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# Network Coding Algorithms with Predetermined Coding Coefficients and Applications for Wireless Networks

Zoltán Király and Erika Renáta Kovács

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**Abstract**

We give new deterministic and randomized algorithms for the wireless model of Avestimehr, Diggavi, Tse for Gaussian relay networks by reducing it to the deterministic network coding problem introduced by Harvey, Karger and Murota. We also give a sufficient condition for a subset of coding coefficients which can be fixed arbitrarily to nonzero values, and the remaining coefficients can be determined in order to have a feasible network code. Finally we present applications to networks with nodes of different transmission properties.

## 1 Introduction

### 1.1 Capacity of Gaussian Relay Networks

Wireless networks have gained considerable attention over the past years. Determining their capacity is a central topic in communication theory. The difficulty of the problem arises from the difference from wired networks: the broadcast nature of the devices, and the presence of interference and noise.

Recently Avestimehr, Diggavi and Tse [1, 2] proposed a deterministic approach for Gaussian relay network model by eliminating the probabilistic factor. They also gave a min-max theorem for the unicast network capacity.

Several polynomial time algorithms were given for the unicast version of the problem. Yazdi and Savari [3] applied submodular flow techniques, Amaudruz and Fragouli [4] used augmenting paths, which Shi and Ramamoorthy [5] accelerated, and Goemans, Iwata and Zenklusen [6] solved the problem with matroid union or intersection. All of the approaches rely on the layered property of the model.

Considering multicast capacity, similarly to the case of wired networks proved by Ahlswede, Cai, Li and Yeung [7], nodes need to be able to perform network coding in order to achieve the maximal multicast capacity. It has been shown in several independent papers that the multicast capacity with network coding equals the minimum of the unicast capacities: Kim and Médard gave a randomized [8] while Yazdi and Savari [9] and Ebrahimi and Fragouli [10] gave deterministic algorithms for the problem.

### 1.2 Deterministic Network Coding Problem in Acyclic Networks

The deterministic network coding framework, introduced by Harvey, Karger and Murota [11] is the classical network coding problem on acyclic networks with the additional constraint that a subset of the coding coefficients are predetermined to a certain value. In [11] both the unicast and multicast versions of the problem were reduced to matrix completion problems and for both cases polynomial algorithms were given, based on matroid union. We give a simple deterministic algorithm for the multicast case when a unicast subroutine is given, and a randomized algorithm for both the unicast and multicast problems over small field.

Ebrahimi and Fragouli in [10] showed that the capacity of the model of Avestimehr, Diggavi, Tse for Gaussian relay networks [1, 2] can be determined based on the deterministic network coding framework. In the wireless model nodes are ordered in input/output layers and every path from the source to a terminal is alternating on them. With the framework of deterministic network coding problem one can model more general, not necessarily layered networks. The underlying graph can be any acyclic graph, and wireless communication can be restricted to only a subset of the nodes.

We also define a new variation of the deterministic network coding problem, called 'fixable pairs problem'. We give a sufficient condition for a subset of coding coefficients which can be fixed *arbitrarily* to nonzero values, and the remaining coefficients can be determined in order to have a feasible network code. Finally, we present an application of this model and give a necessary and sufficient condition for the solvability of network coding problems over networks with nodes of differing transmission properties.

The rest of the paper is organized as follows. In Section 2 the basic definitions of network coding and formulations of some problem types are given. In Section 3 we present deterministic and randomized algorithms for the deterministic network coding problem and give some applications. In Section 4 we discus the problem of fixable pairs.

## 2 Definitions

### 2.1 Network code

**Definition 2.1.** Let $\mathbb{F}_q$ be a finite field of size $q$ and let $\mathbb{F}_q^k$ denote the $k$-dimensional vector space over $\mathbb{F}_q$ and let $\mathbf{e}_i$ denote the $i$th unit vector in $\mathbb{F}_q^k$, $1 \le i \le k$. For a set $W$ of vectors in $\mathbb{F}_q^k$, $\langle W \rangle$ denotes the linear subspace spanned by $W$.

Let $\mathbf{M} = (M_1, M_2, \dots, M_k)$, $M_i \in \mathbb{F}_q$, $1 \le i \le k$ be an ordered set of $k$ messages.

Let $D = (V, A)$ be an acyclic graph with node and arc set $V$ and $A$, respectively, with a single source node $s$, from which the messages are sent, and a set of nodes $T \subseteq V - s$ called **terminals** where the messages are sent. Without loss of generality we may assume that $s$ has $k$ leaving arcs $a_1, \dots, a_k$. A **linear network code** of $k$ messages on $D$ over $\mathbb{F}_q$ is a mapping $\mathbf{c} : A \to \mathbb{F}_q^k$ satisfying $\mathbf{c}(a_i) = \mathbf{e}_i$, and which fulfills the **linear combination property**:

$$\mathbf{c}(uv) = \sum_{wu \in A} \alpha(wu, uv)\mathbf{c}(wu)$$

where $\alpha(wu, uv) \in \mathbb{F}_q$ for all $u \neq s$. Coefficients $\alpha(wu, uv)$ are the **local coefficients** of the network code and function $\mathbf{c}$ denotes the **global coefficients**. That is, on arc $a$ message $\mathbf{c}(a) \cdot \mathbf{M}$ is sent. We will use the notation $\langle \mathbf{c}, u \rangle = \langle \{\mathbf{c}(wu) |\ wu \in A\} \rangle$. For a linear network code $\mathbf{c}$, a node $v$ **can decode** (or receives) message $M_i$, if $\mathbf{e}_i \in \langle \mathbf{c}, v \rangle$.

A network code is **feasible** if every node $t$ in $T$ can decode every message $M_i$, $1 \leq i \leq k$. Note that local and global coefficients can be determined from each other, hence feasibility of local coefficients can be defined accordingly.

## 2.2 Deterministic Network Coding Problems

**Definition 2.2.** Let $L \subseteq A \times A$ be the set of consecutive pairs of arcs: $L = \{(wu, uv) |\ w, u, v \in V,\ wu, uv \in A\}$. For the sake of shortness members of $L$ are called **pairs**. The local coefficients of a network code form a mapping on the pairs: $\alpha : L \to \mathbb{F}_q$.

For a subset of pairs $M \subseteq L$, a mapping $\alpha_0 : M \to \{\mathbb{F}_q - 0\}$ is **extendable**, if it can be extended to the local coefficients of a feasible network code. The pairs in $M$ are called **determined**.

The **unicast deterministic network coding problem** is to decide whether a subset of determined pairs $M \subseteq L$ is extendable for a given network coding problem with a one-element terminal set. The multicast variance of the problem can be defined accordingly.

We say that $M$ is **fixable** if *any* nonzero-valued mapping $\alpha_0 : M \to \{\mathbb{F}_q - 0\}$ is extendable. The **fixable pairs problem** is to decide whether a given set $M \subseteq L$ is fixable or not.

For a pair $\ell = (wu, uv) \in L$, $wu$ and $uv$ are the **first** and **second** arcs of the pair, respectively, and $u$ is the **central node** of the pair. Two pairs $\ell_1$ and $\ell_2$ are **consecutive** if the second arc of $\ell_1$ is the first arc of $\ell_2$. A path contains a pair, if it contains both of its arcs. For a subset of pairs $M \subseteq L$, a node is **influenced** if it is the central node of a pair in $M$. A set of paths is $M$**-independent** if they are pairwise arc-disjoint and any influenced node is contained by at most one of them.

In this paper we use two well-known technical claims, proved for example in [12].

**Claim 2.3.** *Let vectors* $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}_q^k$ *form a basis of* $\mathbb{F}_q^k$ *and let* $\mathbf{v} \in \mathbb{F}_q^k$*. Then*

- *there is at most one value* $\alpha \in \mathbb{F}_q$ not *satisfying that the set* $\{\mathbf{v}'_1 = \mathbf{v}_1 + \alpha\mathbf{v}\} \cup \{\mathbf{v}_i\}_{i=2}^k$ *is also a basis,*
- *there is at most one value* $\beta \in \mathbb{F}_q$ not *satisfying that the set* $\{\mathbf{v}'_1 = \beta\mathbf{v}_1 + \mathbf{v}\} \cup \{\mathbf{v}_i\}_{i=2}^k$ *is also a basis.*

# 3 Algorithms for Deterministic Network Coding

Multicast deterministic network coding problem is equivalent to determining the simultaneous max rank completion of the transfer matrices, and if the field size is greater than the number of matrices given, then the matrices have a simultaneous max rank completion as proved in [8, 11]. This result can be reformulated as follows.

**Theorem 3.1.** *If $q > |T|$, a mapping is extendable over $\mathbb{F}_q$ if and only if for every $t \in T$ it is extendable for the one-element terminal set $\{t\}$.*

Here we give another, very simple, algorithmic proof for this theorem. The advantage of our algorithm is that it alters the value of each local coefficient at most only once. In addition, there is no restriction on the order of determining the coefficients. The idea can be also adapted for cyclic networks and hence accelerates earlier algorithms such as the classical multicast algorithm of Erez and Feder for cyclic networks [13]. Also, we derive a randomized algorithm for both the unicast and multicast problem, based on the idea of the algorithm. We assume that a subroutine is given which solves the unicast problem.

*Proof of Theorem 3.1.* For a terminal $t \in T$, let $\alpha_t$ denote the extension of $\alpha_0$ which is feasible for $t$ and let $\mathbf{c}_t : A \to \mathbb{F}_q^k$ denote the corresponding global coefficients. We start by defining $\alpha(\ell) = \alpha_0(\ell)$ for each $\ell \in M$. Let $\ell_1, \dots, \ell_p$ be an arbitrary order of the pairs in $L \setminus M$. We will determine a value $\alpha(\ell_i)$ for each $\ell_i$ in this order maintaining that the following mapping $\alpha_t^i$ is feasible for every $t$.

$$\alpha_t^i(\ell) = \begin{cases} \alpha(\ell) & \text{if } \ell \in M \cup \{\ell_1, \dots, \ell_i\}, \\ \alpha_t(\ell) & \text{otherwise.} \end{cases}$$

To show the existence of an appropriate $\alpha(\ell_i)$ we prove some lemmata.

**Lemma 3.2.** *Let $\alpha, \mathbf{c}$ denote the local and global coding coefficients of a network code, respectively. By altering a local coefficient $\alpha(uv, vw)$ to $\alpha'(uv, vw) = \alpha(uv, vw) + \beta$, the new global coefficients have the form $\mathbf{c}'(a) = \mathbf{c}(a) + \delta_a \beta \mathbf{c}(uv)$ with an appropriate value $\delta_a \in \mathbb{F}_q$ on every arc $a \in A$.*

*Proof.* We prove by induction on the topological order of the tails of the arcs. If the tail of an arc is earlier in the order than $v$, then $\mathbf{c}'$ remains $\mathbf{c}$ and the claim clearly holds. For arcs leaving $v$, the only arc where $\mathbf{c}$ changes is arc $vw$, and the claim is again clear. Suppose that the claim holds for every arc with tail before $z \in V$ and let $zx \in A$ be an arc. From the linear combination property we have $\mathbf{c}'(zx) = \sum_{yz \in A} \alpha(yz, zx)\mathbf{c}'(yz) = \sum_{yz \in A} \alpha(yz, zx)(\mathbf{c}(yz) + \delta_{yz}\beta\mathbf{c}(uv))$. □

**Lemma 3.3.** *Let vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{F}_q^k$ form a basis of $\mathbb{F}_q^k$ and let $\mathbf{v} \in \mathbb{F}_q^k$, $\delta_1, \dots, \delta_k \in \mathbb{F}_q$. Then there is at most one value $\beta \in \mathbb{F}_q$ satisfying that $\{\mathbf{v}'_i = \mathbf{v}_i + \delta_i \beta \mathbf{v}\}_{i=1}^k$ is not a basis.*

*Proof.* If every $\delta_i$ is zero then the statement is obvious, so without loss of generality we can assume that $\delta_1 \neq 0$. By subtracting $(\delta_i/\delta_1)\mathbf{v}'_1$ from each $\mathbf{v}_i$, we get that vectors $\{\mathbf{v}'_i\}_{i=1}^k$ form a basis if and only if $\{\mathbf{v}_1 + \delta_1\beta\mathbf{v}\} \cup \{\mathbf{v}_i - (\delta_i/\delta_1)\mathbf{v}_1\}_{i=2}^k$ does. Since $\{\mathbf{v}_1\} \cup \{\mathbf{v}_i - (\delta_i/\delta_1)\mathbf{v}_1\}_{i=2}^k$ is a basis, the lemma follows from Claim 2.3. □

Let $\mathbf{c}_t^i$ denote global coefficients corresponding to $\alpha_t^i$.

**Lemma 3.4.** *Suppose that the values $\alpha(\ell_1), \dots, \alpha(\ell_{i-1})$ are chosen so that $\mathbf{c}_t^{i-1}$ is feasible for $t$. Then there is at most one choice of $\alpha(\ell_i)$ such that $\mathbf{c}_t^i$ is not feasible for $t$.*

*Proof.* Since $\mathbf{c}_t^{i-1}$ is feasible, there is a $k$-element arc set $B_t = \{b_1, \dots, b_k\}$ entering $t$ on which the global coefficients of $\mathbf{c}_t^{i-1}$ form a basis. Let $\ell_i = (uv, vw)$. Mappings $\alpha_t^{i-1}$ and $\alpha_t^i$ differ in at most one value (on $\ell_i$), hence from Lemma 3.2 we get that global coefficients on arcs in $B_t$ have the following form: $\mathbf{c}_t^i(b_j) = \mathbf{c}_t^{i-1}(b_j) + \delta_{b_j}\mathbf{c}_t^{i-1}(uv)$. Lemma 3.3 says that there is at most one value of $\ell_i$ so that these vectors do not form a basis. □

Since the size of the field is greater than $|T|$, indeed, a good value can be chosen for each $\alpha(\ell_i)$. If $i = p$ then for every $t$ we have $\alpha_t^p = \alpha$, and we maintained the feasibility. This completes the proof. □

## 3.1 A Randomized Algorithm

Let a multicast deterministic network coding problem be given as described in Section 2.

Applying results in [14], Kim and Médard [8] gave a lower bound on the probability of a random network code to be feasible over $\mathbb{F}_q$ in the model of [1]. Their result can easily be generalized to the following.

**Theorem 3.5.** *If $q > |T|$ and a mapping $\alpha_0 : M \to \mathbb{F}_q$ has a feasible extension then the probability that a random extension is feasible, is at least $\left(1 - \frac{|T|}{q}\right)^{|A'|} \geq 1 - \frac{|T| \cdot |A'|}{q}$, where $A'$ is the subset of arcs which appear as a second arc in a pair in $L \setminus M$.*

The idea of our randomized algorithm is to first construct a random extension over a bigger field $\mathbb{F}_{q^r}$ of size $q^r$ such that $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^r}$. Then we can deterministically modify it to get another extension over $\mathbb{F}_q$ if $q > |T|$. Let $\alpha$ be a random extension of $\alpha_0$ over $\mathbb{F}_{q^r}$. If we choose $r$ so that $q^r > 2|T| \cdot |A'|$, then from Theorem 3.5, $\alpha$ is feasible with probability at least one half. If not, we repeat generating other random extensions till success. The expectation of the number of generations is two. If $\alpha$ is feasible, the extension over $\mathbb{F}_q$ is constructed in the following way.

**for all** $\ell \in L \setminus M$ **do**
  choose a value $f$ from $\mathbb{F}_q$ such that $\alpha$, changed only on $\ell$ to $\alpha(\ell) = f$, remains a feasible network code
**end for**

**Theorem 3.6.** *If $q > |T|$ and there exists a feasible extension over $\mathbb{F}_q$ then the algorithm finds one with probability at least $\left(1 - \frac{|T|}{q^r}\right)^{|A'|} \geq 1 - \frac{|T| \cdot |A'|}{q^r}$.*

*Proof.* We only need to show that a suitable value from $\mathbb{F}_q$ can be chosen for every pair $\ell \in L \setminus M$. In Lemma 3.2 we gave a formula, how the modification of a local coefficient influences global coefficients. Combined with Lemma 3.3, for each $t \in T$

we get that there is at most one value $f \in \mathbb{F}_{q^r}$ so that changing the value of $\alpha(\ell)$ to $f$ destroys feasibility to $t$. Hence in any subset $X \subseteq \mathbb{F}_{q^r}$ with $|X| > |T|$ there exists a value which preserves feasibility for every terminal simultaneously. Since $q > |T|$, subfield $\mathbb{F}_q$ is such a subset, which proves the theorem. □

We point out, that we need to make calculations over a bigger field, which can be time consuming. However these calculations are made only in the code-construction phase, we finally make the code over field $\mathbb{F}_q$ so when using the code the nodes need to make calculations over this small field.

# 4 Fixable pairs

We give a sufficient condition for a subset $M$ of pairs to be fixable.

**Theorem 4.1.** *Let $D = (V, A)$ an acyclic digraph and $T \subseteq V - s$ a terminal set having a feasible network code for $k$ messages over $\mathbb{F}_q$ with $q > |T|$, and let $M \subseteq L$ be a subset of pairs. If for every terminal $t \in T$ there exist $k$ $M$-independent paths from $s$ to $t$, so that none of the paths contains two consecutive pairs in $M$, then $M$ is fixable.*

*Proof.* We follow similar ideas for the network code construction as the ones in [12] but instead of determining the global coefficients one-by-one in a topological order we determine the local coefficients in a special order.

Let $\alpha_0$ be an arbitrary nonzero-valued mapping on $M$. From Theorem 3.1, $M$ is extendable if and only if it is extendable for every one-element terminal set $\{t\}$. Let $t$ be an arbitrary given terminal in $T$. We choose $k$ $M$-independent $st$-paths $P_1, \dots, P_k$ so that no path contains two consecutive pairs in $M$. First we consider the extension $\alpha$ of $\alpha_0$ which is zero on $L \setminus M$. Let $\mathbf{c}$ denote the global coefficients corresponding to $\alpha$. If $\alpha$ alters during the algorithm then $\mathbf{c}$ is modified accordingly. Let us fix a topological order of the nodes and let $\ell_1, \dots, \ell_{|L \setminus M|}$ be an order of the pairs in $L \setminus M$ according to the topological order of the heads of the second arcs. We determine the values of $\alpha$ on the pairs in this order and maintain a set of arcs $B = \{b_1, \dots, b_k\}$, so that $b_i \in P_i$, and $\mathbf{c}(B) = \{\mathbf{c}(b_i)\}_{i=1}^k$ forms a basis, and that $B$ finally contains only arcs entering $t$. First, let $B = \{a_1, \dots, a_k\}$. If there is a pair of the form $(a_i, a)$ in $M$, where $a \in P_i$, then we replace $a_i$ with $a$ in $B$.

**for** $i = 1...m$ **do**
  **if** $\ell_i = (a, a')$, $a \in B$ and $\ell_i$ is contained in path $P_j$ **then**
    $B \leftarrow B - a$
    $\text{arc}_{\text{new}} \leftarrow a'$
    **if** $a'$ is the first arc of a pair $\ell' = (a', a'') \in M$ that is also contained in $P_j$ **then**
      $\text{arc}_{\text{new}} \leftarrow a''$
    **end if**
    $B \leftarrow B + \text{arc}_{\text{new}}$
    if $\mathbf{c}(B)$ is not a basis then $\alpha(\ell_i) \leftarrow 1$.

**end if**
**end for**

**Claim 4.2.** *The modification of $\alpha(\ell_i)$ does not modify any arc in $B - \text{arc}_{\text{new}}$.*

*Proof.* The modification of $\alpha(\ell_i)$ influences an arc $e$ if and only if there is a path $a' = e_0, e_1, e_2, \ldots, e_z = e$ such that $\alpha(e_x, e_{x+1})$ is not zero for $0 \leq x < z$. Note that such a pair cannot be in $L \setminus M$ because these pairs have bigger index in our ordering, so their $\alpha$-values would be still zero. Hence $(e_x, e_{x+1}) \in M$ for each $0 \leq x < z$. Suppose that there exists such an arc $e$ which is in $B$ and is not contained in $P_j$. Since $P_j$ contains the head of $a'$, which is the central node of $(a', e_1)$ and the paths are $M$-independent, $e_1$ cannot be in a path different from $P_j$, so $z \geq 2$. Hence both the tail and head of $e$ are different from the head of $a'$. There was a point when $e$ got into $B$, let $\ell(e)$ be the pair that was being processed at that point. There are two cases: either $e$ is the second arc of $\ell(e)$ or $e$ is an arc following the second arc of $\ell(e)$. In both cases, the head of the second arc of $\ell(e)$ is reachable from the head of the second arc of $\ell_i$. From the choice of the order of processing the pairs, $\ell(e)$ should be processed later than $\ell_i$, which contradicts that $\ell(e)$ was processed earlier than $\ell_i$. □

**Claim 4.3.** *After processing a pair, $\mathbf{c}(B)$ form a basis of $\mathbb{F}_q^k$.*

*Proof.* If $\alpha(\ell)$ remained 0, the claim clearly holds. Using Claim 4.2 we observe that we can apply Claim 2.3, so only one value $\beta$ is wrong. Thus as zero was wrong, value 1 must be good. □

**Claim 4.4.** *After processing a pair $\ell$ in the algorithm, for every arc $b$ in $B$ one of the following hold:*

- *$b$ enters $t$,*
- *for the arc $b'$ following $b$ on $P_j$ the pair $(b, b')$ is not in $M$.*

*Proof.* Suppose that arc $b$ does not enter $t$ and $(b, b') \in M$. Let us take the step when $b$ got into $B$. From the choice of $P_j$, there are no consecutive pairs from $M$ on $P_j$, so $b$ cannot be the second arc of a pair in $M$ contained in $P_j$. Hence $\text{arc}_{\text{new}}$ should have been $b'$ instead of $b$. □

From Claim 4.4 and the choice of the order of pairs the final set $B$ will only contain arcs entering $t$, which proves the theorem.

□

## 4.1 Application of fixable pairs

Let $D = (V, A)$, $s$, $T$, $k$ be as described in Section 2. Suppose that a subset $W \subseteq V \setminus (T \cup \{s\})$ of intermediate nodes can only **broadcast** messages, that is, such a node sends the same message on each of its outgoing arc. To the best of our knowledge, there has been no characterization known on the existence of a feasible network code in such networks.

We say that some $st$-paths are **$W$-disjoint** if they are pairwise arc-disjoint and each node in $W$ is contained in at most one of the paths.

**Theorem 4.5.** *There exists a feasible network code for $T$ where every $W$-node broadcasts if and only if for every $t \in T$ there are $k$ $W$-disjoint $st$-paths.*

*Proof.* Let $D'$ denote the graph attained from $D$ by expanding each node $w \in W$ into two new nodes $w_1$ and $w_2$ with a new arc $w_1 w_2$ such that the incoming and outgoing arcs of $w$ become the incoming and outgoing arcs of $w_1$ and $w_2$, respectively. Clearly, the existence of $k$ $W$-disjoint $st$-paths in $D$ is equivalent with the existence of $k$ arc-disjoint $st$-paths in $D'$. If there is a feasible network code $\mathbf{c}$ on $D$ broadcasting on $W$, then it can be modified to be a feasible network code $\mathbf{c}'$ on $D'$ by setting for $uv \in A: \quad \mathbf{c}'(u_2 v_1) = \mathbf{c}(uv)$ and for $w \in W$ and for any $wv \in A$ we define $\mathbf{c}'(w_1 w_2) = \mathbf{c}(wv)$, this is legal as $w$ is a broadcasting node, consequently $\mathbf{c}(wv)$ is the same on every outgoing arc. This gives that the existence of arc-disjoint paths in $D'$ is necessary, which proves the necessity part of the theorem. For the other direction, let $M$ be the following subset of pairs in $D'$: $M = \{(w_1 w_2,\ w_2 v) \mid w \in W,\ wv \in A\}$. Note that $M$ does not contain consecutive pairs and since every central node of a pair in $M$ has in-degree one, $M$-independentness follows from arc-disjointness. Applying Theorem 4.1 we get that if there exist $W$-disjoint paths in $D$ then $M$ is fixable in $D'$ and so we can take a feasible extension of the constant 1-valued mapping on $M$. Let $\mathbf{c}'$ denote the global coefficients of the network code. One can easily get the global coefficients of a feasible network code on $D$ by contracting arcs in $D'$ of the form $w_1 w_2,\ w \in W$. □

The corresponding theorem for the following model can be similarly proven. We may have restrictions on the incoming arcs, e.g., a node in $W$ always gets the sum of the messages on the incoming arcs. If we choose $q$ to be $2^r$, then elements of $\mathbb{F}_q$ can naturally encoded by length $r$ bit-sequences, and in this case addition in $\mathbb{F}_q$ corresponds to XOR-ing the incoming sequences.

# 5 Conclusion

We investigated variations of the deterministic network coding problem. Given any unicast algorithm as a subroutine, a simple deterministic multicast algorithm has been presented, which can be applied on any acyclic network and hence is more general than former algorithms for linear deterministic relay networks, which work on layered acyclic graphs only. We also gave randomized algorithm over any field $\mathbb{F}_q$ with $q > |T|$ for both the unicast and multicast cases. We proposed the fixable pair problem, gave a sufficient condition and showed an application to networks with varying node transmission properties. A possible topic for further research is the better understanding of the min-max formula derived from [11]. It is defined on an auxiliary matrix, but, to the best of our knowledge, no max-flow–min-cut type formula is known for the general acyclic case of the deterministic network coding problem, like in the special case of deterministic relay networks.

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