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A note on a conjecture on clutters

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Abstract

We prove some partial results on a conjecture of Király on minimally nonideal clutters. We show on one hand that it is true if the core of the clutter (or of its blocker) is cyclic, and on the other hand that it is true if we restrict the conjecture to the cores.

1 Introduction

A set family $\mathcal{C}$ on a finite ground set S is called a *clutter* if no set of it contains another. We will call the sets in a clutter its *edges*. Its *blocker* $b(\mathcal{C})$ is defined as the family of the (inclusionwise) minimal sets that intersect each set in $\mathcal{C}$, in other words the minimal transversals of $\mathcal{C}$. It is known that $b(b(\mathcal{C})) = \mathcal{C}$ for any clutter $\mathcal{C}$. (We regard $\emptyset$ and $\{\emptyset\}$ as clutters too, and they are blockers of each other.)

Definition 1.1. The *covering polyhedron* of a clutter $\mathcal{C}$ is the following:

$$P(\mathcal{C}) = \{x \in \mathbb{R}^S_+ : x(C) \geq 1 \text{ for every } C \in \mathcal{C}\}.$$

The clutter $\mathcal{C}$ is *ideal* if $P(\mathcal{C})$ is an integer polyhedron.

It is easy to see that the $0-1$-elements in $P(\mathcal{C})$ are the transversals of $\mathcal{C}$, and that $\mathcal{C}$ is ideal if and only if $P(\mathcal{C}) = conv\{\chi_B : B \in b(\mathcal{C})\} + \mathbb{R}^S_+$. It is known that a clutter is ideal if and only if its blocker is. For basic results on clutters see [1].

Tamás Király formulated the following conjecture:

Conjecture 1.2. *Let $\mathcal{A}$ be clutter on ground set S and let $\mathcal{B}$ be its blocker. Then $\mathcal{A}$ and $\mathcal{B}$ are nonideal if and only if there exist functions $p : \mathcal{A} \to S$ and $q : \mathcal{B} \to S$ such that $p(A) \in A$ $\forall A \in \mathcal{A}$ and $q(B) \in B$ $\forall B \in \mathcal{B}$ and if $p(A) = q(B)$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $|A \cap B| > 1$.*

In this note we give some partial results about this conjecture.

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2 Results

First we prove that the "if" direction is true, using the following result of Király and the auhor:

Theorem 2.1. *[Király, Pap, [2]] Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets containing the i-th extreme direction do not get the i-th colour, then there is a vertex for which there is a facet incident to it of every colour.*

We shall apply this theorem for the covering polyhedron of an ideal clutter.

Claim 2.2. *Let $\mathcal{A}$ be an ideal clutter on ground set S and let $\mathcal{B}$ be its blocker. Then there are no functions $p : \mathcal{A} \to S$ and $q : \mathcal{B} \to S$ with the desired properties in 1.2.*

Proof. Suppose that there are functions p and q with the above properties. Let us examine the following colouring of the facets of $P(\mathcal{A})$: we colour a facet corresponding to a set $A \in \mathcal{A}$ (namely the facet defined by $x(A) \geq 1$) with colour $p(A)$, and a facet corresponding the i-th nonnegativity constraint gets the i-th colour. This colouring satisfies the condition in Theorem 2.1 since the extreme directions of $P(\mathcal{A})$ are the unit vectors and if a facet has colour i then the i-th coordinate of its normal vector is nonzero, thus the i-th unit vector is not an extreme direction of the facet. Thus we can apply Theorem 2.1 which asserts the existence of a vertex of $P(\mathcal{A})$ which is incident to every colour. Since $\mathcal{A}$ is ideal, we know that the vertex is the characteristic vector of a set in the blocker, say $B \in \mathcal{B}$. It follows that for every $i \in B$ there exists a set A_i for which $|A_i \cap B| = 1$ (i.e. the facet corresponding to A_i is incident to χ_B) and $p(A_i) = i$ (i.e. A_i has colour i). However for $i = q(B)$ there can not be such a set, which is a contradiction. □

We now prove that it is enough to prove Conjecture 1.2 for a special class of clutters. First we need some more definitions. We can define two types of minor operations of clutters corresponding to including or excluding an element in the transversal.

Definition 2.3. Let $\mathcal{C}$ be a clutter and $s \in S$ an element. By *deleting* s we get the clutter $\mathcal{C}\backslash s$ on vertex set $S \backslash \{s\}$ consisting of $\{C : C \in \mathcal{C}, s \notin C\}$. By *contracting* s we get the clutter $\mathcal{C}/s$ on vertex set $S \backslash \{s\}$ containing the minimal sets in $\{C \backslash s : C \in \mathcal{C}\}$. A *minor* of $\mathcal{C}$ is a clutter obtained by these two operations (it is easy to see that the order of the operations does not matter).

It is known that the minor operations act nicely with the blocker operation: $b(\mathcal{C}/s) = b(\mathcal{C})\backslash s$ and $b(\mathcal{C}\backslash s) = b(\mathcal{C})/s$, and that their covering polyhedra can be obtained from the covering polyhedron of $\mathcal{C}$:

$$P(\mathcal{C}/s) = \{x \in \mathbb{R}_+^{S-s} : (x, 0) \in P(\mathcal{C})\} \cong P(\mathcal{C}) \cap \{x \in \mathbb{R}^S : x_s = 0\},$$

$$P(\mathcal{C}\backslash s) = \{x \in \mathbb{R}_+^{S-s} : \exists t : (x, t) \in P(\mathcal{C})\} \cong \mathrm{proj}_s(P(\mathcal{C})).$$

Claim 2.4. *If Conjecture 1.2 holds for $\mathcal{A}/s$ or $\mathcal{A}\backslash s$ then it holds for $\mathcal{A}$ as well.*

Proof. First suppose that the conjecture is true for $\mathcal{A}/s$, and let $p : \mathcal{A}/s \to S$ and $q : \mathcal{B}\backslash s \to S$ be functions satisfying the conditions (in which case $\mathcal{A}/s \neq \{\emptyset\}$).

For a set $A \in \mathcal{A}$ let $p'(A) := p(A')$ for an arbitrary edge $A' \in \mathcal{A}/s$ for which $A' \subseteq A$, and for a set $B \in \mathcal{B}$ let $q'(B) := \begin{cases} s & \text{if } s \in B \\ q(B) & \text{otherwise.} \end{cases}$

So $p'(A) \neq s$ for any $A \in \mathcal{A}$. The conditions $p'(A) \in A$ and $q'(B) \in B$ hold. If $p'(A) = q'(B)$, then this element is not s, so $q'(B) = q(B)$, and for some $A' \subseteq A$, $p'(A) = p(A')$. Thus $|A' \cap B| > 1$ so $|A \cap B| > 1$, so the second condition also holds.

The other case follows from the first because of symmetry. □

Definition 2.5. A clutter $\mathcal{C}$ is called *minimally nonideal* (or mni for short) if it is not ideal but every (other) minor of it is ideal.

Claim 2.4 asserts that it is enough to show Conjecture 1.2 for minimally nonideal clutters.

It follows from the above mentioned facts that a clutter is mni if and only if its blocker is. We note that an excluded minor characterization for mni clutters is not known (which would be a counterpart of the strong perfect graph theorem) but Lehman proved that mni clutters have special structure.

For an integer $t \geq 2$, the clutter $\mathcal{J}_t = \{\{1, 2, \dots t\}, \{0, 1\}, \{0, 2\}, \dots \{0, t\}\}$ on ground set $\{0, 1, \dots t\}$ is called the finite degenerate projective plane. It is known that $\mathcal{J}_t$ is an mni clutter whose blocker is itself.

For a clutter $\mathcal{A}$ we denote its edge-element incidence matrix by $M_{\mathcal{A}}$.

Theorem 2.6 (Lehman's theorem, [3])**.** *Let $\mathcal{A}$ be a minimally nonideal clutter nonisomorphic to $\mathcal{J}_t$ $(t \geq 2)$ and let $\mathcal{B}$ be its blocker. Then $P(\mathcal{A})$ has a unique noninteger vertex, namely $\frac{1}{r}\underline{1}$ where r is the minimal size of an edge in $\mathcal{A}$, and $P(\mathcal{B})$ has a unique noninteger vertex, namely $\frac{1}{s}\underline{1}$ where s is the minimal size of an edge in $\mathcal{B}$. There are exactly n sets of size r in $\mathcal{A}$ and each element is contained in exactly r of them; and similarly for $\mathcal{B}$. Moreover if we denote the clutter of minimum size edges in $\mathcal{A}$ resp. $\mathcal{B}$ by $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$, then the edges of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ can be ordered in such a way that $M_{\bar{\mathcal{A}}}M_{\bar{\mathcal{B}}}^{\mathsf{T}} = M_{\bar{\mathcal{A}}}^{\mathsf{T}}M_{\bar{\mathcal{B}}} = J + dI$, where J is the $n \times n$ matrix of ones, and $d = rs - n$.*

Definition 2.7. The clutter $\bar{\mathcal{A}}$ defined above is called the *core* of the mni clutter $\mathcal{A}$.

Claim 2.8. *Let $\mathcal{A}$ be an mni clutter on ground set $S = \{s_1, s_2, \dots, s_n\}$ and let $\mathcal{B}$ be its blocker. Then there exist functions $p : \bar{\mathcal{A}} \to S$ and $q : \bar{\mathcal{B}} \to S$ such that $p(A) \in A \ \forall A \in \bar{\mathcal{A}}$ and $q(B) \in B \ \forall B \in \bar{\mathcal{B}}$ and if $p(A) = q(B)$ for some $A \in \bar{\mathcal{A}}$ and $B \in \bar{\mathcal{B}}$ then $|A \cap B| > 1$*

Proof. If $\mathcal{A} = \mathcal{J}_t$ then the following functions satisfy the properties: $p(\{1, 2, \dots t\}) = q(\{1, 2, \dots t\}) = 1$, $p(\{0, 1\}) = q(\{0, 1\}) = 0$, $p(\{0, i\}) = q(\{0, i\}) = i$ (for $i \in \{2, 3, \dots t\}$).

If $\mathcal{A}$ is not a degenerate projective plane then due to Lehman's Theorem 2.6 the sets in $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ can be indexed as $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots B_n$ such that $|A_i \cap B_j| > 1$

if and only if $i = j$. So we want to choose $p(A_i) = q(B_i) \in A_i \cap B_i$ so that they are all different. To this end we construct the bipartite graph $G = (S, T; E)$ where $T = \{t_1, t_2, \dots t_n\}$ and $s_i t_j \in E \Leftrightarrow s_i \in A_j \cap B_j$. Lehman's Theorem implies that G is $(d+1)$-regular: on the side of S because $M_{\bar{\mathcal{A}}}^{\mathsf{T}} M_{\bar{\mathcal{B}}} = I + dJ$ which implies that for any element in S there are $d+1$ indices $i \in [n]$ such that $s \in A_i \cap B_i$. And on the side of T because $M_{\bar{\mathcal{A}}} M_{\bar{\mathcal{B}}}^{\mathsf{T}} = I + dJ$ which implies that $|A_i \cap B_i| = d + 1$ for every $i \in [n]$. Therefore the bipartite graph G is regular, so König's Theorem implies that there is a perfect matching in G which gives functions p and q with the desired properties. $\square$

Definition 2.9. A clutter is *cyclic* if it is isomorphic to a clutter of all the sets containing r consecutive elements in cyclic order for some r.

Claim 2.10. *If the core $\bar{\mathcal{A}}$ of the mni clutter $\mathcal{A}$ is cyclic, then Conjecture 1.2 is true for $\mathcal{A}$.*

Proof. Suppose that $S = \{s_{1,2}, \dots s_n\}$ and that the order of the indices is the order for which $\mathcal{A}$ is cyclic. Let us define p and q as follows. For $A \in \bar{\mathcal{A}}$ let $p(A)$ be the last element of A in cyclic order. For $A \in \mathcal{A} \setminus \bar{\mathcal{A}}$ take a modulo r congruence class (here we look at the indices of the elements) which has more than one elements in A and let $p(A)$ be the smallest element among these (there is such a congruence class since in this case $|A| > r$).

For $B \in \mathcal{B}$ let $q(B)$ be the element with the largest index such that the preceding $r - 1$ elements intersect B. There is such an element because otherwise every rth element would be in B but $|B| \geq s = \frac{n+d}{r} > \frac{n}{r}$.

Now suppose that for $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $p(A) = q(B) = s_i$. If $A \in \bar{\mathcal{A}}$, then by the definition of p and q, A and B meet at an element among the $r - 1$ elements preceding s_i. If $A \in \mathcal{A} \setminus \bar{\mathcal{A}}$ then on one hand A contains an element s_j for which $j > i$ and $j \equiv i \pmod{r}$ because of the definition of $p(A)$. On the other hand B contains all these elements since after s_i it has to contain every rth element. So they have another common element, which shows that p and q fulfill the criteria. $\square$

3 An example

Let us examine the clutter $\mathcal{O}_{K_5}$ whose ground set is the edge set of the graph K_5 and which consists of the odd cycles. The blocker $b(\mathcal{O}_{K_5})$ of it consists of the K_4 subgraphs and the triangles together with the edge disjoint from them. These clutters are minimally nonideal as was shown by Seymour [4].

Figure 3 shows functions on the cores $\overline{\mathcal{O}_{K_5}}$ and $\overline{b(\mathcal{O}_{K_5})}$ whose existence is guaranteed by Claim 2.8: it satisfies the conditions for the cores (the graph-edges selected by p_{bad} and q_{bad} are drawn in thick; on the other clutter-edges p_{bad} and q_{bad} act with a rotation symmetry).

However this function can not be extended to the whole clutter, as shown by the 5-cycle of the "outer" edges. But there are functions p_{good} and q_{good} which satisfy all the requirements of Conjecture 1.2; Figure 3 shows such a function (again the clutter-edges not shown have their selected edges symmetrically).

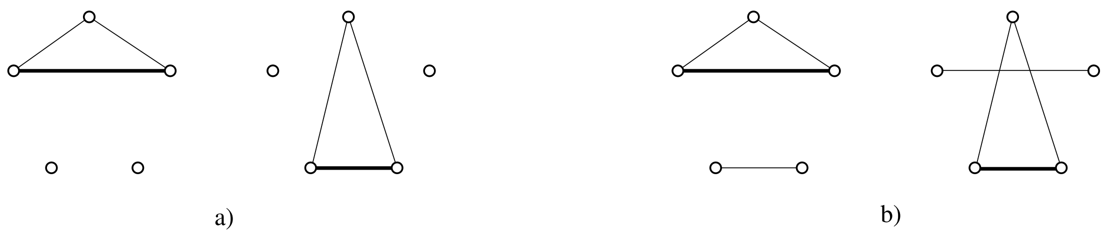

Figure 1: a) Function p_{bad} on $\overline{\mathcal{O}_{K_5}}$, b) function q_{bad} on $\overline{b(\mathcal{O}_{K_5})}$

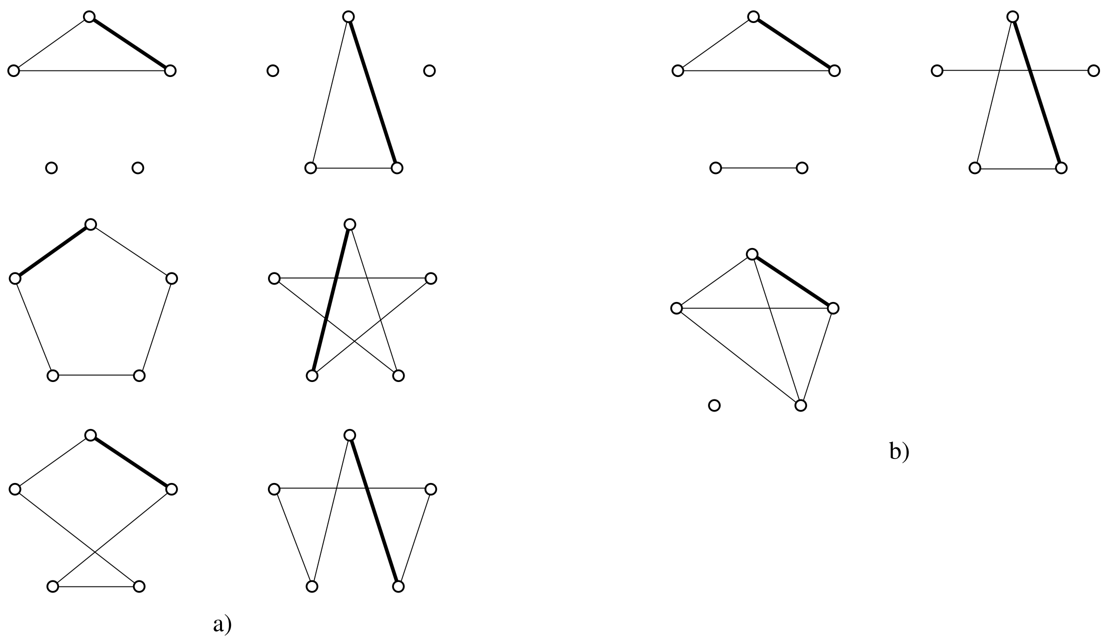

Figure 2: a) Function p_{good} on $\mathcal{O}_{K_5}$, b) function q_{good} on $b(\mathcal{O}_{K_5})$

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