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Matching with partially ordered contracts

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Abstract

In this paper, we study a many-to-many matching model with contracts. We extend the economic model of Hatfield and Milgrom by allowing a partial order on the possible bilateral contracts of the agents in a two-sided market economy. To prove that a generalized stable allocation exists, we use generalized form of properties like path independence and substitutability. The key to our results is the well-known lattice theoretical fixed point theorem of Tarski. The constructive proof of this fixed point theorem for finite sets turns out to be the appropriate generalization of the Gale-Shapley algorithm also in our general setting.

Keywords: stable marriages; Gale-Shapley algorithm; partial orders; lattices

1 Introduction

In a two-sided matching market, the set of players are partitioned into two sets. A matching is a set of pairs of players belonging to opposite sides such that each player appears at most once. Players of a pair in a matching are called partners. A player who has a partner is matched, otherwise she is unmatched, or self-matched, in a different terminology. Stability is a central solution concept for two-sided matching problems. A matching is stable if there exists no pair of players that prefer each other to their match. Examples of real-world applications are radio spectrum auctioning, package auctioning and the National Resident Matching Program (NRMP) of the United States, etc.

The pioneering work in two-sided matching markets is of Gale and Shapley [13], where they proposed a one-to-one (marriage) model in which players of the opposite

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sides (i.e, men and women) seek to form disjoint pairs. In general, a stable matching always exists in the Gale-Shapley model. Further, Gale and Shapley described the college admissions problem and proved that a stable solution always exists if players on one side are allowed to match to multiple partners.

The theory of two-sided matching markets has stimulated extensive research due to its applications to real-world problems. Several variations and extensions of the Gale-Shapley model have been studied in the literature. Job matching markets and auction markets are well studied classical examples of two-sided matching markets. In such markets, the substitutable property of preferences is essential. Without substitutability, the core of the underlying game may be empty, that is, no stable solution exists. The notion of substitutable preferences generalizing the Gale-Shapley model was first introduced by Kelso and Crawford [21] by generalizing the model of Crawford and Knoer [6]. The Crawford-Knoer model itself is a generalization of the Gale-Shapley model. In fact, the firms in this latter model are under two restrictive assumptions that Kelso and Crawford relaxed in their model. In the job matching model by Kelso and Crawford, preferences of firms and workers depend upon two factors: with whom they are matched and what their salaries are. They proposed a variant of the deferred-acceptance algorithm of Gale and Shapley to show that when preferences of the firms obey gross substitutes (GS) condition, the core is nonempty.

Kelso and Crawford's work motivated extensive further research. We find a couple of adaptations of the GS condition in Gul and Stacchetti [14]. In their article, Gul and Stacchetti introduce a single improvement (SI) condition and a no complementaries (NC) condition. They further show that the conditions SI and NC are equivalent to GS when the preferences are quasilinear (The Kelso-Crawford model is more general than the Gul-Stacchetti model since quasilinearity is not assumed there). For further discussion on the GS condition, we refer the reader to Roth and Sotomayor [27], Bevia et al. [4] and Gul and Stacchetti [15]. Until the appearance of the article by Fujishige and Yang [12], the GS condition was regarded as a condition on utility functions. Fujishige and Yang proved that a function defined on a unit hypercube satisfies the GS condition if and only if it is M$^\natural$-concave. We refer the reader to Murota [24] for a detailed discussion on M$^\natural$-concave functions[1]. Murota and Tamura [26] extended the notion of equivalence between GS condition and M$^\natural$-concavity. Danilov et al. [7] introduce a class of functions that satisfy GS condition, and discuss different characteristics of GS-functions. Fujishige and Tamura [10, 11] use GS-functions to design an economic model. They employ M$^\natural$-concave functions as utility functions to express the preferences. A pairwise stable outcome always exists in their models.

A significant step in the research of generalized two-sided matching markets is the observation that there is a close connection between stability and Tarski's well-known fixed point theorem [29]. The first such step was probably done by Adachi who described stable marriages as fixed points of a monotone function in [1]. Fleiner proved that a fairly general class of stability problems defined with the help of choice functions (including many-to-many versions and matroid-generalizations and much more) fits into the Tarski-based framework [8, 9]. Choice functions in this framework

[1] A function f is M$^\natural$-concave if $-f$ is M$^\natural$-convex.

must have the so called comonotone property that is closely related to the well-known substitutability condition. Fleiner pointed out that the well-known theorem by Blair on the lattice structure of generalized stable matchings [5] follows more or less directly from Tarski's fixed point theorem. He also pointed out that the proposal algorithm of Gale and Shapley can be regarded as an iteration method of a monotone function for finding a maximal or a minimal fixed point. A key ingredient in Fleiner's approach is that he considered the set of edges of the underlying bipartite graph as the domain of the key monotone mapping. Fleiner defined the so-called increasing property of a choice function meaning that if we extend the choice set then the number of choices picked cannot decrease. He proved that if choice functions are increasing (beyond comonotone and path-independent) then fixed points of the corresponding monotone function form a sublattice. Hence, the lattice operations on the stable solutions can be calculated directly by the corresponding choice functions. An easy consequence of this is that a natural generalization of the rural hospitals theorem holds for increasing, comonotone and path-independent choice functions. (The "rural hospital theorem" says that if a certain hospital h cannot fill up its quota in some stable matching then hospital h gets the same set of residents in any stable matching.) Based on the Tarski-framework and other well-known results, Fleiner also gave a linear description of several stable matching related polyhedra.

Independently and after Fleiner's work, Alkan [2] and Alkan and Gale [3] proved that if choice functions are cardinal monotone, or more generally size monotone then stable matchings form a lattice and an extension of the rural hospital theorem also holds. These works do not lean on Tarski's theorem and hence the basis of the proofs is a natural generalization of the proposal algorithm of Gale and Shapley.

A major breakthrough in popularizing the Tarski framework was done by Hatfield and Milgrom in [18]. They rediscovered several results of Fleiner and formulated them in a terminology that is much closer to Economists than the former Mathematical approach. In particular, they called the set of edges of the underlying bipartite graph "contracts" and defined substitutable mappings on them. They formulated a stability concept equivalent to Fleiner's in [8, 9] and proved that if contracts are substitutes (that is, if the choice functions are comonotone) for the hospitals and doctors have a linear preference order then in this two-sided one-to-many market stable allocations are basically the fixed points of a monotone function. They also pointed out that the Gale-Shapley algorithm is a monotone function iteration. Another important achievement of [18] is the formulation of the "law of aggregate demand" that corresponds to Fleiner's increasing property and Alkan's cardinal monotonicity. A main result is that if this condition also holds for the hospitals' choice functions then the rural hospital theorem can be generalized and that honest behaviour is a dominant strategy for the doctors if the doctor optimal stable assignment is realized after some bargaining process.

Very recently, Hatfield and Kojima [17] studied further the Hatfield-Milgrom framework of matching with contracts and introduced a bilateral substitutes condition - a less restrictive version of the Hatfield-Milgrom's substitutes condition. They showed that the bilateral substitutes condition is a sufficient condition for the existence of a stable outcome in the Hatfield-Milgrom framework. However, a doctor-optimal stable outcome and a lattice of stable outcomes may not exist if the bilateral substitutes condition is imposed only for hospitals. Hatfield and Kojima [17] further introduced a so-called unilateral substitutes condition - a stronger version of bilateral substitutes condition. They showed that the unilateral substitutes condition is a sufficient condition for the existence of a doctor-optimal stable outcome. The set of stable outcomes, however, may not form a lattice with respect to the preferences of the doctors, even under the unilateral substitutes condition. Klaus and Walzl [22] considered a class of many-to-many matching markets with contracts. They introduced a notion of weak setwise stability and analyze its relationship with other notions of stability. Klaus and Walzl concluded that if the preferences of the players satisfy the Hatfield-Milgrom's substitutes condition then the notions of weak setwise stability and pairwise stability coincide.

The paper is organized as follows. In Section 2, we list some well-known facts about partial orders and lattices and we claim Tarski's fixed point theorem that is key to our results. Section 3 is devoted to the description of our model that is a genuine generalization of the models by Fleiner [8, 9] and Hatfield and Milgrom [18]. The theorems and results included in this section are mostly known or easy to prove. Our main results are contained in Section 4 where we prove the existence of a stable solution and we extend Blair's result on the lattice structure of stable assignments [5] to our model. We point out that the proposal algorithm of Gale and Shapley can be regarded as an iteration of a monotone mapping. We show another related result in this section by demonstrating that a generalization of the proposal algorithm of Gale and Shapley can be used to calculate the lattice operations on stable solutions. We conclude in Section 5 and indicate promising directions of further research.

2 Preliminaries

In this section, we recall some concepts related to partially ordered sets (posets) that are essential in our framework. The reader familiar with posets and lattices might want to skip this part.

A *partially ordered set* (or *poset*) P on ground set X is a pair $P = (X, \preceq)$ where $\preceq$ is a reflexive, antisymmetric and transitive binary relation on X. (That is, for any $x, y, z \in X$ we have $x \preceq x$ and $(x \preceq y \preceq x \Rightarrow x = y)$ and $(x \preceq y \preceq z \Rightarrow x \preceq z$.) Elements x and y of poset $P = (X, \preceq)$ are *comparable* if $x \preceq y$ or $y \preceq x$, otherwise x and y are *incomparable.* If $P = (X, \preceq)$ is a partial order then a *lower ideal* is a set X' of X such that if $y \preceq x \in X'$ then $y \in X'$ holds. Poset $P = (X, \preceq)$ is called *trivial* if $\preceq = \emptyset$, that is if no two different elements of X are comparable. We shall often abuse notation by identifying a poset with its ground set, so for example a mapping $f : P \to P$ means simply a mapping $f : X \to X$ if we want to emphasize the underlying partial order. Or a subset P' of poset $P = (X, \preceq)$ means a poset $P' = (X', \preceq|_{X'})$ for some subset X' of X where $\preceq|_{X'}$ means the restriction of binary relation $\preceq$ on X'.

A subset A of X is an *antichain* of P if no two elements of A are comparable in P, that is, if $a \not\preceq a'$ for different elements a, a' of A. Let $\mathcal{L}(P)$ and $\mathcal{A}(P)$ denote the set of lower ideals and antichains of P, respectively. Note that if P is trivial then $\mathcal{L}(P) = \mathcal{A}(P) = 2^X$. For finite posets P, there is a natural bijection between $\mathcal{L}(P)$ and $\mathcal{A}(P)$. If $L \in \mathcal{L}(P)$ is lower ideal then clearly $\mathrm{Max}(L) = \{x \in L : x \preceq x' \in L \Rightarrow x = x'\}$ is an antichain, so $\mathrm{Max} : \mathcal{L}(P) \to \mathcal{A}(P)$. Moreover, if $A \in \mathcal{A}(P)$ is an antichain then

$$\mathrm{Li}(A) := \{x \in X : \exists a \in A \text{ such that } x \preceq a\}$$

is a lower ideal, hence $\mathrm{Li} : \mathcal{A}(P) \to \mathcal{L}(P)$. It is easy to check that for any finite poset $\mathrm{Li}(\mathrm{Max}(L)) = L$ and $\mathrm{Max}(\mathrm{Li}(A)) = A$ hold for any lower ideal L of $\mathcal{L}(P)$ and any antichain A of $\mathcal{A}(P)$, that is, Max and Li are inverses of one another and both of them define a bijection between $\mathcal{L}(P)$ and $\mathcal{A}(P)$.

Recall that partial order $(\mathcal{L}, \leq)$ is a *lattice* if any two elements x and y of $\mathcal{L}$ has a least upper bound (denoted by $x \vee y$) and a greatest lower bound (denoted by $x \wedge y$), that is, if $(y \leq z \geq x \Rightarrow z \geq x \vee y)$ and $(y \geq t \leq x \Rightarrow t \leq x \wedge y)$ holds. A lattice is called *complete* if any (possibly infinite) subset Y of $\mathcal{L}$ has a least upper bound $\bigvee Y$ and a greatest lower bound $\bigwedge Y$. An example of a complete lattice is $(2^X, \subseteq)$ (denoted by 2^X as a shorthand) consists of all subsets of a ground set X and partial order is given by set inclusion. Clearly, the lattice operations in 2^X are $\cup$ and $\cap$. If $\mathcal{L}$ is a lattice then subset $\mathcal{L}'$ of $\mathcal{L}$ is a *sublattice* if $\mathcal{L}'$ is closed on lattice operations $\vee$ and $\wedge$. If $\mathcal{L}'$ is closed even in the infinite lattice operations $\bigvee$ and $\bigwedge$ then $\mathcal{L}'$ is a *complete sublattice* of $\mathcal{L}$. We shall also need a less restrictive definition of a substructure. A subset $\mathcal{L}'$ of $\mathcal{L}$ is a *(complete) lattice subset* of $\mathcal{L}$ if $\mathcal{L}'$ is a (complete) lattice for the restriction of $\leq$. It is clear from the definition that any (complete) sublattice is a (complete) lattice subset but the converse is not true. It can be easily seen from the example where $X = \{1,2,3\}$ and $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\}$, as $\mathcal{L}$ is a lattice subset of 2^X but it is not a sublattice of it.

Assume that $P = (X, \leq)$ is a partial order on X. Clearly, $\mathcal{L}(P) \subseteq 2^X$, and we have equality if and only if P is trivial. For nontrivial posets P the following is true.

Observation 2.1. $\mathcal{L}(P)$ is a complete sublattice of 2^X, but not all complete sublattices of 2^X are of this form.

Proof. It follows from the definition that both the union and the intersection of any set of lower ideals of P is a lower ideal of P hence $\mathcal{L}(P)$ is a complete sublattice of 2^X. However, if $X = \{1, 2, \ldots, n\}$ and $\mathcal{L} := \{L \subseteq X : 1 \in L \iff 2 \in L\}$ then $\mathcal{L}$ is a complete sublattice of 2^X but $\mathcal{L}$ cannot be represented as $\mathcal{L}(P)$ for a partial order P on X. This is because $1 \in \mathrm{Li}(2)$ and $2 \in \mathrm{Li}(1)$, hence both $1 \leq 2$ and $2 \leq 1$ must hold if $\mathcal{L} = \mathcal{L}(P)$ for some partial order $P = (X, \leq)$. This contradicts antisymmetry. $\square$

We shall lean on Tarski's fixed point theorem, an important result on complete lattices. A mapping $f : X \to X$ on poset $P = (X, \leq)$ is *monotone* if $x \leq y$ implies $f(x) \leq f(y)$.

Theorem 2.2 (Tarski [29])**.** *If $\mathcal{L}$ is a complete lattice and $f : \mathcal{L} \to \mathcal{L}$ is monotone then the set of fixed points $F := \{x \in \mathcal{L} : f(x) = x\}$ forms a nonempty complete lattice subset of $\mathcal{L}$.*

For a finite lattice $\mathcal{L}$, there is a straightforward proof of the existence of a fixed point in Theorem 2.2. Namely, if 0 denotes the smallest element of $\mathcal{L}$ then by monotonicity we have that $0 \leq f(0) \leq f(f(0)) \leq f(f(f(0))) \leq \ldots$ and by finiteness, there has to be an iterated image of 0 that is mapped to itself. It is easy to see that the fixed point of f constructed this way is the a lower bound in $\mathcal{L}$ to any other fixed point of f. Similarly, if we start iterating f from 1 (that denotes the maximal element of $\mathcal{L}$) then we get a decreasing sequence $1 \geq f(1) \geq f(f(1)) \geq \ldots$ that eventually arrives to the maximal fixed point of f.

3 The economic model

In this section, we give a mathematical description of our model that is a genuine extension of that of Hatfield and Milgrom described in [18].

Let D and H be two disjoint sets of agents. We regard D as the set of doctors and H as the set of hospitals. By a contract x, we always mean an agreement between doctor $D(x) \in D$ and hospital $H(x) \in H$. Let X denote the set of all possible contracts in the model. For any subset X' of X, doctor d of D and hospital h of H, $X'(d) = \{x \in X' : D(x) = d\}$ and $X'(h) = \{x \in X' : H(x) = h\}$ denotes all the contracts that involve doctor d and hospital h, respectively.

The main difference between our model and that of Hatfield and Milgrom in [18] is that in our model we allow certain implications between contracts. An example is that if x is a contract that assigns doctor $D(x)$ to hospital $H(x)$ for some i days a week then it is always possible to choose contract x' between $D(x)$ and $H(x)$ that describes the same job as x does except for the total weakly workload is j days for $j < i$. Or, instead of contract x doctor $D(x)$ and hospital $H(x)$ may agree on signing a contract x' for a job that needs a lower qualification than x needs. In these examples, the possibility contract x implies the possibility of contract x' and we denote this fact by $x' \preceq x$. We assume that $P = (X, \preceq)$ is a partially ordered set on the set X of possible contracts[2]. It is easy to check that if there is no implication between contracts whatsoever (that is, if any two contracts are incomparable in poset P, i.e. if P is trivial) then our model reduces to that of Hatfield and Milgrom.

Just like in the Hatfield-Milgrom model, hospitals and doctors have certain preferences on the contracts they participate in. This is described by choice functions as follows. Assume that $X' \subseteq X$ is an lower ideal of P. Then $C_d(X')$ denotes those contracts of $X'(d)$ that doctor d would pick from $X'(d)$ if she is allowed to choose freely. Note that though in the Hatfield-Milgrom model, choice function C_d always selects at most one contract (hence it is a so-called one-to-many matching market), we do not assume this property. For any hospital h, we have a similar choice function C_h that selects the favourite contracts of hospital h from $X'(h)$. We assume that C_d and C_h always select an antichain of P. (That is, if d can work t or t' hours for h

[2] Later we shall see that all our results are true in the more general setting where we do not assume any acyclicity about implications between contracts. One can define "lower ideals" on the transitive closure of the implication digraph and these "lower ideals" form a complete sublattice of 2^X.

according to contracts x and x' then d never wants to sign both contracts x and x' and the same is true for h.) As each agent in our two-sided market has a choice function, we can define two joint choice functions: one for the doctors and one for the hospitals. Formally,

$$C_D(X') = \bigcup\{C_d(X') : d \in D\} \qquad \text{and} \qquad C_H(X') = \bigcup\{C_h(X') : h \in H\}$$

denote the doctors' and hospitals' choice function, respectively. Clearly, each choice function C we defined so far is mapping lower ideals of P into antichains of P such that $C(L) \subseteq L$ holds for any lower ideal L of P. For such a choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$, we define another choice function $C^* : \mathcal{L}(P) \to \mathcal{L}(P)$ by $C^*(L) := \mathrm{Li}(C(L))$. As there is a bijection between antichains and lower ideals of P, not only C determines C^*, but we can calculate C from C^* by $C(L) = \mathrm{Max}(C^*(L))$. Obviously, if P is trivial then $C = C^*$. We can also talk about choice functions in a more general sense. If $\mathcal{L}$ is a subset of 2^X then a choice function on $\mathcal{L}$ is a mapping $C : \mathcal{L} \to \mathcal{L}$ such that $C(L) \subseteq L$ holds for any element L of $\mathcal{L}$. Note that choice functions C^*_D and C^*_H are choice functions in this latter sense, as well.

There are two important properties of choice functions that we shall assume in our model. Choice function $C : \mathcal{L} \to \mathcal{L}$ on subset $\mathcal{L}$ of 2^X is *path independent* if

$$C(L) \subseteq L' \subseteq L \Rightarrow C(L) = C(L') \tag{1}$$

holds for any two members L and L' of $\mathcal{L}$. Note that in the Hatfield-Milgrom model, choice functions are defined by a strict linear order on the subsets X such that $C(Y)$ is that subset of Y that comes first in this linear order. (We shall see an example of such a choice function in Example 3.7.) Clearly, such choice functions are path independent by definition. Note that in the "traditional" definition of path independence is different from ours. Actually, (1) is weaker than that as shown below.

Observation 3.1. If for choice function $C : \mathcal{L} \to \mathcal{L}$ identity $C(A \cup B) = C(C(A) \cup C(B))$ holds for any members A, B of $\mathcal{L}$ then (1) is also true for C.

Proof. For any member A of $\mathcal{L}$ we have $C(A) = C(A \cup A) = C(C(A) \cup C(A)) = C(C(A))$. So if $C(L) \subseteq L' \subseteq L$ then $C(L) = C(L \cup L') = C(C(L) \cup C(L')) = C(C(C(L)) \cup C(L')) = C(C(L) \cup L') = C(L')$. □

In Lemma 3.5 we shall see that assuming substitutability (that we define a bit later) of C then "traditional" path independence is equivalent to (1). The following statement is easy to check.

Observation 3.2. If P is a partial order on X then choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ is path independent if and only if choice function $C^* : \mathcal{L}(P) \to \mathcal{L}(P)$ is path independent. □

From now on, $\mathcal{L}$ denotes a complete sublattice of 2^X. To get some intuition, the reader might simply think that $\mathcal{L} = \mathcal{L}(P)$, but our results that we claim for general complete sublattices are more general than the ones with this restriction. We do think that general complete sublattices still capture some interesting Economics models that do not fit in the poset-framework.

If $C : \mathcal{L} \to \mathcal{L}$ is a choice function then we can compare certain members of $\mathcal{L}$ with the help of C the following way. We say that member L is *C-better* than member L' (denoted by $L' \preceq_C L$) if $C(L \cup L') = L$. We can extend this notion for antichains if choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ maps lower ideals to antichains. This way, antichain A of P is *C-better* than $A' \in \mathcal{A}(P)$ (denoted by $A' \preceq_C A$) if $C(\mathrm{Li}(A \cup A')) = A$. Note that the same notation for lower ideals and antichains does not cause ambiguity as the range of C determines which one we talk about. Note further that $\preceq_C$ is not necessarily a partial order.

The second important property of a choice function is substitutability (or comonotonicity, as called by Fleiner in [9]) that we define here in a somewhat unusual way. A mapping $U : \mathcal{L} \to \mathcal{L}$ is called *antitone* if $U(L') \subseteq U(L)$ holds whenever $L \subseteq L'$ holds for elements L and L' of $\mathcal{L}$. Choice function $C^* : \mathcal{L} \to \mathcal{L}$ is *substitutable* if there exists an antitone mapping $U : \mathcal{L} \to \mathcal{L}$ such that $C^*(L) = L \cap U(L)$ holds for each member L of $\mathcal{L}$. A choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ that selects an antichain is called substitutable if C^* is substitutable. A choice function C in a traditional two-sided market model selects $C(Y)$ from a set Y of alternatives such that $C(Y)$ is the set of all those choices that are undominated by set Y of alternatives. The substitutability property captures the fact that a broader set of alternatives leaves less undominated choices. Or, equivalently, if the choice set is growing then the set of dominated (hence unselected) alternatives is also growing. This phenomenon is used in the definition of substitutability by Hatfield and Milgrom: elements of X are *substitutes* for choice function $C : 2^X \to 2^X$ if the set of rejected elements is a monotone mapping, that is $R(Y) := Y \setminus C(Y) \subseteq Y' \setminus C(Y') = R(Y')$ whenever $Y \subseteq Y' \subseteq X$.

Observation 3.3. If elements of X are substitutes for choice function $C : 2^X \to 2^X$ then C is substitutable.

Proof. As rejection function defined by $R(Y) := Y \setminus C(Y)$ is monotone, its complement $U(Y)$ defined by $U(Y) := X \setminus R(Y)$ is antitone. As partial order of poset P is trivial, $U(Y)$ is a lower ideal. Observe that $Y \cap U(Y) = Y \cap (X \setminus R(Y)) = Y \setminus R(Y) = Y \setminus (Y \setminus C(Y)) = C(Y)$, hence C is indeed substitutable. □

Example 3.4. Assume that hospital h has a linear preference order on $X(h)$ and $C_h(X')$ is the q_h best elements of $X'(h)$. (Here, there is no partial order on X, or if we insist on having one then it is trivial.) It is easy to check that C_h is path independent and contracts in $X(h)$ are substitutes. To see that C_h is substitutable, we define $U : 2^X \to 2^X$ by $U(X')$ denoting the set of those contracts x of $X(h)$ such that $X'(h)$ contains at most $q_h - 1$ contracts that are better than x according to the preference order of h. Clearly, if $X' \subseteq X''$ then $U(X') \supseteq U(X'')$, so U is antitone. It is also clear by the definition of U that $C_h(X') = X' \cap U(X')$, that is, C_h is indeed substitutable.

It is well-known that our definition of path-independence is equivalent to the "traditional" one for substitutable choice functions.

Lemma 3.5. *If choice function $C : \mathcal{L} \to \mathcal{L}$ is substitutable and path-independent then identity $C(A \cup B) = C(C(A) \cup C(B))$ holds for any members A, B of $\mathcal{L}$.*

Proof. By the antitone property of U, we get

$$C(A \cup B) = (A \cup B) \cap U(A \cup B) = (A \cap U(A \cup B)) \cup (B \cap U(A \cup B)) \subseteq \\ (A \cap U(A)) \cup (B \cap U(B)) = C(A) \cup C(B) \subseteq A \cup B$$

and path-independence property (1) of C directly gives that $C(A \cup B) = C(C(A) \cup C(B))$. □

The following theorem points out an interesting property of substitutable choice functions.

Theorem 3.6. *If choice function $C : \mathcal{L}(P) \to \mathcal{A}(P)$ is path-independent and substitutable for some partial order P on a finite ground set X then $\prec_C$ is a partial order on $\{C(L) : L \in \mathcal{L}\}$, that is, on those antichains of P that are in the range of C.*

Proof. Assume that $A = C(L)$ for some $L \in \mathcal{L}$. This means that $A \subseteq \mathrm{Li}(A) \subseteq L$ hence $C(\mathrm{Li}(A)) = A$ by path independence of C. Obviously, $C(\mathrm{Li}(A \cup A)) = C(\mathrm{Li}(A)) = A$, hence $A \preceq_C A$, that is, $\preceq_C$ is reflexive. Now assume that $C(\mathrm{Li}(A')) = A'$ also holds. Clearly, if $A' \preceq_C A$ and $A \preceq_C A'$ then $A = C(\mathrm{Li}(A \cup A')) = A'$, hence $A = A'$, so $\preceq_C$ is antisymmetric. Note that we did not use the substitutable property of C so far.

To prove transitivity, assume that $C(\mathrm{Li}(A'')) = A''$ and $A \preceq_C A' \preceq_C A''$ holds. Define $L := \mathrm{Li}(A), L' := \mathrm{Li}(A')$ and $L'' := \mathrm{Li}(A'')$. From the assumption we have that $C(L \cup L') = A'$ and $C(L' \cup L'') = A''$, or, as P is finite this is equivalent to saying that $C^*(L \cup L') = L'$ and $C^*(L' \cup L'') = L''$. From the definition and the antitone property of U^* we get that

$$\begin{aligned} C^*(L \cup L' \cup L'') = (L \cup L' \cup L'') \cap U^*(L \cup L' \cup L'') = \\ = (L \cup L' \cup L'') \cap U^*(L \cup L' \cup L'') \cap U^*(L \cup L' \cup L'') = \\ = [((L \cup L') \cap U^*(L \cup L' \cup L'')) \cup ((L' \cup L'') \cap U^*(L \cup L' \cup L''))] \cap U^*(L \cup L' \cup L'') \subseteq \\ \subseteq [((L \cup L') \cap U^*(L \cup L')) \cup ((L' \cup L'') \cap U^*(L' \cup L''))] \cap U^*(L \cup L' \cup L'') = \\ = [C^*(L \cup L') \cup C^*(L' \cup L'')] \cap U^*(L \cup L' \cup L'') = \\ = (L' \cup L'') \cap U^*(L \cup L' \cup L'') \subseteq (L' \cup L'') \cap U^*(L' \cup L'') = C^*(L' \cup L'') = L'' \ . \end{aligned}$$

This means that $C^*(L \cup L' \cup L'') \subseteq L'' \subseteq L \cup L' \cup L''$, hence by path independence of C^*, we have $C^*(L \cup L' \cup L'') = C^*(L'') = L''$. So $C^*(L \cup L' \cup L'') = L'' \subseteq L \cup L'' \subseteq L \cup L' \cup L''$ and again, path independence implies $C^*(L \cup L'') = C^*(L \cup L' \cup L'') = L''$. This follows that $C(L \cup L'') = A''$, or, in other words $A \preceq_C A''$. We conclude that $\preceq_C$ is transitive, so it is indeed a partial order. □

The following example shows that our poset-based model is more general than that of Hatfield and Milgrom in [18].

Example 3.7. Assume that we have one hospital h and two doctors d and d'. Contracts x_3, x_4, x_5 and x'_3, x'_4 and x'_5 represent a 3, 4 and 5 days job for d and d' respectively. Assume that h has the following preference order on feasible contract sets (starting from the best):

$$\{x_4, x'_4\}, \{x_5, x'_3\}, \{x_3, x'_5\}, \{x_4, x'_3\}, \{x_3, x'_4\}, \{x_3, x'_3\}, \{x_5\}, \{x'_5\}, \{x_4\}, \{x'_4\}, \{x_3\}, \{x'_3\} .$$

So $C_h(Y)$ is that subset of Y which is the first in the above order. In particular, we have that $C_h(x_5, x_4, x_3, x'_5, x'_3) = \{x_5, x'_3\}$, so $x_4 \in R(x_5, x_4, x_3, x'_5, x'_3)$. Hence, if contracts were substitutes for C_h then R is monotone thus $x_4 \in R(x_5, x_4, x_3, x'_5, x'_4, x'_3) = R(X)$. This means that $x_4 \notin C_h(X)$ contradicting $C_h(X) = \{x_4, x'_4\}$.

However, the above C_h easily fits in our framework if we define poset P by $x_3 \preceq x_4 \preceq x_5$ and $x'_3 \preceq x'_4 \preceq x'_5$. For any lower ideal L, let $U(L) := \{x_3, x_4, x'_3, x'_4\} \cup u(L) \cup u'(L)$ where $u(L) = \emptyset$ if $x_4 \in L$ and $u(L) = \{x'_3, x'_4, x'_5\}$ if $x_4 \notin L$, and similarly $u'(L) = \emptyset$ if $x'_4 \in L$ and $u'(L) = \{x_3, x_4, x_5\}$ if $x'_4 \notin L$. As both u and u' are antitone, U is also such. Hence choice function C^* defined by $C^*(L) = L \cap U(L)$ is substitutable and one can easily check that $C^* = C_h$ on lower ideals of P. As C_h is path independent by definition, our model is indeed a genuine generalization of Hatfield and Milgrom's.

Note that for a substitutable choice function $C^* : \mathcal{L} \to \mathcal{L}$ there might be several antitone functions $U : \mathcal{L} \to \mathcal{L}$ such that $C^*(L) = L \cap U(L)$ holds for any member L of $\mathcal{L}$. The next statement shows that there is a canonical one among these antitone functions and this is in fact the minimal of those. (Actually, there is a maximal such U as well, but we do not need this fact.) For a choice function $C^* : \mathcal{L} \to \mathcal{L}$ define $U^* : \mathcal{L} \to \mathcal{L}$ by

$$U^*(L) := \bigcup\{Y \in \mathcal{L} : Y \subseteq C^*(L \cup Y)\} = \bigcup\{Y \in \mathcal{L} : Y \subseteq U(L \cup Y)\} . \qquad (2)$$

Note that the second equality in (2) holds by the definition of C^*, and this means that the right hand side defines the same U^* no matter which U (that defines C^*) we use.

Observation 3.8. If choice function $C^* : \mathcal{L} \to \mathcal{L}$ is substitutable then U^* in (2) is antitone and for any member L of $\mathcal{L}$ we have $C^*(L) = L \cap U^*(L)$.

Proof. Assume that C^* is substitutable, and $U : \mathcal{L} \to \mathcal{L}$ is an antitone function such that $C^*(L) = L \cap U(L)$ holds for any member L of $\mathcal{L}$. Define

$$U'(L) := \{Y \in \mathcal{L} : Y \subseteq C^*(L \cup Y)\} = \{Y \in \mathcal{L} : Y \subseteq U(L \cup Y)\},$$

that is $U^*(L) = \bigcup U'(L)$. Observe that if L and L' are members of $\mathcal{L}$ with $L \subseteq L'$ and $Y \in U'(L')$ then $Y \subseteq U(L' \cup Y) \subseteq U(L \cup Y)$ by the antitone property of U. This means that $U'(L') \subseteq U'(L)$, hence $U^*(L') = \bigcup U'(L') \subseteq \bigcup U'(L) = U^*(L)$, so U^* is indeed antitone.

For the second part, observe that $C^*(L) \in U'(L)$ by definition, hence $C^*(L) \subseteq U^*(L)$ and $C^*(L) \subseteq L \cap U^*(L)$. Moreover, if $Y \in U'(L)$ then $Y \cap L \subseteq Y \subseteq U(L \cup Y) \subseteq U(L)$, hence $Y \cap L \subseteq L \cap U(L) = C^*(L)$ holds for any $Y \in U'(L)$. This follows that

$$L \cap U^*(L) = L \cap (\bigcup U'(L)) = \bigcup\{L \cap Y : Y \in U'(L)\} \subseteq C^*(L),$$

so $L \cap U^*(L) = C^*(L)$ as we claimed. □

There is another useful fact about the antitone function U^* that defines a path-independent substitutable choice function.

Observation 3.9. If choice function $C^* : \mathcal{L} \to \mathcal{L}$ is path-independent and substitutable then $U^*(L) = U^*(C^*(L))$ holds for any member L of $\mathcal{L}$.

Proof. It follows from the antitone property of U^* and $C^*(L) \subseteq L$ that $U^*(L) \subseteq U^*(C^*(L))$. To show the opposite inclusion, assume that $Y \in U'(C^*(L))$, that is, $Y \subseteq C^*(C^*(L) \cup Y)$ holds. We show that

$$Y \subseteq C^*(L \cup Y) \ , \tag{3}$$

that is $Y \in U'(L)$, hence $U'(C^*(L)) \subseteq U'(L)$ and $U^*(C^*(L)) = \bigcup U'(C^*(L)) \subseteq \bigcup U'(L) = U^*(L)$.

To prove (3), observe that

$$C^*(L \cup Y) = (L \cup Y) \cap U^*(L \cup Y) \subseteq (L \cup Y) \cap U^*(L) \subseteq (L \cap U^*(L)) \cup Y = C^*(L) \cup Y \ ,$$

hence $C^*(L \cup Y) \subseteq C^*(L) \cup Y \subseteq L \cup Y$. Path-independence of C^* gives $C^*(L \cup Y) = C^*(C^*(L) \cup Y)$ and our assumption $Y \subseteq C^*(C^*(L) \cup Y)$ proves (3) that concludes the proof. □

At this point, we can generalize the notion of stability to our framework. Let D and H be the sets of doctors and hospitals, respectively and let X denote the set of possible contracts between doctors and hospitals. Assume that we have given a (complete) sublattice $\mathcal{L}$ of 2^X (for example as the set of lower ideals of a partial order P on X), and let $C^*_D = (C_D)^*$ and $C^*_H = (C_H)^*$ denote the joint choice functions of the doctors and of the hospitals, respectively. For members L_1 and L_2 of $\mathcal{L}$ pair (L_1, L_2) is called a *stable pair* if

$$U^*_D(L_1) = L_2 \qquad \text{and} \qquad U^*_H(L_2) = L_1 \tag{4}$$

holds. If $\mathcal{L} = \mathcal{L}(P)$ for some poset P on X then antichain A of P is called *stable* if

$$U^*_D(\mathrm{Li}(A)) \cap U^*_H(\mathrm{Li}(A)) = \mathrm{Li}(A) \tag{5}$$

Later we shall see that stable pairs are closely related to stable antichains. These latter represent the solution concept of two-sided market situations in our model. What does it mean that an antichain is stable? The first requirement is that if both doctors and hospitals select freely from those contracts that antichain A represents or implies then doctors select $C_D(A) = \mathrm{Max}(\mathrm{Li}(A) \cap U^*_D(\mathrm{Li}(A)) = \mathrm{Max}(\mathrm{Li}(A)) = A$, as $\mathrm{Li}(A) \subseteq U^*_D(\mathrm{Li}(A))$. Similarly follows that $C_H(A) = A$, so hospitals also pick the same antichain A of contracts. Moreover, if there are some further choices available that are represented by antichain Y and both the doctors and the hospitals are happy to pick those (formally, if $Y \subseteq C_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y))$ and $Y \subseteq C_H(\mathrm{Li}(A) \cup \mathrm{Li}(Y)))$ then

$$\begin{aligned} &\mathrm{Li}(Y) \subseteq C^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \cap C^*_H(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \subseteq \\ &\subseteq U^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \cap U^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) \subseteq U^*_D(\mathrm{Li}(A)) \cap U^*_D(\mathrm{Li}(A)) = \mathrm{Li}(A) \ . \end{aligned}$$

So $Y \subseteq C_D(\mathrm{Li}(A) \cup \mathrm{Li}(Y)) = C_D(\mathrm{Li}(A)) = A$. This means that we cannot add further choices to A such that both the doctors and the hospitals will select them.

In the Hatfield-Milgrom model, $A \subseteq X$ is a *stable allocation* if $C_D(A) = C_H(A) = A$ and there exists no hospital h and set of contracts $X'' \neq C_h(X')$ with $X'' = C_h(A \cup X'') \subseteq C_D(A \cup X'')$. Assume that A is a feasible allocation, that is, $C_D(A) = C_H(A) = A$ and A' is a blocking set: $A' \subseteq C_D(A \cup A')$ and $A' \subseteq C_H(A \cup A')$. This means that there is a hospital h that picks a different assignment from A and from $A \cup A'$. Let $X'' = C_h(A \cup A')$ denote the choice of this hospital h. Since $X'' = C_h(A \cup A') \subseteq A \cup X'' \subseteq A \cup A'$, we have $X'' = C_h(A \cup X'')$. Because of $X'' \subseteq C_D(A \cup A')$ and $A = C_D(A)$, each doctor in $\cup_{x \in X''} D(x)$ has the same choice as in $A \cup X''$, that is, $X'' \subseteq C_D(A \cup X'')$. So X'' blocks A in the Hatfield-Milgrom sense. This proves that a stable allocation of contracts in the Hatfield-Milgrom framework is a stable antichain. It is not difficult to see that the other direction is also true: any stable antichain is a stable allocation of contracts for the same model with a trivial underlying partial order.

Example 3.10. Assume we have two hospitals h and $\bar{h}$ and two doctors d and d'. Contract x'_i represents an i-days job of d' at h, and $\bar{x}_j$ stands for a j-days occupation for d at $\bar{h}$, etc. Poset P is defined by relations of type $z_i \preceq z_j$ for $i \leq j$ and $z \in \{x, x', \bar{x}, \bar{x}'\}$. Let

$$X := \{z_i : 1 \leq i \leq 5,\ z \in \{x, x', \bar{x}, \bar{x}'\}\} \ .$$

Assume that d is a famous doctor with a high salary expectation, so each hospital wants to employ her but for a minimum amount of time. Doctor d' can do the same job equally well but she is young and hence costs less to the employer. Assume that each hospital needs 5 days of work and from a given set of options it selects 1 day of work of doctor d, the maximum amount of work for doctor d' up to 5 days altogether and for the missing days it selects d if she is still available. In particular $C_h = C_{\bar{h}}$ and for example, $C_h(x_1, x_2, x_3, x'_1, x'_2, x'_3) = \{x_2, x'_3\}$, $C_h(x_1, x_2, x_3, x'_1, x'_2, x'_3, x'_4) = \{x_1, x'_4\}$ and $C_{\bar{h}}(\bar{x}_1, \bar{x}_2, \bar{x}'_1, \bar{x}'_2) = \{\bar{x}_2, \bar{x}'_2\}$. Assume moreover that $C_d = C_{d'}$ and both doctors d and d' look for 5 days of work, and both of them prefer hospital h to h': $C_d(x_1, x_2, x_3, \bar{x}_1, \bar{x}_2, \bar{x}_3) = \{x_3, \bar{x}_2\}$.

It is easy to check that all four choice functions are substitutable and path-independent on $\mathcal{L}(P)$ and moreover $U^*_H(\mathrm{Li}(A)) = U^*_H(L) = L_1$ and $U^*_D(\mathrm{Li}(A)) = U^*_D(L) = L_2$ holds for

$$A = \{x_1, \bar{x}_4, x'_4, \bar{x}'_1\} \ , \qquad L = \mathrm{Li}(A) = \{x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x'_1, x'_2, x'_3, x'_4, \bar{x}'_1\},$$

$$L_1 = \{x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x'_1, x'_2, x'_3, x'_4, \bar{x}'_1, \bar{x}'_2, \bar{x}'_3, \bar{x}'_4\}, \text{ and}$$

$$L_2 = \{x_1, x_2, x_3, x_4, x_5, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, x'_1, x'_2, x'_3, x'_4, x'_5, \bar{x}'_1\} \ .$$

As $L_1 \cap L_2 = L$, it follows that (L_1, L_2) is a stable pair and A is a stable antichain.

4 Main result

In this section, we prove our main results. Let X be a ground set and define partial order $\sqsubseteq$ on pairs of subsets of X by $(A, B) \sqsubseteq (A', B')$ if $A \subseteq A'$ and $B \supseteq B'$ holds.

It is clear that for any sublattice $\mathcal{L}$ of 2^X, $\sqsubseteq$ defines a lattice on $\mathcal{L} \times \mathcal{L}$ with lattice operations $(A, B) \sqcap (A', B') = (A \cap A', B \cup B')$ and $(A, B) \sqcup (A', B') = (A \cup A', B \cap B')$. The following theorem generalizes some results by Hatfield and Milgrom in [18].

Theorem 4.1. *Let X be a set of possible contracts between set D of doctors and H of hospitals and let $\mathcal{L}$ be a complete sublattice of 2^X. Assume that joint choice functions C^*_D of doctors and C^*_H of hospitals are substitutable. Then stable pairs form a nonempty complete lattice subset of $(\mathcal{L} \times \mathcal{L}, \sqsubseteq)$. In particular, there does exist a stable pair and there is a greatest and a lowest such pair.*

Moreover, if $\mathcal{L} = \mathcal{L}(P)$ is the lattice of lower ideals of some partial order $P = (X, \preceq)$ and both joint choice functions C_D and C_H are substitutatble and path independent then $\preceq_{C_D}$ and $\preceq_{C_H}$ are opposite partial orders on stable antichains and both of them define a lattice.

Note that the 2nd part of Theorem 4.1 generalizes the following well-known result of Blair on the lattice structure of many-to-many stable matchings [5].

Corollary 4.2 (Blair 1988, [5]). *If both doctors' and hospitals' choice functions are substitutable and path independent and moreover no two different contract is possible between the same doctor and hospital then $\preceq_{C_D}$ and $\preceq_{C_H}$ are opposite partial orders on stable assignments and both of them define a lattice.*

There are known proofs for Corollary 4.2 that are based solely on lattices and the Tarski framework. The first one was probably by Fleiner [9] but there also exist simplified versions by Jankó [19, 20]. The proof below uses ideas of the latter works.

Our proof of Theorem 4.1 contains a direct proof of Corollary 4.2.

Proof of Theorem 4.1. Define mapping $f : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \times \mathcal{L}$ by

$$f(L_1, L_2) := (U^*_H(L_2), U^*_D(L_1))$$

where U^*_D and U^*_H are the functions defined according to (2) from C^*_D and C^*_H. By definition, a pair (L_1, L_2) is stable if and only if $f(L_1, L_2) = (L_1, L_2)$. Assume that $(L_1, L_2) \sqsubseteq (L'_1, L'_2)$, i.e. $L_1 \subseteq L'_1$ and $L_2 \supseteq L'_2$. Functions U^*_D and U^*_H are antitone by Observation 3.8, hence $U^*_D(L_2) \subseteq U^*_D(L'_2)$ and $U^*_H(L_1) \supseteq U^*_H(L'_1)$, that is, $f(L_1, L_2) \sqsubseteq f(L'_1, L'_2)$ holds. This means that f is monotone on complete lattice $(\mathcal{L} \times \mathcal{L}, \sqsubseteq)$, hence its fixed points form a nonempty complete lattice subset of $(\mathcal{L} \times \mathcal{L}, \sqsubseteq)$ by Theorem 2.2 of Tarski. This proves the first part of Theorem 4.1.

To show the second part of the Theorem in the special case where $\mathcal{L} = \mathcal{L}(P)$ and choice functions are path independent, we prove that there is a natural bijection between stable pairs and stable antichains in such a way that the partial order on stable antichains induced by the natural bijection and partial order $\sqsubseteq$ coincides with both $\preceq_{C_D}$ and $\succeq_{C_H}$ (the opposite of $\preceq_{C_H}$). As soon as we do so, the second part of Theorem 4.1 immediately follows from the first one.

So assume that $\mathcal{L} = \mathcal{L}(P)$ and choice functions of doctors' and hospitals' are substitutable and path independent. Let (L_1, L_2) be a stable pair of lower ideals of P. Observe that

$$C^*_D(L_1) = L_1 \cap U^*_D(L_1) = L_1 \cap L_2 = U^*_H(L_2) \cap L_2 = C^*_H(L_2)$$

so

$$A(L_1, L_2) := C_D(L_1) = C_H(L_2) = \mathrm{Max}(L) \tag{6}$$

is an antichain, where $L := L_1 \cap L_2$. From (6) and Observation 3.9 it follows that $U^*_D(\mathrm{Li}(A(L_1, L_2))) = U^*_D(C^*_D(L_1)) = U^*_D(L_1) = L_2$ and similarly, $U^*_H(\mathrm{Li}(A(L_1, L_2))) = U^*_H(C^*_H(L_2)) = U^*_H(L_2) = L_1$, hence $U^*_D(\mathrm{Li}(A(L_1, L_2))) \cap U^*_H(\mathrm{Li}(A(L_1, L_2))) = L_1 \cap L_2 = \mathrm{Li}(A(L_1, L_2))$, so $A(L_1, L_2)$ is in indeed stable.

Now assume that A is a stable antichain and define $L := \mathrm{Li}(A)$, $L_1 := U^*_H(L)$ and $L_2 := U^*_D(L)$. We show that (L_1, L_2) is a stable pair such that $A = A(L_1, L_2)$. By stability of antichain A, we have that $L = U^*_D(L) \cap U^*_H(L)$. This means that $L \subseteq U^*_D(L)$, hence $L = L \cap U^*_D(L) = C^*_D(L)$ and $C^*_D(L_1) = L_1 \cap U^*_D(L_1) \subseteq L_1 \cap U^*_D(L) = L_1 \cap L_2 = L$. This means that $C^*_D(L_1) \subseteq L \subseteq L_1$, and path-independence of C^*_D implies that $C^*_D(L_1) = C^*_D(L) = L$. Observation 3.9 yields that $U^*_D(L_1) = U^*_D(L) = L_2$. A similar argument shows that $U^*_H(L_2) = L_1$. We got that (L_1, L_2) is indeed a stable pair, and moreover $A(L_1, L_2) = \mathrm{Max}(L_1 \cap L_2) = \mathrm{Max}(U^*_H(L) \cap U^*_D(L)) = \mathrm{Max}(L) = A$.

To prove the existence of a natural bijection between stable pairs and antichains, we only have to show that the stable pair we construct from $A(L_1, L_2)$ according to the above paragraph is (L_1, L_2) for any stable pair (L_1, L_2). Actually, this stable pair is (L'_1, L'_2) for $L'_1 = U^*_H(\mathrm{Li}(A(L_1, L_2)))$ and $L'_2 = U^*_D(\mathrm{Li}(A(L_1, L_2)))$. We have seen that $C^*_D(L_1) = \mathrm{Li}(A(L_1, L_2)) = C^*_H(L_2)$, so Observation 3.9 implies that $L_1 = U^*_H(L_2) = U^*_H(C^*_H(L_2)) = U^*_H(\mathrm{Li}(A(L_1, L_2))) = L'_1$ and $L_2 = U^*_D(L_1) = U^*_D(C^*_D(L_1)) = U^*_D(\mathrm{Li}(A(L_1, L_2))) = L'_2$. This shows that there is indeed a natural bijection between stable antichains and stable sets.

To finish the proof by justifying the generalization of Theorem 4.2 by Blair, we show that $\sqsubseteq$ and the natural bijection induces a partial order that coincides with $\preceq_{C_D}$ and $\succeq_{C_H}$ on stable antichains. So assume now that (L_1, L_2) and (L'_1, L'_2) are stable pairs that correspond to stable antichains A and A'. This means that $L_1 = U^*_H(\mathrm{Li}(A)), L_2 = U^*_D(\mathrm{Li}(A))$ and $L'_1 = U^*_H(\mathrm{Li}(A')), L'_2 = U^*_D(\mathrm{Li}(A'))$ on one hand and $\mathrm{Li}(A) = L_1 \cap L_2 = C^*_D(L_1) = C^*_H(L_2), \mathrm{Li}(A') = L'_1 \cap L'_2 = C^*_D(L'_1) = C^*_H(L'_2)$ on the other hand.

Assume first that $(L_1, L_2) \sqsubseteq (L'_1, L'_2)$, i.e. $L_1 \subseteq L'_1$ and $L_2 \supseteq L'_2$. Consequently, (using Lemma 3.5)

$$C^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(A')) = C^*_D(C^*_D(L_1) \cup C^*_D(L'_1)) = C^*_D(L_1 \cup L'_1) = C^*_D(L'_1) = \mathrm{Li}(A')$$

and similarly

$$C^*_H(\mathrm{Li}(A) \cup \mathrm{Li}(A')) = C^*_H(C^*_H(L_2) \cup C^*_H(L'_2)) = C^*_H(L_2 \cup L'_2) = C^*_H(L_2) = \mathrm{Li}(A) \ .$$

In other words, $C_D(A \cup A') = A'$ and $C_H(A \cup A') = A$, hence $A \preceq_{C_D} A'$ and $A \succeq_{C_H} A'$.

Now suppose that $A \preceq_{C_D} A'$, that is, $A' = C_D(A \cup A')$. This follows that $\mathrm{Li}(A') = C^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(A'))$. Observation 3.9 and the antitone property of U^* yields that

$$L'_2 = U^*_D(\mathrm{Li}(A')) = U^*_D(\mathrm{Li}(A) \cup \mathrm{Li}(A')) \subseteq U^*_D(\mathrm{Li}(A)) = L_2 \ . \tag{7}$$

As (L_1, L_2) and (L'_1, L'_2) are stable pairs, the antitone property of U^*_H implies that

$$L_1 = U^*_H(L_2) \subseteq U^*_H(L'_2) = L'_1 \ . \tag{8}$$

From (7) and (8), $(L_1, L_2) \sqsubseteq (L'_1, L'_2)$ follows. A similar argument justifies that $(L'_1, L'_2) \sqsubseteq (L_1, L_2)$ holds whenever $A \preceq_{C_H} A'$ and this concludes our proof. $\square$

Theorem 4.1 points out a close connection between the notion of stability and fixed points of a monotone function that always exist by Theorem 2.2 of Tarski. We have already indicated that one can construct the maximal and minimal fixed points by iterating the monotone function starting from the maximum or from the minimum element of the underlying lattice, respectively. Probably, it was Fleiner in [8] who first pointed out that the well-known proposal algorithm of Gale and Shapley that finds a man-optimal stable marriage scheme can be regarded as an iteration of a certain monotone mapping. Later, the same observation was made by Hatfield and Milgrom for a special case of Fleiner's model. Actually, the same connection also holds in our present settings that generalize both Fleiner's and the Hatfield-Milgrom framework. The generalized Gale-Shapley algorithm for finding a stable antichain works as follows.

Let us denote by 0 and 1 the minimal and maximal elements of $\mathcal{L}$, respectively. It is straightforward to check that mapping $f : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \times \mathcal{L}$ on lattice $(\mathcal{L} \times \mathcal{L}, \sqsubseteq)$ defined by $f(L_1, L_2) := (U^*_H(L_2), U^*_D(L_1))$ is monotone by the antitone property of U^*_D and U^*_H. Clearly, $(L_1, L_2) = (U^*_H(L_2), U^*_D(L_1))$ is a fixed point if and only if (L_1, L_2) is a stable pair, and in case of $\mathcal{L} = \mathcal{L}(P)$, it is equivalent to $\mathrm{Max}(L_1 \cap L_2)$ is a stable antichain. So to find the maximal (doctor-optimal) stable antichain, we only have to start to iterate f from the maximal element of $\mathcal{L} \times \mathcal{L}$ to get a $\sqsubseteq$-decreasing sequence $(1, 0) \sqsupseteq f(1, 0) \sqsupseteq f(f(1, 0)) \sqsupseteq \dots$. Hence after at most $2h$ iterations (where h denotes the hight (the length of the longest chain) of poset $\mathcal{L}$) we arrive to a fixed point and find the doctor-optimal stable antichain A_D. If we start the iteration from the bottom of the lattice $\mathcal{L} \times \mathcal{L}$ then the $\sqsubseteq$-minimal fixed point at the "end" of increasing sequence $(0, 1) \sqsubseteq f(0, 1) \sqsubseteq f(f(0, 1)) \sqsubseteq \dots$ represents the hospital-optimal stable antichain A_H. According to Theorem 4.1 for any stable antichain A we have $A_H \preceq_{C_D} A \preceq_{C_D} A_D$ and $A_D \preceq_{C_H} A \preceq_{C_H} A_H$, so for example $C_D(\mathrm{Li}(A \cup A_D)) = A_D$ and $C_H(\mathrm{Li}(A \cup A_H)) = A_H$. This means that if doctors are offered all the choices that the contracts in some stable antichain represent or imply then from this choice set doctors pick contracts of A_D, and a similar property is true for the hospitals with respect to A_H.

It seems that no one observed so far that the monotone function iteration is more powerful than the Gale-Shapley algorithm itself that (in its original form) finds the man-optimal and (with an exchanges of roles) the woman-optimal stable matchings. The iteration method can be used to calculate the lattice operations on the fixed points of the monotone function. Consequently, we can construct the $\preceq_{C_D}$-least and $\preceq_{C_D}$-greatest stable antichains that are the least upper and greatest lower bounds of any given nonempty set of stable antichains. This works as follows: take stable antichains $A_1, A_2, \dots, A_k$ that correspond to stable pairs $(L_1, K_1), (L_2, K_2), \dots, (L_k, K_k)$ and define $L := \bigcup_{i=1}^k L_i$ and $K := \bigcap_{i=1}^k K_i$. By the antitone property of U^*_H and U^*_D

we get

$$U^*_H(K) = U^*_H(\bigcap_{i=1}^{k} K_i) \supset \bigcup_{i=1}^{k} U^*_H(K_i) = \bigcup_{i=1}^{k} L_i = L \text{ and}$$
$$U^*_D(L) = U^*_D(\bigcup_{i=1}^{k} L_i) \subset \bigcap_{i=1}^{k} U^*_D(L_i) = \bigcap_{i=1}^{k} K_i = K \ ,$$

so $(K, L) \sqsubseteq (U^*_H(K), U^*_D(L)) = f(K, L)$. Now monotonicity of f gives that $(K, L) \sqsubseteq f(K, L) \sqsubseteq f(f(K, L)) \sqsubseteq f(f(f(K, L)))\ldots$ and at most $2h$ iterations of f this $\sqsubseteq$-increasing sequence arrives to the $\sqsubseteq$-least stable pair that is $\sqsubseteq$-greater than (K, L). This fixed point clearly corresponds to the stable antichain that is the least upper bound of stable antichains $A_1, A_2, \ldots, A_k$. A similar argument shows that for $L' := \bigcap_{i=1}^{k} L_i$ and $K' := \bigcup_{i=1}^{k} K_i$ we have $(L', K') \sqsupseteq f(L', K') \sqsupseteq f(f(L', K')) \sqsupseteq f(f(f(L', K'))) \sqsupseteq \ldots$ and the "end" of this decreasing sequence corresponds to the meet of stable antichains $A_1, A_2, \ldots, A_k$.

5 Conclusion, open problems

In this work, we described an economic model and proved certain results on stable solutions of the model. We pointed out that our model is a genuine generalization of previous models described by Fleiner [8, 9] and by Hatfield and Milgrom [18]. We tried to illustrate by examples that our framework is not "just" a mathematical generalization but also a practically interesting one. Moreover, we think that the our proofs are more "neat" than the previous ones and they justify that the "right" approach to the stability concept of Gale and Shapley in case of two-sided economies is based on Tarski's fixed point theorem [29]. Previous results along these lines used the Knaster-Tarski fixed point theorem in [23] where the underlying complete lattice is 2^X, the lattice of all subsets of ground set X. In this work, we need the more general version as we talk about sublattices of 2^X. Note that most of our results remain valid if we work on a lattice subset of 2^X, but this kind of generalization does not seem to capture any situation with an interesting practical application.

Our motivation for the above work was to find a practically interesting generalization of the Hatfield-Milgrom results. We think that at least part of this task is completed. However, there are at least two important topics of the Hatfield and Milgrom paper missing from our work. One is a strategy-proofness result saying that reporting true preferences is a dominant strategy for doctors if the doctor-optimal outcome determines the realized contracts. Probably, this result can be generalized to our model. The second such topic has to do with the law of aggregate demand. It follows from Fleiner [9], Alkan and Gale [3] and Hatfield and Milgrom [18] that if choice functions have the property that the number of selected contracts from a greater choice set cannot decrease then a natural generalization of the rural hospital theorem holds. (Recall that such choice function is called "increasing" by Fleiner, "size monotone" by Alkan and Gale and in the Hatfield-Milgrom terminology, it is said to satisfy the "law of aggregate demand".) Actually, from Fleiner's work in [9] it follows that a weaker property, namely w-increasingness is enough for the rural hospital theorem, and there is hope that in our model it is also enough. Answering these questions is subject of ongoing research.

References

[1] Adachi, H.: On a characterization of stable matchings. Econom. Lett. **68**, 43–49 (2000)

[2] Alkan, A.: A class of multipartner matching markets with a strong lattice structure. Econom. Theory **19**, 737–746 (2002)

[3] Alkan, A., Gale, D.: Stable schedule matching under revealed preference. J. Econom. Theory **112**, 289–306 (2003)

[4] Bevia, C., Quinzii, M., Silva, J.: Buying several indivisible goods. Math. Social Sci. **37**, 1–23 (1999)

[5] Blair, C.: The lattice structure of the set of stable matchings with multiple partners. Math. Oper. Res. **13**, 619–628 (1988)

[6] Crawford, V.P., Knoer, E.M.: Job matching with heterogeneous firms and workers. Econometrica **49**, 437–450 (1981)

[7] Danilov,V.I., Koshevoy, G.A., Lang, C.: Gross substitution, discrete convexity, and submodularity. Discrete Appl. Math. **131**, 283–298 (2003)

[8] Fleiner, T.: Stable and crossing structures. Ph.D. thesis, Tecnische Universiteit Eindhoven (2000)

[9] Fleiner, T.: A fixed point approach to stable matching and some applications. Math. Oper. Res. **38**, 103–126 (2003)

[10] Fujishige, S., Tamura, A.: A general two-sided matching market with discrete concave utility functions. Discrete Appl. Math. **154**, 950–970 (2006)

[11] Fujishige, S., Tamura, A.: A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete convex analysis. Math. Oper. Res. **32**, 136–154 (2007)

[12] Fujishige,S., Yang, Z.: A note on Kelso and Crawford's gross substitutes condition. Math. Oper. Res. **28**, 463–469 (2003)

[13] Gale, D., Shapley, L.S.: College admissions and the stability of marriage. Amer. Math. Monthly **69**, 9–15 (1962)

[14] Gul, F., Stacchetti, E.: Walrasian equilibrium with gross substitutes. J. Econom. Theory **87**, 95–124 (1999)

[15] Gul, F., Stacchetti, E.: The English auction with differentiated commodities. J. Econom. Theory **92**, 66–95 (2000)

[16] Hatfield, J.W., Kojima, F.: Matching with contracts: Comment. Amer. Econom. Rev. **98**, 1189–1194 (2008)

[17] Hatfield, J.W., Kojima, F.: Substitutes and stability for matching with contacts. J. Econom. Theory **145**, 1704–1723 (2010)

[18] Hatfield, J.W., Milgrom, P.R.: Matching with contracts. Amer. Econom. Rev. **95**, 913–935 (2005)

[19] Jankó, Z.: Stabil párosítások és egyetemi felvételi ponthatárok. BSc dissertation, Eötvös University (2009)

[20] Jankó, Z.: Generalized Stable Matchings: Theory and applications. MSc dissertation, Eötvös University (2011).
http://www.cs.elte.hu/blobs/diplomamunkak/msc_actfinmat/2011/janko_zsuzsanna.pdf

[21] Kelso, J.A.S., Crawford, V.P.: Job matching coalition formation, and gross substitution. Econometrica **50**, 1483–1504 (1982)

[22] Klaus, B., Walzl, M.: Stable many-to-many matchings with contracts. J. Math. Econom. **45**, 422–434 (2009)

[23] Knaster, B.: Un théorème sur les fonctions d'ensembles. Ann. Soc. Polon. Math. **6**, 133–134 (1928)

[24] Murota, K.: Discrete Convex Analysis. Society for Industrial and Applied Mathematics, Philadelphia (2003)

[25] Murota, K., Shioura, A.: M-convex function on generalized polymatroid, Math. Oper. Res. **24**, 95–105 (1999)

[26] Murota, K., Tamura, A.: New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities. Discrete Appl. Math. **131**, 495–512 (2003)

[27] Roth, A.E., Sotomayor, M.A.O.: Two-sided matching. Cambridge University Press, Cambridge (1990)

[28] Sotomayor, M.: Three remarks on the many-to-many stable matching problem: Dedicated to David Gale on his 75th birthday. Math. Social Sci. **38**, 55–70 (1999)

[29] Tarski, A.: A lattice-theoretical fixpoint theorem and its applications. Pacific J. Math. **5**, 285–309 (1955)