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Special skew-supermodular functions and a generalization of Mader's splitting-off theorem

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Abstract

In this note we give a slight extension of Mader's undirected splitting-off theorem with a simple proof, and we investigate some related questions. Frank used Mader's theorem for solving the local edge-connectivity augmentation problem which is the following: given a graph $G_0 = (V, E_0)$ and a symmetric function $r : V \times V \to \mathbb{Z}_+$, find a graph $G = (V, E)$ with a minimum number of edges such that $\lambda_{G_0+G}(u, v) \geq r(u, v)$ for every pair of nodes u, v. In the solution Frank introduces the set function $R_r(X) = \max\{r(x, y) : x \in X, y \in V - X\}$ for any $X \subseteq V$ and he reproves Mader's theorem using the following three properties of this function: (i) symmetry, (ii) skew-supermodularity and (iii) $R_r(X \cup Y) \leq \max\{R_r(X), R_r(Y)\}$ for every pair of sets $X, Y \subseteq V$. We give a proof where (iii) is only required for disjoint pairs of sets $X, Y \subseteq V$, and we show examples of functions satisfying Properties (i)-(ii) and this weaker form of (iii).

1 Introduction

In this note we give a relatively simple proof of a slight extension of Mader's undirected splitting-off theorem and investigate some related questions. For basic definitions and notation see [4].

Mader proved his lemma in [5]. Frank used this result in order to solve the **local edge-connectivity augmentation problem** in [2]. Our discussion follows that of Frank. The local edge-connectivity augmentation problem is the following.

Problem 1.1 (**Local edge-connectivity augmentation problem**). *Let $G_0 = (V, E_0)$ be a graph and $r : V \times V \to \mathbb{Z}_+$ be a symmetric function (also called the* edge-connectivity requirement*). Find a graph $G = (V, E)$ with a minimum number of edges such that $\lambda_{G_0+G}(u, v) \geq r(u, v)$ for every pair of nodes u, v.*

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Frank [2] showed how to reduce the problem above to the **degree-specified version** of the problem which is the following.

Problem 1.2 (**Local edge-connectivity augmentation problem - degree-specified version**). *Let G_0 be a graph, $r : V \times V \to \mathbb{Z}_+$ be a symmetric function and $m : V \to \mathbb{Z}_+$ be a function on the nodes. Does there exist a graph G such that $\lambda_{G_0+G}(u, v) \geq r(u, v)$ for every pair of nodes u, v and $d_G(v) = m(v)$ for every node $v \in V$?*

We will only deal with this version here. In order to simplify the treatment we will assume that $r(u, v) > 1$ for any pair of nodes $u, v \in V$ (Frank solved the problem above in general, but some technical difficulties arise if we also allow 1 among the requirements, and we want to avoid these difficulties here).

Let us introduce the set function $R_r : 2^V \to \mathbb{Z}$ with $R_r(\emptyset) = R_r(V) = 0$ and

$$R_r(X) = \max\{r(x, y) : x \in X, y \in V - X\} \text{ for any } X \text{ with } \emptyset \neq X \neq V. \tag{1}$$

Note that the graph G with $d_G(v) = m(v)$ for every node $v \in V$ is a solution to Problem 1.2 if and only if $d_G(X) \geq R_r(X) - d_{G_0}(X)$ holds for every $X \subseteq V$ (see also [2]). With these notations Mader's Splitting Theorem states the following.

Theorem 1.3 (Mader's Splitting Theorem). *Let G_0 be a graph, let $r : V \times V \to \mathbb{Z}_{\geq 2}$ be a symmetric function and let $m : V \to \mathbb{Z}_+$ be a degree-specification. There exist a graph G such that $d_G(v) = m(v)$ for every node $v \in V$ and $d_G(X) \geq R_r(X) - d_{G_0}(X)$ holds for every $X \subseteq V$ if and only if $m(V)$ is even and $m(X) \geq R_r(X) - d_{G_0}(X)$ holds for every $X \subseteq V$.*

In Frank's [3] proof of this lemma the following properties of the function R_r play an important role.

1. R_r is symmetric (i.e. $R_r(X) = R_r(V - X)$ for every $X \subseteq V$),
2. R_r is skew-supermodular (to be defined later), and
3. $R_r(X \cup Y) \leq \max\{R_r(X), R_r(Y)\}$ for every pair of sets $X, Y \subseteq V$.

In this note we investigate the relationship between these properties. It turns out that the first and the third properties together already imply the second one, and a function satisfying these 3 properties is essentially of form given above in (1). We also show that we can prove Mader's theorem by relaxing the 3rd property the following way: we only require it for disjoint pairs $X, Y \subseteq V$. We give examples showing that this indeed extends Mader's theorem.

2 Preliminaries

Let V be a finite ground set. For subsets X, Y of V let $\overline{X}$ be $V - X$ (the ground set will be clear from the context). If X has only one element x, then we will not distinguish between X and its only element x. The **characteristic function** $\boldsymbol{\chi_X}$: $V \to \mathbb{Z}$ of the subset X is defined by $\chi_X(v) = 1$ if $v \in X$, $\chi_X(v) = 0$ if $v \notin X$. Sets $X, Y \subseteq V$ are **properly intersecting** if $X \cap Y, X - Y$ and $Y - X$ are all nonempty. If furthermore $X \cup Y \neq V$ then we say that they are **crossing**.

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *skew-supermodular* if at least one of the following two inequalities holds for every $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{$\cap\cup$}$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{$-$}$$

If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a skew-supermodular function and $(\cap\cup)$ holds for some sets $X, Y \subseteq V$ then we say that X *and* Y *satisfy* $(\cap\cup)$: if we do not explicitly say which function is meant then we always mean p. The same notation is used for $(-)$.

A set function is *symmetric* if $p(X) = p(V - X)$ for every $X \subseteq V$. Any function $m : V \to \mathbb{R}$ also induces a set function (that will also be denoted by m) with the definition $m(X) = \sum_{v \in X} m(v)$ for any $X \subseteq V$.

For a graph $G = (V, E)$ and a set $X \subseteq V$ we say that an edge $e \in E$ **enters** X if one endpoint of e is in X and the other endpoint of e is outside of X, and we define $d_G(X) = |\{e \in E : e \text{ enters } X\}|$ (the **degree** of X in G). This is a symmetric submodular function. For two set functions d, p we say that d **covers** p if $d(X) \geq p(X)$ for any $X \subseteq V$ ($d \geq p$ for short). We say that the graph G **covers** p if d_G covers p. For $S, T \subseteq V$ let $\lambda_G(S, T)$ denote the maximum number of edge-disjoint paths starting in S and ending in T (we say that $\lambda_G(S, T) = \infty$ if $S \cap T \neq \emptyset$). By Menger's theorem

$$\lambda_G(S, T) = \min\{d_G(X) : T \subseteq X \subseteq V - S\}.$$

A function $m : V \to \mathbb{Z}_+$ on the nodes will be called a **the degree-specification** if $m(V)$ is even and $m(v) \leq m(V - v)$ for any $v \in V$. A graph G is said to **satisfy the degree-specification** m if $m(v) = d_G(v)$ holds for every $v \in V$. It is easy to check that if G satisfies the degree-specification m then we can transform G into a loopless graph G' such that G' satisfies m and $d_{G'}(X) \geq d_G(X)$ holds for every $X \subseteq V$. This way we will avoid loop edges and the problem whether they should or should not count in the degree of singleton sets.

Given a symmetric skew-supermodular function p and a degree-specification $m : V \to \mathbb{Z}_+$, we want to decide whether there exists a graph G covering p and satisfying the degree-specification m. An important necessary condition is the following:

$$m(X) \geq p(X) \text{ holds for every } X \subseteq V. \tag{2}$$

We say that the degree-specification m is **admissible (with respect to p)** if (2) holds. For a node $v \in V$ we say that v is **positive** if $m(v) > 0$, and **neutral** otherwise. The set of positive nodes will be denoted by V^+. Let $u, v \in V^+$ be two distinct positive nodes. The operation **splitting-off (at u and v)** is the following: let

$$m' = m - \chi_{\{u,v\}} \text{ and } p' = p - d_{(V,\{uv\})}, \tag{3}$$

where $d_{(V,\{uv\})}$ is the degree function of the graph $(V, \{uv\})$ having only one edge. If $m'(X) \geq p'(X)$ for any $X \subseteq V$, then we say that the splitting off is **admissible**. We will usually use the notation that after the splitting off operation we substitute p by p' and m by m'. This way an (admissible) splitting-off preserves the relevant properties of an (admissible) degree-specification.

A set X is **dangerous** if $m(X)-p(X) \leq 1$ and it is called **tight** if $m(X)-p(X) = 0$. Clearly, splitting off at u and v is admissible if and only if there is no dangerous set X containing both u and v. We will also say that such a dangerous set X **blocks the splitting at** u **and** v, or simply that X **blocks** u **and** v.

Mader's splitting-off theorem can be formulated with a special skew-supermodular function. In this note we investigate how special this function is and we show that we can prove Mader's theorem for a a slightly broader class of functions, too.

Let $M_p = \max\{p(X) : X \subseteq V\}$. A set X with $p(X) = M_p$ will be called p-**maximal**. Clearly, if $M_p \leq 0$ then any splitting-off is admissible. Note that by the skew-supermodularity of p, for two p-maximal sets X and Y either both of $X \cap Y$ and $X \cup Y$ or both of $X - Y$ and $Y - X$ are also p-maximal.

We will also be interested in another inequality for a set function R:

$$R(X \cup Y) \leq \max\{R(X), R(Y)\}. \tag{4}$$

Mader's splitting theorem is concerned with the special set function $R = R_r$ defined in (1). Observe that the function R_r is symmetric and skew-supermodular, and it satisfies $R_r(X \cup Y) \leq \max\{R_r(X), R_r(Y)\}$ for **every pair** of sets $X, Y \subseteq V$.

3 Special skew-supermodular functions

In this subsection we describe symmetric set functions R that satisfy (4) for **every** pair of subsets $X, Y \subseteq V$. Observe that (4) applied for an arbitrary subset $X \subseteq V$ and $Y = V - X$ gives that $R(V) \leq R(X)$, i.e. V (and $\emptyset$) minimizes such a function R. Thus we will assume that $R(\emptyset) = R(V) = 0$, since we can always achieve this by adding a constant.

Let us give an example of such a function.

Definition 3.1. *Let P be a path with node set $V(P) = V$ (in other words, a linear ordering of V; P will denote the edge set of the path) and assume that we are given arbitrary nonnegative numbers $c : P \to \mathbb{R}_+$ on the edges of this path. Define the function $R_{P,c}$ with $R_{P,c}(X) = \max\{c(xy) : xy \in P, x \in X, y \in V - X\}$ for any X with $\emptyset \neq X \neq V$ and $R_{P,c}(\emptyset) = R_{P,c}(V) = 0$.*

It turns out that this definition gives every function satisfying our requirements. Note that $R_{P,c}$ is also of form R_r if we take $r(x, y) = \min\{c(uv) : uv \in P[x, y]\}$, where $P[x, y]$ denotes the subpath of P between x and y (note that this function value $r(x, y)$ can also be considered as the maximum flow value between x and y in the very simple undirected path-network P with edge-capacities c).

Lemma 3.2. *If the function $R : 2^V \to \mathbb{R}$ is symmetric (with $R(\emptyset) = R(V) = 0$) and it satisfies* (4) *for every pair of subsets $X, Y \subseteq V$, then R is of form $R_{P,c}$ with a suitably chosen path P and values $c : P \to \mathbb{R}_+$.*

Proof. First we prove a proposition whose origin is not clear, but which appeared as a question in the Schweitzer mathematics competition of ELTE. The current proof (to the best of our knowledge) is due to Gyula Pap.

Claim 1. *The function R can only take at most n different values.*

Proof. Let the different values of the function be $0 = a_1 < a_2 < \dots$. For any i let $\mathcal{H}(i) = \{X \subseteq V : R(X) \leq a_i\}$ (for example $\mathcal{H}(1)$ contains the minimizers of R, and $\mathcal{H}(i) \subsetneq \mathcal{H}(i+1)$ for every i). By the symmetry and (4), $\mathcal{H}(i)$ is closed under complementation and union, i.e. $X, Y \in \mathcal{H}(i)$ implies that $V - X$ and $X \cup Y$ are also in $\mathcal{H}(i)$ (note that this also implies that $X \cap Y, X - Y, Y - X \in \mathcal{H}(i)$). Let us define $\mathcal{P}(i)$ to be the system containing the (inclusionwise) minimal members of $\mathcal{H}(i)$ for every i. Observe that $\mathcal{P}(i)$ is a partition and $\mathcal{H}(i) = \{\bigcup \mathcal{X} : \mathcal{X} \subseteq \mathcal{P}(i)\}$. Since $\mathcal{H}(i) \subsetneq \mathcal{H}(i+1)$, the partition $\mathcal{P}(i+1)$ is a (proper) refinement of $\mathcal{P}(i)$ for every i. But we cannot refine a partition more than $|V| = n$ times. ■

In fact the above proof also shows how to prove the lemma. We define the total ordering of the elements (i.e. the path) the following way: initially all the nodes of V are incomparable and we refine this partial ordering step by step. For sake of simplicity, let $\mathcal{P}(-1) = \{V\}$ be the trivial partition. If a member $X \in \mathcal{P}(i)$ is the union of $X_1, X_2, \dots, X_t$ of some members of $\mathcal{P}(i+1)$ (where $i \geq -1$) then refine the partial ordering by saying that x_j is less than x_k for every $x_j \in X_j, x_k \in X_k$ and $1 \leq j < k \leq t$ (note that the indexing of the sets $X_1, X_2, \dots, X_t$ can be arbitrarily chosen). The assignment of the c values is also straightforward: $c(uv) = \min\{a_i : \mathcal{P}(i)$ separates u from $v\}$. ■

Finally we give an example of a set function that is symmetric and skew-supermodular, satisfies (4) for every pair of **disjoint** sets $X, Y \subseteq V$, but which cannot be written in the form R_r (since it fails to satisfy (4) for a pair of non-disjoint sets X, Y). This example shows that the proof we give for Mader's theorem proves indeed a more general result.

Definition 3.3. *Let $1 = a_1, a_2, \dots, a_{k+1} = |V|$ be positive integers such that a_j divides a_{j+1} for every $j = 1, 2, \dots, k$, and $b_1 > b_2 > \dots > b_k \geq 0$ are nonnegative reals. Define the set function $R_F : 2^V \to \mathbb{Z}$ as follows: let $R_F(X) = b_j$ if and only if a_j divides $|X|$, but a_{j+1} does not divide $|X|$ (where $j \leq k$, and let $R_F(\emptyset) = R_F(V) = 0$).*

Note that R_F is indeed symmetric and that (4) holds for any pair of **disjoint** sets $X, Y \subseteq V$ (however it might fail for non-disjoint pairs).

Lemma 3.4. *The function R_F is skew-supermodular.*

Proof. In order to show the skew-supermodularity of R_F, observe that we only need to check the case when X and Y cross each other. Assume that $R_F(X) = b_j$ and $R_F(Y) = b_l$, where $1 \leq j \leq l \leq k$, i.e. a_j divides $|X|$ and a_l divides $|Y|$, but a_{j+1} does not divide $|X|$ and a_{l+1} does not divide $|Y|$. Observe that a_{j+1} cannot divide both $|X - Y|$ and $|X \cap Y|$, since that would imply $a_{j+1} \mid |X|$. For the same reason, a_{j+1} cannot divide both of $|Y - X|$ and $|X \cup Y|$. If a_{j+1} does not divide $|X \cup Y|$

and $|X \cap Y|$, then $(\cap\cup)$ holds, so we are done. If a_{j+1} does not divide $|X - Y|$ and $|Y - X|$, then again we are done since $(-)$ holds. So assume that a_{j+1} divides, say, $|X \cup Y|$ and $|X - Y|$, then it does not divide $|X \cap Y|$ and $|Y - X|$ (the other case is analogous). But then we use the obesrvation that a_{l+1} does not divide both of $|Y - X|$ and $|X \cap Y|$, so either $(\cap\cup)$ or $(-)$ holds again, finishing the proof. ■

Let us investigate some special cases of the function R_F defined above. If $k = 1$ then the function is the one used in global edge-connectivity augmentation: $R_F - d_{G_0}$ is the function to be covered if G_0 is to be made b_1-edge-connected. Assume that $k = 2$ and $a_2 = |V|/2$, where $|V|$ is even. One can immediately see that a graph $H = G + G_0$ covers R_F if and only if H is either b_1-edge-connected, or H consists of two b_1-edge-connectivity components V_1, V_2 of equal size (i.e. $|V_1| = |V_2|$ and $V_2 = V - V_1$), and $d_H(V_1) \geq b_2$. Thus the problem of covering the function $R_F - d_{G_0}$ is equivalent with the problem where we want to augment G_0 to become a b_1-edge-connected graph, but we allow one cut of smaller capacity, if that cut exactly halves the node set.

4 Generalization of Mader's Theorem

In this section we prove Theorem 4.2 which slightly extends Mader's Theorem (Theorem 1.3). The proof we give is similar to the proof of Mader's theorem (proof of Lemma 4.2 on page 58) given in [1]. A different (but also simple) proof of the same result (i.e. Theorem 4.2) was given in [7] (Section 5.1.3 on page 88).

First we prove the following lemma. This proof can also be found in [1]: we only include it to be self-contained. This lemma was also found by Nutov [6].

Lemma 4.1. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function and $m : V \to \mathbb{Z}_+$ be an admissible degree-specification. If $M_p = \max\{p(X) : X \subseteq V\} > 1$, then there is an admissible splitting-off.*

Proof. Let Y be a minimal set satisfying $p(Y) = M_p$. By symmetry, $p(V - Y) = M_p$ too, so we can choose a minimal set $Z \subseteq V - Y$ satisfying $p(Z) = M_p$. Since $M_p \geq 1$ we can choose $y \in Y, z \in Z$ with $m(y), m(z) > 0$. We claim that the splitting at y and z is admissible. Suppose for contradiction that it is not and consider a dangerous set X containing y and z. Since X is dangerous and $p(Y) = M_p$, we have $p(X) + p(Y) \geq m(X) - 1 + M_p$. On the other hand, $p(X - Y) \leq m(X - Y) \leq m(X) - 1$ and $p(Y - X) < M_p$ by the choice of Y, so $p(X - Y) + p(Y - X) < m(X) - 1 + M_p$. Thus $p(X) + p(Y) > p(X - Y) + p(Y - X)$, and, by the skew-supermodularity of p, X and Y must satisfy $(\cap\cup)$, which means that $p(X \cap Y) + p(X \cup Y) \geq p(X) + p(Y) \geq m(X) - 1 + M_p$.

Using $m(X \cap Y) = m(X) - m(X - Y) \leq m(X) - m(z) \leq m(X) - 1$ and $p(X \cup Y) \leq M_p$, we obtain that $p(X \cap Y) + p(X \cup Y) \leq m(X) - 1 + M_p$, hence equality holds, which implies that $m(X - Y) = 1$ and $p(X \cup Y) = M_p$. Now $X \cup Y$ and Z cannot satisfy $(-)$ since this would give $p(Z - (X \cup Y)) = M_p$, contradicting the minimality of Z. Therefore $X \cup Y$ and Z satisfy $(\cap\cup)$ implying that $p(Z \cap (X \cup Y)) = M_p$, which, by the minimality of Z, is only possible if $Z \subseteq X \cup Y$. But then $Z \subseteq X - Y$, so $2 \leq M_p = p(Z) \leq m(Z) \leq m(X - Y) = 1$ gives a contradiction. ■

Finally we give a simple proof of a slight extension of Mader's classical Splitting Theorem. The example in Definition 3.3 shows that this theorem is indeed more general than Theorem 1.3 of Mader.

Theorem 4.2. *Let $R : 2^V \to \mathbb{Z}_{\geq 2}$ be a symmetric skew-supermodular function satisfying $R(X \cup Y) \leq \max\{R(X), R(Y)\}$ for any* ***disjoint*** *pair $X, Y \subseteq V$. Let furthermore G_0 be a graph and $m : V \to \mathbb{Z}_+$ be a degree-specification. Then there exists a loopless graph G such that $d_G(v) = m(v)$ for every node $v \in V$ and $d_G(X) \geq R(X) - d_{G_0}(X)$ holds for every $X \subseteq V$ if and only if $m(X) \geq R(X) - d_{G_0}(X)$ holds for every $X \subseteq V$.*

Proof. The necessity of the conditions is clear so we prove the sufficiency. Let $p(X) = R(X) - d_{G_0}(X)$ for any $X \subseteq V$: clearly p is symmetric and skew-supermodular, and our condition says that m is p-admissible. If we prove that there exists an admissible splitting-off then the theorem is proved. First note that if there exists a node $v \in V$ with $m(v) = m(V - v)$ then the splitting-off at v and an arbitrary other positive node u is admissible: if there was a dangerous set X containing both u and v then $p(V - X) = p(X) \geq m(X) - 1 \geq m(v) + m(u) - 1 \geq m(v)$ and $m(V - X) \leq m(V - v - u) \leq m(v) - 1$ would contradict the admissibility of m (for the set $V - X$). This also implies that the number of positive nodes is at least 3. Suppose that there is no admissible splitting-off: by Lemma 4.1, we can assume that $p(X) \leq 1$ for any $X \subseteq V$. Furthermore observe that $m(v) \leq 1$ holds for every $v \in V$: if $m(v) \geq 2$ for some $v \in V$ then any set X containing v and some other positive node has $m(X) \geq 3$, so it cannot be dangerous by $p(X) \leq 1$, contradicting our assumption. Let x, y, z be 3 distinct positive nodes and let X, Y, Z be 3 **maximal** dangerous sets with $y, z \in X, z, x \in Y$ and $x, y \in Z$. Consider the following two cases.
Case I.: Assume that two of these three sets (wlog. X and Y) satisfy $(\cap\cup)$. Note that this implies that $p(X \cup Y) = p(X \cap Y) = 1$. Substitute Z by a minimal dangerous set Z' containing x and y. Observe that $(-)$ cannot hold for Z' and $X \cup Y$, since that would imply $p(Z' - (X \cup Y)) = 1$, contradicting $m(Z' - (X \cup Y)) = 0$. Therefore $(\cap\cup)$ for Z' and $X \cup Y$ and the minimality of Z' gives that $Z' \subseteq X \cup Y$. Similarly, $(\cap\cup)$ cannot hold for Z' and $X \cap Y$, since that would imply $p(Z' \cap X \cap Y) = 1$, contradicting $m(Z' \cap X \cap Y) = 0$. Therefore $(-)$ for Z' and $X \cap Y$ and the minimality of Z' gives that $Z' \cap X \cap Y = \emptyset$. We can assume that $R(Y \cap Z') \leq R(X \cap Z')$ and thus by by our assumption (4) for the disjoint sets $X \cap Z'$ and $Y \cap Z'$ we have $R(Z') \leq R(X \cap Z')$. Since $p(X \cap Z') \leq 1$, $d_{G_0}(X \cap Z') \geq R(X \cap Z') - 1$. Similarly, $p(Y \cap Z') \leq 1$ and $R(Y \cap Z') \geq 2$ together give that $d_{G_0}(Y \cap Z') \geq 1$. If $d_{G_0}(X \cap Z', Y \cap Z') = 0$ then these two give that $d_{G_0}(Z') = d_{G_0}(X \cap Z') + d_{G_0}(Y \cap Z') \geq R(X \cap Z') \geq R(Z')$, contradicting the fact that $p(Z') = 1$. Therefore $d_{G_0}(X \cap Z', Y \cap Z') > 0$, but then X and Y cannot satisfy $(\cap\cup)$ with equality, a contradiction.
Case II.: Assume that X, Y and Z pairwise satisfy $(-)$. This implies that $p(X - Y) = 1$, consequently Z and $X - Y$ cannot satisfy $(-)$, since $m((X - Y) - Z) = 0$. Thus they satisfy $(\cap\cup)$ which implies by the maximality of Z that $X - (Y \cup Z) = \emptyset$. Similarly we can prove that $Y - (Z \cup X) = Z - (X \cup Y) = \emptyset$. Using that $m(V - (X \cup Y \cup Z)) \geq 1$ we can deduce that $R(X \cup Y \cup Z) \geq 2$. However, since X, Y and Z pairwise satisfy $(-)$ with equality, there must not be an edge of $G[V]$ leaving $X \cup Y \cup Z$. But this would imply that $p(X \cup Y \cup Z) \geq 2$, contradicting Lemma 4.1. ■

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