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# A Note On Strongly Edge-Disjoint Arborescences

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# A Note On Strongly Edge-Disjoint Arborescences

Kristóf Bérczi and Erika Renáta Kovács <sup>\*</sup>

## Abstract

In [1], Colussi, Conforti and Zambelli conjectured that in a rooted  $k$ -edge-connected digraph there exist  $k$  strongly edge-disjoint arborescences, and also gave a proof for  $k = 2$ . In this paper, we give a generalization of the case  $k = 2$  and show that the conjecture does not hold for  $k \geq 3$ .

## 1 Introduction

Let  $D = (V + r, A)$  be a directed graph with designated root-node  $r$ . A spanning arborescence  $F$  of  $D$  rooted at  $r$  is called an  **$r$ -arborescence**. A node  $u$  is an  **$F$ -ancestor** of another node  $v$  if there is a directed path from  $u$  to  $v$  in  $F$ . We denote this unique path by  $F(u, v)$ . For example, in an  $r$ -arborescence  $F$  the root is the  $F$ -ancestor of all other nodes. We call  $D$  **rooted  $k$ -edge-connected** if for each  $v \in V$ , there exist  $k$  edge-disjoint directed paths from  $r$  to  $v$ . The maximum number of edge-disjoint  $r - v$  paths is denoted by  $\lambda(r, v)$ . A fundamental theorem on packing arborescences is due to Edmonds who gave a characterization of the existence of  $k$  edge-disjoint spanning arborescences rooted at the same node [2].

**Theorem 1.1** (Edmonds' theorem). *Let  $D = (V + r, A)$  be a digraph with root  $r$ .  $D$  has  $k$  edge-disjoint spanning  $r$ -arborescences if and only if  $D$  is rooted  $k$ -edge-connected.*

A natural idea is to reformulate the problem to the node-connected case. Let  $D$  and  $r$  denote a digraph and a root-node as previously, then  $D$  is called **rooted  $k$ -node-connected** (or rooted  $k$ -connected, for short) if there exist  $k$  internally node-disjoint directed paths from  $r$  to  $v$  for each  $v \in V$ , that is, any two of the paths have only  $r$  and  $v$  in common. The maximum number of node-disjoint  $r - v$  paths is denoted by  $\kappa(r, v)$ .

Note that two  $r$ -arborescences  $F_1$  and  $F_2$  are edge-disjoint if and only if for each  $v \in V$  the two paths  $F_1(r, v)$  and  $F_2(r, v)$  are edge-disjoint. That gives the idea of the following definition: we call two spanning  $r$ -arborescences  $F_1$  and  $F_2$  **independent** if  $F_1(r, v)$  and  $F_2(r, v)$  are internally node-disjoint for each  $v \in V$ .

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As a node-disjoint counterpart of Edmonds' theorem, Frank conjectured that in a rooted  $k$ -connected graph there exist  $k$  independent arborescences (see eg. [12]). The case  $k = 2$  was verified by Whitty [13], but for  $k \geq 3$  the statement does not hold as was shown by Huck [4]. However, Huck also proved that the conjecture is true for simple acyclic graphs [5] and verified the statement for planar multigraphs except for a few values of  $k$  [6].

**Theorem 1.2.**

- (i) (Whitty) Let  $D = (V + r, A)$  be a digraph with root  $r$ .  $D$  has two independent spanning  $r$ -arborescences if and only if  $D$  is rooted 2-connected.
- (ii) (Huck) Let  $D = (V + r, A)$  be an acyclic digraph with root  $r$  such that  $D - r$  is simple.  $D$  has  $k$  independent spanning  $r$ -arborescences if and only if  $D$  is rooted  $k$ -connected.
- (iii) (Huck) Let  $D = (V + r, A)$  be a directed multigraph with root  $r$  and  $k \in \{1, 2\} \cup \{6, 7, 8, \dots\}$  such that  $D$  is planar if  $k \geq 6$ .  $D$  has  $k$  independent spanning  $r$ -arborescences if and only if  $D$  is rooted  $k$ -connected.

In [1], Colussi, Conforti and Zambelli introduced another type of disjointness concerning arborescences, which put slightly stronger restrictions on the paths than edge-disjointness. In a digraph we call two arcs **symmetric** if they share the same end-nodes but have opposite orientations. Two arborescences  $F_1, F_2$  rooted at  $r$  are called **strongly edge-disjoint** if the paths  $F_1(r, v), F_2(r, v)$  do not contain a pair of symmetric arcs. In [1], the following strengthening of Edmonds' theorem was proposed.

**Conjecture 1.3** (Colussi, Conforti, Zambelli). Let  $D = (V + r, A)$  be a digraph with root  $r$ .  $D$  has  $k$  strongly edge-disjoint spanning  $r$ -arborescences if and only if  $D$  is rooted  $k$ -edge-connected.

For  $k = 2$ , the conjecture was verified in [1]. As Colussi et al. note, the motivation of the problem is the following. It is easy to see that a similar statement holds for strongly edge-disjoint directed  $s - t$  paths. Hence the conjecture, if it were true, could be considered as a common generalization of Edmonds' disjoint arborescences theorem and Menger's theorem. Note that the arborescences in the conjecture are allowed to contain pairs of symmetric arcs, only the paths in question are required not to do so.

Throughout the paper, we use the following notation. A directed graph is denoted by  $D = (V + r, A)$  where  $V$  and  $A$  stand for the set of nodes and arcs, respectively, and  $r$  is a root-node. We always assume that each node  $v \in V - r$  is reachable from  $r$  on a directed path. By an arborescence, if not stated otherwise, we mean a spanning  $r$ -arborescence. The in-degree of a set  $X \subseteq V$  is denoted by  $\varrho(X)$ . For a singleton  $v$ , we abbreviate  $\varrho(\{v\})$  by  $\varrho(v)$ . We say that a node  $w$  **dominates** a node  $v$  if every path from  $r$  to  $v$  includes  $w$ . We denote the set of nodes dominating  $v$  by  $dom(v)$ . Clearly,  $r$  and  $v$  are in  $dom(v)$ . Sometimes we use these notations with subscripts when only a subset  $F \subseteq E$  is considered or we work with different graphs simultaneously.

The rest of the paper is organized as follows. Section 2 gives a short overview of the corresponding results concerning Steiner-arborescences and describes a special ordering of the nodes. As a consequence, we get a new proof of a theorem of Georgiadis and Tarjan [3]. Based on this theorem, we prove a generalization of Conjecture 1.3 for  $k = 2$  in Section 3. In Section 4, we give a disproof of the conjecture for  $k \geq 3$ . Finally, in Section 5, we propose a new conjecture concerning strongly edge-disjoint arborescences.

## 2 Disjoint Steiner-arborescences

For a digraph  $D = (V + r, A)$  with root  $r$  and terminal set  $T \subseteq V$ , an  $r$ -arborescence spanning  $T$  is called a **Steiner-arborescence**. Two Steiner-arborescences  $F_1$  and  $F_2$  are called **edge-independent** if the paths  $F_1(r, t), F_2(r, t)$  are edge-disjoint for every terminal  $t \in T$ . **Independent** Steiner-arborescences can be defined in a straightforward manner. Note that paths corresponding to non-terminal nodes are allowed to violate the disjointness condition hence the arborescences are not necessarily edge-disjoint.

Z. Király asked [10] whether the existence of  $k$  edge-independent Steiner-arborescences is ensured by  $\lambda(r, t) \geq k$  for each  $t \in T$ . As Frank's conjecture would follow from such a result, Huck's counterexample shows that  $k = 2$  is the only case when this statement may hold. Indeed, the following theorem appeared in [11].

**Theorem 2.1.** *Let  $D = (V + r, A)$  be a digraph with root  $r$ , terminal set  $T \subseteq V$  and  $\lambda(r, t) \geq 2$  for each  $t \in T$ . Then there exist two edge-independent Steiner-arborescences.*

*Proof.* We may assume that all of the terminal nodes have in-degree two. Indeed, if  $\varrho(t) \geq 3$  for some  $t \in T$ , then take 2 edge-disjoint  $r - t$  paths. We claim that any edge entering  $t$  and not used by these paths can be left out without violating the conditions of the theorem. Indeed, the only problem may arise if after the deletion of an edge  $e$  we have  $\lambda_{D-e}(r, t') \leq 1$  for some  $t' \in T$ , that is, there exist a set  $X \subseteq V$  such that  $\varrho_{D-e}(X) \leq 1$ . Clearly,  $e$  enters  $X$  so  $t \in X$ , contradicting the fact that there are two  $r - v$  paths in  $D - e$ . We may also assume that  $T$  contains all the nodes for which  $\lambda(r, v) \geq 2$ , hence the non-terminal nodes have in-degree one.

We prove by induction on the number of nodes plus edges. Assume first that there is a terminal node  $t$  for which  $\text{dom}(t) - \{r, t\}$  is not empty and take a node  $x \in \text{dom}(t) - \{r, t\}$ . Let  $M$  denote the set of nodes that are cut from  $r$  by  $x$ , excluding  $x$ .

As  $\lambda(r, t) \geq 2$ , we have  $\lambda(r, x) \geq 2$  and so  $x \in T$ . That means that in the subgraph spanned by  $M + x$  with root  $x$ , the set of terminal nodes (that is, the set of nodes with  $\lambda(x, t) \geq 2$ ) is exactly  $T \cap M$ . As  $r$  is not included in this graph, the number of nodes and edges together is strictly smaller than that of in  $D$ , hence the theorem can be applied for this smaller graph. Let  $F_1^x$  and  $F_2^x$  denote the arborescences obtained this way.

On the other hand, let  $D'$  denote the graph obtained from  $D$  by contracting  $M$  to a single node  $m$  and deleting all but one  $xm$  edges. As  $M$  contained a terminal, the number of nodes plus edges in  $D'$  is strictly smaller than in  $D$  so we can apply the theorem to  $D'$ . Let  $F'_1$  and  $F'_2$  denote the independent arborescences in  $D'$ . Note that  $\kappa_{D'}(r, t)$  remains 2 for each  $t \in T \setminus M$ .

Now take the two pairs of edge-disjoint arborescences and match them arbitrarily, say,  $F'_1, F'_1^x$  and  $F'_2, F'_2^x$ . Now we can build a Steiner arborescence of  $D$  from  $F'_i$  and  $F'_i^x$  by simply leaving out  $m$  from  $D'$  and unifying the arborescences at  $x$ . The resulting edge set is clearly an arborescence  $F_i$  which spans all the terminals. It only remains to verify that the paths  $F_1(r, t)$  and  $F_2(r, t)$  are edge-disjoint for each  $t \in T$ .

This clearly holds for  $t \in M + x$ . Since  $\rho_{D'}(m) = 1$ , for each terminal  $t \in T \setminus M$ , at most one of its paths uses  $m$  in  $D'$  so at most one of the paths is modified when constructing  $F_i$  from  $F'_i$  and  $F'_i^x$ . As we only add new edges to the graph, the disjointness still holds after this step.

So assume that  $\kappa(r, t) \geq 2$  for every terminal  $t \in T$ . Now modify the graph in the following way. If the set of non-terminal nodes spans an arc then contract its ends. Then for each non-terminal node  $u$ , if its ancestor is  $r$  then add another  $ru$  edge to the graph; if its ancestor  $v$  is not  $r$  (which also means that it is a terminal node) then contract  $u$  with  $v$ .

The resulting graph is rooted 2-connected, hence Whitty's theorem can be applied to get two node-independent spanning arborescences. By inverting the modifications on the non-terminals, the arborescences naturally extend to edge-independent Steiner-arborescences in the original graph.  $\square$

The node-independent version of the theorem is also of interest. However, the result of Georgiadis and Tarjan in [3] is a generalization of Theorem 1.2 (i).

**Theorem 2.2** (Georgiadis and Tarjan). *Let  $D = (V + r, A)$  be a digraph with root  $r$ , terminal set  $T \subseteq V$  and  $\kappa(r, t) \geq 2$  for each  $t \in T$ . Then there exists two independent Steiner-arborescences.*

In fact, it can be showed that the two versions are equivalent.

**Claim 2.3.** *Theorems 2.2 and 2.1 are equivalent.*

*Proof.* Theorem 2.2 follows from Theorem 2.1 by substituting every node by an edge in the usual way. That is, for each  $v \in V$  we substitute  $v$  by two new nodes  $v_1, v_2$  where the ancestors of  $v_1$  are the ancestors of  $v$ , the children of  $v_2$  are the children of  $v$  and we add the arc  $v_1v_2$  to the graph. It is easy to see that if there are two internally node-disjoint paths from  $r$  to  $v$  in the original graph then there are two edge-disjoint paths from  $r$  to  $v_1$  in the new graph. Let  $T_1 = \{v_1 : v \in T\}$ . Then, by Theorem 2.1, there exist two edge-disjoint Steiner-arborescences w.r.t.  $T_1$ . It is easy to see that their restriction to the original graph gives two independent Steiner-arborescences w.r.t.  $T$ .

To see the opposite direction, consider an extension of the edge-graph of  $D$ . More precisely, take the following graph:  $r$  and nodes in  $T$  belong to the new graph. Also, for each edge  $uv$  we add a new node to the graph denoted by  $n_{uv}$ . The edge set consists

of edges of form  $rn_{rv}$  for  $rv \in A$ ,  $n_{vt}t$  for some  $t \in T$  and  $n_{uv}n_{vw}$  for edges  $uv, vw$ . In the graph thus obtained,  $\kappa(r, t) \geq 2$  for each  $t \in T$ . Then, by Theorem 2.2, there exist two independent Steiner-arborescences w.r.t.  $T$ . If we take their corresponding image in the original digraph in the natural way, we get two edge-disjoint Steiner-arborescences as required.  $\square$

Whitty's proof of Theorem 1.2 (i) is based on a special ordering of the nodes.

**Lemma 2.4.** *Let  $D = (V + r, A)$  be a digraph with root  $r$  and  $\kappa(r, v) \geq 2$  for each  $v \in V$ . There is an ordering  $r = v_0, v_1, \dots, v_n, v_{n+1} = r$  of the nodes so that, for each  $v_i \in V$ , there is an edge  $v_h v_i$  with  $h < i$  and an edge  $v_i v_j$  with  $i < j$ .*

The proof of Theorem 2.2 in [3] uses the properties of depth-first search (DFS) to find the two arborescences in question. Huck's proof for Theorem 1.2 (ii) is based on the following lemma which is a variant of Lemma 2.4 for acyclic graphs.

**Lemma 2.5.** *Let  $D = (V + r, A)$  be a simple acyclic graph with  $\varrho(r) = 0$  and  $\varrho(v) \geq 1$  for each  $v \in V$ . There is an ordering  $o : V + r \rightarrow \mathbb{Z}$  of the nodes and an  $r$ -arborescence  $F$  such that for each  $uv \in A$ , we have  $uv \in F$  if and only if  $o(u) < o(v)$ , that is, the set of edges going forward is exactly  $F$ .*

*Proof.* Consider the following version of the DFS algorithm. For each node  $v \in V$  let  $w(v)$  denote the length of the longest directed path from  $r$  to  $v$ . The only restriction is that when being at node  $v$ , the algorithm steps to one of its child

$$\text{with the largest } w \text{ value.} \quad (\mathbf{I})$$

We start the search from the root-node  $r$  with  $o(r) = 0$  and an arborescence  $F$  containing only the single node  $r$ . We extend the ordering and build up the arborescence while running the search as follows: when we arrive at a node  $v$  not reached yet we add it to the end of the ordering. Also, if we reach it from node  $u$ , we add the arc  $uv$  to the arborescence.

We claim that the ordering and the arborescence thus obtained satisfy the conditions of the Lemma. First of all, the edges of  $F$  are going forward as if  $uv \in F$  then  $v$  was reached from  $u$  during the search so  $o(u) < o(v)$ . Hence it suffices to show that no arc in  $A \setminus F$  goes forward.

Let  $uv \in A \setminus F$ . If  $o(u) < o(v)$ , then, by the rule of the DFS algorithm, there must be a directed path  $u = v_1, v_2, \dots, v_q = v$  such that  $o(v_i) < o(v_{i+1})$  and  $v_i v_{i+1} \in F$  for  $i = 1, \dots, q - 1$ . As  $D$  is acyclic,  $w(v) = w(v_q) > w(v_{q-1}) > \dots > w(v_2)$ , contradicting  $uv_2 \in F, uv \notin F$  and  $(\mathbf{I})$ .  $\square$

With the help of Lemma 2.4 and using the idea of the above proof of Theorem 2.1, the following ordering of the nodes immediately shows the existence of proper Steiner-arborescences.

**Theorem 2.6.** *Let  $D = (V + r, A)$  be a digraph with root  $r$ ,  $\rho(v) = \lambda(r, v) \leq 2$  for each  $v \in V$  and assume that the set of nodes with in-degree 1 is stable. Then there exists an ordering  $v_0, v_1, \dots, v_{n+1}$  of the nodes for which*

- (i)  $v_0 = v_{n+1} = r$
- (ii) Cutting nodes appear twice, other nodes appear once.
- (iii) Entering edges of nodes with in-degree 1 appear twice, other edges appear once.
- (iv) For a cutting node  $p$ , if  $v_i = v_j = p$  and  $i < j$  then there is an edge entering  $v_i$  from the left and there is an edge entering  $v_j$  from the right, and all the copies of nodes cut by  $p$  from  $r$  lie between them.
- (v) For every non-cutting node  $v$ , there is an edge entering  $v$  from the left and one from the right.
- (vi) If  $F_1$  and  $F_2$  denote the sets of edges going forward and backward, respectively, then  $F_1$  and  $F_2$  are independent Steiner-arborescences with terminal set  $T = \{v \in V : \lambda(r, v) = 2\}$ .

*Proof.* We prove by induction and follow the main steps of the proof of Theorem 2.1. If there is a terminal node  $t \in T$  and  $x \in V$  with  $x \in \text{dom}(t) - \{r, t\}$ , then we can apply the induction step for the two smaller graphs that was described in the proof. The obtained orderings can be put together as  $x$  is a cutting node and so it appears twice in the final ordering.

If there is no cutting node, we can eliminate the nodes with in-degree 1 in the same way as before. The resulting graph is rooted 2-connected, hence Whitty's theorem can be applied. The ordering provided by Lemma 2.4 can be modified to a proper ordering of the original graph: we make two copies of the nodes that are ancestors of a node with in-degree 1 (the edges leaving this node to the right and entering it from the left will belong to the left copy of the node, and the other way around), then the nodes with in-degree 1 can be placed between the two copies of its ancestor.

In such an ordering the set of edges going right and the set of edges going left form two edge-disjoint Steiner-arborescences. To see this, it suffices to show that the edges appearing twice can not belong to both the left and right path of a node. These edges are exactly the edges entering a node with in-degree 1. As we assumed that the set of nodes with in-degree 1 is stable, none of them is a cutting node so they appear in the ordering once. Hence the two copies of the entering edge of a node can appear in both paths only if these paths cross each other, that is, they share a common node (namely the node with in-degree 1 in question). But the left and right paths of a node cannot cross each other, so they use different edges.  $\square$

The most important consequence of the existence of the above ordering is the following. Note, that each non-cutting node appears only once in the ordering. This observation immediately implies the following theorem, which was also proved in [3].

**Theorem 2.7.** *Let  $D = (V, A)$  be a digraph with root  $r$ . There exist two arborescences  $F_1$  and  $F_2$  such that for each  $v \in V - r$ , the paths  $F_1(r, v)$  and  $F_2(r, v)$  intersect only at the nodes of  $\text{dom}(v)$ .*

This theorem is the base of our proof for a slight generalization of Conjecture 1.3 when  $k = 2$ .

The following example shows that even acyclicity is not satisfactory for the existence of edge-independent Steiner-arborescences.

**Theorem 2.8.** *There is an acyclic graph for which there are three internally node-disjoint paths to all of the terminals but there are no three edge-independent Steiner-arborescences.*

*Proof.* The terminal set of the example consists of two nodes  $t_1, t_2$  (see Figure 1). It can be easily checked that three edge-disjoint paths can be chosen only one way for both terminals but these cannot be partitioned into three arborescences.  $\square$

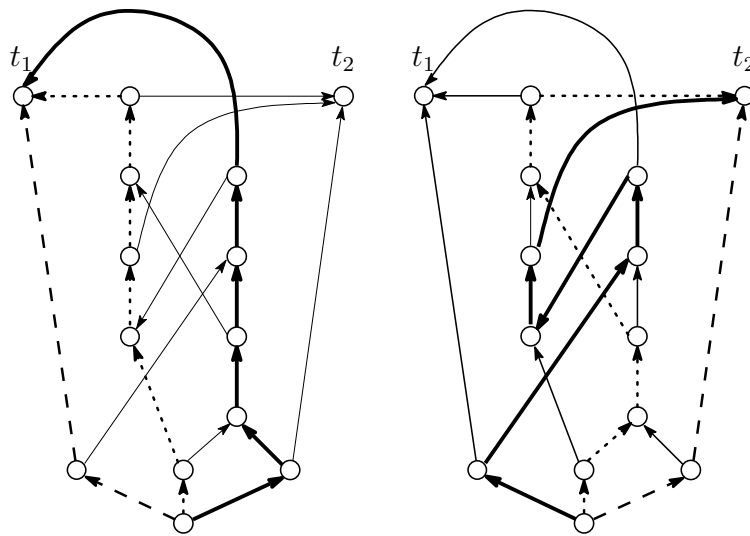


Figure 1: Example for an acyclic graph where there are no three edge-independent Steiner-arborescences

### 3 A generalization

Note that a pair of symmetric arcs can be considered as a directed cycle. This gives the idea of the following definition. Let  $D = (V + r, A)$  be a digraph with root  $r$  and terminal set  $T \subseteq V$ . We call two Steiner-arborescences  $F_1$  and  $F_2$  **dicycle-disjoint** if for each  $t \in T$  the union  $F_1(r, t) \cup F_2(r, t)$  does not contain a directed cycle. The motivation of this definition is the following: if  $T = V$  and the arborescences are dicycle-disjoint then they are also strongly edge-disjoint.

The following theorem generalizes the theorem of Colussi, Conforti and Zambelli for  $k = 2$ .

**Theorem 3.1.** *Let  $D = (V, A)$  be a directed graph with root  $r$  and terminal set  $T$ . There exist two dicycle-disjoint Steiner-arborescences if and only if  $\lambda(r, t) \geq 2$  for each  $t \in T$ .*



*Proof.* The necessity is clear, we prove sufficiency. Consider the arborescences provided by Theorem 2.7. We claim that these arborescences are dicycle-disjoint.

Assume indirectly that there is a node  $t \in T$  such that the union of the paths  $F_1(r, t)$  and  $F_2(r, t)$  contains a directed cycle. Let  $r = x_1, x_2, \dots, x_p = t$  and  $r = y_1, y_2, \dots, y_q = t$  denote the nodes along these paths. As the union of the paths contains a cycle, there are indices  $i_1, i_2, j_1, j_2$  such that  $x_{i_1} = y_{j_2}$ ,  $x_{i_2} = y_{j_1}$  and  $i_1 < i_2, j_1 < j_2$ . Let  $x_{i_1} = y_{j_2} = w$  and  $x_{i_2} = y_{j_1} = z$ . The choice of  $F_1$  and  $F_2$  implies  $w, z \in \text{dom}(t)$ . Now consider the graph  $G - z$ . Then the union  $F_1(r, w) \cup F_2(w, t)$  contains a path from  $r$  to  $t$ , which contradicts to  $z \in \text{dom}(t)$ .  $\square$

## 4 Disproof of Conjecture 1.3 for $k \geq 3$

We give a counterexample for  $k = 3$  based on a graph given by Huck [4], for other values a similar construction works. Let  $D$  be the graph of Figure 2. It is easy to check that  $D$  is rooted 3-edge-connected. The set of nodes in  $V - r$  is partitioned into three blocks  $B_1, B_2$  and  $B_3$ . There is one arc from  $r$  to  $B_i$ , and there are two arcs from  $B_i$  to  $B_{i+1}$  for each  $i$  (the indices are meant modulo 3 plus 1) such that together they form two directed cycles of length three. The edges of these triangles are denoted by  $e_{12}, e_{23}, e_{31}$  and  $f_{12}, f_{23}, f_{31}$ , respectively (see Figure 2).

Assume that there exist three strongly edge-disjoint arborescences  $F_1, F_2$  and  $F_3$ . Clearly, each  $F_i$  contains an edge from  $r$  to one of the blocks, say  $F_i$  contains the one that goes to  $B_i$ , and it uses exactly one of  $e_{ii+1}$  and  $f_{ii+1}$  and the same holds for  $e_{i+1i+2}$  and  $f_{i+1i+2}$ . Also, at least one of the arborescences has to use the pair  $e_{ii+1}, f_{i+1i+2}$  or  $f_{ii+1}, e_{i+1i+2}$ . Assume that  $F_1$  does so. But that implies that  $F_1$  and  $F_2$  can not be strongly edge-disjoint as they have to share a symmetric pair in  $B_2$  that they use when going to  $B_3$ , so for any node  $v \in B_3$  the paths  $F_1(r, v)$  and  $F_2(r, v)$  contain a pair of symmetric arcs.

## 5 Conclusion

We have given a generalization of a theorem of Colussi et al. about packing strongly edge-disjoint arborescences. We also showed that Conjecture 1.3 does not hold for  $k \geq 3$ .

Concluding the results, Edmonds' theorem gives a characterization of the existence of  $k$  edge-disjoint arborescences. On the other hand, we have seen that the analogue statement about independent arborescences does not hold. The notion of strongly edge-disjointness somehow lies between these two types of disjointness, but, as we showed, the conditions of Edmonds' theorem do not ensure the existence of such arborescences. So a natural idea is to turn to the other 'extremity' concerning the necessary conditions, and formulate the following conjecture.

**Conjecture 5.1.** *Let  $D = (V + r, A)$  be a digraph with root  $r$  and assume that  $\kappa(r, v) \geq k$  for each  $v \in V$ . Then there exist  $k$  dicycle-disjoint arborescences.*

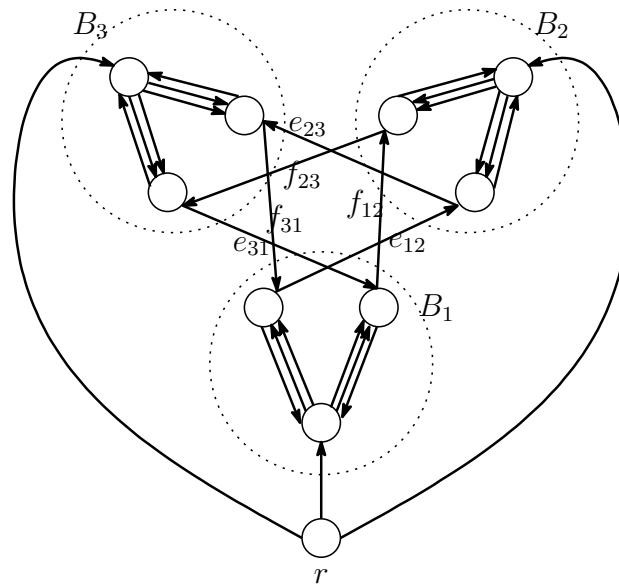


Figure 2: Counterexample for Conjecture 1.3

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