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Tibor Jordán, Gábor Domokos, and Krisztina Tóth

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Tibor Jordán[⋆], Gábor Domokos[⋆⋆], and Krisztina Tóth[⋆⋆⋆]

Abstract

Let (G, p) be an infinitesimally rigid d-dimensional bar-and-joint framework and let L be an equilibrium load on p. The load can be resolved by appropriate stresses $w_{i,j}$, $ij \in E(G)$, in the bars of the framework. Our goal is to identify the following parts (zones) of the framework:

(i) when the location of an unloaded joint v is slightly perturbed, and the same load is applied, the stress will change in some of the bars. We call the set of these bars the *influenced zone of* v (with respect to L, p and the modified configuration p'),

(ii) let S be a designated set of joints and suppose that each joint with a non-zero load belongs to S. The *active zone* of S (with respect to p and L) is the set of those bars in which the stress, which resolves L, is non-zero.

We show that if (G, p) is generic and $d = 2$ then, for almost all loads, these zones depend only on the graph G of the framework and can be computed by efficient combinatorial methods.

1 Introduction

Let (G, p) be an infinitesimally rigid d-dimensional bar-and-joint framework and let L be an equilibrium load on p. The load can be resolved by appropriate stresses $w_{i,j}$, $ij \in E(G)$, in the bars of the framework. Our goal is to identify the following parts (zones) of the framework:

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[⋆]Department of Operations Research, Eötvös University, Pázmány sétány 1/C, 1117 Budapest, Hungary. e-mail: jordan@cs.elte.hu

[⋆⋆]Department of Mechanics, Materials and Structures, Budapest University of Technology and Economics, Műegyetem rkp. 3-9, H-1111 Budapest, Hungary. e-mail: domokos@iit.bme.hu

[⋆⋆⋆]Department of Mechanics, Materials and Structures, Budapest University of Technology and Economics, Műegyetem rkp. 3-9, H-1111 Budapest, Hungary. e-mail: tothk@szt.bme.hu

We show that if (G, p) is generic and $d = 2$ then, for almost all loads, these zones depend only on the graph G of the framework and are independent of the configurations p, p' and the load L. See Figure 1.

We give an efficient combinatorial algorithm for finding these zones. As a corollary we also obtain that the influenced zone of a vertex is monotone increasing when new constraints (bars) are added to the framework.

These results may be useful when one needs to recompute the stresses due to minor changes in the geometry of the framework. Identifying the influenced zone by a quick combinatorial algorithm can serve as a preprocessing step which reduces the size of the problem.

Our results may also be used in the analysis and design of highly geometrically sensitive (non-sensitive, resp.) generic frameworks (graphs), in which a small perturbation of any vertex results in the change of stresses in the whole (resp. in just a small part of the) framework. For example, if the goal is to attach (a small number of) devices to some of the bars that can observe a small perturbation of some joint by sensing the change of stress in the bar, a highly sensitive framework seems more advantageous. On the other hand, recomputing the stresses is easier if the framework is non-sensitive.

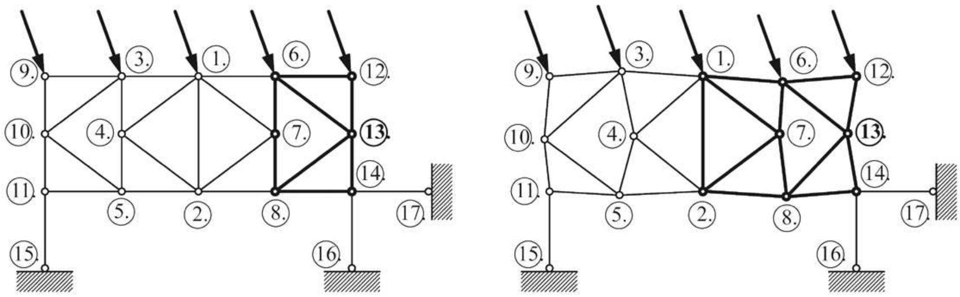

Figure 1: Two loaded pinned frameworks with the same underlying graph. The non-zero components of the loads are denoted by arrows. In the first framework, which is non-generic, the influenced zone (the thick bars) of vertex 13 is smaller than its influenced zone in the second framework, which is generic.

We note that some of the notions and ideas we shall use are from [11, 12], where the notions of influenced and active zones (in minimally rigid resp. rigid frameworks) as well as some of the results with proof ideas appeared.

The organization of the paper is as follows. In Section 2 we recall some basic definitions and results related to bar-and-joint frameworks. In Section 3 we give bounds on these zones which are valid for all minimally rigid d-dimensional frameworks and loads. After a few combinatorial lemmas, given in Section 4, we show that for minimally rigid two-dimensional generic frameworks these bounds are tight and give rise to combinatorial characterizations for the influenced zones and active zones. These results are in Section 5. In Section 6 we extend our results to arbitrary rigid two-dimensional generic frameworks. Section 7 is devoted to some concluding remarks.

1.1 Notation

Graphs in this paper are undirected and simple (that is, no loops and multiple edges are allowed). Let $G = (V, E)$ be a graph. For $X \subseteq V$ let $G[X]$ be the induced subgraph of G on vertex set X and $E_G(X)$ be the set of edges of $G[X]$. The number of edges in $G[X]$ is denoted by $i_G(X)$. The *degree* of X in G, denoted by $d_G(X)$, is the number of edges of G with exactly one end-vertex in X. For $v \in V$, $d_G(v)$ denotes the degree of v and $N_G(v)$ the set of neighbours of v in G. We will suppress the subscript G when the graph is clear from the context.

Let (G, p) be a framework and let H be a subgraph of G. For simplicity we may use (H, p) to denote the framework with underlying graph H and for which the configuration of the vertices of H is determined by the restriction of p to $V(H)$.

2 Preliminaries

In this section we introduce some basic notions of rigidity theory along with some preliminary lemmas. See [3, 13, 14] for a detailed introduction to the theory of rigid graphs and frameworks.

A d-dimensional (bar-and-joint) *framework* (G, p) is a pair, where $G = (V, E)$ is a graph and $p : V \to \mathbb{R}^d$ is a map. We say that p is a *configuration* of V and (G, p) is a *realization* of G in $\mathbb{R}^d$. The *rigidity matrix* of the framework is the matrix $R(G, p)$ of size $|E| \times d|V|$, where, for each edge $v_i v_j \in E$, in the row corresponding to $v_i v_j$, the entries in the d columns corresponding to vertices i and j contain the d coordinates of $(p(v_i) - p(v_j))$ and $(p(v_j) - p(v_i))$, respectively, and the remaining entries are zeros.

Lemma 2.1. *[13, Lemma 11.1.3] Let (G, p) be a framework in $\mathbb{R}^d$. Then $\text{rank}\, R(G, p) \leq S(n, d)$, where $n = |V(G)|$ and*

$$S(n,d) = \begin{cases} nd - \binom{d+1}{2} & \text{if } n \geq d+2 \\ \binom{n}{2} & \text{if } n \leq d+1. \end{cases}$$

A framework (G, p) for which $R(G, p)$ has rank $S(n, d)$ is said to be *infinitesimally rigid* in $\mathbb{R}^d$. We say that (G, p) is *independent* if the rows of $R(G, p)$ are linearly independent. An independent and infinitesimally rigid framework is called *minimally infinitesimally rigid*.

Infinitesimal rigidity can also be characterized by equilibrium loads and infinitesimal motions as follows. An *equilibrium load* on a configuration p of point set V is an assignment $L : V \to \mathbb{R}^d$ of vectors L_i to the vertices "without net translational or rotational component". More precisely, an equilibrium load is any vector in $\mathbb{R}^{dn}$ orthogonal to the kernel of $R(K_n, p)$, where K_n is the complete graph on n vertices. In particular, the d-tuples of any row of the rigidity matrix $R(K_n, p)$ form an equilibrium load on p. Thus the row space of $R(G, p)$ is a subset of the space of equilibrium loads. The equilibrium loads form a subspace of $\mathbb{R}^{dn}$ of dimension $S(n, d)$ (provided that the affine span of the points is $\mathbb{R}^d$, or they are affine independent).

A *resolution* of equilibrium load L on (G, p) is a *stress*, which is an assignment of scalars $\omega : E \to \mathbb{R}$ to the edges such that for each vertex $i \in V$:

$$L_i + \sum_{j:ij \in E} w_{i,j}(p_i - p_j) = 0. \tag{1}$$

Let $R_{i,j}(p)$ denote the row of $R(G, p)$ corresponding to edge (i, j). With this notation we have that

$$L + \sum_{ij \in E} w_{i,j} R_{i,j}(p) = 0. \tag{2}$$

By definition, (G, p) is infinitesimally rigid if the dimension of the row space equals the dimension of the equilibrium loads. Since the row space is contained in the space of equilibrium loads, the two spaces are the same for infinitesimally rigid frameworks.

A *self-stress* on framework (G, p) is an assignment $\omega : E \to \mathbb{R}$ such that, for each vertex $i \in V$:

$$\sum_{j:ij \in E} w_{i,j}(p_i - p_j) = 0. \tag{3}$$

Thus a self-stress is a resolution of the zero equilibrium load. The self-stresses are the row dependencies of the rigidity matrix $R(G, p)$. If the framework is independent then the resolution of an equilibrium load is unique. However, if the framework is dependent then we can add any multiple of a self-stress to a given resolution to get another resolution.

An *infinitesimal motion* of (G, p) is an assigment of *velocities* to the vertices $u : V \to \mathbb{R}^d$, such that for each edge $ij \in E$ we have $(p_i - p_j)(u_i - u_j) = 0$. Equivalently, an infinitesimal motion is a solution to the system of linear equations $R(G, p)x = 0$. An infinitesimal motion is *trivial* if it belongs to the kernel of $R(K_n, p)$.

Let $\mathcal{S}(G, p)$ be the vector space of self-stresses of (G, p) and let $\mathcal{M}(G, p)$ be the vector space of infinitesimal motions of (G, p). We shall use the following well-known fact: for a d-dimensional framework (G, p) we have

$$|E| - dim(\mathcal{S}(G, p)) = d|V| - dim(\mathcal{M}(G, p)) \tag{4}$$

We also need the following basic result.

Theorem 2.2. *[13, Theorem 3.1.1] Let (G, p) be a d-dimensional framework. Then the following are equivalent:*
(i) (G, p) is infinitesimally rigid,
(ii) every equilibrium load on p has a resolution in the bars of (G, p),
(iii) every infinitesimal motion of (G, p) is trivial.

The next lemma gives a similar connection between stresses and motions for a designated pair of vertices.

Lemma 2.3. *[13, Theorems 3.1.3, 9.3.1] Let (G, p) be a d-dimensional framework and $h, k \in V(G)$. Then the following are equivalent:*
(i) $R_{h,k}(p)$ cannot be resolved,
(ii) every self-stress ω on $E \cup \{hk\}$ is zero on hk,
(iii) there is an infinitesimal motion u on (G, p), such that $(p_h - p_k)(u_h - u_k) \neq 0$.

We shall also use the following simple operation. Given a graph $G = (V, E)$, the *vertex d-addition* operation adds a new vertex v_0 and d new edges $v_0v_1, ..., v_0v_d$ for some $v_i \in V$, $1 \leq i \leq d$. The corresponding geometric operation on (G, p) adds a new vertex positioned at p_0 and inserts d new bars from p_0 to p_i, $1 \leq i \leq d$.

Lemma 2.4. *[13, Lemma 11.1.1] Let (G, p) be a d-dimensional framework and let (G', p) be obtained from (G, p) by a vertex d-addition. If $p_0, p_1, ..., p_d$ are in general position in d-space then rank $R(G', p) =$ rank $R(G, p) + d$.*

We shall also need the next simple fact.

Lemma 2.5. *Let p be a configuration of point set P in $\mathbb{R}^d$ and let $Q \subseteq P$. Let $q = p|Q$ denote the corresponding configuration of point set Q. Suppose that L is an equilibrium load on p such that $L(v) = 0$ for all $v \in P - Q$. Then $L|q$ is an equilibrium load on q.*

Generic rigidity

The rigidity matrix of (G, p) defines the *rigidity matroid* of (G, p) on the ground set E by linear independence of the rows of the rigidity matrix. (See [9, 10] for basic concepts in matroid theory.) A *framework* (G, p) is *generic* if for each subset of edges (rows) the corresponding submatrix of $R(G, p)$ has maximum rank (over all realizations of G)[1]. Thus the infinitesimal rigidity of a generic framework (G, p) depends only on G and any two generic frameworks (G, p) and (G, p') have the same rigidity matroid. We call this the d-dimensional *rigidity matroid* $\mathcal{R}_d(G)$ of graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

We say that a graph $G = (V, E)$ is *rigid* in $\mathbb{R}^d$ if $r_d(G) = S(n, d)$. We say that G is *M-independent* or an *M-circuit* in $\mathbb{R}^d$ if E is independent or a circuit, respectively, in $\mathcal{R}_d(G)$. If G is both independent and rigid then G is said to be *minimally rigid*.

Lemma 2.1 implies the following necessary condition for G to be M-independent.

Lemma 2.6. *If $G = (V, E)$ is M-independent in $\mathbb{R}^d$ then $i(X) \leq S(|X|, d)$ for all $X \subseteq V$.*

Laman proved that for $d = 2$ the above sparsity condition is also sufficient.

Theorem 2.7. *[7] A graph $G = (V, E)$ is M-independent in $\mathbb{R}^2$ if and only if*

$$i(X) \leq 2|X| - 3 \text{ for all } X \subset V \text{ with } |X| \geq 2. \tag{5}$$

It follows that G is minimally rigid in $\mathbb{R}^2$ if and only if $|E| = 2|V| - 3$ and (5) holds.

It remains an open problem to find good characterizations for independence or, more generally, the rank function in the d-dimensional rigidity matroid of a graph when $d \geq 3$.

[1] For example, if the set of coordinates of the points $p(v)$, $v \in V$, is algebraically independent over the rationals (that is, if the coordinates do not satisfy any non-zero polynomial with rational coefficients) then (G, p) is generic.

Stresses and internal forces

The definition of a stress above is different from what the word means in the engineering literature. The engineering definition of a stress in a bar is the magnitude of the internal force per unit area of the cross section of the bar (in trusses). By assuming that the cross sections are equal, we have

$$\omega_{i,j}(p_i - p_j) = \sigma_{i,j}\frac{(p_i - p_j)}{|p_i - p_j|},$$

where $\sigma_{i,j}$ is the "engineering stress".

It is easy to see that the active zone of a set of vertices in a minimally rigid graph remains unchanged if we use σ instead of ω: it is due to the uniqueness of the stress which resolves an equilibrium load. The influenced zone of a vertex, however, may be different when we consider σ instead of ω.

3 Influenced zones and active zones in minimally rigid frameworks

Let (G, p) be a minimally infinitesimally rigid framework in $\mathbb{R}^d$ and let $v \in V(G)$ be a designated vertex. Consider an equilibrium load $L : V \to \mathbb{R}^d$ with $L(v) = 0$. Let ω be the stress in (G, p) which resolves L. Let (G, p') be another minimally infinitesimally rigid realization of G in which $p'(v) \neq p(v)$ but $p'(u) = p(u)$ for all $u \in V(G)$ with $u \neq v$. Then L is also an equilibrium load on (G, p'). Let ω' be the stress in (G, p') which resolves L.

The *influenced zone* of v, denoted by $I_v(G, L, p, p')$, is the set of those edges $ij \in E(G)$ for which $\omega_{ij} \neq w'_{ij}$.

Theorem 3.1. *Let H be a subgraph of G with $(N_G(v) \cup \{v\}) \subseteq V(H)$ and suppose that the subframework (H, p) is minimally infinitesimally rigid. Then*

$$I_v(G, L, p, p') \subseteq E(H). \tag{6}$$

Proof: Let G^* be the graph obtained from G by adding a new vertex v' and new edges $v'u$ for all $u \in N_G(v)$. Consider the framework (G^*, p^*), where $p^*(u) = p(u)$ for all $u \in V(G)$ and $p^*(v') = p'(v)$. We may extend the stress ω to G^* by defining the stress to be zero on the edges incident with v'. By using a similar extension, we may also think of ω' as a stress on G^*. Then we can define $\omega^* = \omega - \omega'$ and deduce that

$$\omega^* \text{ is a self-stress on } (G^*, p^*). \tag{7}$$

Since (G, p') is infinitesimally rigid, $d_G(v) \geq d$ and the points p_i, $i \in N_G(v) \cup \{v\}$, are not in a hyperplane. Hence it follows from Lemma 2.4 that (H, p) can be extended to an infinitesimally rigid subframework (H^*, p^*) of (G^*, p^*) by adding v' and a properly chosen set of d edges incident with v'. It also follows that (G^*, p^*) is infinitesimally rigid.

Hence the remaining $d_G(v) - d$ edges incident with v' are all redundant in (G^*, p^*) (that is, deleting such an edge does not destroy infinitesimal rigidity), which implies that for all edges $e \in E(G) - E(H)$ we must have $rank\ R(G^*, p^*) = rank\ R(G^* - e, p^*) + 1$. Thus we must also have $\omega^*(e) = 0$ and the theorem follows. •

Let (G, p) be a minimally infinitesimally rigid framework and let $S \subseteq V$ be a designated vertex-set. Let L be an equilibrium load on (G, p) with $\{v \in V : L(v) \neq 0\} \subseteq S$. The *active zone* of S, with respect to p and L, denoted by $A_S(G, L, p)$, is the set of those edges in which the stress, which resolves L, is non-zero.

Theorem 3.2. *Let H be a subgraph of G with $S \subseteq V(H)$ and suppose that the subframework (H, p) is minimally infinitesimally rigid. Then*

$$A_S(G, L, p) \subseteq E(H). \tag{8}$$

Proof: Since L_H, the restriction of L to $V(H)$, is an equilibrium load on (H, p), there is a stress ω on the edge set $E(H)$ which resolves L_H. We may extend ω to the edges of $E(G) - E(H)$ by zeros to obtain a stress on $E(G)$ which resolves L. Since the resolution of L is unique, the theorem follows. •

We shall prove that if (G, p) is a two-dimensional generic framework then there is a *unique* smallest subgraph H satisfying the conditions of Theorems 3.1 and 3.2, respectively. Furthermore, for typical loads, we have equality in (6) and (8) for this smallest subgraph H. To verify these claims we need a few combinatorial lemmas which are given in the next section.

4 Cores in two-dimensional minimally rigid graphs

In the rest of this paper we shall consider generic frameworks and the case when $d = 2$. We shall suppress explicit reference to the dimension.

Let $G = (V, E)$ be a minimally rigid graph. For a given $S \subseteq V$ with $|S| \geq 2$ let C_S be a minimal minimally rigid subgraph of G with $S \subseteq V(C_S)$. It follows from the next well-known lemma (see e.g. [4]) that C_S is unique.

Lemma 4.1. *[4, Lemma 2.3] Let $G = (V, E)$ be an M-independent graph and let G_1, G_2 be minimally rigid subgraphs of G with $|V(G_1) \cap V(G_2)| \geq 2$. Then $G_1 \cap G_2$ is minimally rigid.*

The unique minimal minimally rigid subgraph C_S of G with $S \subseteq V(C_S)$ is called the *rigid core* (or simply the *core*) of vertex set S in G and is denoted by $C_S(G)$. When $S = \{a, b\}$ for some $a, b \in V$ then we may also use the notation $C_{a,b}(G)$.

We say that a pair a, b is *loose* in (a not necessarily rigid) graph G if there is no rigid subgraph H of G with $a, b \in V(H)$. It can be seen that a, b is loose if and only if for every generic realization (G, p) there is an infinitesimal motion u with $(p_a - p_b)(u_a - u_b) \neq 0$.

Before proving the main combinatorial lemma, we recall another well-known result.

Lemma 4.2. *[4, Corollary 2.14] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two rigid graphs with $|V_1 \cap V_2| \geq 2$. Then $G_1 \cup G_2$ is rigid.*

Lemma 4.3. *Let G be a minimally rigid graph and let $S \subseteq V$ with $|S| \geq 2$. Let B be a set of new edges on S for which $\bar{S} = (S, E_G(S) \cup B)$ is minimally rigid. Then*
(i) $C_S(G) = \cup_{ab \in E(\bar{S})} C_{a,b}(G)$;
(ii) For each $e \in E(C_S(G)) - E(S)$ there is a pair $a, b \in S$ with $ab \in B$ such that a, b is loose in $C_S(G) - e$.

Proof: Let $C = \cup_{ab \in E(\bar{S})} C_{a,b}(G)$. To prove (i) first observe that, by definition, $C_{a,b}(G) \subseteq C_S(G)$ and hence $C \subseteq C_S(G)$. Thus, since $S \subseteq C$, it remains to prove that C is minimally rigid. To see this note that C can be obtained from the minimally rigid graph $\bar{S}$ by attaching the subgraphs $C_{a,b}(G)$, for all $ab \in B$ (which preserves rigidity by Lemma 4.2 and makes all edges of B redundant) and then deleting the edges of B.

Next consider (ii). For a contradiction suppose that each pair a, b with $ab \in B$ is part of a rigid subgraph in $G - e$. Hence e is not an edge of $C_{a,b}(G)$, for all pairs a, b with $ab \in B$. Now (i) implies $e \notin E(C_S(G))$, a contradiction. This proves (ii). •

Note that (i) holds for any set B of new edges for which $\bar{S} = (S, E_G(S) \cup B)$ is minimally rigid. It also implies that to find the core of S it suffices to have a subroutine for finding the core $C_{a,b}(G)$ of a given vertex pair a, b (and a proper set B of new edges). Furthermore $C_{a,b}(G) + ab$ is the unique M-circuit of $G + ab$[2]. These observations imply that there is an efficient algorithm for finding the core $C_S(G)$, see Section 7.

See Figure 2 for an illustration.

A rigid graph G is said to be *redundantly rigid* if $G - e$ is rigid for all $e \in E(G)$.

Lemma 4.4. *Let H be a minimally rigid graph with a designated vertex v of degree at least three. Let H^* be obtained from H by adding a new vertex v' and edges from v' to each vertex in $N_H(v)$. Suppose that $C_{N(v)}(H) = H$. Then H^* is redundantly rigid.*

Proof: Since H is rigid, it follows from Lemma 2.4 that H^* is rigid. In fact, by adding an arbitrary pair of edges incident with v' to H we obtain a rigid spanning subgraph of H^*, by Lemma 2.4. Since the degree of v' in H^* is at least three, it follows that there is a rigid spanning subgraph of H^* not containing e, for all edges e incident with v'. Thus $H^* - e$ is rigid. By symmetry, the same conclusion holds for all edges e incident with v.

Next consider an edge e not incident with v or v' and suppose, for a contradiction, that $H^* - e$ is not rigid. Let $v'a, v'b, v'w$ be three edges. Since $H - e + v'a + v'b$ is M-independent and $H' = H - e + v'a + v'b + v'w$ is not (for otherwise $H^* - e$ would be rigid) it follows that $v'w$ belongs to an M-circuit C of H'. Since C has minimum

[2]This follows from (5). The count implies that the following three statements are equivalent: (a) $H = (V, E)$ is an M-circuit; (b) $|E| = 2|V| - 2$ and $H - e$ is minimally rigid for all $e \in E$; (c) $|E| = 2|V| - 2$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $2 \leq |X| \leq |V| - 1$. For example, K_4, $K_{3,3}$ plus an edge, and $K_{3,4}$ are all M-circuits.

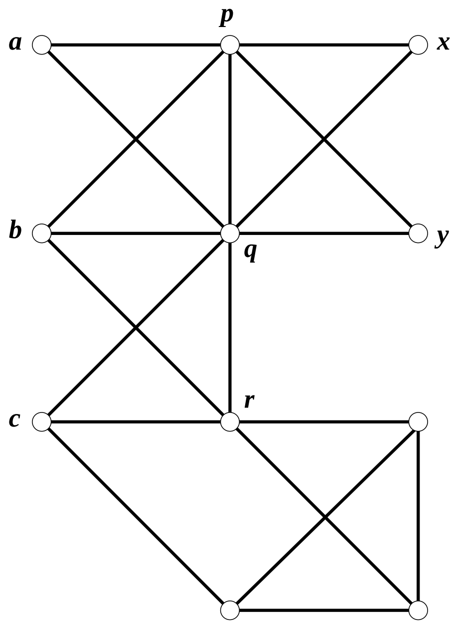

Figure 2: This graph G is minimally rigid. Let $S = \{a, b, c\}$. For $B = \{ab, bc, ca\}$ we have that $\bar{S} = (S, E_G(S) \cup B)$ is minimally rigid (a triangle). The vertex sets of the subgraphs $C_{a,b}$ and $C_{b,c}$ are $\{a, b, p, q\}$ and $\{b, c, q, r\}$, respectively, while $C_{a,c}$ has vertex set $\{a, p, b, q, r, c\}$. Thus C_S is the minimally rigid subgraph of G on vertices $\{a, b, c, p, q, r\}$. By adding an edge xy to G we obtain a rigid graph $\bar{G}$ in which another minimally rigid spanning subgraph is $H = \bar{G} - pq$. The core of S in H is the minimally rigid subgraph of H on vertex set $T = \{a, b, c, p, q, r, x, y\}$. The graph $\bar{G} + B$ has seven M-connected components: the subgraph induced by T and the remaining six edges as singleton components. Thus the active zone of S in $\bar{G}$ is the edge set of $\bar{G}[T]$.

degree three, C must contain $v'a$ and $v'b$. Now $K = C - v'w$ is a rigid subgraph of H' (and of $H^* - e$) which contains vertices a, b. Furthermore, since $d_K(v') = 2$, $K - v'$ is a rigid subgraph of $H - e$ which contains a, b. Thus $e \notin C_{a,b}(H)$ for all pairs $a, b \in N_H(v)$. Hence, by Lemma 4.3(i), $e \notin C_{N(v)}(H)$, which contradicts our assumption. This completes the proof. •

5 Influenced zones and active zones in minimally rigid graphs

Consider a minimally rigid generic framework (G, p) and let $v \in V(G)$ be a designated vertex. The equilibrium loads L on (G, p) satisfying $L(v) = 0$ form a subspace $\mathcal{L}_v$ of dimension $2n - 5$ in the space of all equilibrium loads on (G, p), where $n = |V(G)|$.

Recall from the proof of Theorem 3.1 that we used G^* to denote the graph obtained from G by adding a new vertex v' and new edges $v'u$ for all $u \in N_G(v)$ and (G^*, p^*)

to denote the framework, in which $p^*(u) = p(u)$ for all $u \in V(G)$ and $p^*(v') = p'(v)$, where $p'(v)$ is the modified position of v. We also defined the self-stress $\omega^* = \omega - \omega'$ on (G^*, p^*), where ω and ω' denoted the stresses resolving a given equilibrium load $L \in \mathcal{L}_v$ on (G, p) and (G, p'), respectively.

Theorem 5.1. *Let (G, p) be a minimally rigid generic framework and let (G, p') be obtained from (G, p) by relocating v so that (G^*, p^*) is also generic. Suppose that $d_G(v) \geq 3$. Then for all equilibrium loads $L \in \mathcal{L}_v$, with the exception of finitely many proper subspaces, each edge $e \in E(C_{N(v)}(G))$ belongs to the influenced zone $I_v(G, L, p, p')$.*

Proof: Let $k = d_G(v)$. Let $H = C_{N(v)}(G)$ and let H^* be obtained from H by adding a new vertex v' and edges from v' to each vertex of $N_H(v)$. Thus H^* is a subgraph of G^*. Since $k \geq 3$, we have $v \in C_{N(v)}(G)$. Clearly, $H = C_{N(v)}(H)$.

By Theorem 3.1, and since (G, p) is generic, $\omega^*(e) = 0$ for all $e \in E(G^*) - E(H^*)$. Thus the restriction of ω^* to $E(H^*)$ is a self-stress on (H^*, p^*). Note that for a given $L \in \mathcal{L}_v$ the values of the unique stress, which resolves L, on the edges of $\delta(v)$ belong to the projection of $\mathcal{S}(H^*, p^*)$ to $\delta(v)$. Since (H^*, p^*) is infinitesimally rigid and has $n + 1$ vertices and $2n - 3 + k$ edges, we have $dim\ \mathcal{S}(H^*, p^*) = k - 2$ by (4).

Now suppose that $\omega^*(f) = 0$ for some edge $f \in E(H^*)$. Lemma 4.4 and the fact that (G^*, p^*) is generic imply that $(H^* - f, p^*)$ is infinitesimally rigid. Thus $dim\ \mathcal{S}(H^* - f, p^*) = k - 3$.

Consider the projection $\mathcal{S}_v^f$ of the space $\mathcal{S}(H^* - f, p^*)$ to $\delta(v)$. Let $\mathcal{L}_v^f$ denote the space of those equilibrium loads $L \in \mathcal{L}_v$ for which the projection of ω, the unique stress in (G, p) which resolves L, to the edges of $\delta(v)$, belongs to $\mathcal{S}_v^f$. For this space we have $dim\ \mathcal{L}_v^f \leq dim\ \mathcal{S}_v^f + |E(G - v)| \leq k - 3 + 2n - 3 - k = 2n - 6$. Thus $\mathcal{L}_v^f$ is a proper subspace of $\mathcal{L}_v$,as claimed. •

Note that the subspaces $\mathcal{L}_v^f$ depend on $p'(v)$. We say that an equilibrium load $L \in \mathcal{L}_v$ is *typical* (with respect to G, p, and p') if L does not belong to $\mathcal{L}_v^f$ for any $f \in E(H^*)$. Note that all loads in the vector space $\mathcal{L}_v$, except the members of a finite number of smaller dimensional proper subspaces, are typical. It follows from the proof of the theorem that if $k = 3$ then all non-zero loads are typical.

By Theorems 3.1 and 5.1 we have

Theorem 5.2. *Let (G, p) be a minimally rigid generic framework and let (G, p') be obtained from (G, p) by relocating v so that (G^*, p^*) is also generic. Suppose that $d_G(v) \geq 3$ and let $L \in \mathcal{L}_v$ be typical with respect to G, p, and p'. Then*

$$I_v(G, L, p, p') = E(C_{N(v)}(G)). \tag{9}$$

Thus, when the framework is generic and the load is typical with respect to G, p, and p', the influenced zone depends only on the graph and we may simply denote it by $I_v(G)$ and call it the *influenced zone of v in G*.

Note that the case when $d_G(v) = 2$ is straightforward: in this case the stress on each edge incident with v is zero, for all $L \in \mathcal{L}_v$, and hence the influenced zone is empty.

5.1 Active zones in minimally rigid graphs

Consider a minimally rigid generic framework (G,p) and a designated set $S \subseteq V$ with $|S| \geq 2$. Let $H = G[S]$. The equilibrium loads L on (G,p) with $\{v \in V : L(v) \neq 0\} \subseteq S$ correspond to equilibrium loads on (H,p) and form a vector space $\mathcal{L}_S$ of dimension $2|S|-3$. Let $C = C_S(G)$ be the core of S in G. For each $e \in E(C)$ the equilibrium loads of $\mathcal{L}_S$ that can be written as $\sum_{ij \in E(C)-e} w_{i,j} R_{i,j}(p)$ form a subspace $\mathcal{L}_S^e$ of dimension at most $2|S|-4$, since there is a pair $a,b \in S$ for which the equilibrium load $R_{a,b}(p)$ cannot be obtained in this form. For edges $e \in E(H)$ this follows from the independence of the framework, while for edges $e \in E(C) - E(H)$ it follows by Lemmas 2.3 and 4.3(ii). We say that an equilibrium load $L \in \mathcal{L}_S$ is *typical* with respect to G, S and p if L does not belong to $\mathcal{L}_S^e$ for any $e \in E(C)$. Note that all loads in the vector space $\mathcal{L}_S$, except the members of a finite number of smaller dimensional proper subspaces, are typical. Thus we have:

Theorem 5.3. *Let (G,p) be a minimally rigid generic framework. Then for all equlibrium loads $L \in \mathcal{L}_S$, with the exception of finitely many proper subspaces, the stress ω on (G,p) which resolves L satisfies $\omega(e) \neq 0$ for all edges $e \in E(C_S(G))$.*

By Theorems 3.2 and 5.3 we obtain the following characterization of active zones in minimally rigid generic frameworks.

Theorem 5.4. *Let (G,p) be a minimally rigid generic framework and let $L \in \mathcal{L}_S$ be an equilibrium load which is typical with respect to G, S, and p. Then*

$$A_S(G,L,p) = E(C_S(G)).$$

Thus, when the framework is generic and the load is typical with respect to G, S and p, the active zone depends only on the graph and we may simply denote it by $A_S(G)$ and call it the *active zone of S in G*.

By comparing Theorems 5.2 and 5.4 we can deduce that the influenced zone of v equals the active zone of its neighbour set.

Corollary 5.5. *Suppose that $d_G(v) \geq 3$. Then $I_v(G) = A_{N(v)}(G)$.*

6 Rigid graphs

In this section we consider the general case of rigid two-dimensional generic frameworks. Let $G = (V,E)$ be a rigid graph and let $S \subseteq V$ with $|S| \geq 2$. We say that $e \in E$ is in the *active zone* of S in G if there is a minimally rigid spanning subgraph H of G for which e is in the active zone of S in H. To characterize the active zones of rigid graphs and to develop an efficient algorithm for identifying them we need some basic concepts from matroid theory.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, we define a relation on E by saying that $e, f \in E$ are related if $e = f$ or if there is a circuit C in $\mathcal{M}$ with $e, f \in C$. It is well-known that this is an equivalence relation. The equivalence classes are called the *components* of $\mathcal{M}$. If $\mathcal{M}$ has at least two elements and only one component then $\mathcal{M}$ is said to be *connected*. If $\mathcal{M}$ has components $E_1, E_2, \ldots, E_t$ and $\mathcal{M}_i$ is the matroid restriction of $\mathcal{M}$ onto E_i then $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \ldots \oplus \mathcal{M}_t$, where $\oplus$ denotes the direct sum of matroids. A matroid $\mathcal{M}$ is connected if and only if for every proper subset $F \subset E$ there is a circuit C with $C \cap F \neq \emptyset \neq C - F$. Let B be a base of $\mathcal{M}$ and let $e \in E - B$. Then the *fundamental circuit* of e with respect to B is the unique circuit in $B + e$. See [9] for more details.

We say that a graph $G = (V, E)$ is *M-connected* if $\mathcal{R}(G)$ is connected. For example, $K_{3,m}$ is M-connected for all $m \geq 4$. The *M-components* of G are the subgraphs of G induced by the components of $\mathcal{R}(G)$. The M-components are also vertex-induced subgraphs of G. It is not hard to see that if G is M-connected then G is (redundantly) rigid. See [4, Section 3] for further properties of M-connected graphs.

We shall need the following lemma.

Lemma 6.1. *Let $\mathcal{M} = (E \cup D, \mathcal{I})$ be a connected matroid with rank function r, where $D \neq \emptyset$, and suppose that $r(E \cup D) = r(E)$. Then for every $f \in E$ there exists a base $H \subseteq E$ such that f is in the fundamental circuit of some element $e \in D$ with respect to H.*

Proof: Since $\mathcal{M}$ is connected, there is a circuit C_f with $f \in C_f$ and $C_f \cap D \neq \emptyset$. Suppose that C_f, with these properties, is chosen so that $|C_f \cap D|$ is as small as possible. We claim that $|C_f \cap D| = 1$ must hold. For a contradiction suppose that $|C_f \cap D| \geq 2$ and let $e \in C_f \cap D$. Take some base $H' \subseteq E$ and let C_e be the fundamental circuit of e with respect to H'. Now $f \in C_e$ would contradict our assumption, so $f \in C_f - C_e$ must hold. By applying the strong circuit axiom[3] to circuits C_f, C_e, we obtain that there is a circuit $C \subseteq C_f \cup C_e$ with $f \in C$ and $e \notin C$. By the choice of C_f we must have $C \cap D = \emptyset$. Clearly, $C \cap C_e \neq \emptyset$ and $(C \cup C_e) \cap D = \{e\}$. Thus, using the fact that the restriction of $\mathcal{M}$ to $C \cup C_e$ is connected, we obtain that there is a circuit $C' \subseteq C \cup C_e$ with $f, e \in C'$. The claim follows, since $|C' \cap D| = 1$.

Now let $H \subseteq E$ be a base with $(C_f - e) \subseteq H$. Then f is in the fundamental circuit of $e \in D$ with respect to H, as required. •

Theorem 6.2. *Let $G = (V, E)$ be a rigid graph, $S \subseteq V$ with $|S| \geq 2$, and let B be a set of new edges on S for which $(S, E_G(S) \cup B)$ is rigid. Then an edge $f \in E$ belongs to the active zone of S in G if and only if the M-component containing f in $G' = (V, E \cup B)$ intersects $E_G(S) \cup B$.*

Proof: Let f be an edge in the active zone of S in G. If $f \in E_G(S)$ then the M-component of f in G' clearly intersects $E_G(S) \cup B$. Now suppose that $f \notin E_G(S)$. Let H be a minimally rigid spanning subgraph of G for which f is in the active zone of S in H. Thus by Lemma 4.3 and Theorem 5.3 $f \in C_{a,b}(H)$ for some $a, b \in S$ with $ab \in B'$, where B' is some arbitrarily chosen set of edges which makes $H[S]$ minimally rigid. Since $(S, E_G(S) \cup B)$ is rigid, we may suppose that $B' \subseteq E_G(S) \cup B$. Since

[3]Let C, C' be two circuits of some matroid with $e \in C \cap C'$ and $f \in C - C'$. Then there is a circuit $C'' \subseteq C \cup C'$ with $e \notin C''$ and $f \in C''$.

$C_{a,b}(H) + ab$ is an M-circuit in G', it follows that the M-component of G' containing f contains ab and hence intersects $E_G(S) \cup B$.

Conversely, suppose that the M-component K containing the edge $f \in E$ in G' intersects $E_G(S) \cup B$. Since each edge of G is in some minimally rigid spanning subgraph H of G and $E_H(S)$ belongs to the active zone of S in H, every edge $f \in E_G(S)$ belongs to the active zone of S in G. So we may suppose that $f \notin E_G(S)$.

First suppose that $K \cap B = \emptyset$. Then there is an M-circuit C in G' with $f \in E(C) \subseteq E$ and $E_G(S) \cap E(C) \neq \emptyset$. Let $e = ab \in E_G(S) \cap E(C)$. Consider a minimally rigid spanning subgraph H of G containing $C - e$. Since there is a unique M-circuit in $H + e$ (namely, C) and $f \notin E_G(S)$, it is easy to see that $H[S] + e$ is M-independent and hence there is a set B' with $e \in B'$ such that $(S, E_H(S) + B')$ is minimally rigid. Since $ab \in B'$ and $f \in C_{a,b}(H)$, Lemma 4.3 and Theorem 5.3 imply that f belongs to the active zone of S in H, and therefore also in G.

Next suppose that $K \cap B \neq \emptyset$. It is easy to see that $G[V(K)]$ is rigid[4]. By applying Lemma 6.1 (with $\mathcal{M} = \mathcal{R}(K)$, $E = E(K) - B$, and $D = E(K) \cap B$) we can deduce that there exists an M-circuit C in K with $f \in E(C)$ and $|E(C) \cap B| = 1$. Let $e = ab$ be the common edge of C and B. Consider a minimally rigid spanning subgraph H of G containing $C - e$. As above, we get that $f \in C_{a,b}$ in H. Hence, by Lemma 4.3 and Theorem 5.3, f belongs to the active zone of S in H and also in G. •

Since the M-components in a graph are monotone increasing with respect to edge addition, we obtain the following corollary.

Theorem 6.3. *Let $G = (V, E)$ be a rigid graph, $S \subseteq V$ with $|S| \geq 2$, and let u, v be a pair of non-adjacent vertices in G. Then the active zone of S in G is a subset of the active zone of S in $G + uv$.*

We can also deduce an equivalent characterization of the active zones which will be used in the next subsection.

Theorem 6.4. *Let $G = (V, E)$ be a rigid graph, $S \subseteq V$ with $|S| \geq 2$, and let H be a minimally rigid spanning subgraph of G. Let $\mathcal{K} = \{K_1, K_2, ..., K_t\}$ be the family of M-components of G. Then an edge $f \in E$ belongs to the active zone of S in G if and only if the M-component of G containing f intersects $E(C_S(H))$.*

Proof: Consider an M-component K_i of G for which $e \in E(K_i)$ holds for some $e \in E(C_S(H))$. Let $f \in E(K_i)$ be an edge. Since $E(C_S(H)) \subseteq A_S(G)$ by Theorem 5.4, we may assume that $f \notin E(C_S(H))$. Since e, f belong to the same M-component of G, there is a circuit C in G with $e, f \in C$. Let B be a set of new edges on S for which $(S, E_G(S) \cup B)$ is rigid. Since $e \in A_S(G)$, e belongs to an M-component K of $G' = (V, E \cup B)$ intersecting $E_G(S) \cup B$ by Theorem 6.2. Hence, by the existence of C, f must belong to the same M-component K of G'. Thus f also belongs to the active zone of S in G by Theorem 6.2.

[4]If $V(K) = V$ then the claim follows from the fact that G is rigid. Otherwise G' has at least two M-components $K_1, K_2, ..., K_t$ and we have $2|V| - 3 = r(G') = \sum_1^t r(K_i) = \sum_1^t 2|V(K_i)| - 3$. Since the M-components of G form a refinement of the M-components of G', we have $2|V| - 3 = r(G) = \sum_1^t r(G[V(K_i)]$, which implies that $r(G[V(K)] = 2|V(K)| - 3$. Thus $G[V(K)]$ is rigid.

Conversely, consider an edge f in the active zone of S in G. For all edges $uv \in E$ with $\{u, v\} \subset V(C_S(H))$ the M-component of G containing uv intersects $E(C_S(H))$, thus we may assume that at least one end-vertex of f is not included in the vertex set of $C_S(H)$. In particular, f is not induced by S in G. By definition and Theorem 5.4 there is a minimally rigid spanning subgraph H' of G for which $f \in E(C_S(H'))$. Let B' be a set of new edges on S for which $(S, E_{H'}(S) \cup B')$ is minimally rigid. It follows from Lemma 4.3(i) that there is a circuit C_f in $H' + b$, for some edge $b = xy \in B'$, with $f, b \in C_f$. Now either $xy \in E(C_S(H))$ or there is a circuit C_e in $C_S(H) + b$ with $e, b \in C_e$, for some $e \in E(C_S(H))$. In both cases we can deduce (by applying the strong circuit axiom in the latter case) that there is a circuit C in G containing f and at least one edge of $E(C_S(H))$. Thus the M-component containing f in G intersects $E(C_S(H))$. •

Note that the active zone of S does not depend on the choice of H.

6.1 Influenced zones in rigid graphs

Let $G = (V, E)$ be a rigid graph and let $v \in V$ with $d_G(v) \geq 3$. Based on Corollary 5.5 we define the *influenced zone* $I_v(G)$ of v to be the active zone $A_{N_G}(G)$ of its set of neighbours. By applying Theorem 6.4 we can now deduce:

Theorem 6.5. *Let $G = (V, E)$ be a rigid graph and let $v \in V$ with $d_G(v) \geq 3$. Let H be a minimally rigid spanning subgraph of G. Then an edge $f \in E$ belongs to the influenced zone of v in G if and only if the M-component of G containing f intersects $E(C_{N_G(v)}(H))$.*

We remark that, since the set $\delta(v)$ of edges incident with v in G is independent, we may choose a minimally rigid spanning subgraph H of G which contains $\delta(v)$. Also note that if $d_G(v) = 2$ holds then the influenced zone of v in G is clearly empty, as in the minimally rigid case.

7 Concluding remarks

We note that our results about influenced and active zones easily extend to frameworks in which some vertices are pinned down. This can be verified by considering the unpinned framework obtained by adding new bars connecting the pinned joints so that the new bars form a minimally rigid subframework on the set of pinned vertices, see Figure 3. We omit the details.

Algorithms

¿From the algorithmic point of view our results imply that in order to find the active zone of a subset or the influenced zone of a vertex we need subroutines for testing rigidity, finding a minimally rigid spanning subgraph in a rigid graph, or finding the M-components of a graph. There exist efficient algorithms for these algorithmic problems that run in $O(n^2)$ time, see e.g [1].

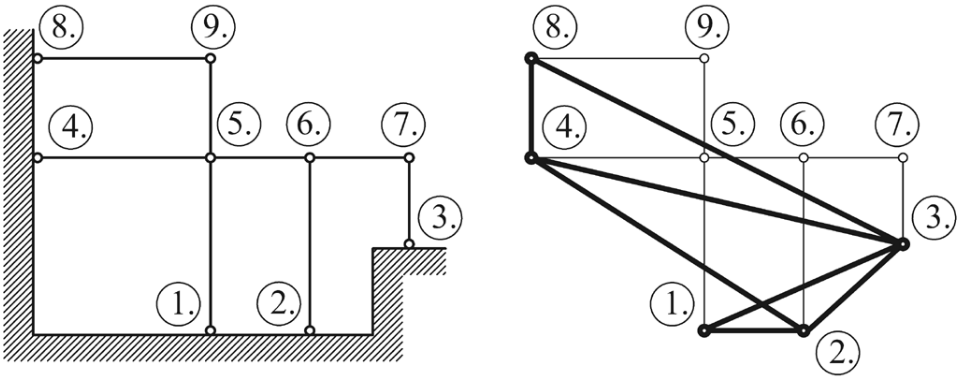

Figure 3: A pinned framework and the corresponding unpinned framework.

The sensitivity index

One way to measure the geometric sensitivity of a generic framework is as follows. As in [11] define the *joint sensitivity index* of G by

$$s(G) = \frac{\sum_{v \in V} |I_v(G)|}{|V|^2}.$$

Following [5] we call a minimally rigid graph G *special* if every proper rigid subgraph H of G is complete (and hence is a complete graph on two or three vertices). The graphs $K_{3,3}$ and the prism are both special, as well as all graphs which can be obtained from $K_{3,3}$ by the following operation: replace two incident edges ab, bc by six edges $aa', a'b, bc', c'c, ac', a'c$, where a', c' are new vertices. Thus the family of special graphs is infinite. A special graph on at least six vertices has minimum degree three. Thus, by Theorem 5.2, for all $v \in V(G)$ we have that $I_v(G)$ is a minimally rigid subgraph on at least four vertices and hence $V(I_v(G)) = V(G)$ must hold. This implies that special graphs are highly sensitive.

Lemma 7.1. *Let G be a special graph on at least six vertices. Then $s(G) = 1$.*

Not only special graphs may have the highest joint sensitivity index. For example, the minimally rigid graph G' obtained by connecting the minimally rigid graph $K_4 - e$ and a four-cycle C_4 by four disjoint edges has $s(G') = 1$, but is not special.

To find families of lowest sensitivity first observe that in a minimally rigid graph $G = (V, E)$ with $|V| \geq 4$ we cannot have adjacent vertices of degree two. Thus, since G has no cut-vertices, we must have $|V_3| \geq 2$, where V_3 is the set of vertices of degree at least three. Furthermore, for each $v \in V_3$ we have $|I_v(G)| \geq d_G(v) + 1$. Hence we have

$$\sum_{v \in V} |I_v(G)| \geq \sum_{v \in V} (d_G(v) - 2) + 3|V_3| \geq 2|E| - 2|V| + 6 = 4|V| - 6 - 2|V| + 6 = 2|V|.$$

This bound is attained for the graph obtained from $K_{2,n-2}$ by adding an edge to the smaller colour class.

For graphs with minimum degree at least three we have

$$\sum_{v \in V} |I_v(G)| \geq \sum_{v \in V} (d_G(v) + 1) = 5|V| - 6.$$

To get close to this bound we need graphs in which for all vertices v (except possibly a few vertices of small degree) $G[N(v) + v]$ is rigid. Such graphs can easily be obtained from a graph which is constructed from a triangle by repeated applications of the operation which adds a new vertex v and two edges va, vb, where ab is an edge incident with a degree two vertex.

Open questions

Finally we discuss a few open questions which remain unsolved. The first one is related to Theorem 5.2, which asserts that for generic frameworks (G, p) and typical loads L the influenced zone depends only on the graph G, where typical is defined with respect to G, p, and p'. Now suppose that the load L is given, which is in some sense typical. For which set of configurations p' can we deduce (9)?

A challenging problem is to prove (at least partial) results about the case when $d \geq 3$. For $d = 3$ we can show, by combinatorial methods, that the minimal minimally rigid subgraph C_S of a minimally rigid graph G satisfying $S \subseteq V(C_S)$ is unique, where $S \subseteq V(G)$ with $|S| \geq 2$ is a given vertex set. Thus we may define the core of S as in the two-dimensional case.

On the other hand, finding the active zone of S seems to be at least as hard as finding the fundamental circuit of an edge with respect to a minimally rigid graph, which is a difficult unsolved problem for $d \geq 3$. This follows from the fact that when $S = \{a, b\}$, the active zone of S equals the unique M-circuit H of $G + ab$, minus ab.

Furthermore, as the following example shows, the core $C_{a,b}$ and $H - ab$ may be different, unlike the two-dimensional case. Take the well-known double banana graph. Denote the hinge pair by x, y and the internal vertices of the bananas by $\{a, b, c\}$ and $\{p, q, r\}$, respectively. Let G be the graph obtained by deleting the edge ax and adding a new vertex v and new edges va, vx, vp, vq. This graph G is minimally rigid in $\mathbb{R}^3$. We have $C_{a,x} = G$ while the unique M-circuit in $G + ax$ is the double banana.

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