e \le 1 && (e \in E), \\ (ii)\ & x(\dot\delta(v)) \le b(v) && (v \in V), \\ (iii)\ & x(E[K]) + x(F) \le \left\lfloor \tfrac{b(K)+|F|}{2} \right\rfloor && ((K, F)\ \textit{odd}). \end{aligned} \qquad (P_3)$$

We refer the reader to Part III, in particular, Chapters 30-33 of Schrijver [14] for a detailed discussion of b-matchings and b-factors.

Results on b-factors can be reduced to perfect matchings via a simple construction. Given a graph $G = (V, E)$, construct a new graph $G' = (V', E')$ as follows. Introduce $b(v)$ vertices for each node $v \in V$. For each edge $e = uv \in E$, introduce two vertices $p_{e,u}$ and $p_{e,v}$, an edge $p_{e,u}p_{e,v}$, and edges connecting $p_{e,u}$ to all $b(u)$ copies of u and connecting $p_{e,v}$ to all $b(v)$ copies of v. It is not difficult to see that G' contains a perfect matching if and only if G contains a b-factor. Using this correspondence, results on matchings can be extended to b-factors, including Theorem 1.2, which thus deduces from Theorem 1.1. To the extent of our knowledge, all previous proofs of Theorem 1.3 used this correspondence.

An important subclass of b-factors are 2-factors, decompositions of a graph to disjoint union of cycles. Hamiltonian cycles being 2-factors, it is a natural question looking at special 2-factors not containing short cycles. A $C_{\le k}$**-free** 2**-factor** is a 2-factor not containing cycles of length at most k. Cornuéjols and Pulleyblank [5] showed that deciding the existence of such a subgraph is NP-complete for $k \geq 5$. On the other hand, Hartvigsen gave a difficult algorithm for the triangle-free case ($k = 3$) in his Ph.D. thesis [8]. The existence of a $C_{\leq 4}$-free or C_4-free 2-matching is still open (in the latter problem, triangles are allowed). Yet imposing the condition that the graph is subcubic (that is, the maximum degree of G is 3), these problems become solvable, see [2, 3].

Considering the maximum weight version of the problems, there is a firm difference between triangle- and square-free 2-factors. Király showed [11] that finding a maximum weight square-free 2-factor is NP-complete even in bipartite graphs with $0-1$ weights. On the other hand, for subcubic graphs, polynomial-time algorithms were given by Hartvigsen and Li [9], and by Kobayashi [12] for the weighted C_3-free 2-factor problem with an arbitrary weight function. The former result implies that we should not expect a nice polyhedral description of the square-free 2-factor polytope. However, solvability of the triangle-free case was a main motivation of our result.

The existence of triangle-free 2-matchings becomes significantly harder without assuming the graph is subcubic. Yet if instead of (simple) 2-factors, we look at the problem of **uncapacitated 2-factors**, when we are allowed to use two copies of the same edge, there exists a polyhedral description for arbitrary graphs, given by Cornuéjols and Pulleyblank [6]. Let $\mathcal{T}$ be a set consisting of triangles of G. The node-set and the edge-set of a triangle $T \in \mathcal{T}$ are denoted by V_T and E_T, respectively. An (uncapacitated) 2-factor is called **$\mathcal{T}$-free** if it contain at most two edges (counted by multiplicity) of any member of $\mathcal{T}$. Cornuéjols and Pulleyblank proved the following.

Theorem 1.4. *The convex hull of $\mathcal{T}$-free uncapacitated 2-factors is determined by*

$$\begin{aligned} &(i)\ 0 \leq x_e && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) = 2 && (v \in V), \\ &(iii)\ x(E_T) \leq 2 && (T \in \mathcal{T}). \end{aligned} \tag{P_4}$$

[6] also proves that this description is totally dual integral.

Returning to our subject, Hartvigsen and Li gave a polyhedral description of the triangle-free 2-factor polytope for subcubic simple graphs [9].

Theorem 1.5. *The $\mathcal{T}$-free 2-factor polytope of a simple subcubic graph is determined by*

$$\begin{aligned} &(i)\ 0 \leq x_e \leq 1 && (e \in E), \\ &(ii)\ x(\dot{\delta}(v)) = 2 && (v \in V), \\ &(iii)\ x(\delta(K) \setminus F) - x(F) \geq 1 - |F| && (K \subseteq V, F \subseteq \delta(K), \\ & && |F|\ odd), \\ &(iv)\ x(E_T) = 2 && (T \in \mathcal{T}). \end{aligned} \tag{P_5}$$

Their proof is based on shrinking triangles and on a variation of the Basic Polyhedral Theorem of [4]. In the same paper, they gave a description of the 2-matching polytope as well and gave a sketch of the proof, which was published in its full version in [10].

As we have seen, the b-matching and b-factor polytopes have a similar description. Unexpectedly, the same does not hold in the triangle-free case. We say that a triangle T **1-fits** (resp. **2-fits**) a set $K \subseteq V$ if $|V_T \cap K| = 1$ (resp. 2). The **special edge** of a triangle T 1-fitting (resp. 2-fitting) the set K is the edge $e \in E_T$ having exactly 0 (resp. 2) end-nodes in K, and is denoted by e_T. Given a set $\mathcal{T}$ of forbidden triangles, the set of triangles 1-fitting (resp. 2-fitting) K is denoted by $\mathcal{T}_K^1$ (resp. $\mathcal{T}_K^2$) while $\mathcal{T}_K$ stands for $\mathcal{T}_K^1 \cup \mathcal{T}_K^2$. $(K, F, \mathfrak{T})$ is called a **triplet of Type i** if $K \subseteq V$, $F \subseteq \delta(K)$, $\mathfrak{T} \subseteq \mathcal{T}_K^i$ are such that $F \cap E_{\mathfrak{T}} = \emptyset$, the triangles in $\mathfrak{T}$ are edge-disjoint and F does not contain loops. A triplet is called **odd** if $b(K) + |F| + |\mathfrak{T}|$ is odd. The **deficiency** of a triplet is defined by $\mathtt{def}(K, F, \mathfrak{T}) = x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - \lfloor \frac{1}{2}(b(K) + |F| + 3|\mathfrak{T}|) \rfloor$.

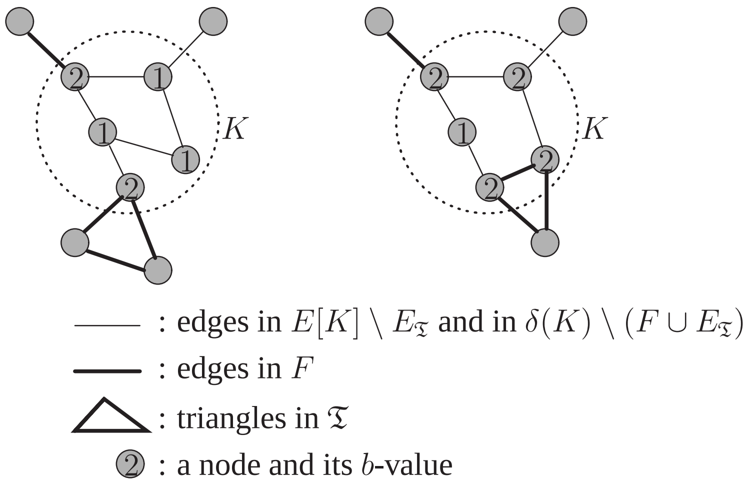

Figure 1: Odd triplets of Type 1 and 2

The fundamental result of Hartvigsen and Li is the following ([9, 10]).

Theorem 1.6. *The $\mathcal{T}$-free 2-matching polytope of a simple subcubic graph is determined by*

$$
\begin{aligned}
&(i)\ 0 \le x_e \le 1 && (e \in E),\\
&(ii)\ x(\dot{\delta}(v)) \le 2 && (v \in V),\\
&(iii)\ x(E[K]) + x(F) + \textstyle\sum_{T \in \mathfrak{T}} x(E_T) \le |K| + \lfloor \frac{|F| + 3|\mathfrak{T}|}{2} \rfloor && ((K, F, \mathfrak{T}) \text{ odd triplet of Type 2}),\\
&(iv)\ x(E_T) \le 2 && (T \in \mathcal{T}).
\end{aligned} \qquad (P_6)
$$

Their proof is algorithmic and uses, in some sense, an Edmonds-style matching algorithm consisting of clever triangle alteration and alternating forest growing. The algorithm alternates between a primal and a dual phase, and a quite complex dual change is performed whenever no improvement is found during the forest growing. The algorithm stops when the primal and dual solutions (that are feasible throughout) satisfy complementary slackness.

In this paper we give new proofs of Theorems 1.5 and 1.6 in a slightly more general form. Let $G = (V, E)$ be a graph and $b : V \to \mathbb{Z}_+$ an upper bound on the node-set such that for any $T \in \mathcal{T}$ and any node v of T,

$$d_G(v) \leq 3, \tag{1}$$
$$b(v) = 2. \tag{2}$$

These settings clearly includes and generalizes the triangle-free 2-factor and 2-matching problems in subcubic graphs.

Theorem 1.7. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (1) and (2). Moreover, assume that there are no two forbidden triangles on the same node-set. The $\mathcal{T}$-free b-factor polytope is determined by*

$$\begin{array}{lr} (i)\ 0 \leq x_e \leq 1 & (e \in E), \\ (ii)\ x(\dot{\delta}(v)) = b(v) & (v \in V), \\ (iii)\ x(\delta(K) \setminus F) - x(F) \geq 1 - |F| & ((K, F) \text{ odd}), \\ (iv)\ x(E_T) = 2 & (T \in \mathcal{T}). \end{array} \tag{P_7}$$

Note that the assumption that no two forbidden triangles share the same node-set is not a restricting one. Indeed, if $V_{T_1} = V_{T_2}$ then, by (1) and (2), no b-factor exists in G.

Our main result is the proof of the following theorem which gives a slight generalization of Theorem 1.6. The method we use here is also inspired by Edmonds' matching algorithm, but different from that of [10] and is based on a new shrinking method. We hope that our proof can be extended to the non-subcubic case as well which is the sole remaining open problem concerning triangle-free 2-matchings.

Theorem 1.8. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (1) and (2). The $\mathcal{T}$-free b-matching polytope is determined by*

$$\begin{array}{lr} (i)\ 0 \leq x_e \leq 1 & (e \in E), \\ (ii)\ x(\dot{\delta}(v)) \leq b(v) & (v \in V), \\ (iii)\ x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) \leq & ((K, F, \mathfrak{T}) \text{ odd} \\ \quad \lfloor \frac{b(K)+|F|+3|\mathfrak{T}|}{2} \rfloor & \text{triplet of Type 2}), \\ (iv)\ x(E_T) \leq 2 & (T \in \mathcal{T}), \\ (v)\ x(E_{T_1} \cup E_{T_2}) \leq 2 & (T_1, T_2 \in \mathcal{T}, V_{T_1} = V_{T_2}). \end{array} \tag{P_8}$$

Our proof is a natural extension of the proof of Theorem 1.1 given by Aráoz, Cunningham, Edmonds, and Green-Krótki [1] and Schrijver [13]. Assumption (1) here is essential: the theorem is false if we remove the degree bound $d_G(v) \leq 3$ on nodes of forbidden triangles. An example is shown in Section 8.

Throughout the paper we use the following notation. For an undirected graph $G = (V, E)$ and set $X \subseteq V$, the subgraph induced by X is denoted by $G[X]$. The sets of edges induced by and having exactly one end in X are denoted by $E[X]$ and $\delta(X)$, respectively. For disjoint subsets X, Y of V, $E[X, Y]$ denotes the set of edges between X and Y. Also, $d(X, Y)$ stands for the number of edges going between disjoint subsets X and Y. For a set $X \subseteq V$, $\bar{X}$ denotes $V - X$.

Recall that $\dot{\delta}(v)$ stands for the collection of edges incident to v in which loops at v (whose set is denoted by $l(v)$) are included twice. We define $d(v) = |\delta(v)|$, that is, loops on v are counted twice. For a set $X \subseteq V$, the set of loops induced by X is denoted by $l(X)$, that is, $l(X) = \cup_{v \in X} l(v)$. For a node $v \in V$, we abbreviate the set $\{v\}$ by v, for example, $X - v$ stands for $X \setminus \{v\}$. For a node $v \in V$ and edges $F \subseteq E$, $F_v = \delta(v) \cap F$.

We use $b(U) = \sum_{v \in U} b(v)$ for a function $b : V \to \mathbb{Z}_+$ and a set $U \subseteq V$.

For a set $\mathfrak{T}$ of triangles, $V_{\mathfrak{T}}$ and $E_{\mathfrak{T}}$ stand for the set of nodes and edges contained by the triangles in $\mathfrak{T}$, respectively. For an edge $e \in E$, we denote its ends by e^u and e^v. For a triangle T with $V_T = \{u, v, w\}$ (resp. $\{t_1, t_2, t_3\}$) we denote its edges by e^T_{uv}, e^T_{vw} and e^T_{uw} (resp. e^T_{12}, e^T_{23} and e^T_{13}) where e^T_{ij} is the edge between i and j (resp. t_i and t_j). Recall that the special edge of a triangle 1-fitting (resp. 2-fitting) a set K is its edge having 0 (resp. 2) end-nodes in K.

Sometimes we use these notations with subscripts when only a subset $F \subseteq E$ is considered or we work with different graphs simultaneously.

For vectors $x, y \in \mathbb{N}^n$ we say that x is **lexicographically larger** than y if and only if there is an $1 \leq i \leq n$ such that $x_j = y_j$ for each $1 \leq j < i$ and $x_i > y_i$.

The above notation may seem to be overcomplicated. The reason for using such complex notation is that we do not want to forbid the existence of parallel edges and loops in the graph. Apart from making the proofs more difficult, these indicate that the introduction of a precise notation is crucial.

The rest of the paper is organized as follows. In Section, 2 we define a shrinking operation which is then used in Section 3 to prove Theorem 1.2. As an introduction into our method, we reprove Theorem 1.7 in Section 4. Section 5 extends the shrinking procedures introduced earlier. With the help of the new operations, we prove Theorem 1.8 in Section 6. Section 7 exhibits an example showing why (1) cannot be omitted from Theorem 1.8. We also define a new class of inequalities and give a conjecture for the description of the triangle-free 2-matching polytope of simple -not necessarily subcubic- graphs.

2 Shrinking odd pairs

We prove Theorem 1.2 by induction on $b(V)$, $|V|$ and $|E|$. In the proof we use a shrinking operation to get a smaller graph on which the induction step can be applied. Note that condition (iii) in Theorems 1.2 and 1.7 is required for odd pairs. If $b(V)$ is odd then $(V, \emptyset)$ is an odd pair and thus (P_2) and (P_7) are infeasible. In the sequel we assume that $b(V)$ is even.

Definition 2.1 (Shrinking an odd pair). **Shrinking** an odd pair (K, F) consists of the following operations:

- replace K by an edge pq with $b^\circ(p) = |F|$ and $b^\circ(q) = 1$,
- define $b^\circ(v) = b(v)$ for each $v \in V - K$,
- replace each edge e with $e^u \in K, e^v \in V - K$ by an edge qe^v if $e \in F$, otherwise by pe^v.

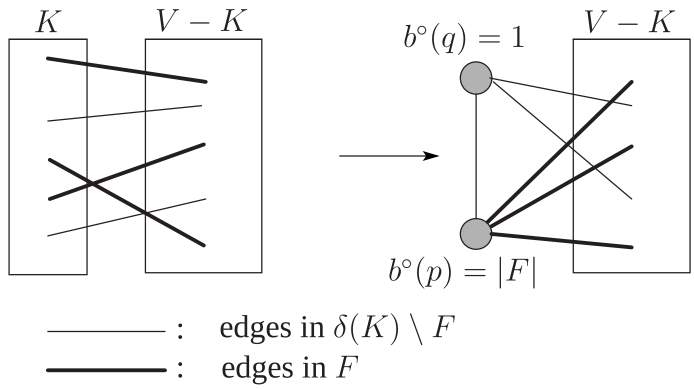

Figure 2: Shrinking an odd pair (K,F)

We usually denote the graph obtained by shrinking an odd pair by $G^\circ=(V^\circ,E^\circ)$. By abuse of notation, each edge $e\in\delta(K)$ is denoted by e again after shrinking the pair and is called the **image** of the original edge. Hence the intersection $E\cap E^\circ$ stands for the set of all edges not induced by K, in other words, $E^\circ-pq\subseteq E$. Similarly, $V^\circ\setminus\{p,q\}\subseteq V$.

Assume that $x\in\mathbb{R}^E$ satisfies (P_2). An odd pair (K,F) is called x-**tight** if it satisfies (iii) with equality. When shrinking an x-tight pair, we use the notation x° for the image of x, namely

$$x^\circ(e)=\begin{cases}x(e) & \text{if } e\in E^\circ-pq,\\ |F|-x(F) & \text{if } e=pq.\end{cases}$$

The main advantage of the shrinking operation is the following.

Lemma 2.2. *Let $G=(V,E)$ be a graph with $b:V\to\mathbb{Z}_+$. Assume that $x\in\mathbb{R}^E$ satisfies (P_2) and (K,F) is an x-tight pair. Then x° satisfies (P_2) in $G^\circ=(V^\circ,E^\circ)$ with b°.*

Proof. (i) clearly holds for edges different from pq. Concerning pq, $x^\circ(pq)=|F|-x(F)\geq 0$. The tightness of (K,F) implies $x^\circ(pq)=|F|-x(F)=1-x(\delta(K)\setminus F)\leq 1$.

For a node v in $V^\circ-\{p,q\}$, by the definition of shrinking, $x^\circ(\dot\delta(v))=x(\dot\delta(v))=b(v)=b^\circ(v)$. Also, $x^\circ(\dot\delta(p))=x(F)+x^\circ(pq)=|F|=b^\circ(p)$. By the tightness of (K,F), $x^\circ(\dot\delta(q))=x(\delta(K)\setminus F)+x^\circ(pq)=1=b^\circ(q)$.

It only remains to show that x° satisfies (iii) in G°. First, observe that -assuming $b(V)$ is even- (Z,H) is an odd pair if and only if $(\bar Z,H)$ is also an odd pair. For these two pairs, condition (iii) is identical.

(iii) immediately follows for odd pairs (Z,H) with $Z\subseteq V^\circ\setminus\{p,q\}$ as x satisfied (iii) in the original problem. By taking $(\bar Z,H)$ instead, it also holds if $p,q\in Z$. Again by possibly changing Z to $\bar Z$, it remains to show that (iii) is satisfied if $p\in Z, q\notin Z$.

If $pq\in H$, then add q to Z and delete pq from H. We have previously seen that the odd pair $(Z',H')=(Z+q,H-pq)$ satisfies (iii), thus

$$\begin{aligned}x(\delta(Z)\setminus H)-x(H)&=x(\delta(Z')\setminus H')-x(H')-x(\delta(q))\\&\geq(1-|H'|)-1\\&=1-|H|.\end{aligned}$$

If $pq \notin H$, then first consider the case when $F \cap (\delta(Z) \setminus H) \neq \emptyset$. Let f be an edge in this set. Define $(Z', H') = (Z + q, H + f)$, which is again an odd pair satisfying (iii). Then

$$\begin{aligned} x(\delta(Z) \setminus H) - x(H) &= x(\delta(Z') \setminus H') + 2x(pq) - x(\delta(q)) + 2x(f) \\ &\geq (1 - |H'|) + 2(x(pq) + x(f)) - 1 \\ &= 1 - |H| + 2(x(pq) + x(f) - 1) \\ &\geq 1 - |H|. \end{aligned}$$

For the last inequality, we use that $x(\delta(p)) = |F|$, and the degree of p is $|F| + 1$. Hence pq and f, two edges incident to p must have x value together at least 1.

If $F \cap (\delta(Z) \setminus H) = \emptyset$, then let $F_1 = F \cap H$, $F_2 = F \setminus H$. Define $Z' = Z - p$, $H' = (H \setminus F_1) \cup F_2$. (Z', H') is odd since $b(Z') + |H'| = b(Z) + |H| - |F| - |F_1| + |F_2| = b(Z) + |H| - 2|F_1|$. As we have seen, the pair (Z', H') satisfies (iii).

$$\begin{aligned} x(\delta(Z) \setminus H) - x(H) &= x(\delta(Z') \setminus H') - x(H') + x(F_2) + x(pq) - x(F_1) \\ &\geq (1 - |H'|) + x(\dot{\delta}(p)) - 2x(F_1) \\ &\geq (1 - |H'|) + |F| - 2|F_1| \\ &= 1 - |H|. \end{aligned}$$

This completes the proof. □

3 Proof of Theorem 1.2

It is easy to see that each b-factor satisfies (i) and (ii). To show that (iii) indeed holds for a b-factor $M \subseteq E$, add all equalities $d_M(v) = b(v)$ for $v \in K$. This gives

$$2|M \cap E[K]| + |M \cap \delta(K)| = b(K). \tag{3}$$

Adding the inequalities $|M \cap F| \leq |F|$ and $-|M \cap (\delta(K) \setminus F)| \leq 0$, we get $2|M \cap E[K]| + 2|M \cap F| \leq b(K) + |F|$. This yields $|M \cap E[K]| + |M \cap F| \leq \lfloor \frac{1}{2}(b(K) + |F|) \rfloor = \frac{1}{2}(b(K) + |F| - 1)$ since (K, F) is odd. Subtracting the double of this from (3), we get $|M \cap (\delta(K) \setminus F)| - |M \cap F| \geq 1 - |F|$, as required.

Recall that we may assume that $b(V)$ is even since otherwise there exists no b-factor, and also the polytope (P_2) is empty.

It remains to show that (i), (ii) and (iii) completely determine the b-factor polytope, that is, any $x \in \mathbb{R}^E$ satisfying (P_2) is a convex combination of incidence vectors of b-factors. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_2) not contained in the b-factor polytope.

We choose this counterexample in such a way that $(l(V), b(V), |V|, |E|)$ is lexicographically minimal. This implies that $0 < x < 1$ as edges with $x_e = 0$ could be deleted, while if $x_e = 1$ we can delete e and decrease the b values on its ends by one (if e is a loop on v then decrease $b(v)$ by 2). It is easy to see that the x' and b' thus obtained would satisfy $(i) - (iii)$ thus giving a smaller counterexample, a contradiction. Also, it can be shown that, in presence of parallel edges, the total x value of parallel edges between two nodes should be strictly smaller than one.

As $b(v) \geq 1$ for each $v \in V$, each node has degree at least 2 in G, so $|E| \geq |V|$. G is connected, otherwise one of its components would be a smaller counterexample. If $|E| = |V|$, then G is an even cycle as it implies that $b \equiv 1$ and $b(V)$ is even. By (ii), x is alternately μ and $1 - \mu$ for some value $0 < \mu < 1$ on the edges of this cycle, hence it is the convex combination of the two perfect matchings of the graph, a contradiction.

So $|E| > |V|$. As x is a vertex, it satisfies $|E|$ linearly independent constraints among (P_2) with equality. From $|E| > |V|$, there is a tight odd pair (K, F) linearly independent from the equalities of form (ii).

Proposition 3.1. *For any tight odd pair (K, F) independent from equalities of form (ii), the shrinking of (K, F) results in a lexicographically smaller problem, and the same holds for $(\bar{K}, F)$.*

Proof. The second part follows by complementing K and by the observation that (K, F) is independent from equalities of form (ii) if and only if $(\bar{K}, F)$ does so.

What we have to prove is that either **(A)** $l(K) \neq \emptyset$, or **(B)** $l(K) \neq \emptyset$ and $b(K) > |F| + 1$, or **(C)** $l(K) = \emptyset, b(K) = |F| + 1$ and $|K| > 2$, or **(D)** $l(K) = \emptyset, b(K) = |F| + 1, |K| = 2$ and $E[K] > 1$ as $(b(V), l(V), |V|, |E|)$ decreases only in these cases. However, we will show that either **(A)**, **(B)** or **(C)** is satisfied.

We claim that $G[K]$ is connected. Indeed, assume indirectly that $K = K_1 \cup K_2$ where $K_1 \cap K_2 = \emptyset$ and there is no edge between K_1 and K_2. Define $F_i = F \cap \delta(K_i)$ for $i = 1, 2$. Then one of the pairs $(K_1, F_1), (K_2, F_2)$ is odd while the other is not, say (K_1, F_1) is odd. We have

$$\begin{aligned} 1 - |F| &= x(\delta(K) \setminus F) - x(F) \\ &= x(\delta(K_1) \setminus F_1) - x(F_1) + x(\delta(K_2) \setminus F_2) - x(F_2) \\ &\geq 1 - |F_1| - |F_2| \\ &= 1 - |F|, \end{aligned}$$

thus we have equality everywhere. That means that $x(\delta(K_2) \setminus F_2) - x(F_2) = -|F_2|$, which is only possible (by $0 < x < 1$) if $\delta(K_2) = \emptyset$, contradicting the connectivity of G. Hence $G[K]$ must be connected.

Assume that **(A)** does not hold, so $l(K) = \emptyset$ and **(B)** does not hold either, so $b(K) \leq |F| + 1$. We show that $b(K) = |F| + 1$ in this case. Otherwise $b(K) \leq |F| - 1$ as (K, F) is an odd pair. As $x(F) \geq |F| - 1$, only $b(K) = |F| - 1$ ispossible. By $0 < x < 1$, $E[K] = \emptyset$ and so $|K| = 1$ by the previous observation. If $F = \delta(v)$, the tightness of (K, F) is identical to $x(\dot{\delta}(v)) = b(v)$, contradicting linear independence. Hence $\delta(v) \setminus F \neq \emptyset$ and thus $x(\delta(v) \setminus F) > 0$. Also, $x(F) \leq b(v) \leq |F| - 1$. Consequently, $x(\delta(v) \setminus F) - x(F) > 1 - |F|$, a contradiction.

Now we show that $|K| \geq 2$. If $K = \{v\}$ then $x(\delta(v) \setminus F) \geq 1$ as $l(v) = \emptyset$. If $F \neq \emptyset$ then $x(F) < |F|$ as $x < 1$, so (iii) cannot hold with equality. Hence $F = \emptyset$ and $x(\delta(v)) = 1 = b(v)$, so the tightness of (K, F) is identical to $x(\dot{\delta}(v)) = b(v)$, contradicting independence.

Assume that **(C)** does not hold either, so $l(K) = \emptyset, b(K) = |F| + 1$ and $|K| = 2$. We show that this leads to contradiction. Let $K = \{u, v\}$, and let C be the set of parallel edges between u and v. Then we have

$$x(\delta(K) \setminus F) - x(F) = b(u) + b(v) - 2x(C) - 2x(F_u) - 2x(F_v).$$

As $b(u) + b(v) = |F| + 1$, either $b(u) \leq |F_u|$ or $b(v) \leq |F_v|$, say the first holds. In this case $x(C) + x(F_u) \leq b(u) \leq |F_u|$, so $x(C) + x(F_u) + x(F_v) \leq |F_u| + |F_v|$. Here $F_v = \emptyset$, otherwise strict inequality holds by $x < 1$, contradicting the tightness of (K, F). Then the tightness of the pair can be reformulated as $x(\delta(u) \setminus (C)) - 2x(F_u) = 1 - |F_u|$. By subtracting this from equality $2x(C) + x(\delta(K)) = |F| + 1$, we get $2x(C) + x(\delta(K) \setminus \delta(u)) + 2x(F_u) = 2|F_u| = 2b(u)$. But $x(C) + x(F_u) \leq b(u)$, hence $\delta(K) \setminus \delta(u) = \emptyset$ and $x(C) + x(F_u) = b(u) = |F_u|$, $b(v) = 1$. That means that the tightness of (K, F) is identical to $x(\delta(u)) = b(u)$, contradicting linear independence. $\square$

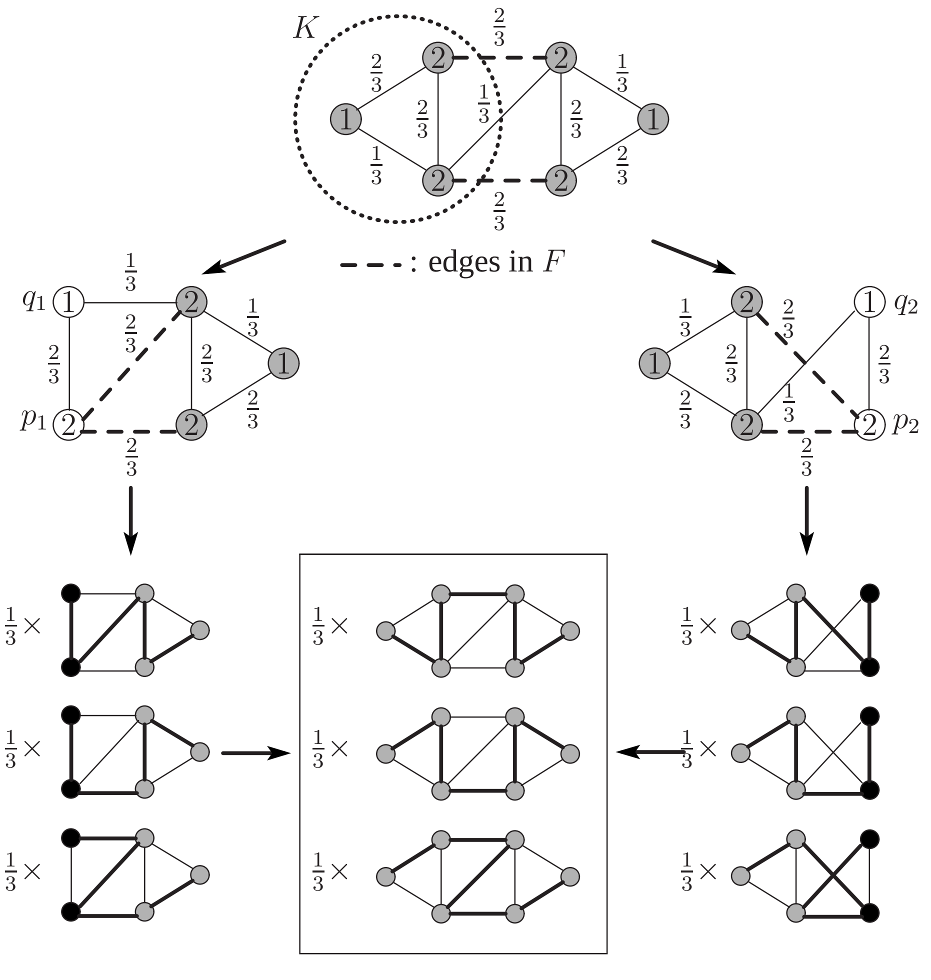

Figure 3: Illustration of the shrinking method

Note that $(\bar{K}, F)$ is also x-tight. Let $G_1^\circ = (V_1^\circ, E_1^\circ), b_1^\circ, x_1^\circ$ and $G_2^\circ = (V_2^\circ, E_2^\circ), b_2^\circ, x_2^\circ$ denote the problems we get after shrinking (K, F) and $(\bar{K}, F)$, respectively. By Proposition 3.1, the induction step can be applied, and -by the minimality of G- x_i° is the convex combination of incidence vectors of b_i°-factors of G_i°. Note, that a b_i°-factor contains either each edge of F and exactly one edge from $\delta(K) \setminus F$, or all but one edges of F, the edge $p_i q_i$ and none of the edges of $\delta(K) \setminus F$. We can write these combinations in the form $x_1^\circ = \frac{1}{k} \sum \chi_{M_i}$ and $x_2^\circ = \frac{1}{k} \sum \chi_{N_j}$ for some $k \in \mathbb{Z}_+$, where the M_i's and N_j's are (not necessarily distinct) b_1°- and b_2°-factors, respectively (note that x° is rational, being a vertex of a rational polytope).

Then each edge $e \in \delta(K) \setminus F$ is contained in exactly $kx(e)$ number of M_i's and N_j's. Each of them contains the entire F. We can pair these b-factors and 'glue' them together to get $kx(e)$ b-factors of G containing the edge e. This can be done for each edge $e \in \delta(K) \setminus F$.

Similarly, for each edge $e \in F$ there are exactly $k(1 - x(e))$ M_i's and N_j's that does not contain e. Notice that these contain all edges in $F - e$ and none in $\delta(K) - F$. Again, pair and glue these together to get b-factors of G not containing e.

These b-factors altogether yield x as a convex combination of b-factors of G, a contradiction.

Remark 3.2. Note that the above proof also gives a new proof of Theorem 1.3 by using the well-known construction given below.

Take a copy of G denoted by G' and for each $v \in V$ add $b(v)$ new edges between v and v'. Let G^* be the graph thus arising and define $b^*(v) = b^*(v') = b(v)$. Theorem 1.3 follows as the restriction of a b^*-matching of G^* to G gives a b-matching in G, and the restriction of the b^*-factor polytope of G^* to G gives exactly the polytope described by P_3.

4 Triangle-free b-factors

In this section, we extend the proof of Theorem 1.2 to Theorem 1.7. Besides shrinking odd pairs, we also need to shrink triangles. The following shrinking operation appeared in [2].

Definition 4.1 (Shrinking a triangle)**.** Assume G, b and $\mathcal{T}$ satisfy (1) and (2). **Shrinking** a triangle $T \in \mathcal{T}$ consists of the following operations:

- replace T by a node t,
- replace each edge $e \in E \setminus E_T$ with $e^u \in V_T, e^v \in V \setminus V_T$ by an edge te^v, and each edge $e \in E \setminus E_T$ with $e^u, e^v \in V_T$ by a loop e on t,
- let $b^\circ(t) = 2$ and define $b^\circ(v) = b(v)$ if $v \neq t$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from T.

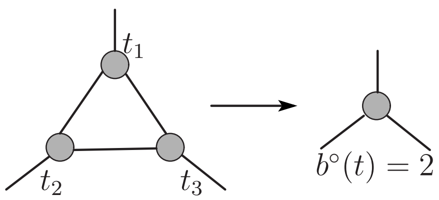

Figure 4: Shrinking a triangle

Similarly to Definition 2.1, we use the notation $G^\circ = (V^\circ, E^\circ)$ for the shrunk graph with $E^\circ \subseteq E$ and $V^\circ - t \subseteq V$. It is easy to see that G°, b° and $\mathcal{T}^\circ$ also satisfy (1) and (2).

Assume that $x \in \mathbb{R}^E$ satisfies (P_7). When shrinking an x-tight triangle, we use the notation x° for the image of x, that is, $x^\circ(e) = x(e)$ for each $e \in E^\circ$.

Lemma 4.2. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (1) and (2). Moreover, assume that there are no two forbidden triangles on the same node-set. Assume that $x \in \mathbb{R}^E$ satisfies (P_7) and $T \in \mathcal{T}$ is a forbidden triangle. Then x° satisfies (P_2) in $G^\circ = (V^\circ, E^\circ)$ with b° and $\mathcal{T}^\circ$.*

Proof. $(i), (iii)$ and (iv) easily follow from the same inequalities in the original graph. Also, (ii) holds for nodes different from t. As T is x-tight, $x^\circ(\dot{\delta}(t)) = x(\delta(V_T)) = \sum x(\dot{\delta}(t_i)) - 2x(E_T) = 2 = b^\circ(t)$. □

Now we turn to the proof of Theorem 1.7. It is clear that a $\mathcal{T}$-free b-factor satisfies $(i) - (iv)$ ((iii) can be verified as in the proof of Theorem 1.2).

It remains to show that $(i) - (iv)$ completely determine the polytope in question, that is, any $x \in \mathbb{R}^E$ satisfying (P_7) is a convex combination of incidence vectors of $\mathcal{T}$-free b-factors. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_7) not contained in the $\mathcal{T}$-free b-factor polytope.

We choose this counterexample in such a way that $(|V|, |E|)$ is lexicographically minimal. This immediately implies that $\mathcal{T} = \emptyset$. Indeed, if there is a triangle $T \in \mathcal{T}$ then it is automatically tight, that is, $x(E_T) = 2$. Shrink T to a single node t as in Definition 4.1, obtaining G°, b°, $\mathcal{T}^\circ$, x°. By Lemma 4.2, these satisfy (P_7). As $|V^\circ| < |V|$, x° is a convex combination of $\mathcal{T}^\circ$-free b°-factors M_i of G°. Note that $b^\circ(t) = 2$ and $d_{G^\circ}(t) \leq 3$ follows by (1). Let $x^\circ = \frac{1}{k}\sum \lambda_i \chi_{M_i^\circ}$. For each i, $|M_i^\circ \cap \delta(t)| = 2$. Moreover, $|M_i^\circ \cap \delta(t_j)| \leq 1$ for $j = 1, 2, 3$. We extend M_i° to a $\mathcal{T}$-free b-matching of G as follows: if $|M_i^\circ \cap \delta(t_j)| = |M_i^\circ \cap \delta(t_{j+1})| = 1$ (indices are meant modulo 3) then $M_i = M_i^\circ \cup \{e^T_{j,j+2}, e^T_{j+1,j+2}\}$.

Proposition 4.3. *M_i is a $\mathcal{T}$-free b-factor of G.*

Proof. Assume that $|M_i^\circ \cap \delta(t_1)| = |M_i^\circ \cap \delta(t_2)| = 1$. M_i cannot contain a triangle in $\mathcal{T}^\circ$, and neither contains T due to the construction. It suffices to check that it does not contain a triangle $T' \in \mathcal{T}$ which shares a node with T. By (1), T and T' must have an edge in common. If the common edge is e^T_{12}, then M_i does not contain T' since $e^T_{12} \notin M_i$. If the common edge is e^T_{13} then $e^T_{13}, e^T_{23} \in M_i$ and (2) implies that the edge of T' not incident to t_1 is not in M_i. The same argument works if the common edge of T and T' is e^T_{23}. □

As $b(t_j) = 2$ for $j = 1, 2, 3$ and $x(E_T) = 2$, an easy computation shows that $x(e^T_{j,j+1}) = x(\dot{\delta}(t_{j+2}) \setminus E_T)$. This implies that $x = \frac{1}{k}\sum \chi_{M_i}$, a contradiction. So $\mathcal{T} = \emptyset$ indeed holds and the theorem follows from Theorem 1.2.

5 Extending the shrinking operations

Theorem 1.7 turned out to easily follow from Theorem 1.2 thanks to the fact that a forbidden triangle is always tight if (1) and (2) hold. Not surprisingly, the latter does not hold for b-matchings. In this section, we extend the notion of shrinking to triplets. To prove Theorem

1.8, we also need to slightly modify the notion of shrinking a triangle. We start with the latter one.

Definition 5.1 (Shrinking a triangle - extended). Assume G, b and $\mathcal{T}$ satisfy (1) and (2). **Shrinking** a triangle $T \in \mathcal{T}$ consists of the following operations:

- replace T by two nodes t, t',
- replace each edge $e \in E \setminus E_T$ with $e^u \in V_T, e^v \in V \setminus V_T$ by an edge te^v, and each edge $e \in E \setminus E_T$ with $e^u, e^v \in V_T$ by a loop e on t,
- add three edges between t and t' denoted by g_1, g_2 and g_3,
- let $b^\circ(t) = 2$, $b^\circ(t') = 2$ and define $b^\circ(v) = b(v)$ if $v \neq t, t'$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from T.

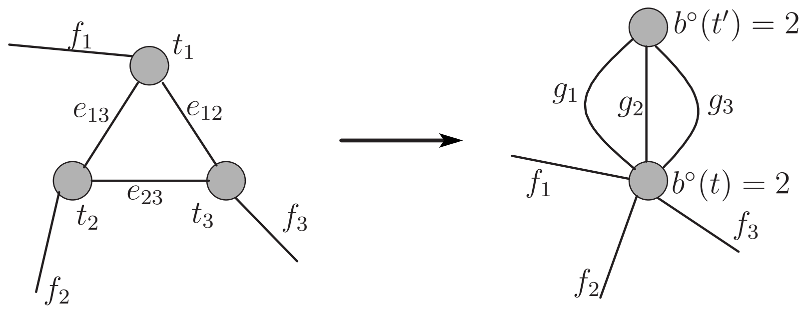

Figure 5: Shrinking a triangle - extended

We use the notation $G^\circ = (V^\circ, E^\circ)$ for the shrunk graph with $E^\circ \setminus \{g_1, g_2, g_3\} \subseteq E$ and $V^\circ \setminus \{t, t'\} \subseteq V$. It is easy to see that G°, b° and $\mathcal{T}^\circ$ also satisfy (1) and (2).

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). A triangle $T \in \mathcal{T}$ is called x-**tight** if it satisfies (iv) with equality. Let $T \in \mathcal{T}$ be a tight triangle with $V_T = \{t_1, t_2, t_3\}$ and $\delta(t_1) \setminus E_T = f_1$, $\delta(t_2) \setminus E_T = f_2$ and $\delta(t_3) \setminus E_T = f_3$ (two of these edges may coincide). When shrinking T, we use the notation x° for the image of x, namely

$$x^\circ(e) = \begin{cases} x(e) & \text{if } e \in E^\circ \setminus E^\circ[t, t'], \\ x(e^T_{i+1,i+2}) - x(f_i) & \text{if } e = g_i \text{ for } i = 1, 2, 3. \end{cases}$$

Remark 5.2. In case of x being a b-factor, $x(g_i) = 0$ for each i, making the presence of edges g_1, g_2, g_3 unnecessary. That is the reason for the simpler definition of shrinking a triangle when proving Theorem 1.7.

Lemma 5.3. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (1) and (2). Assume that $x \in \mathbb{R}^E$ satisfies (P_8) and T is an x-tight triangle. Then x° satisfies (P_8) in $G^\circ = (V^\circ, E^\circ)$ with b° and $\mathcal{T}^\circ$.*

Proof. Let $V_T = \{t_1, t_2, t_3\}$ and $\delta(t_1) \setminus E_T = f_1$, $\delta(t_2) \setminus E_T = f_2$ and $\delta(t_3) \setminus E_T = f_3$ again. Then $(i), (iv)$ and (v) easily follow from the same inequalities in the original graph and from $x(g_i) = x(e^T_{i+1,i+2}) - x(f_i) \geq 0$. Also, (ii) holds for nodes different from t and t'. Clearly, $x^\circ(\dot{\delta}(t)) = x(E_T) = 2 = b^\circ(t)$. As for t', $x^\circ(\dot{\delta}(t')) = x(E_T) - \sum_i x(\delta(t_i) \setminus E_T) \leq 2 = b^\circ(t')$.

Concerning (iii), for a triplet $(Z, H, \mathfrak{R})$ with $Z \subseteq V^\circ, H \subseteq \delta(Z), \mathfrak{R} \subseteq \mathcal{T}^\circ$ the required inequality follows from the same inequality for $(Z \setminus \{t, t'\}, H \setminus (\delta(t) \cup \delta(t'), \mathfrak{R})$ in the original graph. $\square$

As mentioned earlier, forbidden triangles are not automatically tight in case of b-matchings. This phenomenon lead us to extend the notion of shrinking to more complex structures than odd pairs, namely to triplets, already introduced in Section 1.

Definition 5.4 (Shrinking a triplet of Type 1)**.** **Shrinking** a triplet $(K, F, \mathfrak{T})$ of Type 1 consists of the following operations:

- replace K by an edge pq with $b^\circ(p) = |F| + |\mathfrak{T}|$ and $b^\circ(q) = 1$,
- replace each triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u\}$ by edges $pr_T, r_Ts_T, r_Tt_T, s_Tv, t_Tw$ where r_T, s_T and t_T are new nodes with $b^\circ(r_T) = 2, b^\circ(s_T) = b^\circ(t_T) = 1$, and we also set $b^\circ(v) = b^\circ(w) = 1$,
- define $b^\circ(v) = b(v)$ for each $v \in V \setminus (K \cup V_{\mathfrak{T}})$,
- replace each edge $e \in E$ with $e^u \in K, e^v \in V \setminus K$ by an edge pe^v if $e \in F$, and by qe^v if $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from $K \cup V_{\mathfrak{T}}$.

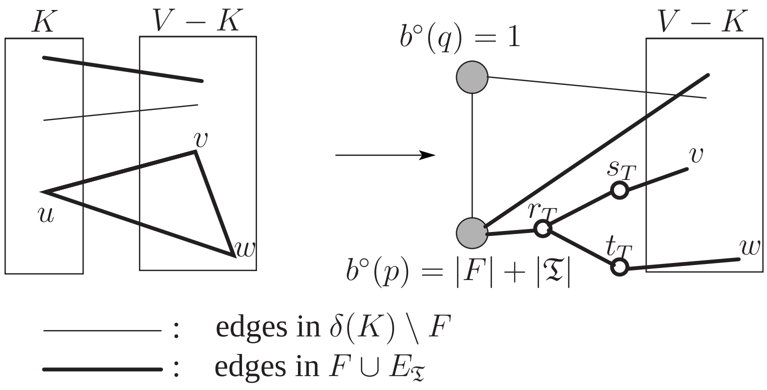

Figure 6: Shrinking a triplet of Type 1

We usually denote the graph obtained by shrinking a triplet of Type 1 by $G^\circ = (V^\circ, E^\circ)$. By abuse of notation, each edge $e \in \delta(K) \setminus E_{\mathfrak{T}}$ is denoted by e again after shrinking the triplet and is called the **image** of the original edge. Hence the intersection $E \cap E^\circ$ stands for the set of all edges not induced by K nor by a triangle in $\mathfrak{T}$.

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). When shrinking a triplet of Type 1, we use the notation x° for the image of x, namely

- for an edge $e \in E \cap E^\circ$ let $x^\circ(e) = x(e)$,
- for a triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u\}$ consider the new edges mentioned in Definition 5.4, and define

$$\begin{aligned}
x^\circ(pr_T) &= 2x(e^T_{vw}) + x(e^T_{uv}) + x(e^T_{uw}) - 2,\\
x^\circ(r_T s_T) &= 2 - x(e^T_{vw}) - x(e^T_{uv}),\\
x^\circ(r_T t_T) &= 2 - x(e^T_{vw}) - x(e^T_{uw}),\\
x^\circ(s_T v) &= x(e^T_{vw}) + x(e^T_{uv}) - 1,\\
x^\circ(t_T w) &= x(e^T_{vw}) + x(e^T_{uw}) - 1,
\end{aligned}$$

- define $x^\circ(pq) = |F| + 3|\mathfrak{T}| - x(F) - \sum_{T\in\mathfrak{T}} x(E_T) - \sum_{T\in\mathfrak{T}} x(e_T)$.

Recall that e_T denotes the special edge of triangle T, that is, the edge in E_T having no end in K.

Definition 5.5 (Shrinking an odd triplet of Type 2)**.** **Shrinking** a triplet $(K, F, \mathfrak{T})$ of Type 2 consists of the following operations:

- replace K by an edge pq with $b^\circ(p) = |F| + |\mathfrak{T}|$ and $b^\circ(q) = 1$,
- replace each triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u, v\}$ by an edge pr_T, a loop l_T on r_T, and two parallel edges between r_T and w_T (denoted by $r_T w_1$ and $r_T w_2$) where r_T is a new node with $b^\circ(r) = 2$,
- define $b^\circ(v) = b(v)$ for each $v \in V \setminus K$,
- replace each edge edge $e \in E$ with $e^u \in K, e^v \in V \setminus K$ by an edge pe^v if $e \in F$, and by qe^v if $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$,
- let $\mathcal{T}^\circ$ denote the set of triangles in $\mathcal{T}$ node-disjoint from K.

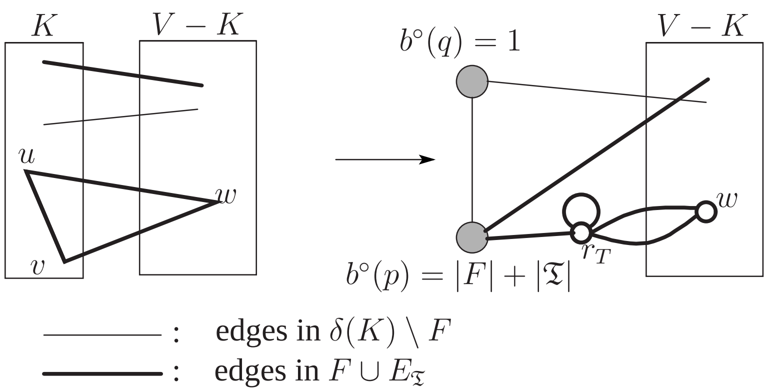

Figure 7: Shrinking a triplet of Type 2

We usually denote the graph obtained by shrinking a triplet of Type 2 by $G^\circ = (V^\circ, E^\circ)$. Again, by abuse of notation, each edge $e \in \delta(K) \setminus E_{\mathfrak{T}}$ is denoted by e again after shrinking the triplet.

Assume that $x \in \mathbb{R}^E$ satisfies (P_8). When shrinking a triplet of Type 2, we use the notation x° for the image of x, namely

- for an edge $e \in E \cap E^\circ$ let $x^\circ(e) = x(e)$,
- for a triangle $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$ and $V_T \cap K = \{u, v\}$ consider the new edges mentioned in Definition 5.5, and define

$$\begin{aligned} x^\circ(pr_T) &= 2x(e^T_{uv}) + x(e^T_{vw}) + x(e^T_{uw}) - 2, \\ x^\circ(l_T) &= 2 - x(e^T_{uv}) - x(e^T_{vw}) - x(e^T_{uw}), \\ x^\circ(r_T w_1) &= x(e^T_{uw}), \\ x^\circ(r_T w_2) &= x(e^T_{vw}), \end{aligned}$$

- define $x^\circ(pq) = |F| + 3|\mathfrak{T}| - x(F) - \sum_{T \in \mathfrak{T}} x(E_T) - \sum_{T \in \mathfrak{T}} x(e_T)$.

Recall that e_T denotes the special edge of triangle T, that is, the edge in E_T having both ends in K.

An odd triplet of Type 2 $(K, F, \mathfrak{T})$ is called x**-tight** (or tight, for short) if it satisfies (iii) with equality. If $\mathfrak{T} = \emptyset$ then (K, F) is called a **tight pair** instead.

The following simple observation will be useful later.

Proposition 5.6. *Let $(K, F, \mathfrak{T})$ be an x-tight triplet of any type for some $0 < x < 1$. Then for any $F' \subseteq F, \mathfrak{T}' \subseteq \mathfrak{T}, \mathfrak{T}'' \subseteq \mathfrak{T}$ and $H \subseteq \delta(K) \setminus (F \cup E_{\mathfrak{T}})$ we have*

$$x(H) \leq 1$$

and

$$|F| + 2|\mathfrak{T}'| + |\mathfrak{T}''| - 1 \leq x(F) + \sum_{T \in \mathfrak{T}'} x(E_T) + \sum_{T \in \mathfrak{T}''} x(e_T) \leq |F| + 2|\mathfrak{T}'| + |\mathfrak{T}''|.$$

Moreover, if at least one of F and $\mathfrak{T}''$ is nonempty then the upper bound hold with strict inequality.

Proof. As $(K, F, \mathfrak{T})$ is x-tight, we have

$$x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) = \tfrac{b(K) + |F| + 3|\mathfrak{T}| - 1}{2}.$$

On the other hand, $2x(E[K]) + x(\delta(K)) \leq b(K)$, together implying the first part of the proposition. By (iv), $\sum_{T \in \mathfrak{T}'} x(E_T) \leq 2|\mathfrak{T}'|$ hold. For nonempty F' and $\mathfrak{T}''$ we have $x(F') < |F'|$ and $\sum_{T \in \mathfrak{T}''} x(e_T) < |\mathfrak{T}''|$ by $x < 1$, proving the upper bound. □

In the sequel, we will refer to the following special case of Proposition 5.6 several times.

Corollary 5.7. *If v is a node without loops and $x(\delta(v)) = b(v) = d(v) - 1$ then $x(F) \geq |F| - 1$ for any $F \subseteq \delta(v)$.*

The main advantage of shrinking odd pairs was that the arising graph and vector still satisfied (P_2). The above definitions also have this useful property, as shown in the following lemma. The proof is rather technical and needs a lot of computation, the reader may skip it in order to follow the main idea of the proof of Theorem 1.8.

Lemma 5.8. *Let $G = (V, E)$, $b : V \to \mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (1) and (2). Assume that $x \in \mathbb{R}^E,\ 0 < x < 1$ satisfies (P_8) and $(K, F, \mathfrak{T})$ is an x-tight triplet of Type 2. Then either shrinking $(K, F, \mathfrak{T})$ or $(\bar{K}, F, \mathfrak{T})$, (1) and (2) hold for $G^\circ = (V^\circ, E^\circ)$. Moreover, $b^\circ, \mathcal{T}^\circ$ and x° satisfies (P_8).*

Proof. The validity of (1) and (2) can be checked easily in both cases. We discuss the second part separately for K and $\bar{K}$.

(I) Shrinking $(\bar{K}, F, \mathfrak{T})$:

We use the notation of Definition 5.5. (i) clearly holds for edges different from pq and not contained in $\delta(K) \cap E_{\mathfrak{T}}$. For the rest of the edges the required inequalities follow from Proposition 5.6. As an example, we show this for pq. We have

$$x(F) + \sum_{T \in \mathfrak{T}} x(E_T) + \sum_{T \in \mathfrak{T}} x(e_T) \leq |F| + 2|\mathfrak{T}| + |\mathfrak{T}| = |F| + 3|\mathfrak{T}|,$$

that is, $x^\circ(pq) \geq 0$. On the other hand,

$$x(F) + \sum_{T \in \mathfrak{T}} x(E_T) + \sum_{T \in \mathfrak{T}} x(e_T) \geq |F| + 2|\mathfrak{T}| + |\mathfrak{T}| - 1 = |F| + 3|\mathfrak{T}| - 1$$

by Proposition 5.6, so $x^\circ(pq) \leq 1$.

The validity of (ii) is straightforward for nodes different from q. However, the tightness of the triplet implies

$$\begin{aligned}
x^\circ(\dot{\delta}(q)) &= x^\circ(pq) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\
&= |F| + 3|\mathfrak{T}| - x(F) - \sum_{T \in \mathfrak{T}} x(E_T) - \sum_{T \in \mathfrak{T}} x(e_T) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\
&= 2x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) + 1 - b(K) \\
&\quad - \sum_{T \in \mathfrak{T}} x(e_T) + x(\delta(K) \setminus (F \cup E_{\mathfrak{T}})) \\
&= 2x(E[K]) + x(\delta(K)) + 1 - b(K) \\
&\leq 1 \\
&= b^\circ(q).
\end{aligned}$$

(iv) and (v) remain valid for triangles in $\mathcal{T}^\circ$ as the same inequalities were true in the original graph. So it remains to show that (iii) is indeed satisfied in G°. Choose an odd triplet $(Z, H, \mathfrak{R})$ of G° with $(\texttt{def}(Z, H, \mathfrak{R}), |\bar{Z} \cup \{p, q\}|, |H|)$ lexicographically maximal. Our aim is to show that $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$, which would prove (iii) for all odd triplets.

Proposition 5.9. *Let $v \in Z \setminus \{p,q\}$ be a node with $l(v) = \emptyset$, $b^\circ(v) = d^\circ(v) - 1$, $x^\circ(\dot\delta(v)) = b^\circ(v)$ and $v \notin V^\circ_{\mathfrak{R}}$. Then $\delta(Z)_v \subseteq H$. If $u \in \{p,q\}$ and $\delta(Z)_u \setminus H \neq \emptyset$ then $H_u = \emptyset$.*

Proof. The conditions on v imply that for any two edges $e, f \in \delta(v)$ we have $x^\circ(e) + x^\circ(f) \geq 1$. If $|\delta(Z)_v \setminus H| \geq 2$ then the addition of two of these edges to H would result in a lexicographically larger triplet, a contradiction.

Assume that $|\delta(Z)_v \setminus H| = 1$. Define $Z' = Z - v$, $H' = (H \setminus H_v) \cup E^\circ[v, Z - v]$. The triplet $(Z', H', \mathfrak{R})$ thus arising has deficiency

$$\begin{aligned}\mathtt{def}(Z', H', \mathfrak{R}) &= \mathtt{def}(Z, H, \mathfrak{R}) - x^\circ(H_v) + \tfrac{b^\circ(v) + |H_v| - |E^\circ[v, Z-v]|}{2} \\ &= \mathtt{def}(Z, H, \mathfrak{R}) - x^\circ(H_v) + \tfrac{b^\circ(v) + |H_v| - d^\circ(v) + |H_v| + 1}{2} \\ &= \mathtt{def}(Z, H, \mathfrak{R}) - x^\circ(H_v) + |H_v|.\end{aligned}$$

That is, the deficiency is not decreased and $|Z \setminus \{p,q\}|$ decreased by 1, a contradiction. The second part immediately follows from $x > 0$. Indeed, the same computation shows that the deficiency would strictly decrease in case of $H_u \neq \emptyset$, contradicting the choice of the triplet. □

Proposition 5.10. *There is no $v \in Z \setminus \{p,q\}$ with $b^\circ(v) = d^\circ(v) - 1 = 1$.*

Proof. The deletion of v from Z decreases $x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E^\circ_T)$ by at most 1 while $\lfloor \frac{1}{2}(b^\circ(Z) + |H| + 3|\mathfrak{R}|) \rfloor$ always decreases by 1 unless $|H_v| = 0$, so we may assume that the latter holds.

If $|\delta(Z)_v| = 2$ then the deletion of v from Z, while if $|\delta(Z)_v| \leq 1$ then the deletion of v from Z and the addition of an edge from $\delta(v) \setminus \delta(Z)$ to H would result in a lexicographically larger triplet, a contradiction. □

The above propositions indicate the following simple but useful observation.

Corollary 5.11. *Let $T \in \mathfrak{T}$ be a triangle with $V_T = \{u, v, w\}$. Then $r_T, s_T, t_T, v, w \notin Z$.*

Corollary 5.11 reduces the number of cases to be checked.

<u>Case 1:</u> $p, q \notin Z$

In this case (iii) holds for $(Z, H, \mathfrak{R})$ as the same inequality is true in the original graph. We use here that, by Corollary 5.11, $(Z, H, \mathfrak{R})$ is contained in G in an unchanged form.

<u>Case 2:</u> $p, q \in Z$

We prove Case 2 with the help of Case 1. First of all note that $|H_p| \geq |\delta(Z)_p| - 1$. To prove this, assume that $|\delta(Z)_p \setminus H_p| \geq 2$. We have $x^\circ(\dot\delta(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$. Hence any two edges incident to p must have x° value together at least 1. The addition of two of these edges to H would result in a lexicographically larger triplet, a contradiction.

We distinguish two subcases.

<u>Subcase 2.1:</u> $\delta(Z)_p = H_p$

If $|H_q| \geq 1$ then let $F_1 = H_p$, $F_2 = \delta(p) \setminus (F_1 + pq)$. Take $Z' = Z \cap K$, $H' = (H \setminus (F_1 \cup H_q)) \cup F_2$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(pq) \\
&\qquad + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q, Z']) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z)-1-|F|-|\mathfrak{T}|+|H|-|F_1|+|F_2|-1+3|\mathfrak{R}|}{2} \rfloor \\
&\qquad + x^\circ(E^\circ[q, Z]) + x^\circ(pq) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor - |F_1| - 1 + x^\circ(E^\circ[q, Z]) + x^\circ(pq) + x^\circ(H_q) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor,
\end{aligned}$$

as $x^\circ(E^\circ[q, Z]) + x^\circ(pq) + x^\circ(H_q) \leq x^\circ(\delta(q)) \leq 1$. This implies $\mathtt{def}(Z, H, \mathfrak{R}) \leq 0$.

Now assume that $|H_q| = 0$. If $Z = \{p, q\}$ then $\mathfrak{R} = \emptyset$ and $H = \delta(p) - pq$. Hence $x^\circ(E^\circ[Z]) + x^\circ(H) = x^\circ(\delta(p)) = |F| + |\mathfrak{T}| \leq \lfloor \frac{|F|+|\mathfrak{T}|+1+|F|+|\mathfrak{T}|}{2} \rfloor = \lfloor \frac{b^\circ(p)+b^\circ(q)+|H|}{2} \rfloor$, so (iii) holds.

So assume that $Z \neq \{p, q\}$ and let $Z' = Z \cap K$. Define $K' = V^\circ \setminus \{p, q\}$ and $F' = \delta(p) - pq$. It is easy to see that the tightness of $(K, F, \mathfrak{T})$ implies the tightness of (K', F'). Using this and that (iii) holds if $Z = \{p, q\}$, we have the following

$$\begin{aligned}
&x^\circ(E^\circ[K']) + x^\circ(F') + E^\circ[Z] + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[K' \setminus Z]) + x^\circ(E^\circ[Z \setminus K']) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(F') \\
&\qquad + 2x^\circ(E^\circ[Z']) + x^\circ(E^\circ[K' \setminus Z', Z']) + x^\circ(E^\circ[\{p, q\}, Z']) \\
&\quad \leq \lfloor \tfrac{b^\circ(K'\setminus Z)+|H|+3|\mathfrak{R}|}{2} \rfloor + \lfloor \tfrac{b^\circ(Z\setminus K')+|F'|}{2} \rfloor + 2x^\circ(E^\circ[Z']) + x^\circ(\delta(Z')) \\
&\quad = \tfrac{b^\circ(K')+|F'|-1}{2} + \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2} - b^\circ(Z') + 2x^\circ(E^\circ[Z']) + x^\circ(\delta(Z')) \\
&\quad \leq \tfrac{b^\circ(K')+|F'|-1}{2} + \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2}.
\end{aligned}$$

The tightness of (K', F') implies $\mathtt{def}(Z, H, \mathfrak{R}) \leq 0$. In the proof we used that $(K' \setminus Z, H, \mathfrak{R})$ and $(Z \setminus K', F')$ are also odd. This can be seen by $b^\circ(K' \setminus Z) + |H| + 3|\mathfrak{R}| = b^\circ(K') - b^\circ(Z) + 1 + |F'| + |H| + |\mathfrak{R}|$ which is odd as (K', F') and $(Z, H, \mathfrak{R})$ are odd, and $b^\circ(Z \setminus K') + |F'| = 1 + 2|F'|$.

Subcase 2.2: $|\delta(Z)_p| = |H_p| + 1$

By Proposition 5.9, $H_p = \emptyset$. Let $\delta(Z)_p = f$ and $F_2 = \delta(p) - f$. Take $Z' = Z \cap K$, $H' = (H \setminus \delta(q)) \cup F_2$. Then

$$
\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q) \\
&\quad = \lfloor \tfrac{b^\circ(Z)-1-|F|-|\mathfrak{T}|+|H|+|F_2|+3|\mathfrak{R}|}{2} \rfloor + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q) \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor - 1 + x^\circ(pq) + x^\circ(E^\circ[q,Z'])) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor,
\end{aligned}
$$

as $x^\circ(pq) + x^\circ(E^\circ[q,Z]) + x^\circ(H_q) \le x^\circ(\dot\delta(q)) \le 1$. This implies $\texttt{def}(Z,H,\mathfrak{R}) \le 0$.

<u>Case 3:</u> $p \in Z, q \notin Z$

If $pq \in H$, then add q to Z and delete H_q - including pq - from H. We have previously seen that the triplet $(Z', H', \mathfrak{R})$ thus obtained satisfies (iii), so

$$
\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) - x^\circ(E^\circ[q,Z]) + x^\circ(H_q) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+1+|H|-1+3|\mathfrak{R}|}{2} \rfloor \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}
$$

This implies $\texttt{def}(Z,H,\mathfrak{R}) \le 0$.

If $pq \notin H$, then first consider the case when $\delta(Z)_p \setminus (H_p + pq) \neq \emptyset$. Let f be an edge in this set. Define again $Z' = Z + q$, delete H_q from H and add f to it. For the new triplet $(Z', H', \mathfrak{R})$, we have

$$
\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(H_q) - x^\circ(E^\circ[q,Z]) - x^\circ(f) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(f) \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+1+|H|+3|\mathfrak{R}|}{2} \rfloor - 1 \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}
$$

For the last inequality, we used Corollary 5.7 ($x^\circ(\dot\delta(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$, hence pq and f, two edges incident to p must have x° value together at least 1). This implies $\texttt{def}(Z,H,\mathfrak{R}) \le 0$.

If $\delta(Z)_p \setminus (H_p + pq) = \emptyset$, then let $F_1 = H_p - pq$, $F_2 = \delta(p) \setminus (H + pq)$. Define $Z' = Z - p$, $H' = (H \setminus F_1) \cup F_2$. Note that $(Z', H', \mathfrak{R})$ is odd since $b^\circ(Z') + |H'| + |\mathfrak{R}| = b^\circ(Z) + |H| - |F| - |\mathfrak{T}| - |F_1| + |F_2| + |\mathfrak{R}| = b^\circ(Z) + |H| + |\mathfrak{R}| - 2|F_1|$. Hence

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) - |F| - |\mathfrak{T}| + |H| - |F_1| + |F_2| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| - 2|F_1| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(F_1) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

<u>**Case 4:**</u> $p \notin Z, q \in Z$

If $H_q \neq \emptyset$, then delete q from Z and H_q from H. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(E^\circ[q, Z - q]) + x^\circ(H') + x^\circ(H_q) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(\delta(q)) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) - 1 + |H| - 1 + 3|\mathfrak{R}|}{2} \rfloor + 1 \\
&\quad = \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor.
\end{aligned}$$

This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $H_q = \emptyset$, then first consider the case when $E^\circ[p, Z - q] \setminus H \neq \emptyset$. Let f be an edge in this set. Delete q from Z and take $H' = H + f$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) \\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T \in \mathfrak{R}} x^\circ(E_T^\circ) + x^\circ(E^\circ[q, Z - q]) - x^\circ(f) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z') + |H'| + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(E_1^\circ[q, Z - q]) - x^\circ(f) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) - 1 + |H| + 1 + 3|\mathfrak{R}|}{2} \rfloor + x^\circ(\dot{\delta}(q)) - x^\circ(pq) - x^\circ(f) \\
&\quad \leq \lfloor \tfrac{b^\circ(Z) + |H| + 3|\mathfrak{R}|}{2} \rfloor
\end{aligned}$$

by Proposition 5.7. This implies $\texttt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $E^\circ[p, Z - q] \setminus H = \emptyset$ then let $F_1 = H_p - pq$ and $F_2 = \delta(p) \setminus (H + pq)$. Define

$Z' = Z + p$ and $H' = (H \setminus F_1) \cup F_2$. For the triplet $(Z', H', \mathfrak{R})$

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) \\
&\quad = x^\circ(E^\circ_1[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) - x^\circ(pq) - x^\circ(F_2) \\
&\quad \le \lfloor \tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(F_2) \\
&\quad = \lfloor \tfrac{b^\circ(Z)+|F|+|H|-|F_1|+|F_2|+3|\mathfrak{R}|}{2} \rfloor - x^\circ(pq) - x^\circ(F_2) \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor + |F_2| - x^\circ(pq) - x^\circ(F_2) \\
&\quad \le \lfloor \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2} \rfloor
\end{aligned}$$

by Proposition 5.7. This implies $\texttt{def}(Z, H, \mathfrak{R}) \le 0$.

(II) Shrinking $(K, F, \mathfrak{T})$:

The verification of $(i), (ii), (iv)$ and (v) goes in the same way as in the previous case. Choose an odd triplet $(Z, H, \mathfrak{R})$ of G° with $(\texttt{def}(Z, H, \mathfrak{R}), |\bar{Z} \cup \{p, q\}|, |H|)$ lexicographically maximal. We start again with some technical propositions. These are only easy observations but they greatly help us to reduce the number of cases to be checked.

Proposition 5.12. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. Then $x(e^T_{uv}) + x(e^T_{uw}) \ge 1$ and $x(e^T_{uv}) + x(e^T_{vw}) \ge 1$.*

Proof. Assume that one of the mentioned sums, say $x(e^T_{uv}) + x(e^T_{uw})$, is strictly less than 1. Then $(K, F + e^T_{uw}, \mathfrak{T} - T)$ violates (iii), a contradiction. □

Proposition 5.13. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If both $p, w \notin Z$ then $r_T \notin Z$.*

Proof. It is easy to see, by using Proposition 5.12, that otherwise $(Z - r_T, H \setminus H_{r_T}, \mathfrak{R})$ would be a lexicographically larger triplet, a contradiction. □

Proposition 5.14. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If $p, w \in Z$ then $r_T \in Z$.*

Proof. It is easy to see, by using Proposition 5.12, that otherwise $(Z + r_T, H \setminus H_{r_T}, \mathfrak{R})$ would be a lexicographically larger triplet, a contradiction. □

Proposition 5.15. *Let $T \in \mathfrak{T}$ with $V_T = \{u, v, w\}$, $V_T \cap K = \{u, v\}$. If $p \notin Z$ and $w \in Z$ then $r_T \notin Z$.*

Proof. Let $wz = \delta(w) \setminus E_T$ and assume indirectly that $r_T \in Z$. If $pr_T \notin H$ then $(Z - r_T, H, \mathfrak{R})$, if $pr_T \in H$ and $z \in Z$ then $(Z - r_T - w, H - pr_T + wz, \mathfrak{R})$, and finally if $pr_T \in H$ and $z \notin Z$ then $(Z - r_T - w, H - pr_T, \mathfrak{R})$ has deficiency at most $\texttt{def}(Z, H, \mathfrak{R})$ and smaller $|Z|$, a contradiction. □

Case 1: $p, q \notin Z$

By Propositions 5.13 and 5.15, $r_T \notin Z$ for $T \in \mathfrak{T}$. Hence (iii) follows from the same inequality in the original graph.

Case 2: $p, q \in Z$

For a forbidden triangle $T \in \mathfrak{T}$ let $V_T = \{u_T, v_T, w_T\}$ with $u_T, v_T \in K$. Define

$$\begin{aligned}
\mathfrak{T}_1 &= \{T \in \mathfrak{T} :\ r_T, w_T \in Z\},\\
\mathfrak{T}_2 &= \{T \in \mathfrak{T} :\ r_T \in Z, w_T \notin Z\},\\
\mathfrak{T}_3 &= \{T \in \mathfrak{T} :\ r_T, w_T \notin Z,\ pr_T \in H\},\\
\mathfrak{T}_4 &= \{T \in \mathfrak{T} :\ r_T, w_T \notin Z,\ pr_T \notin H\}.
\end{aligned}$$

Propositions 5.13, 5.14 and 5.15 imply $\mathfrak{T} = \mathfrak{T}_1 \cup \mathfrak{T}_2 \cup \mathfrak{T}_3 \cup \mathfrak{T}_4$. However, $|\mathfrak{T}_4| \leq 1$. Indeed, $x^\circ(\dot{\delta}(p)) = |F| + |\mathfrak{T}|$, and the degree of p is $|F| + |\mathfrak{T}| + 1$, so any two edges incident to p must have x° value together at least 1. If $|\delta(Z)_p \setminus H_p| \geq 2$, then the addition of two edges from this set to H would not decrease the deficiency of the triplet, not increase $|Z|$ but increase $|H|$, a contradiction.

If $\mathfrak{T}_4 = \emptyset$ then let $S = K \cup (Z \cap \bar{K})$, $I = \{u_T w_T :\ r_T w_{T_1} \in H\} \cup \{v_T w_T :\ r_T w_{T_2} \in H\} \cup (H \cap E)$ and $\mathfrak{P} = \mathfrak{R} \cup \mathfrak{T}_3$. Then

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T \in \mathfrak{R}} x^\circ(E^\circ_T)\\
&\quad = x(E[S]) + x(I) + \sum_{T \in \mathfrak{P}} x(E_T) - x(E[K]) + x^\circ(pq) + \sum_{T \in \mathfrak{T}_1 \cup \mathfrak{T}_2 \cup \mathfrak{T}_3} x(e_T)) - 2|\mathfrak{T}_3|\\
&\quad = x(E[S]) + x(I) + \sum_{T \in \mathfrak{P}} x(E_T) - x(E[K]) + |F| + 3|\mathfrak{T}|\\
&\qquad - x(F) - \sum_{T \in \mathfrak{T}} x(E_T) - 2|\mathfrak{T}_3|\\
&\quad \leq \left\lfloor \tfrac{b(S)+|I|+3|\mathfrak{P}|}{2} \right\rfloor - \tfrac{b(K)-|F|-3|\mathfrak{T}|-1}{2} - 2|\mathfrak{T}_3|\\
&\quad = \tfrac{b(K)+b^\circ(Z)-1-|F|-|\mathfrak{T}|-2|\mathfrak{T}_1 \cup \mathfrak{T}_2|+|H|-|\mathfrak{T}_3|+3|\mathfrak{R}|+3|\mathfrak{T}_3|-1}{2} - \tfrac{b(K)-|F|-3|\mathfrak{T}|-1}{2} - 2|\mathfrak{T}_3|\\
&\quad = \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2} - |\mathfrak{T}_1 \cup \mathfrak{T}_2 \cup \mathfrak{T}_3| + |\mathfrak{T}|\\
&\quad = \tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-1}{2}.
\end{aligned}$$

This implies $\mathtt{def}(Z, H, \mathfrak{R}) \leq 0$.

If $|\mathfrak{T}_4| = 1$ then take $Z' = Z \cap (\bar{K} \cup \{r_T :\ T \in \mathfrak{T}\})$, $F_2 = \{pr_T :\ T \in \mathfrak{T}_2\}$ and $H' = (H \setminus H_q) \cup F_2$. Thus

$$\begin{aligned}
&x^\circ(E^\circ[Z]) + x^\circ(H) + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T)\\
&\quad = x^\circ(E^\circ[Z']) + x^\circ(H') + \sum_{T\in\mathfrak{R}} x^\circ(E^\circ_T) + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q)\\
&\quad \le \lfloor\tfrac{b^\circ(Z')+|H'|+3|\mathfrak{R}|}{2}\rfloor + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q)\\
&\quad \le \lfloor\tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|-|F^\circ|-1+|F_2|}{2}\rfloor + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q)\\
&\quad \le \lfloor\tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2}\rfloor - 1 + x^\circ(pq) + x^\circ(E^\circ[q,Z']) + x^\circ(H_q)\\
&\quad \le \lfloor\tfrac{b^\circ(Z)+|H|+3|\mathfrak{R}|}{2}\rfloor.
\end{aligned}$$

This implies $\mathtt{def}(Z,H,\mathfrak{R}) \le 0$.

Case 3: $p \notin Z, q \in Z$ The proof of this case, by using the above propositions, goes exactly the same way as in case $(I)/3$.

Case 4: $p \in Z, q \notin Z$ The proof of this case, by using the above propositions, goes exactly the same way as in case $(I)/4$. □

Remark 5.16. In the above, we only defined shrinking for triplets either of Type 1 or 2. The definition could be easily generalized to shrink gadgets with both triangles 1-fitting and 2-fitting them. The reason for not introducing shrinking in that way was the form of description (P_8).

6 Proof of Theorem 1.8

It is easy to see that each $\mathcal{T}$-free b-matching satisfies (i), (ii), (iv) and (v). To show that (iii) indeed holds for a $\mathcal{T}$-free b-matching $M \subseteq E$, take an odd triplet $(K,F,\mathfrak{T})$ and add up inequalities $d_M(v) \le b(v)$ for $v \in K$, $|M\cap F| \le |F|$, $|M \cap E_T| \le 2$ and $|M \cap e_T| \le 1$ for $T \in \mathfrak{T}$. This gives

$$2|M\cap E[K]| + |M\cap\delta(K)| + |M\cap F| + \sum_{T\in\mathfrak{T}}(|M\cap E_T| + |M\cap e_T|) \le b(K) + |F| + 3|\mathfrak{T}|.$$

Clearly, $|M\cap F| + |M\cap E_{\mathfrak{T}}| \le |M\cap\delta(K)| + \sum_{T\in\mathfrak{T}}|M\cap e_T|$, so $|M\cap E[K]| + |M\cap F| + \sum_{T\in\mathfrak{T}}|M\cap E_T| \le \lfloor\frac{1}{2}(b(K)+|F|+3|\mathfrak{T}|)\rfloor$, as required. The above proof easily implies that (iii) is also valid for even triplets.

It remains to show that $(i)-(v)$ completely determine the $\mathcal{T}$-free b-matching polytope, that is, any $x \in \mathbb{R}^E$ satisfying (P_8) is a convex combination of incidence vectors of $\mathcal{T}$-free b-matchings. Assume that this does not hold. Let us choose x to be a vertex of the polytope described by (P_8) not contained in the $\mathcal{T}$-free b-matching polytope.

We choose this counterexample in such a way that $(|\mathcal{T}|, l(V), b(V), |V|, |E|)$ is lexicographically minimal. G is connected, otherwise one of its components would be a smaller counterexample. As x is a vertex, it satisfies $|E|$ linearly independent constraints among

(P_8) with equality. We call a node, a triplet or a triangle x-**tight** if the corresponding inequality, which is of type (ii), (iii) or (iv), respectively, is satisfied with equality. Also, the corresponding inequality is called x-tight. We also use this notation for even triplets satisfying (iii) with equality.

From now on, our aim is to show that there is a tight triplet or triangle whose shrinking results in a lexicographically smaller problem. Then we show that a proper convex combination for the smaller problem can be transformed into a convex combination for the original problem giving x, thus leading to contradiction.

Proposition 6.1. *For each $T \in \mathcal{T}$, V_T does not span parallel edges.*

Proof. Assume to the contrary that $V_T = \{u, v, w\}$ spans parallel edges, say between v and w. By (1), $d(u), d(v), d(w) \leq 3$. We claim that G is in fact consists of these three nodes, or these three nodes plus an edge incident to u. Indeed, $d(u) \leq 3$ implies that if $|V| \geq 4$ then u has a third neighbour different from v and w, say z, and uz is a cutting edge in G. Let G_1 and G_2 denote the graphs consisting of a component of $G - uz$ plus uz. We denote by $x_1, b_1, \mathcal{T}_1$ and $x_2, b_2, \mathcal{T}_2$ the natural restriction of x, b and $\mathcal{T}$ to G_1 and G_2, respectively. If both of these graphs have at least two nodes then x_i is a convex combination of $\mathcal{T}_i$-free b_i-matchings of G_i. These could be glued together as to get a convex combination of $\mathcal{T}$-free b-matchings of G giving x, a contradiction.

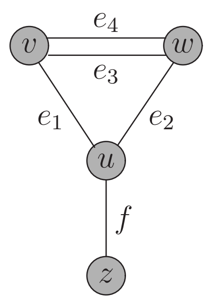

Figure 8: V_T spanning parallel edges

So G is in fact consists of four or three nodes. Let us consider the first case, the second can be handled similarly (by using (v) of (P_8). We use the notation of Figure 8. First assume that both triangles are forbidden. Delete z from G. The graph thus arising is not a counterexample, hence the restriction of x to $G - z$ is a convex combination of $\mathcal{T}$-free b-matchings of $G - z$. Let $\frac{1}{k}\sum \chi_{M_i}$ denote this combination and let $\lambda_I = \frac{1}{k}|\{i : M_i = \{e_j : j \in I\}\}|$ for $I \subseteq \{1, 2, 3, 4\}$. Moreover, take a convex combination with λ_{12} as small as possible. That means that $\lambda_{12} = 0$ or $\lambda_3 = \lambda_4 = \lambda_{34} = 0$. If $\lambda_{12} = 0$ then f can be added to any of these b-matchings, a contradiction. So $\lambda_3 = \lambda_4 = \lambda_{34} = 0$ and $\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{23}+\lambda_{24}+\lambda_1+\lambda_2 = 1$. If $\lambda_{12} \leq 1-x_f$ then we can add the edge f to some of these b-matchings with total coefficients x_f and so get a proper convex combination in the original graph, a contradiction. Hence $x(\delta(v)) = x_f + 2\lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} + \lambda_{34} + \mu_1 + \mu_2 > 2$, a contradiction.

Now assume that only one of the triangles, say $\{e_1, e_2, e_3\}$, is forbidden. Delete z from G. The graph thus arising is not a counterexample, hence the restriction of x to $G - z$ is a convex combination of $\mathcal{T}$-free b-matchings of $G - z$. Let $\frac{1}{k}\sum \chi_{M_i}$ denote this combination and let $\lambda_I = \frac{1}{k}|\{i : M_i = \{e_j : j \in I\}\}|$ for $I \subseteq \{1,2,3,4\}$. Moreover, take a convex combination with λ_{12} as small as possible, and beside this, λ_{124} as small as possible. That means that $\lambda_{12} = 0$ or $\lambda_3 = \lambda_4 = \lambda_{34} = 0$, and also $\lambda_{124} = 0$ or $\lambda_3 = \lambda_4 = \lambda_{34} = 0$. If both $\lambda_{12} = \lambda_{124} = 0$ then f can be added to any of these b-matchings, a contradiction. If at least one of λ_{12} and λ_{124} is greater than 0 then if $\lambda_{12} + \lambda_{124} \leq 1 - x_f$ then we can add the edge f to some of these b-matchings with total coefficients x_f and so get a proper convex combination in the original graph, a contradiction. If $\lambda_{12} + \lambda_{124} > 1 - x_f$ then $x(\dot{\delta}(v)) = x_f + 2\lambda_{12} + 2\lambda_{124} + \lambda_{13} + \lambda_{14} + \lambda_{23} + \lambda_{24} + \lambda_1 + \lambda_2 > 2$, a contradiction. $\square$

Proposition 6.2. $0 < x_e < 1$ *for each* $e \in E$.

Proof. Clearly, edges with $x_e = 0$ could be deleted, contradicting minimality.

If $x_e = 1$ and $\mathcal{T} = \emptyset$, delete e and decrease b on its end-nodes by 1 (if e is a loop on v then decrease $b(v)$ by 2). However, the situation is more complicated if $\mathcal{T} \neq \emptyset$. If $e \in E_T$ for some $T \in \mathcal{T}$, it may happen that there is a proper convex combination in the smaller graph, but it can not be extended to the original problem because a triangle may arise. Hence we use a simple trick here to show $x_e < 1$.

Assume that $x_{uv} = 1$ and let $\mathcal{T}_{uv} \subseteq \mathcal{T}$ denote the set of triangles containing uv (there are at most two such triangles as (1) holds). Note that the edge uv is well-defined as there exist no parallel edges between u and v by Proposition 6.1. For a triangle $T \in \mathcal{T}_{uv}$, let t_T denote its third node.

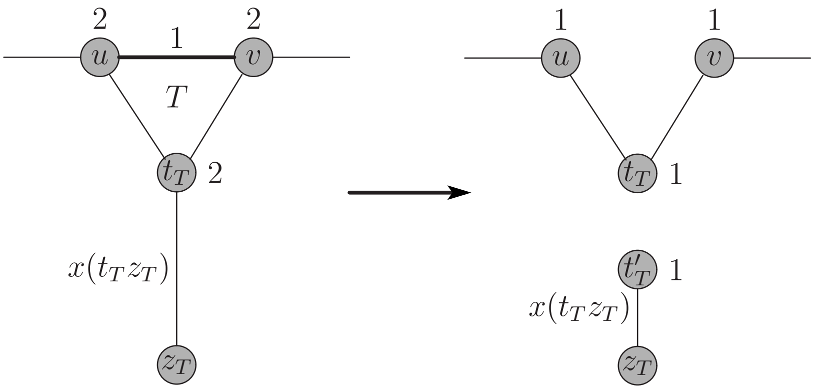

Figure 9: Excluding saturated edges

By (1), t_T has at most one neighbour different from u and v, denoted by z_T (if exists). Delete e from G, decrease $b(u)$ and $b(v)$ by one, for each $T \in \mathcal{T}_{uv}$ decrease $b(t_T)$ by one, delete -if exists- $t_T z_T$ and add a new edge $t'_T z_T$ where t'_T is a new node. The graph thus arising will be denoted by $G' = (V', E')$. The modified degree prescription is denoted by b' (with $b'(t'_T) = 1$ for a new node) and the natural image of x on E' is denoted by x' (that is, $x'(t'_T z_T) = x(t_T z_T)$). Let $\mathcal{T}' \subseteq \mathcal{T}$ denote the set of triangles disjoint from the triangles in $\mathcal{T}_{uv}$. The degree condition implies that two triangles are node-disjoint if and only if they are edge-disjoint. It is easy to check that x' satisfies (P_8) in G' with b' and $\mathcal{T}'$.

As $|\mathcal{T}'| < |\mathcal{T}|$, x' is a convex combination of incidence vectors of $\mathcal{T}'$-free b'-matchings of G', say $x' = \frac{1}{k}\sum \chi_{M'_i}$. These b'-matchings use at most one of $e^T_{ut_T}, e^T_{vt_T}$ for each $T \in \mathcal{T}_{uv}$. If we extend M_i' by uv and edges $\{t_T z_T : T \in \mathcal{T}_{uv},\ t'_T z_T \in M_i'\}$, we get a $\mathcal{T}$-free b-matching M_i of G by (2) and Proposition 6.1.

It is easy to see that $x = \frac{1}{k}\sum \chi_{M_i}$, hence x is a convex combination of $\mathcal{T}$-free b-matchings of G, a contradiction. □

So we may assume that $0 < x_e < 1$ for each edge $e \in E$.

Proposition 6.3. *For each* $u, v \in V$, $x(E[u,v]) < 1$.

Proof. If $|E[u,v]| = 1$ then the proposition follows from Proposition 6.2. Otherwise no edge in $E[u,v]$ is contained in a forbidden triangle by Proposition 6.1 and we can decrease the x-values on them by one in total and also decrease $b(u), b(v)$ by one, thus obtaining a smaller counterexample, a contradiction. □

Lemma 6.4. *There is no x-tight triangle $T \in \mathcal{T}$.*

Proof. Assume that there exists a tight triangle T and let $V_T = \{t_1, t_2, t_3\}$. Shrink T to a single node t as in Definition 5.1, obtaining G°, b°, $\mathcal{T}^\circ$, x°. By Lemma 5.3, these satisfy (P_8).

Note that $b^\circ(t) = 2$ and $d_{G^\circ}(t) \le 3$ follows by (1). As $|\mathcal{T}^\circ| < |\mathcal{T}|$, x° is a convex combination of $\mathcal{T}^\circ$-free b°-matchings M_i° of G°. Let $x^\circ = \frac{1}{k}\sum \chi_{M_i^\circ}$ and let $\alpha_{jl} = \frac{1}{k}|\{i : f_j, f_l \in M_i\}|$, $\beta_{jk} = \frac{1}{k}|\{i : f_j, g_l \in M_i\}|$ and finally $\gamma_{jk} = \frac{1}{k}|\{i : g_j, g_l \in M_i\}|$ where $f_1, f_2, f_3, g_1, g_2, g_3$ are as in Definition 5.1. As $x^\circ(\dot{\delta}(t)) = 2$, we have $\sum \alpha_{jl} + \sum \beta_{jl} + \sum \gamma_{jl} = 1$.

Proposition 6.5. *There exist a proper convex combination for what* $\sum \beta_{jj} = 0$.

Proof. Take such a combination in which $\sum \beta_{jj}$ is minimal and assume that $\beta_{11} > 0$. This immediately implies that $\beta_{22}, \beta_{23}, \beta_{32}, \beta_{33}, \gamma_{23} = 0$ as otherwise we could easily modify the b°-matchings and decrease $\sum \beta_{jj}$.

We have the following equalities.

$$\begin{aligned}
\alpha_{12} + \alpha_{13} + \beta_{11} + \beta_{12} + \beta_{13} &= x(f_1),\\
\alpha_{12} + \alpha_{23} + \beta_{21} &= x(f_2),\\
\alpha_{13} + \alpha_{23} + \beta_{31} &= x(f_3),\\
\beta_{11} + \beta_{21} + \beta_{31} + \gamma_{12} + \gamma_{13} &= x(t_2t_3) - x(f_1),\\
\beta_{12} + \gamma_{12} &= x(t_1t_3) - x(f_2),\\
\beta_{13} + \gamma_{13} &= x(t_1t_2) - x(f_3).
\end{aligned}$$

From these and from $x(E_T) = 2$ we get $\alpha_{23} - \beta_{11} = 1 - x(t_2t_3) > 0$. Hence there is an M_i, say M_1, with $f_1, g_1 \in M_1$ and another one, say M_2, with $f_2, f_3 \in M_2$. The proof of Theorem 4.1 of [12] implies that we can take an alternating path P in $M_1 \triangle M_2$ starting at t' such that $M_1 \triangle P$ and $M_2 \triangle P$ are also $\mathcal{T}^\circ$-free b°-matchings of G°. Hence β_{11} can be decreased, and the proposition follows. □

Take a convex combination $\frac{1}{k}\sum \chi_{M_i}$ described in Proposition 6.5. We extend the M_i°'s to $\mathcal{T}$-free b-matchings of G as follows: if $M_i^\circ \cap \delta(t) = \{f_j, f_l\}$ or $\{f_j, g_l\}$ or $\{g_j, g_l\}$ where $j \neq l$ then define $M_i = M_i^\circ \cup (E_T - e_{j,l}^T)$.

It suffices to verify that the b-matchings thus arising are $\mathcal{T}$-free b-matchings of G. Indeed, they cannot contain any triangle in $\mathcal{T}^\circ$, and neither contain T due to the construction. For a triangle $T' \in \mathcal{T}$ which shares a node with T, by (1), T and T' must have an edge in common. By Proposition 6.1, they do not have the same node-set but then (2) implies that at least one of the edges of T' is not in M_i.

The convex combination of the M_i's gives x. To see this, it suffices to check that the combination gives $x(e_{j,j+1}^T)$ in total for each $j = 1, 2, 3$. This is assured by the choice of the coefficients as T is tight. □

If x is a b-factor, that is, $x(\dot{\delta}(v)) = b(v)$ for each $v \in V$ then each $T \in \mathcal{T}$ is tight. By Theorem 1.2 and Lemma 6.4, x is not a b-factor. So our aim is now to show that there is an x-tight odd triplet $(K, F, \mathfrak{T})$ of Type 2 whose shrinking lexicographically decreases $(|\mathcal{T}|, b(V), l(V), |V|, |E|)$, and the same holds for $(\bar{K}, F, \mathfrak{T})$.

The next proposition states that, as one would expect, $b \leq d$ can be assumed.

Proposition 6.6. $b(v) \leq \min\{d(v), \lceil x(\dot{\delta}(v))\rceil + 1\}$ *for each* $v \in V$.

Proof. Assume that $b(v) > d(v)$ for some $v \in V$. Set $b(v) := d(v)$. We claim that the inequalities of (P_8) remain valid, contradicting the minimal choice of the counterexample. Assume indirectly that there is a triplet $(K, F, \mathfrak{T})$ with $v \in K$ violating (iii) after the modification. However, for the triplet $(K - v, F \setminus F_v \cup E[v, K - v], \mathfrak{T})$ the left hand side of (iii) decreases by $x(l(v)) + x(F_v)$ while the right decreases by $\frac{1}{2}(d(v) + |F_v| - |E[v, K - v]|) = |l(v)| + |F_v|$ (compared to $(K, F, \mathfrak{T})$ after the modification), hence $(K - v, F \setminus F_v \cup E[v, K - v], \mathfrak{T})$ is a violating triplet even in the original problem.

If we set $b'(v) := \lceil x(\dot{\delta}(v))\rceil$ for each $v \in V$ then $(i), (ii), (iv)$ and (v) clearly remains valid in (P_8). Assume that there is an odd triplet $(K, F, \mathfrak{T})$ violating (iii) after the modification. Inequalities of form (iii) are obtained by summing up inequalities of from (i) and (ii), then dividing by two and taking the floor of the right hand side. But until the very last step the inequality remains valid, so the violation, that is, the deficiency of the triplet can be at most $\frac{1}{2}$. Hence setting $b'(v) := \min\{b(v), \lceil x(\dot{\delta}(v))\rceil + 1\}$ assures that no violating triplet arises.

The proposition follows by the choice of the counterexample. □

Since G is connected, $|E| \geq |V| - 1$. If $|E| = |V| - 1$ or $|E| = |V|$ then G may contain a triangle if and only if G itself is a triangle or a triangle and a node connected by an edge. These cases can be easily seen not to give a counterexample (similarly to the proof of Proposition 6.1), while the remaining cases follow from Theorem 1.3. Hence we may assume that $|E| > |V|$.

Proposition 6.7. *Let* (K, F) *be a tight pair,* $v \in \bar{K}$. *If* $b(v) \leq |F_v|$ *then* $(K + v, F \setminus F_v)$ *is also tight. Moreover,* $\dot{\delta}(v) \setminus F = \emptyset$.

Proof. By adding v to K, the left hand side of (iii) increases by at least $x(\dot{\delta}(v) \setminus F)$ while the right hand side may only decrease. The statement follows by Proposition 6.2. □

If there is an x-tight odd triplet $(K, F, \mathfrak{T})$ such that $\mathfrak{T} \neq \emptyset$, then $|\mathcal{T}|$ decreases when shrinking either $(K, F, \mathfrak{T})$ or $(\bar{K}, F, \mathfrak{T})$, and we are done. So assume that this is not the case. Recall that a tight triplet $(K, F, \mathfrak{T})$ with $\mathfrak{T} = \emptyset$ was called a tight pair.

We have already seen that there is no tight constraint of form $(i), (iv)$ or (v), and now we assumed that neither of form (iii) with $\mathfrak{T} \neq \emptyset$. Let us call an x-tight constraint **bad** if it is of form (ii) for some $v \in V$, or it is of form (iii) for some odd pair (K, F) and one of the followings hold.

(I) $l(K) = \emptyset, b(K) \leq |F|$

(II) $l(K) = \emptyset, b(K) = |F| + 1, |K| = 1$

(III) $l(K) = \emptyset, b(K) = |F| + 1, |K| = 2, |E[K]| \leq 1$

(IV) $l(\bar{K}) = \emptyset, b(\bar{K}) \leq |F|$

(V) $l(\bar{K}) = \emptyset, b(\bar{K}) = |F| + 1, |\bar{K}| = 1$

(VI) $l(\bar{K}) = \emptyset, b(\bar{K}) = |F| + 1, |\bar{K}| = 2, |E[\bar{K}]| \leq 1$

If the shrinking of (K, F) or the shrinking of $(\bar{K}, F)$ does not result in a lexicographically smaller problem then (K, F) must be bad (however, it may happen that we get a smaller problem even in case of a bad pair as $\mathcal{T}_K \neq \emptyset$ or $l(K), l(\bar{K}) \neq \emptyset$ would also assure that).

As we may assume that $|E| > |V|$, the existence of a tight odd pair (K, F) whose shrinking results in a lexicographically smaller problem and the same holds for $(\bar{K}, F)$ is assured by the following fundamental lemma.

Lemma 6.8. *Under the assumption that there is no tight constraint of form (iii) with $\mathfrak{T} \neq \emptyset$, the maximum number of linearly independent bad constraints is at most $|V|$.*

Proof. Take a maximal independent set of tight equalities of form (ii), and extend this to a maximal independent set with bad equalities of type (IV) with $|K| = 1$, and then with equalities of type (V). Let $\mathcal{L}$ denote the set of equalities thus obtained.

Claim 6.9. *There is no bad pair (K, F) independent from $\mathcal{L}$.*

Proof. In the proof we will strongly rely on Proposition 5.6 several times without mentioning it.

Assume that (K, F) is of type (I) independent from $\mathcal{L}$. First of all, $b(K) \geq |F| - 1$ as otherwise $x(E[K]) + x(F) = \lfloor \frac{1}{2}(b(K) + |F|) \rfloor \leq |F| - 2$, contradicting $x(F) \geq |F| - 1$. If $b(K) = |F| - 1$ then from $x(E[K]) + x(F) = |F| - 1$ we get $x(E[K]) = 0$ and $x(F) = b(K)$ which in turn imply $E[K] = \emptyset$ and $F = \delta(K)$, so $x(\delta(v)) = b(v)$ for each $v \in K$. But this is a contradiction as (K, F) is supposed to be independent from equalities of form (ii). $b(K) = |F|$ is not possible as (K, F) is an odd pair.

Assume that (K, F) is a bad pair of type (II), so $K = \{v\}, F \subseteq \delta(v), l(v) = \emptyset$ and $b(v) = |F| + 1$. Then the tightness of (v, F) means $x(F) = |F|$, which is only possible if $F = \emptyset$ by $x < 1$, contradicting independence.

Assume that (K, F) is a bad pair of type (III) independent from $\mathcal{L}$ and let $K = \{u, v\}$. Let C be the set of parallel edges between u and v.

As $b(u) + b(v) = |F_u| + |F_v| + 1$, either $b(u) \leq |F_u|$ or $b(v) \leq |F_v|$, say the first one. In this case $x(C) + x(F_u) \leq b(u) \leq |F_u|$, so $x(C) + x(F_u) + x(F_v) \leq |F_u| + |F_v|$. Here $F_v = \emptyset$, otherwise even strict inequality holds by $x(F_v) < |F_v|$, contradicting the tightness of (K, F). So $F_v = \emptyset$. By the tightness of the pair, $x(C) + x(F_u) = |F_u|$. We assumed that $b(u) \leq |F_u|$, so $b(u) = |F_u|$ and $b(v) = 1$ implying $\delta(u) \setminus (C \cup F_u) = \emptyset$. But then the tightness of the pair (K, F) is equivalent to $x(\dot{\delta}(u)) = b(u)$, contradicting linear independence.

Assume now that (K, F) is of type (IV) independent from $\mathcal{L}$ with $|K| \geq 2$. It can be seen similarly to the earlier cases that $b(\bar{K}) \geq |F| - 1$ must hold. If $b(\bar{K}) = |F| - 1$ then $x(E[\bar{K}]) + x(\delta(K) \setminus F) = 0$, hence $E[\bar{K}] = \emptyset$ and $\delta(K) = F$. So we have $x(E) = x(E[K]) + x(\delta(K)) = x(E[K]) + x(F) = \frac{1}{2}(b(K) + |F| - 1) = \frac{1}{2}b(V)$. That is, x is in fact a b-factor, a contradiction.

If $b(\bar{K}) = |F|$ then $x(E) \geq x(E[K]) + x(F) + x(E[\bar{K}]) = \frac{1}{2}(b(K) + |F| - 1) + x(E[\bar{K}]) = \lfloor \frac{1}{2}b(V) \rfloor + x(E[\bar{K}])$. But $x(E) \leq \lfloor \frac{1}{2}b(V) \rfloor$ so $E[\bar{K}] = \emptyset$ and also $\delta(K) = F$. That means that $\bar{K}$ consists of isolated nodes $v_1, ..., v_k$ and $\delta(K) = F = \delta(v_1) \cup ... \cup \delta(v_k)$. Let $F_i = \delta(v_i)$. We claim that $b(v_i) = |F_i|$ for each i. Indeed, otherwise there is an i with $b(v_i) \geq |F_i| + 1 > d(v_i)$, contradicting Proposition 6.6. So $b(v_i) = |F_i|$ for each i. Then $(K \cup \{v_1, ..., v_{k-1}\}, F_k)$ is also tight, and the tightness of (K, F) is identical to the tightness of this pair, a contradiction.

Now assume that (K, F) is a bad pair of type (VI) independent from $\mathcal{L}$ and let $\bar{K} = \{u, v\}$. As $b(u) + b(v) = |F_u| + |F_v| + 1$, either $b(u) \leq |F_u|$ or $b(v) \leq |F_v|$, say the first one. By Proposition 6.7, $(K + v, F_u)$ is also tight and $\dot{\delta}(v) \setminus F = \emptyset$, hence the tightness of (K, F) is equivalent to the tightness of $(K + v, F_u)$, contradicting linear independence. $\square$

Claim 6.9 implies that an upper bound for $|\mathcal{L}|$ is also an upper bound for the maximum number of independent bad constraints. Hence it suffices to bound $|\mathcal{L}|$. We say that a bad constraint in $\mathcal{L}$ **corresponds** to a node $v \in V$ if it is either of type $x(\dot{\delta}(v)) = b(v)$, or of type (IV) or (V) with $\bar{K} = \{v\}$. We give a bound on the number of bad constraints in $\mathcal{L}$ corresponding to a node $v \in V$.

Proposition 6.10. *If (K, F) is in $\mathcal{L}$ then $(K, F') \notin \mathcal{L}$ for $F' \subset F$.*

Proof. Assume indirectly that (K, F') is in $\mathcal{L}$ for some $F' \subset F$. Then $x(F \setminus F') = \frac{|F \setminus F'|}{2}$ from what $F' = \emptyset, |F| = 2, x(F) = 1$ follow by Propositions 5.6 and 6.2. But then each node is saturated in K and $(K, F') = (K, \emptyset)$ is not independent from equalities of form (ii). $\square$

Claim 6.11. *If $x(\dot{\delta}(v)) = b(v)$ then there is no bad constraint of type (IV) or (V) in $\mathcal{L}$ corresponding to v.*

Proof. Let v be such that $x(\dot{\delta}(v)) = b(v)$ and $x(E[K])) + x(F) = \frac{b(K)+|F|-1}{2}$ for some $F \subseteq \delta(K)$ where $K = V - v$. Recall that $l(v) = \emptyset$ is assumed.

Assume first that $b(v) = |F|$. By Proposition 6.7, $\dot{\delta}(v) \setminus F = \emptyset$. Hence $x(\dot{\delta}(v)) = b(v)$ is identical to $x(F) = |F|$, a contradiction.

Assume now that $b(v) = |F| + 1$. As $x(\delta(v)) = b(v) = |F| + 1$ and $x(F) \leq |F|$, $x(\delta(v) \setminus F) \geq 1$ must hold. Hence we have $x(E) = x(E[K]) + x(F) + x(\delta(v) \setminus F) \geq \frac{b(K)+|F|-1}{2} + 1 = \frac{b(V)}{2}$, which is only possible if x is a b-factor, a contradiction. $\square$

Observe that if there is a bad constraint of type (IV) corresponding to v then this constraint is unique (namely $(V - v, \delta(v))$). Moreover, there is no bad constraint of type (V) corresponding to v by Proposition 6.10.

Claim 6.12. *For each $v \in V$, there is at most one bad constraint of type (V) in $\mathcal{L}$ corresponding to v.*

Proof. Assume that v is such that $x(E[K]))+x(F_1) = \frac{b(K)+|F_1|-1}{2}$ and $x(E[K]))+x(F_2) = \frac{b(K)+|F_2|-1}{2}$ for different $F_1, F_2 \subseteq \delta(K)$ where $K = V - v$.

Proposition 6.13. $|F_1| = |F_2|$.

Proof. Assume to the contrary that $|F_1| > |F_2|$. $(F_1 \setminus F_2) \subseteq F_1$ hence $x(F_1 \setminus F_2) \geq |F_1 \setminus F_2| - 1$. On the other hand, $(F_1 \setminus F_2) \subseteq (\delta(K) \setminus F_2)$, hence $x(F_1 \setminus F_2) \leq 1$. These imply $|F_1 \setminus F_2| \leq 2$. By parity arguments, $F_2 \subseteq F_1$, contradicting Proposition 6.10. □

Proposition 6.14. $|F_1 \cap F_2| = 0$.

Proof. Assume that $F_1 \cap F_2 = F \neq \emptyset$. From the tightness of (K, F_1) and (K, F_2) we get $2x(E[K]) + 2x(F) + x(F_1 \triangle F_2) = b(K) + |F| + \frac{|F_1 \triangle F_2|}{2} - 1 \geq b(K) + |F|$. On the other hand, we know that $2x(E[K]) + x(\delta(K)) \leq b(K)$ and $x(F) < |F|$ implying $2x(E[K]) + 2x(F) + x(\delta(K) \setminus F) < b(K) + |F|$, a contradiction. □

Proposition 6.15. $|F_1| = |F_2| = 1$

Proof. By Proposition 5.6, $x(F_1) \leq 1$ as $F_1 \subseteq \delta(K) \setminus F_2$, hence $|F_1| \leq 2$ by the same proposition.

Assume that $|F_1| = 2$. From the tightness of (K, F_1) and (K, F_2) we get

$$2x(E[K]) + x(F_1) + x(F_2) = b(K) + 1.$$

On the other hand, we know that $2x(E[K]) + x(\delta(K)) \leq b(K)$, a contradiction. □

Let $F_1 = f_1, F_2 = f_2$. Clearly, $x(f_1) = x(f_2)$.

Proposition 6.16. $\delta(v) = \{f_1, f_2\}$

Proof. We have $x(E[K])+x(f_1) = \frac{1}{2}b(K)$ and $x(E[K])+x(f_2) = \frac{1}{2}b(K)$, so $2x(E[K]) + x(f_1) + x(f_2) = b(K)$. That means that each node is saturated in K by the x-values on $E[K]$ and $\{f_1, f_2\}$, hence there is no edge $f \in \delta(K) \setminus \{f_1, f_2\}$ by Proposition 6.2. □

Proposition 6.16 implies that there are at most two bad constraints of type (V) in $\mathcal{L}$ corresponding to a node. Assume that v is such that there are exactly two such constraints. The proof of Proposition 6.16 implies that all the other nodes are saturated by x, hence v is unique with this property by Claim 6.11.

We claim that $\mathcal{T} = \emptyset$. Indeed, assume first that there is a forbidden triangle $T \in \mathcal{T}$ containing v. Let $f_1 = vu$ and $f_2 = vw$ be the two edges incident to v. Both u and w have degree 3 as they are saturated and $x < 1$. Let $e_1 = \delta(u) \setminus E_T$ and $e_2 = \delta(w) \setminus E_T$. It is easy to see that $x(e_1) = x(e_2) > x(f_1) = x(f_2)$. Also, $x(e_i) > \frac{1}{2}$ by $x < 1$, the previous observation and $x(e_i) + x(f_i) + x(uw) = 2$.

Edges e_1, e_2, uw do not form the edge-set of a forbidden triangle T' as otherwise $x(E_T) + x(E_{T'}) = x(\delta(u)) + x(\delta(w)) = 4$, hence both T and T' are tight, a contradiction.

Delete the edges uv, uw from G, shrink u and w in a single node z with $b(z) = 2$ and add a new edge vz to the graph with $x(vz) = 2 - x(e_1) - x(e_2)$. Let $G', b', \mathcal{T}', x'$ denote the lexicographically smaller problem thus arising. An easy case-check shows that x' satisfies (P_8) in G' with b' and $\mathcal{T}'$ hence it is a convex combination of $\mathcal{T}'$-free b'-matchings of G'. This convex combination can be extended to the original problem in a straightforward manner thus giving x, a contradiction.

Proposition 6.17. *There is no triangle $T \in \mathcal{T}$ whose nodes are all saturated.*

Proof. Assume that $x(\delta(v)) = 2$ for each $v \in V_T$ for some $T \in \mathcal{T}$. Recall that V_T does not span parallel edges by Proposition 6.1. Then $2x(E_T) + x(\delta(V_T)) = 6$, and so $x(E_T) + x(\delta(V_T)) \geq 5-2 = 4$. On the other hand, $(V_T, \delta(V_T))$ is an odd pair, so $x(E_T)+x(\delta(V_T)) \leq \lfloor \frac{6+3}{2} \rfloor) = 4$. Hence we have equality everywhere, implying $x(E_T) = 2$, a contradiction. ☐

By Claim 6.17, there is no $T \in \mathcal{T}$ with $V_T \subseteq V - v$ either. Let $f_1 = vu$ and $f_2 = vw$ be the two edges incident to v. Delete v from G and add a new edge between u and w with x-value $x(f_1) = x(f_2) = C$. Let G', x' denote the graph and vector thus arising.

Proposition 6.18. *x' satisfies (P_8) in G'.*

Proof. It only suffices to verify (iii). Assume that there is an odd pair (Z, H) with $Z \subseteq V - v, H \subseteq \delta(Z) \setminus \{f_1, f_2\}$ violating (iii) in G'. It is easy to see that $u, w \in Z$ must hold otherwise there would be a violating pair in the original problem, too. That means that $x(E[Z]) + x(H) > \frac{b(Z)+|H|-1}{2} - C$. In other words, as each node different from v is saturated, $b(Z) - x(E[Z]) - x(\delta(Z) \setminus H) > \frac{b(Z)+|H|-1}{2} - C$, so $x(E[Z]) + x(\delta(Z) \setminus H) < \frac{b(Z)-|H|+1}{2} + C$. If (Z, H) is odd then $(V \setminus (Z+v), H)$ is also odd and $x(E[V \setminus (Z+v)]) + x(H) \leq \frac{(V \setminus (Z+v))+|H|-1}{2}$. Summing up these we get $x(E) < \frac{b(V-v)}{2} + C$.

As both $(V - v, f_1)$ and (V_v, f_2) are tight, $2x(E[V - v]) + x(\{f_1, f_2\}) = b(V - v)$, that is, $2x(E) = b(V - v) + 2C$, a contradiction. ☐

As G', x' provides a lexicographically smaller problem, x' is a convex combination of b-matchings (in fact factors) of G'. These b-matchings easily extends to G giving x, a contradiction. ☐

Claims 6.9, 6.11 and 6.12 imply that $|\mathcal{L}| \leq |V|$, and we are done. ☐

As $|E| > |V|$, Lemma 6.8 implies that there exists a tight odd triplet $(K, F, \mathfrak{T})$ whose shrinking lexicographically decreases the problem, and the same holds for $(\bar{K}, F, \mathfrak{T})$. More precisely, there is a tight triplet $(K, F, \mathfrak{T})$ with either $\mathfrak{T} \neq \emptyset$ or being independent from $\mathcal{L}$ defined earlier. Take such a triplet with $|K|$ being minimal and let $G_1^\circ = (V_1^\circ, E_1^\circ), b_1^\circ, x_1^\circ, \mathcal{T}_1^\circ$ and $G_2^\circ = (V_2^\circ, E_2^\circ), b_2^\circ, x_2^\circ, \mathcal{T}_2^\circ$ denote the problems arising through shrinking $(K, F, \mathfrak{T})$ and $(\bar{K}, F, \mathfrak{T})$, respectively. We refer to the new nodes p, q in these graphs by p_1, q_1 and p_2, q_2, respectively. By the minimality of the counterexample, x_i° is a convex combination of ${\mathcal{T}^\circ}_i$-free b_i°-matchings of G_i°, say, $x_1^\circ = \frac{1}{k} \sum \chi_{M_i}$ and $x_2^\circ = \frac{1}{2} \sum \chi_{N_j}$ for some $k \in \mathbb{Z}_+$ (note that x_i° is rational, being a vertex of a rational polytope). The following proposition is an easy observation.

Proposition 6.19. *The tightness of* $(K, F, \mathfrak{T})$ *implies that exactly one of the followings holds for each* M_i*:*

$$(\delta(p_1) - p_1q_1) \subseteq M_i,\ |(\delta(q_1) - p_1q_1) \cap M_i| \leq 1,\ \textit{or}$$
$$|(\delta(p_1) - p_1q_1) \setminus M_i| = 1,\ p_1q_1 \in M_i,\ (\delta(q_1) - p_1q_1) \cap M_i = \emptyset.$$

Similarly, for N_j*'s:*

$$(\delta(p_2) - p_2q_2) \subseteq N_j,\ |(\delta(q_2) - p_2q_2) \cap N_j| \leq 1,\ \textit{or}$$
$$|(\delta(p_2) - p_2q_2) \setminus N_j| = 1,\ p_2q_2 \in N_j,\ (\delta(q_2) - p_2q_2) \cap N_j = \emptyset.$$

Each edge $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$ is contained in exactly $kx(e)$ number of M_i's and N_j's. By the above observation, each of these M_i's contains the entire F and edges pr_T, r_Tw_1 or pr_T, r_Tw_2 for each $T \in \mathfrak{T}$, while each of the N_j's contains the entire F and edges pr_T, r_Ts_T, t_Tw or pr_T, r_Tt_T, s_Tv. However, it is easy to see that, as they are parallel, the role of edges r_Tw_1 and r_Tw_2 can be 'exchanged' in such a way that the total number of M_i's with $pr_T, r_Tw_1 \in M_i$ is equal to the number of N_j's with $pr_T, r_Tt_T, s_Tv \in N_j$. This makes possible to pair these b_i°-matchings and 'glue' them together to get $kx(e)$ b-matchings of G containing the edge e. The b-matching obtained by gluing an M_i with $pr_T, r_Tw_1 \in M_i$ and an N_j's with $pr_T, r_Tt_T, s_Tv \in N_j$ contains e_{uv}^T and e_{uw}^T from E_T, and similarly to the other case where $pr_T, r_Tw_2 \in M_i$ and $pr_T, r_Ts_T, t_Tv \in N_j$. This can be done for each edge $e \in \delta(K) \setminus (F \cup E_{\mathfrak{T}})$.

Similarly, for each edge $e \in F$ there are exactly $k(1 - x(e))$ M_i's and N_j's that does not contain e. Notice that these contain all edges in $\delta(p_i) - e$ and none in $\delta(K) - (F \cup E_{\mathfrak{T}})$. Again, pair and glue these together to get b-matchings of G not containing e.

The number of M_i's with $l_T \in M_i$ or $r_Tw_1, r_Tw_2 \in M_i$ for some $T \in \mathfrak{T}$ is equal to the number of N_j's with $r_Ts_T, r_Tt_T \in N_j$. However, we have to pair these matchings together carefully. Note, that $\mathcal{T}_2^\circ$ consists of triangles disjoint from K. Hence it may happen that there is a forbidden triangle $T' \in \mathcal{T}$ such that $V_{T'} \subseteq K$ and there is a triangle $T \in \mathfrak{T}$ with $|V_T \cap V_{T'}| = 2$. In this case, we are not allowed to pair an M_i and an N_j for what $l_T \in M_i$ and the two remaining edges of T' not contained by T is in N_j. We can easily avoid this unless the sum of the coefficients of these N_j's is more than $1 - x_1^\circ(l_T) = x(E_T) - 1$. Consider a convex combination in which the sum of the coefficients of b_2°-matchings containing the edges of T' different from e_T is minimal. If this value is positive then there is no N_j containing none of these two edges. But this implies that $x(E_{T'}) > 2(x(E_T) - 1) + (1 - (x(E_T) - 1)) + x(e_T) = x(E_T) + x(e_T) \geq 2$, a contradiction. The last inequality follows from Proposition 5.6.

So the pairing can be done. However, it is left to prove that the b-matchings thus arising are also $\mathcal{T}$-free.

Lemma 6.20. *The* b*-matchings thus obtained are* $\mathcal{T}$*-free.*

Proof. The only triangles possibly contained in one of the b-matchings could be those in $\mathcal{T} - (\mathcal{T}_1^\circ \cup \mathcal{T}_2^\circ)$. Moreover, by the above, a bad triangle should have nodes both in K and $\bar{K}$.

Due to the construction, a triangle $T \in \mathfrak{T}$ is not contained in the b-matchings obtained. Also, a T with $E_T \cap E_{\mathfrak{T}} \neq \emptyset$ is not contained by (1), (2) and Proposition 6.19. Assume that T shares no edge with triangles in $\mathfrak{T}$.

If $|E_T \cap F| = 0$ then each M_i contains at most one of T's edges going between K and $\bar{K}$ as $|M_i \cap (\delta(K) \setminus (F \cup E_{\mathfrak{T}}))| \leq 1$, hence T is not contained by the b-matchings.

Let $V_T = \{r, s, t\}$. Recall that $(K, F, \mathfrak{T})$ is such that either $\mathfrak{T} \neq \emptyset$ or it is independent from $\mathcal{L}$. The following proposition will be useful.

Proposition 6.21. *There is no tight even triplet in G.*

Proof. Assume to the contrary that $(Z, H, \mathfrak{R})$ is a tight even pair, that is, $x(E[Z]) + x(H) + \sum_{T \in \mathfrak{R}} x(E_T) = \frac{b(Z)+|H|+3|\mathfrak{R}|}{2}$. By $0 < x < 1$, this immediately implies $H = \delta(Z) = \emptyset$, which is only possible if $Z = V$ as G is connected. But $x(E) = \frac{b(V)}{2}$ means that x is a b-factor, a contradiction. □

We distinguish the following cases.

Case 1: $|E_T \cap F| = 1, |V_T \cap K| = 1$

Assume that $V_T \cap K = r$ and $rt \in F$. Let u be the third neighbour of r, if exists. If $u \in K$ then $x(E[K - r]) + x(F - rt + ru) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K - r) + |F - rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K - r, F - rt + ru, \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $ru \in F$ then $x(E[K - r]) + x(F - rt - ru) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 2$ while $b(K - r) + |F - rt - ru| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K - r, F \setminus \delta(r), \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K - r]) + x(F - rt) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K - r) + |F - rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 3$, a contradiction as $(K - r, F - rt, \mathfrak{T})$ is an even triplet that would violate (iii) which is not possible.

Case 2: $|E_T \cap F| = 1, |V_T \cap K| = 2$

Assume that $K \cap V_T = \{r, s\}$ and $rt \in F$. Let u be the third neighbour of s, if exists. If $u \in K$ then $x(E[K - s]) + x(F + su + rs) + \sum_{T \in \mathfrak{T}} x(E_T) = x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T)$ while $b(K - s) + |F + su + rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}|$. Hence $(K - s, F + su + rs, \mathfrak{T})$ is also tight and its tightness is identical to that of the original triplet. However, $|K|$ decreased and $\mathfrak{T}$ did not change, contradicting the minimality of K.

If $u \in \bar{K}$ and $su \in F$ then $x(E[K - s]) + x(F - su + rs) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K - s) + |F - su + rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K - s, F - su + rs, \mathfrak{T})$ would violate (iii), a contradiction.

If $u \in \bar{K}$ and $su \notin F$ or s has no third neighbour then $x(E[K - s]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T \in \mathfrak{T}} x(E_T) - 1$ while $b(K - s) + |F| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K - s, F, \mathfrak{T})$ would violate (iii), a contradiction.

Case 3: $|E_T \cap F| = 2, |V_T \cap K| = 1$

Assume that $V_T \cap K = r$ and $rs, rt \in F$. Let u be the third neighbour of r, if exists. If $u \in K$ then $x(E[K-r]) + x(F-rs-rt) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt| + 3|\mathfrak{T}| \leq b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence we must have equality everywhere, so $x(\delta(r)) = 2$ and $(K-r, F-rs-rt, \mathfrak{T})$ is tight. The tightness of $(K-r, F-rs-rt, \mathfrak{T})$ is identical to that of the original triplet, while $\mathfrak{T}$ did not change. However, $|K|$ decreased, contradicting the minimality of K.

If $u \in \bar{K}$ and $ru \in F$ then $x(E[K-r]) + x(F-rs-rt-ru) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt-ru| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 5$. We must have equality everywhere as otherwise $(K-s, F-rs-rt-ru, \mathfrak{T})$ would be an even triplet violating (iii). That is, $x(\delta(r)) = 2$ and $(K-s, F-rs-rt-ru, \mathfrak{T})$ is tight. Note that $|K| \neq 1$ as either $\mathfrak{T} \neq \emptyset$ or the triplet is independent from $\mathcal{L}$. Hence $(K-s, F-rs-rt-ru, \mathfrak{T})$ is a tight even triplet, contradicting Proposition 6.21.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K-r]) + x(F-rs-rt) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-rs-rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K-r, F-rs-rt, \mathfrak{T})$ would violate (iii), a contradiction.

Case 4: $|E_T \cap F| = 2, |V_T \cap K| = 2$

Assume that $K \cap V_T = \{r, s\}$ and $rt, st \in F$. Let u be the third neighbour of r, if exists. If $u \in \bar{K}$ and $ru \in F$ then $x(E[K-r]) + x(F-ru-rt) + \sum_{T\in\mathfrak{T}} x(E_T) \geq x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r) + |F-ru-rt| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $x(\delta(r)) = 2$, $(K-r, F-ru-rt, \mathfrak{T})$ is also tight and is independent from $\mathcal{L}$ if the original triplet was so (note that $K - r \neq \emptyset$). However, $|K|$ decreased and $\mathfrak{T}$ did not change, contradicting the minimality of K.

If $u \in \bar{K}$ and $ru \notin F$ or r has no third neighbour then $x(E[K-r]) + x(F-rt+rs) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 1$ while $b(K-r) + |F-rt+rs| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 2$. Hence $(K-r, F-rt+rs, \mathfrak{T})$ would violate (iii), a contradiction.

The same can be told about the third neighbour of s denoted by v, if exists. So the only remaining case is when both $u, v \in K$. Then $x(E[K-r-s]) + x(F-rs-rt+ru+sv) + \sum_{T\in\mathfrak{T}} x(E_T) > x(E[K]) + x(F) + \sum_{T\in\mathfrak{T}} x(E_T) - 2$ while $b(K-r-s) + |F-rs-rt+ru+sv| + 3|\mathfrak{T}| = b(K) + |F| + 3|\mathfrak{T}| - 4$. Hence $(K-r-s, F-rs-rt+ru+sv, \mathfrak{T})$ would violate (iii), a contradiction. $\square$

By Lemma 6.20, the b-matchings constructed above altogether yield x as a convex combination of $\mathcal{T}$-free b-matchings of G, a contradiction.

7 Inequalities for arbitrary graphs

The problem of giving a complete description of the triangle-free 2-matching polytope of arbitrary graphs is still open. As mentioned in Section 1, assumption (1) is essential: Theorem 1.8 is false if we remove the degree bound $d_G(v) \leq 3$ on nodes of forbidden triangles, as shown by the following example.

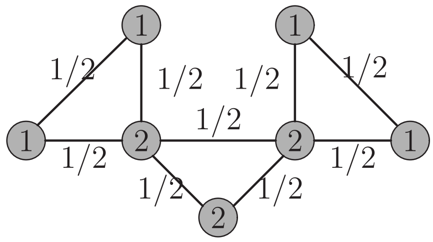

Figure 10: A counterexample for the non-subcubic case

The values on the nodes and on the edges represent b and x, respectively, and $\mathcal{T}$ contains the triangle in the center. One may check that x satisfies (P_8) with total value $\frac{9}{2}$, but the maximum size of a $\mathcal{T}$-free b-matchings is 4, hence x is definitely not contained in the $\mathcal{T}$-free b-matching polytope.

Again, instead of the triangle-free 2-matching problem we investigate the slightly more general $\mathcal{T}$-free b-matching problem under restriction (2). In the sequel, we define a new class of inequalities valid for the $\mathcal{T}$-free b-matching polytope.

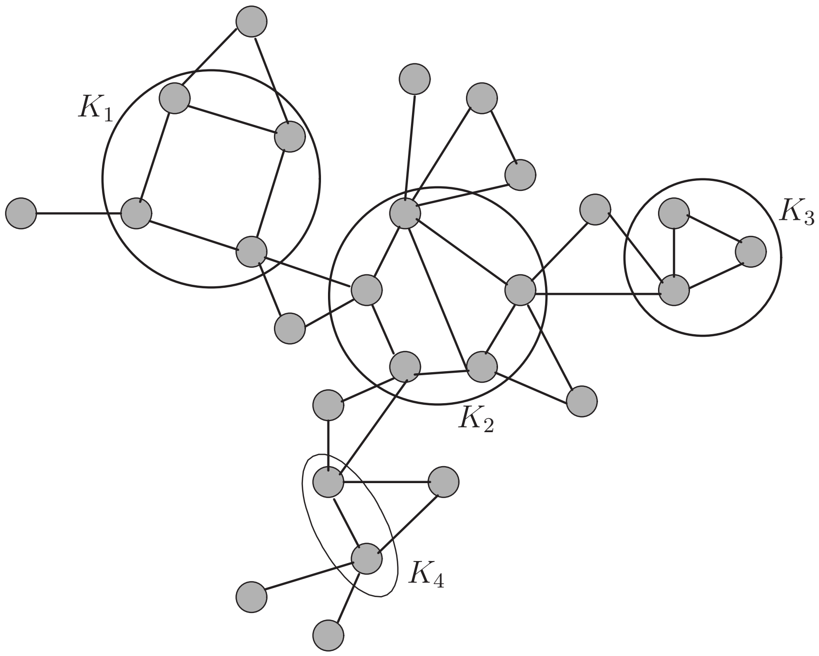

Figure 11: A puli

A **puli** (named after the famous breed of Hungarian dog) is a triplet $(\mathcal{K}, F, \mathfrak{T})$ where $\mathcal{K}$ is a collection of disjoint subsets of V, F is a subset of edges leaving $\mathcal{K}$ (that is, $F \subseteq \delta(\cup_{K\in\mathcal{K}} K)$), $\mathfrak{T}$ is a set of edge-disjoint forbidden triangles such that $F \cap E_{\mathfrak{T}} = \emptyset$. More precisely, $\mathfrak{T}$ is the union of $\mathfrak{T}^c$ and $\mathfrak{T}^f$. Here $\mathfrak{T}^c$ is a set of triangles connecting the members of $\mathcal{K}$, that is, $|\mathfrak{T}^c| = |\mathcal{K}| - 1$, each triangle $T \in \mathfrak{T}^c$ 2-fits $\cup_{K\in\mathcal{K}} K$ but none of $K \in \mathcal{K}$, and there is a $T \in \mathfrak{T}^c$ with $V_T \cap K \neq \emptyset$ for each $K \in \mathcal{K}$. $\mathfrak{T}^f$ contains triangles 1- or 2-fitting

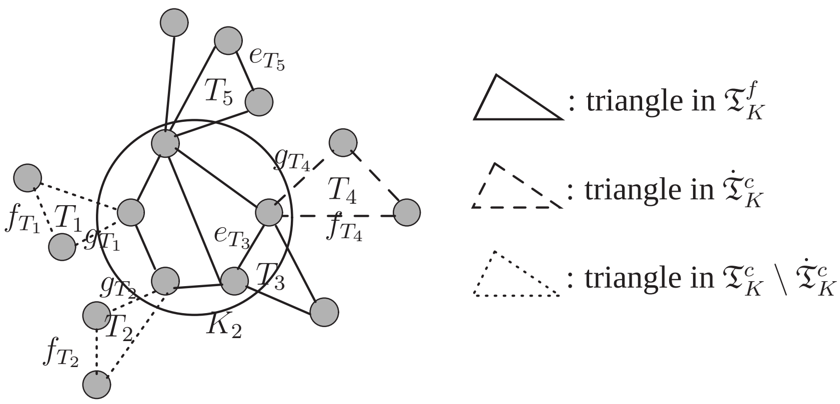

Figure 12: A tuft of the puli of Figure11

one of the members of $\mathcal{K}$ and is divided into disjoint subsets $\mathfrak{T}^f_1$ and $\mathfrak{T}^f_2$ according to this. Figure 11 shows a puli as an example.

Let $K \in \mathcal{K}$, $\mathfrak{T}^f_K \subseteq \mathfrak{T}^f$ be the set of non-connecting, $\mathfrak{T}^c_K \subseteq \mathfrak{T}^c_{\mathcal{K}}$ be the set of connecting triangles incident to K and $F_K = F \cap \delta(K)$. We denote $\mathfrak{T}^f_K \cup \mathfrak{T}^c_K$ by $\mathfrak{T}_K$. Moreover, we assign a subset $\dot{\mathfrak{T}}^c_K$ of $\mathfrak{T}^c_K$ to K. The quartet $(K, F_K, \mathfrak{T}_K, \dot{\mathfrak{T}}^c_K)$ is then called a **tuft** of the puli. For a triangle $T \in \mathfrak{T}^f_K$, we use again the notion of special edge e_T, that is, for a triangle $T \in \mathfrak{T}^f_1$ we define $e_T = E_T \setminus \delta(K)$, while for a triangle $T \in \mathfrak{T}^f_2$ we define $e_T = E_K \cap E_T$. For a triangle $\mathfrak{T}^c_K$, the unique edge in $E_T \setminus \delta(K)$ is denoted by f_T while its edge leaving K and not entering any other member of $\mathcal{K}$ is denoted by g_T. Using these notations, the following inequalities are clearly valid for the $\mathcal{T}$-free b-matching polytope.

$$
\begin{array}{ll}
(i)\ x_e \le 1 & (e \in E),\\
(ii)\ -x_e \le 0 & (e \in E),\\
(iii)\ x(\delta(v)) \le b(v) & (v \in V),\\
(iv)\ x(E_T) \le 2 & (T \in \mathfrak{T}).
\end{array}
$$

By summing up (i) for $e \in F_K$, (ii) for $e \in \delta(K) \setminus (F_K \cup E_{\mathfrak{T}_K})$, (iii) for $v \in K$, (iv) for $T \in \mathfrak{T}^f_K \cup \dot{\mathfrak{T}}^c_K$, (i) for edges of form e_T for some $T \in \mathfrak{T}^f_K$ or of form f_T for some $T \in \dot{\mathfrak{T}}^c_K$ and (i) for edges $e \in (E_T - f_T)$ for some $T \in \mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K$, we get

$$
\begin{aligned}
2x(E[K]) + 2x(F_K) + 2\sum_{T \in \mathfrak{T}_K} x(E_T) - 2\sum_{T \in \mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K} x(f_T) \le \\
\le b(K) + |F_K| + 3|\mathfrak{T}^f_K| + 2|\mathfrak{T}^c_K| + |\dot{\mathfrak{T}}^c_K|.
\end{aligned}
$$

That is,

$$
\begin{aligned}
x(E[K]) + x(F_K) + \sum_{T \in \mathfrak{T}_K} x(E_T) - \sum_{T \in \mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K} x(f_T) \le \\
\le \lfloor \tfrac{b(K) + |F_K| + 3|\mathfrak{T}^f_K| + 2|\mathfrak{T}^c_K| + |\dot{\mathfrak{T}}^c_K|}{2} \rfloor
\end{aligned}
\tag{4}
$$

is a valid inequality for the $\mathcal{T}$-free b-matching polytope.

By summing up (i) for $e \in F_K$, (ii) for $e \in \delta(K) \setminus (F_K \cup \{g_T| \ T \in \dot{\mathfrak{T}}^c_K\})$, (iii) for $v \in K$, (iv) for $T \in \mathfrak{T}^f_K$, (i) for edges of form e_T for some $T \in \mathfrak{T}^f_K$ and (i) for edges g_T for some $T \in \dot{\mathfrak{T}}^c_K$, we get

$$2x(E[K]) + 2x(F_K) + 2\sum_{T \in \mathfrak{T}^f_K} x(E_T) + 2\sum_{T \in \dot{\mathfrak{T}}^c_K} x(g_T) \le b(K) + |F_K| + 3|\mathfrak{T}^f_K| + |\dot{\mathfrak{T}}^c_K|.$$

That is,

$$x(E[K]) + x(F_K) + \sum_{T \in \mathfrak{T}^f_K} x(E_T) + \sum_{T \in \dot{\mathfrak{T}}^c_K} x(g_T) \le \lfloor \tfrac{b(K)+|F_K|+3|\mathfrak{T}^f_K|+|\dot{\mathfrak{T}}^c_K|}{2} \rfloor \tag{5}$$

is also a valid inequality for the $\mathcal{T}$-free b-matching polytope.

Let $(\mathcal{K}, F, \mathfrak{T})$ be a puli and assume that $(K_i, F_{K_i}, \mathfrak{T}_{K_i}, \dot{\mathfrak{T}}^c_{K_i})$ are tufts $(i = 1, ..., |\mathcal{K}|)$ such that sets $\dot{\mathfrak{T}}^c_{K_i}$ are disjoint and $\cup_i \dot{\mathfrak{T}}^c_{K_i} = \mathfrak{T}^c_{\mathcal{K}}$. By summing up inequalities (4) and (5) for a tuft we get

$$\begin{aligned} 2\sum_i x(E[K_i]) + 2\sum_i x(F_{K_i}) + 2\sum_{T \in \mathfrak{T}_{\mathcal{K}}} x(E_T) \le \\ \le \sum_i \lfloor \tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+2|\mathfrak{T}^c_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|}{2} \rfloor + \sum_i \lfloor \tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|}{2} \rfloor. \end{aligned}$$

For an i,

$$\begin{aligned} \lfloor \tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+2|\mathfrak{T}^c_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|}{2} \rfloor + \lfloor \tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|}{2} \rfloor = \\ = b(K_i) + |F_{K_i}| + 3|\mathfrak{T}^f_{K_i}| + |\mathfrak{T}^c_{K_i}| + |\dot{\mathfrak{T}}^c_{K_i}| - \chi_i, \end{aligned}$$

where χ_i is 1 if $b(K_i) + |F_{K_i}| + 3|\mathfrak{T}^f_{K_i}| + |\dot{\mathfrak{T}}^c_{K_i}|$ is odd and 0 otherwise. We call the tuft $(K_i, F_{K_i}, \mathfrak{T}_{K_i}, \dot{\mathfrak{T}}^c_{K_i})$ **odd** in the first case.

However, we can use a trick if the tuft is even but $\mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K \neq \emptyset$, called a **pre-odd** tuft. Consider such a tuft and let T_{spec} be a triangle in $\mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K$. Let e_1 and e_2 denote the edges in $E_{T_{spec}} - f_{T_{spec}}$. By a simple modification of the above we get inequalities

$$\begin{aligned} x(E[K]) + x(F_K) + \sum_{T \in \mathfrak{T}_K} x(E_T) - \sum_{T \in \mathfrak{T}^c_K \setminus \dot{\mathfrak{T}}^c_K} x(f_T) - x(e_2) \le \\ \le \lfloor \tfrac{b(K)+|F_K|+3|\mathfrak{T}^f_K|+2|\mathfrak{T}^c_K|+|\dot{\mathfrak{T}}^c_K|-1}{2} \rfloor \end{aligned}$$

and

$$x(E[K]) + x(F_K) + \sum_{T \in \mathfrak{T}^f_K} x(E_T) + \sum_{T \in \dot{\mathfrak{T}}^c_K} x(g_T) + x(e_2) \le \lfloor \tfrac{b(K)+|F_K|+3|\mathfrak{T}^f_K|+|\dot{\mathfrak{T}}^c_K|+1}{2} \rfloor.$$

The sum of the right sides gives

$$\lfloor\tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+2|\mathfrak{T}^c_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|-1}{2}\rfloor+\lfloor\tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|+1}{2}\rfloor=$$
$$=b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+|\mathfrak{T}^c_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|-1$$

Hence we have

$$\sum_i x(E[K_i])+\sum_i x(F_{K_i})+\sum_{T\in\mathfrak{T}_{\mathcal{K}}} x(E_T)\le$$
$$\le\lfloor\sum_i \tfrac{b(K_i)+|F_{K_i}|+3|\mathfrak{T}^f_{K_i}|+|\mathfrak{T}^c_{K_i}|+|\dot{\mathfrak{T}}^c_{K_i}|-\chi_i}{2}\rfloor,$$

where χ_i is 1 if the ith tuft is odd or pre-odd and 0 otherwise.

Given a puli $(\mathcal{K},F,\mathfrak{T})$, fix a $K_{fix}\in\mathcal{K}$. We can assign the triangles in $\mathfrak{T}^c$ to the members of $\mathcal{K}\setminus K_{fix}$ in such a way that each $K\in\mathcal{K}\setminus K_{fix}$ corresponds to an odd tuft (this is possible as the members of $\mathcal{K}$ together with the triangles in $\mathfrak{T}^c$ have a 'tree-structure', hence we may start from the leaves). After that $\dot{\mathfrak{T}}^c_{K_{fix}}$ also becomes fixed. It is easy to check that if K_{fix} corresponds at the end to a tuft which is not odd nor pre-odd then $b(K_{fix})+|F_{K_{fix}}|+|\mathfrak{T}_{K_{fix}}|$ must be even. However, we choose K_{fix} arbitrarily, so if there is a $K\in\mathcal{K}$ with $b(K)+|F_K|+|\mathfrak{T}_K|$ odd then

$$\sum_i x(E[K_i])+\sum_i x(F_{K_i})+\sum_{T\in\mathfrak{T}_{\mathcal{K}}} x(E_T)\le$$
$$\le\lfloor\tfrac{b(\mathcal{K})+|F|+3|\mathfrak{T}^f|+2|\mathfrak{T}^c|-1}{2}\rfloor$$

holds as $|\mathcal{K}|=|\mathfrak{T}^c|+1$, or for short,

$$x(\cup_i E[K_i])+x(F)+x(E_{\mathfrak{T}})\le\lfloor\tfrac{b(\mathcal{K})+|F|+3|\mathfrak{T}^f|+2|\mathfrak{T}^c|-1}{2}\rfloor.$$

We call a puli $(\mathcal{K},F,\mathfrak{T})$ **essential** if $b(\mathcal{K})+|F|+|\mathfrak{T}^f|$ is even and there is a $K\in\mathcal{K}$ with $b(K)+|F_K|+|\mathfrak{T}_K|$ odd. The above observations suggest the following conjecture.

Conjecture 7.1. *Let $G=(V,E)$ be a simple graph, $b:V\to\mathbb{Z}_+$ and $\mathcal{T}$ a collection of triangles satisfying (2). The $\mathcal{T}$-free b-matching polytope is determined by*

$$\begin{array}{lll}
(i)\ 0\le x_e\le 1 & (e\in E), & \\
(ii)\ x(\dot{\delta}(v))\le b(v) & (v\in V), & \\
(iii)\ x(E[K])+x(F)+x(E_{\mathfrak{T}})\le & ((K,F,\mathfrak{T})\ \textit{odd} & \\
\quad\lfloor\tfrac{b(K)+|F|+3|\mathfrak{T}|}{2}\rfloor & \textit{triplet of Type 2}), & (P_9)\\
(iv)\ x(\cup_i E[K_i])+x(F)+x(E_{\mathfrak{T}})\le & ((\mathcal{K},F,\mathfrak{T}) & \\
\quad\lfloor\tfrac{b(\mathcal{K})+|F|+3|\mathfrak{T}^f|+2|\mathfrak{T}^c|-1}{2}\rfloor & \textit{essential puli}), & \\
(v)\ x(E_T)\le 2 & (T\in\mathcal{T}). &
\end{array}$$

Let us return to the example of Figure 10. The graph itself consists of an essential puli, as shown on Figure 13 where the central triangle -denoted by T- can be assigned either to K_1 or K_2. Then (iii) gives $x(E[K_1])+x(E[K_2])+x(E_T)\le\lfloor\frac{8+2-1}{2}\rfloor=4$, while we have $x(E[K_1])+x(E[K_2])+x(E_T)=4.5$, showing that x is indeed not contained in the $\mathcal{T}$-free b-matching polytope.

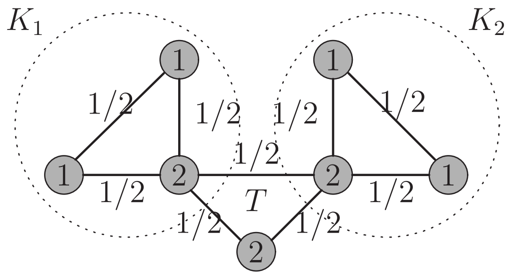

Figure 13: A violating puli

8 Conclusion

We gave a new proof of the polyhedral description of b-factors, based on a newly introduced contraction operation. The proof easily extended to the polyhedral description of $\mathcal{T}$-free b-factors under assumptions (1) and (2). The description (P_7) is what one would naturally expect having the description (P_2) of b-factors and (P_4) of uncapacitated $\mathcal{T}$-free 2-factors at hand. Hartvigsen and Li showed that the polyhedral description of $\mathcal{T}$-free 2-matchings is far more complicated, and proved (P_6) in [9]. We gave a slight generalization of their nice result by extending our contraction techniques.

Yet giving a polyhedral description of triangle-free (or, more generally, $\mathcal{T}$-free) 2-factors and 2-matchings of arbitrary graphs is still open. One might wonder whether (P_8) could possibly be a valid description for the general case, without the restrictive assumption that the degree of nodes incident to triangles is at most three. Unfortunately, the answer is negative as shown by the counterexample of Figure 10.

The examination of the general case led us to the notion of puli-inequalities which form a new class of valid inequalities for the $\mathcal{T}$-free b-matching polytope. Based on our observations, we proposed a polyhedral description of $\mathcal{T}$-free b-matchings for simple graphs satisfying (2). We believe that our method can be applied to the general case by further extending the notion of shrinking.

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