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Augmenting the edge-connectivity of a hypergraph by adding a multipartite graph

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Abstract

Given a hypergraph, a partition of its vertex set and an integer k, find a minimum number of graph edges to be added between different members of the partition in order to make the hypergraph k-edge-connected. This problem is a common generalization of the following two problems: edge-connectivity augmentation of graphs with partition constraints (J. Bang-Jensen, H. Gabow, T. Jordán, Z. Szigeti, Edge-connectivity augmentation with partition constraints, *SIAM J. Discrete Math.* Vol. 12, No. 2 (1999) 160-207) and edge-connectivity augmentation of hypergraphs by adding graph edges (J. Bang-Jensen, B. Jackson, Augmenting hypergraphs by edges of size two, *Math. Program.* Vol. 84, No. 3 (1999) 467-481). We give a min-max theorem for this problem, that implies the corresponding results on the above mentioned problems, and our proof yields a polynomial algorithm to find the desired set of edges.

1 Introduction

Since Watanabe and Nakamura [6] solved the problem of edge-connectivity augmentation of a graph, that is given a graph and an integer k find a minimum set of edges whose addition makes the graph k-edge-connected, many generalizations have been studied. For a survey, we refer to [5].

An important breakthrough in the area of connectivity augmentation came with Frank's algorithm [4]. It led to an efficient approach to this kind of problems. It consists of the following two steps. First, add a special vertex s to the starting graph, and a minimum set of edges between s and the graph in order to satisfy the desired connectivity property. Second, apply the technique of splitting off, that is replace

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two edges incident to s by an edge between the corresponding vertices of the original graph if the desired connectivity property remains valid. Repeat this operation in order to get rid of the edges incident to s and finally delete the isolated vertex s. The set of new edges obtained provides an optimal solution of the problem.

In [2], the authors are given not only a graph and an integer k, but also a partition of the vertex set and they ask for the new edges, whose addition results in a k-edge-connected graph, to connect distinct members of this partition. We may see that it contains the first problem by chosing the partition composed of singletons. They efficiently solve it, and show that the natural lower bound $\max\{\alpha, \beta\}$ is almost always the correct answer. They also characterize graphs that fail the lower bound and show that one more edge is sufficient for them. The definition of α, β and that of the configurations are given in Section 3.

Theorem 1.1 (Bang-Jensen, Gabow, Jordán, Szigeti [2]). *Let $G = (V, E)$ be a graph, $\mathcal{P}$ a partition of V and $k \geq 2$. Then the minimum number of edges connecting distinct members of $\mathcal{P}$ to be added to G in order to make it k-edge-connected is equal to* $\max\{\alpha, \beta\}$ *unless G contains a C_4- or C_6-configuration, in which case it is equal to* $\max\{\alpha, \beta\} + 1$.

Another possible generalization is to study the problem for hypergraphs. Bang-Jensen and Jackson solved the problem of making a hypergraph k-edge-connected by adding a minimum number of graph edges in [1]. Namely, they showed that the natural lower bound $\max\{\alpha, \omega(\mathcal{H})\}$ (see the definitions in Sections 3) can always be achieved.

Theorem 1.2 (Bang-Jensen, Jackson [1]). *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and k an integer. Then the minimum number of graph edges to be added to $\mathcal{H}$ in order to make it k-edge-connected is* $\max\{\alpha, \omega(\mathcal{H})\}$.

In the present paper, we provide a common generalization of these two theorems. More precisely, we give an algorithmic proof for a min-max theorem solving the following. Given a hypergraph $\mathcal{H}$, a partition $\mathcal{P}$ of its vertex set and an integer k, find a minimum set of graph edges between different members of $\mathcal{P}$ to be added to $\mathcal{H}$ in order to make it k-edge-connected. We emphasize that in his thesis [3] Cosh solved the special case when $\mathcal{P}$ is a bipartition.

The outline of the paper is as follows. In Section 2 we recall basic definitions and state some useful facts. In Section 3 we provide the lower bound for our problem and we give a complete description of the hypergraphs failing the lower bound. A useful theorem about the number of splitting off one may choose is shown in Section 4.1, which helps us to prove the splitting off theorem of Section 4.2. We solve the main problem, and provide the augmentation theorem in Section 4.3. Finally we provide the algorithmic details why the proof of our main theorem yields a strongly polynomial algorithm.

2 Preliminaries

2.1 Definitions

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, where V is a finite set and $\mathcal{E}$ is a set of subsets of V, called **hyperedges**. A hyperedge of cardinality 2 is a **graph edge**. For a set $X \subseteq V$, let $\boldsymbol{\delta_{\mathcal{H}}(X)}$ be the set of hyperedges containing at least one vertex in X and at least one in $V - X$. The cardinality of this set is called the **degree** of X and is denoted by $\boldsymbol{d_{\mathcal{H}}(X)}$. When no confusion may arise we shall omit the subscript. Two sets X and Y are **crossing** when none of $X-Y, Y-X, X\cap Y, V-(X\cup Y)$ is empty. For a family $\mathcal{M} = \{M_1, \dots, M_l\}$ of subsets of V, let $\boldsymbol{M_0^\star}= \bigcap_{i=1}^{l} M_i$ and $\boldsymbol{M_i^\star}= M_i - \bigcup_{j\neq i} M_j$. For a partition $\{A_1, \dots, A_k\}$ of V and for $i = 1, \dots, k$, sets A_i and A_{i+1} are called **consecutive**, where $A_{k+1} = A_1$. We denote by $\boldsymbol{X \subset V}$ that $X \subseteq V$ and $X \neq V$. A hypergraph is **k-edge-connected** when $d(X) \geq k$ for all nonempty $X \subset V$. It is well known that the degree function satisfies the following equality for any subsets X and Y of V, where $\boldsymbol{d_0(X,Y)}$ (respectively $\boldsymbol{d_1(X,Y)}$) is the number of hyperedges intersecting $X-Y$ and $Y-X$ and none (resp. exactly one) of $X\cap Y$ and $V-(X\cup Y)$. $\boldsymbol{d(X,Y)}$ is the number of graph edges between $X-Y$ and $Y-X$.

$$d(X) + d(Y) \quad = \quad d(X\cap Y) + d(X\cup Y) + 2d_0(X,Y) + d_1(X,Y). \tag{1}$$

Let $\mathcal{G} = (V+s, \mathcal{E})$ be a hypergraph where s is incident only to graph edges. Let $\boldsymbol{\Gamma(s)}$ denote the set of **neighbors** of s, that is the vertices of V that are adjacent to s. If a set $X \subset V$ contains a unique neighbor of s, then this neighbor will be denoted by $\boldsymbol{x}$. We say that $\mathcal{G}$ is **k-edge-connected in V** if, for any nonempty $X \subset V$, $d(X) \geq k$. In this section let $\mathcal{G}$ be such a hypergraph with $d(s) > 0$ even. Let su, sv be edges of $\mathcal{G}$. We denote $\mathcal{G} - su - sv + uv$ by $\mathcal{G}_{uv}$. Replacing $\mathcal{G}$ by $\mathcal{G}_{uv}$ is called the **splitting off** su, sv and uv is a **split edge** of $\mathcal{G}_{uv}$. A pair or a splitting off su, sv is **admissible** if $\mathcal{G}_{uv}$ is still k-edge-connected in V. We will also say that su is admissible with sv. For a split edge uv of $\mathcal{G}$, $\boldsymbol{\mathcal{G}^{uv}}$ will denote the hypergraph where we **unsplit** uv, that is we undo the splitting off su, sv. For split edges e and f and edges su, sv of $\mathcal{G}$, we will use $\mathcal{G}^{e,f}$ and $\mathcal{G}^{e}_{uv}$ instead of $(\mathcal{G}^e)^f$ and $(\mathcal{G}^e)_{uv}$. Suppose $\hat{\mathcal{P}}$ is a partition of $\delta(s)$ such that $|\hat{P}| \leq \frac{d(s)}{2}$ for all $\hat{P} \in \hat{\mathcal{P}}$. We call $\hat{P} \in \hat{\mathcal{P}}$ **dominating** if $|\hat{P}| = \frac{d(s)}{2}$. The partition $\hat{\mathcal{P}}$ can be considered as a coloring of the edges incident to s. For an edge e incident to s, $\boldsymbol{c(e)}$ denotes the color of e. A **rainbow pair** is an admissible pair su, sv of different colors so that any dominating color class contains one of su and sv. A **complete rainbow splitting off** is a sequence of rainbow splittings that decreases the degree of s to zero. A set $X \subset V$ is called **tight** if $d(X) = k$ and **dangerous** if $d(X) \leq k+1$. For a set $T \subset V$, let $\boldsymbol{\mathcal{G}/T}$ be the hypergraph obtained from $\mathcal{G}$ by contracting T.

2.2 Tight sets

Recall that $\mathcal{G} = (V+s, \mathcal{E})$ is a hypergraph that is k-edge-connected in V and s is incident only to graph edges with $d(s) > 0$ is even.

The following claim can be proved by applying (1) for X and Y and for X and $V + s - Y$.

Claim 2.1. *Let X, Y be tight sets in $\mathcal{G}$. Then 1. if $X \cap Y \neq \emptyset$ and $X \cup Y \neq V$, then they are tight, 2. if $X - Y, Y - X \neq \emptyset$, then they are tight and $d(s, X \cap Y) = 0$.*

We may define, by Claim 2.1, for a vertex $u \in \Gamma(s)$ that belongs to some tight sets, $\boldsymbol{X_u}$ as the minimal tight set containing u. The hypergraph $\mathcal{G}$ can be modified without destroying k-edge-connectivity in V as follows.

Claim 2.2. *Let $u \in \Gamma(s)$ and $u' \in X_u$. Then $\mathcal{G} - su + su'$ is k-edge-connected in V.*

Proof. Otherwise there exists a set Y of degree less than k in the new hypergraph. Then Y contains u and but not u' and it was tight in $\mathcal{G}$. Thus $X_u \cap Y \subset X_u$ is tight by Claim 2.1, a contradiction. □

Claim 2.3. *Let $D \subseteq \delta_{\mathcal{G}}(s)$. Assume that each edge of D enters a tight set. Then there exists a partition $\mathcal{X}$ of $\bigcup_{su \in D} X_u$ such that $\sum_{X \in \mathcal{X}} (k - d_{\mathcal{G}-s}(X)) \geq |D|$.*

Proof. By Claim 2.1, $\{X_u : su \in D\}$ form a laminar family. Let $\mathcal{X}$ be the maximal sets of this family. Then $\mathcal{X}$ is a partition of $\bigcup_{su \in D} X_u$ and $|D| \leq d_{\mathcal{G}}(s, \bigcup_{su \in D} X_u) = \sum_{X \in \mathcal{X}} d_{\mathcal{G}}(s, X) = \sum_{X \in \mathcal{X}} (k - d_{\mathcal{G}-s}(X))$. □

Tight sets can be contracted without violating k-edge-connectivity in V, so, by the following well known lemma, we will sometimes make the following assumption.

$$\text{Every tight set is a singleton.} \tag{2}$$

Lemma 2.4. *[1] For a tight set T of $\mathcal{G}$, $\{su, sv\}$ is admissible in $\mathcal{G}$ if and only if $\{su, sv\}$ is admissible in $\mathcal{G}/T$.*

2.3 Dangerous sets

We start this subsection by the characterization of admissible pairs, see [1]. In the light of Lemma 2.5 it is natural to study the properties of dangerous sets. The following technical lemmas will be applied throughout the paper.

Lemma 2.5. *[1] A pair of edges su, sv is admissible in $\mathcal{G}$ if and only if no dangerous set contains both u and v.*

Claim 2.6. *For a dangerous set Y, 1. $d(s, Y) \leq d(s, V - Y)$, and 2. if for a tight set X, $X \cap Y \neq \emptyset$ and $X \cup Y \neq V$, then $Y \cup X$ is dangerous.*

Proof. By Y dangerous, $d(V - Y) \geq k$, and (1) applied to Y and s, we have $1 = k + 1 - k \geq d(Y) - d(V - Y) = 2d(s, Y) - d(s)$, and then by $d(s)$ is even, 2.6.1 is satisfied. By (1) applied to X and Y and by $d(X \cap Y) \geq k$, 2.6.2 is satisfied. □

Note that Claim 2.6.2 implies that if Y is a maximal dangerous set and X a tight set, then X and Y do not cross.

Claim 2.7. *Let $\mathcal{M} := \{M_1, M_2\}$ be a family of maximal dangerous sets. If $M_i^\star \cap \Gamma(s) \neq \emptyset$ for $i = 0, 1, 2$, then 1. $M_i^\star$ is tight for $i = 0, 1, 2$, 2. $d(s, M_0^\star) = 1$, 3. there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}|$ is odd and $\mathcal{F} = \delta(M_1^\star) \cap \delta(M_2^\star) = \delta(M_0^\star) \cap \delta(V - M_1 - M_2)$.*

Proof. Note that, by Claim 2.6.1, $M_1 \cup M_2 \neq V$. By maximality of M_1, $d(M_1 \cup M_2) \geq k+2$, then apply (1) to M_1 and M_2 and then to M_1 and $V+s-M_2$ to get 2.7.1, 2.7.2, $d(M_1) = k+1, d_0(M_1, M_2) = d_1(M_1, M_2) = d_1(M_1, V-M_2) = 0$ and $d_0(M_1, V+s-M_2) = 1$. Thus $\delta(M_1^\star) \cap \delta(M_2^\star) = \delta(M_0^\star) \cap \delta(V-M_1-M_2)$. Let $\mathcal{F}$ be this hyperedge set. By (1) applied to $M_0^\star$ and $M_1^\star$, $k+k = d(M_0^\star)+d(M_1^\star) = d(M_1)+2d_0(M_0^\star, M_1^\star)+d_1(M_0^\star, M_1^\star) = k+1+2d_0(M_0^\star, M_1^\star)+|\mathcal{F}|$, so $k-|\mathcal{F}|$ is odd, and 2.7.3 is proved. $\square$

Claim 2.8. *Let $\mathcal{M} = \{M_1, \dots, M_l\}$ be a family of maximal dangerous sets with $l \geq 3$. If $\Gamma(s) \bigcap M_i^\star \neq \emptyset$ for $i = 0, \dots, l$, then for $i = 0, \dots, l$, 1. $M_i^\star$ is tight, 2. $d(s, M_i^\star) = 1$, 3. $M_i = M_i^\star \cup M_0^\star$, 4. there exists a set $\mathcal{F}$ of $k-1$ hyperedges of $\mathcal{E}$ intersecting every $M_i^\star$, 5. $d(M_j^\star \cup M_{j'}^\star) = k+1$ for $1 \leq j < j' \leq l$.*

Proof. By applying Claim 2.7 to two distinct pairs of sets in $\{M_1, M_2, M_3\}$, by the maximality of M_i and by the remark after Claim 2.6, we have 2.8.3, 2.8.4 and then 2.8.2 follows for $i = 1, 2, 3$. It is obvious that 2.8.5 is implied by 2.8.1-2.8.4. Repeat the above argument to every triplet of $\mathcal{M}$. $\square$

Corollary 2.9. *If $\mathcal{G}$ has no admissible pair, then there exists a partition $V_1, \dots, V_l$ of V and a hyperedge set $\mathcal{F} \subseteq \mathcal{E}$ of cardinality $k-1$ such that V_i is tight, $d_{\mathcal{G}}(s, V_i) = 1$ and $\delta(V_i) = \mathcal{F} \cup sv_i$ with some $v_i \in V_i$ for every i.*

Proof. We assume that (2) holds. For $t \in \Gamma(s)$, by Lemma 2.5, let $\mathcal{M}_t = \{M_1, \dots, M_l\}$ be a family of maximal dangerous sets containing t and covering $\Gamma(s)$. By Claim 2.6.1, $|\mathcal{M}_t| \geq 2$, and then by Claim 2.7 applied to M_1 and M_2 in $\mathcal{M}_t$, $d(s,t) = 1$ and $M_i = \{m_i, t\}$ for $i = 1, 2$. $|\mathcal{M}_x| = 2$ for some $x \in \Gamma(s)$ would imply $d(s) = 3$, a contradiction, thus $|\mathcal{M}_t| \geq 3$. By Claim 2.8 applied to $\mathcal{M}_t$, $\{V_i := M_i^\star : i = 0, \dots, l\}$ is a subpartition of V satisfying the corollary. It is in fact a partition because if $Z := V - \bigcup_i V_i \neq \emptyset$, then every hyperedge $e \in \delta(Z)$ belongs to some $\delta(V_i)$ and hence to $\mathcal{F}$, so we have $k \leq d(Z) \leq |\mathcal{F}| = k-1$, a contradiction. $\square$

3 Ingredients

In this section, $\mathcal{H} = (V, \mathcal{E})$ will be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Let $\boldsymbol{OPT(\mathcal{H}, \mathcal{P}, k)}$ be the minimum number of desired edges in our augmentation problem. First, we provide the natural lower bound for $OPT(\mathcal{H}, \mathcal{P}, k)$ in subsection 3.1. Following Frank's algorithm, we then describe in subsection 3.2 how to do an optimal extension, that is how to add a new vertex s to $\mathcal{H}$ and a minimum number of graph edges between s and V in order to satisfy the partition and connectivity requirements. Afterwards, the aim is to split off rainbow pairs incident to s to get rid of s. In subsection 3.3, we will characterize the hypergraphs for which this is impossible. To do so we will introduce obstacles. Finally, we characterize in subsection 3.4 the hypergraphs for which the lower bound may not be achieved. We will introduce configurations, the structures that force to have an obstacle in any optimal extension.

3.1 Lower bound

Definition 3.1. Let $\mathbf{\Phi}$ be the maximum of the following values.

$$\begin{aligned}\alpha &= \max\{\lceil \frac{1}{2}\sum_{X\in\mathcal{X}}(k-d_{\mathcal{H}}(X))\rceil : \mathcal{X} \text{ subpartition of } V\},\\ \beta &= \max\{\sum_{Y\in\mathcal{Y}}(k-d_{\mathcal{H}}(Y)) : \mathcal{Y} \text{ subpartition of } P, P\in\mathcal{P}\},\\ \omega(\mathcal{H}) &= \max\{\#\text{component}(\mathcal{H}-\mathcal{F})-1 : \mathcal{F}\subseteq\mathcal{E}, |\mathcal{F}|=k-1\}.\end{aligned}$$

Lemma 3.2. $OPT(\mathcal{H},\mathcal{P},k)\geq\Phi$.

Proof. The first value is a lower bound because at least $k-d(X)$ new edges must enter a set $X\subset V$ with $d(X)<k$ and a new edge may enter at most two sets of the subpartition $\mathcal{X}$. The second value arises as we may not add edges within a member of $\mathcal{P}$. The third value captures the fact that to make a graph G connected, we need $\#\text{component}(G)-1$ edges. If we remove $k-1$ hyperedges of $\mathcal{H}$, we must add a tree connecting the resulting connected components in order to make $\mathcal{H}$ k-edge-connected. This argument shows that Φ is a lower bound for $OPT(\mathcal{H},\mathcal{P},k)$. □

3.2 Optimal extension

Given a hypergraph $\mathcal{H}=(V,\mathcal{E})$, a partition $\mathcal{P}$ of V and an integer k, we describe how to extend $\mathcal{H}$ that is how to add a new vertex s to $\mathcal{H}$ and a minimum number of graph edges between s and V in order to satisfy the partition and connectivity requirements. We will also extend the partition $\mathcal{P}$ of V for a partition $\hat{\mathcal{P}}$ of the set of edges incident to s in the extended graph. Considering the partition $\hat{\mathcal{P}}$ instead of the partition $\mathcal{P}$ allows us to contract the tight sets.

Definition 3.3. An **optimal extension** of $(\mathcal{H},\mathcal{P})$ is a pair $(\hat{\mathcal{H}},\hat{\mathcal{P}})$ where $\hat{\mathcal{H}}=(V+s,\mathcal{E}+\delta_{\hat{\mathcal{H}}}(s))$ is a hypergraph and $\hat{\mathcal{P}}=\{\delta_{\hat{\mathcal{H}}}(s)\cap\delta_{\hat{\mathcal{H}}}(P):P\in\mathcal{P}\}$ is a partition of $\delta_{\hat{\mathcal{H}}}(s)$ such that

1. $\hat{\mathcal{H}}$ is k-edge-connected in V,
2. $\delta_{\hat{\mathcal{H}}}(s)$ consists of 2Φ graph edges,
3. $|\hat{P}|\leq\frac{1}{2}d_{\hat{\mathcal{H}}}(s)$ for all $\hat{P}\in\hat{\mathcal{P}}$.

Theorem 3.4. *There exists an optimal extension of* $(\mathcal{H},\mathcal{P})$.

Proof. We may find such an extension as follows. Recall that X_u is a minimal tight set containing u.

1. Introduce a new vertex s.
2. Add a minimum set of graph edges F between s and V such that $(V+s,\mathcal{E}+F)$ is k-edge-connected in V.
3. If $d(s)$ is odd, add an arbitrary edge incident to s to make $d(s)$ even. (Then $d(s)=2\alpha$.)
4. Add some other edges incident to s if necessary so that $d(s)=\max\{2\alpha,2\omega(\mathcal{H})\}$.

5. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then proceed as follows.
(a) If there exists an edge $su, u \in P$ such that $X_u \not\subseteq P$, then replace su by su', for some $u' \in X_u - P$.
Note that the number of edges between s and P is decreased by 1. Repeat 5.
(b) Otherwise for all $su, u \in P$, we have $X_u \subseteq P$. Add $2d(s,P) - d(s)$ edges between s and $V - P$.
6. Stop. Let $\hat{\mathcal{H}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{H}}}(s) \cap \delta_{\hat{\mathcal{H}}}(P) : P \in \mathcal{P}\}$.

By construction and by Claim 2.2, 3.3.1 and 3.3.3 are satisfied. Lemma 3.5 ensures that 3.3.2 is satisfied and hence we have obtained an optimal extension of $(\mathcal{H}, \mathcal{P})$.

Lemma 3.5. $d_{\hat{\mathcal{H}}}(s) = 2\Phi$.

Proof. Let $\mathcal{G}_i$ be the hypergraph obtained after Step i. By Claim 2.3 applied to $D = \delta_{\mathcal{G}_2}(s)$, $d_{\mathcal{G}_3}(s) \leq 2\alpha$. By Lemma 3.2, we have equality. If we do not add edges in Step 5 then $2\Phi \leq d_{\mathcal{G}_5}(s) = \max\{2\alpha, 2\omega(\mathcal{H})\} \leq 2\Phi$. Otherwise we added $2d_{\mathcal{G}_{5a}}(s,P) - d_{\mathcal{G}_{5a}}(s)$ edges so $d_{\hat{\mathcal{H}}}(s) = 2d_{\mathcal{G}_{5a}}(s,P)$. Then, by Claim 2.3 applied to $D = \delta_{\mathcal{G}_{5a}}(s) \cap \delta_{\mathcal{G}_{5a}}(P)$, there exists a subpartition $\mathcal{Y}$ of P such that $2\Phi \geq 2\sum_{Y \in \mathcal{Y}}(k - d_{\mathcal{H}}(Y)) \geq 2|D| = 2d_{\mathcal{G}_{5a}}(s,P) = d_{\hat{\mathcal{H}}}(s)$. Note that in this case, for some subpartition $\mathcal{Y}'$ of some $P' \in \mathcal{P}$, by 3.3.1 and 3.3.3, $\Phi = \beta = \sum_{Y \in \mathcal{Y}'}(k - d_{\mathcal{H}}(Y)) \leq \sum_{Y \in \mathcal{Y}'} d_{\hat{\mathcal{H}}}(s,Y) \leq d_{\hat{\mathcal{H}}}(s,P) = |\hat{P}| \leq \frac{1}{2} d_{\hat{\mathcal{H}}}(s)$, and we have equality. □

3.3 Obstacles

Let $\mathcal{G} = (V + s, \mathcal{E}')$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d_{\mathcal{G}}(s)$ is even, and $\hat{\mathcal{P}}$ a partition of $\delta_{\mathcal{G}}(s)$ such that $|\hat{P}| \leq \frac{d_{\mathcal{G}}(s)}{2}$ for all $\hat{P} \in \hat{\mathcal{P}}$. Below we describe two structures when no complete rainbow splitting off may be found.

Definition 3.6. A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a $\boldsymbol{\mathcal{C}_4}$**-obstacle** of $\mathcal{G}$ if

1. $d_{\mathcal{G}}(A_i) = k$, for $i = 1, \ldots, 4$,
2. there exists $\mathcal{F} \subseteq \mathcal{E}'$ such that $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta_{\mathcal{G}}(A_1) \cap \delta_{\mathcal{G}}(A_3) = \delta_{\mathcal{G}}(A_2) \cap \delta_{\mathcal{G}}(A_4)$,
3. there exist $l \in \{1, 2\}$ and a dominating $\hat{P} \in \hat{\mathcal{P}}$ such that $\delta_{\mathcal{G}}(A_l \cup A_{l+2}) \cap \delta_{\mathcal{G}}(s) = \hat{P}$.

A $\mathcal{C}_4$-obstacle is called **simple** if $d(s, A_i) = 1$ for $i = 1, \ldots, 4$. The set $A_l \cup A_{l+2}$ in 3.6.3 is called **dominating**.

Definition 3.7. A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a $\boldsymbol{\mathcal{C}_6}$**-obstacle** of $\mathcal{G}$ if

1. $d_{\mathcal{G}}(A_i) = k$, $d_{\mathcal{G}}(s, A_i) = 1$, $d_{\mathcal{G}}(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. there exists $\mathcal{F} \subseteq \mathcal{E}'$ so that $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta_{\mathcal{G}}(A_j) \cap \delta_{\mathcal{G}}(A_l)$ for all distinct non consecutive A_j and A_l,
3. there exist three distinct classes $\hat{P}_1, \hat{P}_2, \hat{P}_3 \in \hat{\mathcal{P}}$ such that $\delta_{\mathcal{G}}(A_j \cup A_{j+3}) \cap \delta_{\mathcal{G}}(s) = \hat{P}_j$ for $j = 1, 2, 3$.

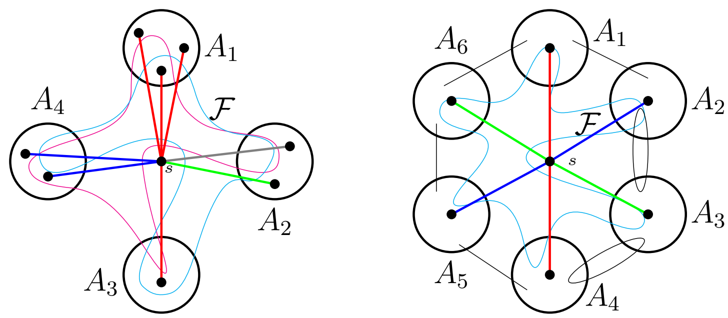

Figure 1: A $\mathcal{C}_4$-obstacle and a $\mathcal{C}_6$-obstacle

An **obstacle** is either a $\mathcal{C}_4$- or a $\mathcal{C}_6$-obstacle. An **uncolored** $\mathcal{C}_4$- (respectively $\mathcal{C}_6$-) obstacle is a partition satisfying Definition 3.6.1-2 (resp. 3.7.1-2). Let $\mathcal{A}$ be an obstacle of $\mathcal{G}$. It is important to keep in mind that edges of the same color cannot enter consecutive sets of $\mathcal{A}$. We emphasize that 3.6.2 and 3.7.2 imply that the hyperedge set of $\mathcal{G}$ is composed of the following hyperedges: the edges incident to s, the set $\mathcal{F}$ of hyperedges intersecting every A_i, hyperedges intersecting only two consecutive sets and no others, and hyperedges lying inside the sets A_i.

Claim 3.8. *If $\mathcal{A}$ is an uncolored obstacle, then $d(s, A_i) \geq 1$. Moreover, if $\mathcal{A}$ is a simple uncolored $\mathcal{C}_4$-obstacle or an uncolored $\mathcal{C}_6$-obstacle, then $d_0(A_i, A_{i+1}) = \frac{k-|\mathcal{F}|-1}{2} \geq 1$ and $d(A_i \cup A_{i+1}) = k + 1$ for all i.*

Proof. For an uncolored $\mathcal{C}_6$-obstacle, (1) applied to A_i and A_{i+1}, and 3.7.1 imply the claim. Let $\mathcal{A}$ be an uncolored $\mathcal{C}_4$-obstacle. (1) applied to A_i and A_{i+1}, 3.6.1 and 3.6.2 imply that $d(A_i \cup A_{i+1}) - k$ is odd and then, by k-edge-connectivity in V, $d(A_i \cup A_{i+1}) \geq k+1$. It also follows that $d_0(A_i, A_{i+1}) \leq \frac{k-|\mathcal{F}|-1}{2}$. Then $0 = d(A_i) - k = |\mathcal{F}| + d(s, A_i) + d_0(A_i, A_{i+1}) + d_0(A_i, A_{i-1}) - k \leq |\mathcal{F}| + d(s, A_i) + 2\frac{k-|\mathcal{F}|-1}{2} - k = d(s, A_i) - 1$. Thus $d(s, A_i) \geq 1$ and if $d(s, A_i) = 1$ then we have equality everywhere and the claim follows by 3.6.2. $\square$

Claim 3.9. *In an uncolored obstacle $\mathcal{A}$, no dangerous set may intersect distinct non consecutive sets A_i and A_j.*

Proof. Suppose that a maximal dangerous set Y intersects non consecutive A_i and A_j. We show that $A_i \cup A_j = Y$. By Claim 2.6.2, $A_i \cup A_j \subseteq Y$. If Y intersected an other A_k, then by Claim 2.6.2, $A_k \subseteq Y$. Suppose that $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle. Then, by 3.6.3 and Claim 2.6.1, $\frac{d(s)}{2} < d(s, Y) \leq \frac{d(s)}{2}$, is a contradiction. Now suppose that $\mathcal{A}$ is an uncolored $\mathcal{C}_6$-obstacle. Then, by Y is dangerous and by Claim 3.8, $k + 1 \geq d(Y) \geq |\mathcal{F}| + 2\frac{k-|\mathcal{F}|-1}{2} + 3 = k + 2$, is a contradiction. Thus $A_i \cup A_j = Y$. By Y is dangerous, (1) applied to A_i and A_j and by 3.6.1-2 or 3.7.1-2, $k + 1 \geq d(Y) = d(A_i) + d(A_j) - 2d_0(A_i, A_j) - d_1(A_i, A_j) = 2k - 0 - |\mathcal{F}| \geq k + 2$, a contradicton. $\square$

Claim 3.10. *In an uncolored obstacle, A_i is a maximal tight set for all i.*

Proof. Suppose that a maximal tight set X intersects A_i and A_j for some $i < j$. By Claims 3.9 and 2.1, $j = i + 1$ and $X = A_i \cup A_{i+1}$. Then by (1) applied to A_i and A_{i+1} and by 3.6.1 or 3.7.1, $k - |\mathcal{F}| = 2d_0(A_i, A_{i+1})$ is even, contradicting 3.6.2 or 3.7.2. □

Claim 3.11. *If $\mathcal{A} = \{A_1, \dots, A_6\}$ is a $\mathcal{C}_6$-obstacle, then splitting off any rainbow pair gives rise to a simple $\mathcal{C}_4$-obstacle.*

Proof. By Claim 3.9 and Lemma 2.5, 3.7.1 and 3.7.3, the only rainbow pairs are sa_{i-1}, sa_{i+1} for all i. By (1) applied to $A_i \cup A_{i-1}$ and $A_i \cup A_{i+1}$, after splitting such a pair, $A_{i-1} \cup A_i \cup A_{i+1}$ is tight and $\{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a $\mathcal{C}_4$-obstacle. □

Claim 3.12. *If $\mathcal{A} = \{A_1, \dots, A_4\}$ is a $\mathcal{C}_4$-obstacle then there exists a rainbow splitting off, unless $\mathcal{A}$ is simple.*

Proof. We can assume that (2) holds and therefore $A_i = a_i$ by Claim 3.10 for every i. Note that by 3.6.3, a rainbow pair can only be of form sa_i, sa_{i+1} for some i. If neither of sa_i, sa_{i+1} and sa_i, sa_{i-1} is admissible then $\{a_i, a_{i+1}\}$ and $\{a_i, a_{i-1}\}$ are both maximal tight sets and, by Claim 2.7.2, $d(s, a_i) = 1$. Applying this for every i we get that $\mathcal{A}$ is simple. □

Lemma 3.13. *If $\mathcal{G}$ contains an obstacle, then there is no complete rainbow splitting off, but there is a complete admissible splitting off.*

Proof. By Claims 3.11 and 3.12, it is enough to note that if $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle, then, by 3.6.3, splitting off any rainbow pair gives rise to a $\mathcal{C}_4$-obstacle. Hence, by Claim 3.8, there is no complete rainbow splitting off in $\mathcal{G}$. Since by Claim 3.9 there exists an admissible splitting off in a simple $\mathcal{C}_4$-obstacle, we get the last statement of the lemma. □

Corollary 3.14. *If $\mathcal{G}$ contains an obstacle then $d_{\mathcal{G}}(s) \geq 2\omega(\mathcal{G} - s)$.*

3.4 Configurations

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a partition $\mathcal{P}$ of V and an integer k, we describe the structures of $\mathcal{H}$ for which the lower bound may not be achieved and then in the following lemma we make a link between configurations and obstacles.

Definition 3.15. A partition $\{A_1, \dots, A_4\}$ of V is a **$\mathcal{C}_4$-configuration** of $\mathcal{H}$ if

1. $k - d_{\mathcal{H}}(A_i) > 0$ for $i = 1, \dots, 4$,
2. there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}|$ is odd and $\mathcal{F} = \delta_{\mathcal{H}}(A_1) \cap \delta_{\mathcal{H}}(A_3) = \delta_{\mathcal{H}}(A_2) \cap \delta_{\mathcal{H}}(A_4)$,
3. there exist $l \in \{1, 2\}$, $P \in \mathcal{P}$ and a subpartition $\mathcal{X}_j$ of $A_j \cap P$ such that $\sum_{X \in \mathcal{X}_j}(k - d_{\mathcal{H}}(X)) = k - d_{\mathcal{H}}(A_j)$ for $j = l, l+2$,
4. $\Phi = k - d_{\mathcal{H}}(A_1) + k - d_{\mathcal{H}}(A_3) = k - d_{\mathcal{H}}(A_2) + k - d_{\mathcal{H}}(A_4)$.

Definition 3.16. A partition $\{A_1, \dots, A_6\}$ of V is a $\boldsymbol{\mathcal{C}_6}$**-configuration** of $\mathcal{H}$ if

1. $k - d_{\mathcal{H}}(A_i) = 1$, $k - d_{\mathcal{H}}(A_i \cup A_{i+1}) = 1$ for $i = 1, \dots, 6$,

2. there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}|$ is odd and $\mathcal{F} = \delta_{\mathcal{H}}(A_j) \cap \delta_{\mathcal{H}}(A_l)$ for all distinct non consecutive A_j and A_l,

3. there exist $A'_i \subseteq A_i$ and three distinct classes $P_1, P_2, P_3 \in \mathcal{P}$ such that $k - d_{\mathcal{H}}(A'_i) = 1$ for $i = 1, \dots, 6$ and
   $A'_j \cup A'_{j+3} \subseteq P_j$ for $j = 1, \dots, 3$,

4. $\Phi = 3$.

A **configuration** is either a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration. We mention that specializing these definitions to graphs we get the original definitions of C_4- and C_6-configurations given in [2].

Lemma 3.17. *Every optimal extension of* $(\mathcal{H}, \mathcal{P})$ *contains an obstacle if and only if* $\mathcal{H}$ *contains a configuration.*

Proof. (of sufficiency) We show that if $\mathcal{A}$ is a $\mathcal{C}_4$-configuration (respectively $\mathcal{C}_6$-configuration) of $\mathcal{H}$ and $(\hat{\mathcal{H}}, \hat{\mathcal{P}})$ is an optimal extension of $(\mathcal{H}, \mathcal{P})$, then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle (resp. $\mathcal{C}_6$-obstacle) of $\hat{\mathcal{H}}$. By 3.3.2, 3.15.4 (resp. 3.16.1 and 3.16.4) and 3.3.1, we have $\sum_i d_{\hat{\mathcal{H}}}(s, A_i) = d_{\hat{\mathcal{H}}}(s) = 2\Phi = \sum_i (k - d_{\mathcal{H}}(A_i)) \leq \sum_i d_{\hat{\mathcal{H}}}(s, A_i)$. Hence $d_{\hat{\mathcal{H}}}(s, A_i) = k - d_{\mathcal{H}}(A_i)$ so $d_{\hat{\mathcal{H}}}(A_i) = k$ for all i, providing 3.6.1 (resp. the first part of 3.7.1. The second part of 3.7.1 comes from 3.16.1 which implies $d_{\hat{\mathcal{H}}}(A_i \cup A_{i+1}) = d_{\mathcal{H}}(A_i \cup A_{i+1}) + d_{\hat{\mathcal{H}}}(s, A_i) + d_{\hat{\mathcal{H}}}(s, A_{i+1}) = (k-1) + 1 + 1 = k + 1$). Note that $k - |\mathcal{F}| \neq 1$ otherwise $\frac{1}{2}|\mathcal{A}| = \Phi \geq \omega(\mathcal{H}) \geq \#\text{component}(\mathcal{H} - \mathcal{F}) - 1 = |\mathcal{A}| - 1 \geq \frac{1}{2}|\mathcal{A}| + 1$, a contradiction. Hence 3.15.2 (resp. 3.16.2) implies 3.6.2 (resp. 3.7.2). By 3.15.3-4 (resp. 3.16.3-4), the edges between s and $A_l \cup A_{l+2}$ (resp. $A_j \cup A_{j+3}$) are between s and $\mathcal{X}_l \cup \mathcal{X}_{l+2}$ (resp. $A'_j \cup A'_{j+3}$), hence between s and P (resp. P_j), implying 3.6.3 (resp. 3.7.3) by 3.3.3.

Proof. (of necessity) Suppose that $(\mathcal{H}, \mathcal{P})$ contains no configuration. By Theorem 3.4 there exists an optimal extension $(\hat{\mathcal{H}}, \hat{\mathcal{P}})$ of $(\mathcal{H}, \mathcal{P})$. Suppose that $\hat{\mathcal{H}}$ contains a $\mathcal{C}_4$-obstacle (respectively $\mathcal{C}_6$-obstacle) $\mathcal{A}$. 3.6.1-2 (resp. 3.7.1-2) imply 3.15.1-2 (resp. 3.16.1-2). 3.3.2, 3.6.1 and 3.6.3 (resp. 3.7.1) imply 3.15.4 (resp. 3.16.4). Therefore, since $(\mathcal{H}, \mathcal{P})$ contains no configuration, 3.15.3 (resp. 3.16.3) does not hold. That is, by Claim 2.3, for any dominating $\hat{P} \in \hat{\mathcal{P}}$ (resp. there exists $\hat{P} \in \hat{\mathcal{P}}$) there exists $su \in \hat{P}$ such that $X_u - P \neq \emptyset$ and we may replace su by su' with $u' \in X_u - P$ without violating $|\hat{P}'| \leq \frac{d(s)}{2}$ for all $\hat{P}' \in \hat{\mathcal{P}}'$ and k-edge-connectivity in V by Claim 2.2. In the new hypergraph $\hat{\mathcal{H}}'$, 3.6.3 (resp. 3.7.3) is not satisfied so $\mathcal{A}$ is not a $\mathcal{C}_4$-obstacle (resp. $\mathcal{C}_6$-obstacle) anymore. Since X_u remains tight in $\hat{\mathcal{H}}'$ and by Claim 3.10, $\mathcal{A}$ is a partition of V into maximal tight sets in $\hat{\mathcal{H}}$, it is also in $\hat{\mathcal{H}}'$. Thus no obstacle can exist in $(\hat{\mathcal{H}}', \hat{\mathcal{P}}')$ by Claim 3.10 and it is an optimal extension of $(\mathcal{H}, \mathcal{P})$. □

4 Main results

4.1 A new theorem on admissible pairs

In this section we generalize and refine Theorem 2.12(b) of [2] on admissible pairs. It will help us to find a rainbow pair when no simple $\mathcal{C}_4$-obstacle exists but an admissible pair exists. Note that the partition constraints are not considered in the following theorem.

Theorem 4.1. *Let $\mathcal{G} = (V+s, \mathcal{E})$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d(s)$ is even. Suppose there is an admissible pair incident to s. Then either (i) there is an edge st that belongs to at least $\frac{d(s)}{2}$ distinct admissible pairs or (ii) $\mathcal{G}$ contains a simple uncolored $\mathcal{C}_4$-obstacle.*

Proof. For $t \in \Gamma(s)$, let $S_t \subseteq \delta(s)$ be the set of edges admissible with st, and, by Lemma 2.5, let $\mathcal{M}_t = \{M_1, \dots, M_l\}$ be a minimal family of maximal dangerous sets such that $t \in M_0^*$ and $\delta(s) - S_t = \delta(s) \cap \delta(\bigcup_{i=1}^{l} M_i)$. Suppose that (i) is not satisfied that is $(*)$ $|S_t| \leq \frac{d(s)}{2} - 1$ for all $t \in \Gamma(s)$. We may assume that (2) holds.

Claim 4.2. *For all $t \in \Gamma(s)$, $|\mathcal{M}_t| = 2$, $d(t) = k$, $d(s,t) = 1$ and $M_i = \{t, t_i\}$ for all $M_i \in \mathcal{M}_t$ with some $t_i \in \Gamma(s)$.*

Proof. If for some $t \in \Gamma(s)$, $|\mathcal{M}_t| = 1$, then by Claim 2.6.1 and M_1 is dangerous, $d(s) - |S_t| = d(s, M_1) \leq d(s, V - M_1) = |S_t|$ that contradicts $(*)$. Thus $|\mathcal{M}_t| \geq 2$ for all $t \in \Gamma(s)$. Claim 2.7, applied to pairs of sets of $\mathcal{M}_t$, and (2) implies that $M_i = \{t_i, t\}$ for all i, $d(t) = k$ and $d(s,t) = 1$. Suppose that for some $t_0 \in \Gamma(s)$, $l = |\mathcal{M}_{t_0}| \geq 3$. Then, by (2) and Claim 2.8, there is a set $\mathcal{F}$ of $k-1$ hyperedges each containing t_i hence $\delta(t_i) = \mathcal{F} \cup st_i$ for all $i = 0, 1 \dots, l$. By Claim 2.8.5 and Lemma 2.5, $S_{t_i} = S_{t_0}$ for all $i = 1, \dots, l$. The existence of an admissible pair implies that there exists $u \in S_{t_0}$. Since $st_0 \in S_u$, $\{st_0, st_1, \dots, st_l\} \subseteq S_u$. Then $(*)$ applied to u and t_0 implies that $\frac{d(s)}{2} - 1 \geq |S_u| \geq l + 1 = d(s) - |S_{t_0}| \geq \frac{d(s)}{2} + 1$, contradiction. $\square$

By Claim 4.2 and $(*)$, for all $t \in \Gamma(s)$, $3 = d(s, \bigcup \mathcal{M}_t) \geq \frac{d(s)}{2} + 1$. Then, since $d(s) \geq 3$ is even, $d(s) = |\Gamma(s)| = 4$. Let $\Gamma(s) = \{a_1, a_2, a_3, a_4\}$ so that $\mathcal{M}_{a_1} = \{\{a_1, a_2\}, \{a_1, a_4\}\}$. By the claim below, (ii) is satisfied and the theorem is proved.

Claim 4.3. *$\mathcal{A} = \{a_1, a_2, a_3, a_4\}$ is a simple uncolored $\mathcal{C}_4$-obstacle.*

Proof. By Claim 4.2 and $\mathcal{M}_{a_1} = \{\{a_1, a_2\}, \{a_1, a_4\}\}$, $\mathcal{M}_{a_3} = \{\{a_2, a_3\}, \{a_3, a_4\}\}$. Since $\{a_1, a_2\}$ is dangerous, so is $V - \{a_1, a_2\}$ by $d(s) = 4$. By maximality, $V - \{a_1, a_2\} = \{a_3, a_4\}$ so $V = \{a_1, a_2, a_3, a_4\}$. Claim 4.2 implies 3.6.1. Claim 2.7.3 applied to $\mathcal{M}_{a_1}$ provides 3.6.2, except $k - |\mathcal{F}| \neq 1$. It also holds because, by (1), $k + 2 \leq d(\{a_1, a_3\}) = d(a_1) + d(a_3) - 2d_0(a_1, a_3) - d_1(a_1, a_3) = k + k - 0 - |\mathcal{F}|$. $\square$

Corollary 4.4. *Let $\mathcal{G} = (V+s, \mathcal{E})$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d(s)$ is even, and $\hat{\mathcal{P}}$ a partition of $\delta(s)$ with $|\hat{P}| \leq \frac{1}{2} d_{\mathcal{G}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$. Suppose there is an admissible pair but no rainbow pair incident to s. Then $\mathcal{G}$ contains a simple $\mathcal{C}_4$-obstacle.*

Proof. Theorem 4.1 applies to $\mathcal{G}$. Since no rainbow pair exists and $|\hat{P}| \leq \frac{1}{2}d_{\mathcal{G}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$, 4.1(i) does not hold. Then, by 4.1(ii), $\mathcal{G}$ contains a simple uncolored $\mathcal{C}_4$-obstacle. 3.6.3 also holds, otherwise, by Claim 3.9 and Lemma 2.5, there exists a rainbow pair, a contradiction. □

4.2 A new splitting off theorem

In this section, we prove the following splitting off result.

Theorem 4.5. *Let $\mathcal{G} = (V + s, \mathcal{E})$ be a hypergraph, where s is incident only to graph edges, and $\hat{\mathcal{P}}$ a partition of $\delta(s)$. There is a complete rainbow splitting off in $\mathcal{G}$ if and only if $\mathcal{G}$ is k-edge-connected in V, $d_{\mathcal{G}}(s) \geq 2\omega(\mathcal{G} - s)$ is even, $|\hat{P}| \leq \frac{1}{2}d_{\mathcal{G}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$ and $\mathcal{G}$ contains no obstacle.*

Proof. Suppose there is a complete rainbow splitting off. By Lemma 3.13, all conditions must clearly be satisfied.

Suppose now that all the conditions are satisfied. Let G and $\hat{\mathcal{P}}'$ be the hypergraph and the partition of $\delta_G(s)$ obtained from $\mathcal{G}$ and $\hat{\mathcal{P}}$ by performing any longest sequence of rainbow splittings. We must show that $d_G(s) = 0$. We suppose that this is not the case. Clearly, $d_G(s) = 2$ cannot be the case, so $d_G(s) \geq 4$.

Lemma 4.6. *G contains an admissible pair.*

Proof. Suppose not and let $\{V_1, \ldots, V_l\}$ be the partition and $\mathcal{F}$ the set of hyperedges provided by Corollary 2.9.

Claim 4.7. *Each split edge uv of G is a cut edge in $G - s - \mathcal{F}$.*

Proof. We may assume $u, v \subseteq V_1$. Since we performed a longest sequence of rainbow splitting off, in G^{uv} we can not split consecutively admissible pairs su, sv_i and sv, sv_j for any sv_i and sv_j with suitable colors $(i, j \neq 1)$. Then there exists a dangerous set of G containing either u, v, v_i and v_j or exactly one of u and v, and at least one of v_i and v_j. Take a maximal such set Y. We may assume that $u, v_i \in Y$. By Claim 2.6, Y contains the tight set V_i, is disjoint from some V_k and if $u, v \in Y$, then $V_1 \subset Y$. Then $\mathcal{F} \cup sv_i \subseteq \delta(Y)$ and either $sv_1, sv_j \subseteq \delta(Y)$ or $uv \subseteq \delta(Y)$. Since $|\mathcal{F}| = k - 1$ and $d_G(Y) \leq k + 1$, we have $\mathcal{F} \cup sv_i \cup uv = \delta(Y)$ and uv is a cut edge in $G - s - \mathcal{F}$. □

Since $\delta(V_i) = \mathcal{F} \cup sv_i, G - s - \mathcal{F}$ has at least l connected components. Let F be the set of split edges in G. Then, by Claim 4.7, $G - s - \mathcal{F} - F = \mathcal{G} - s - \mathcal{F}$ has at least $l + |F|$ connected components. As $|\mathcal{F}| = k - 1$ and $l = d_G(s) \geq 4$, we get $1 + \omega(\mathcal{G} - s) \geq l + |F| = \frac{d_G(s)}{2} + \frac{d_G(s)+2|F|}{2} \geq 2 + \frac{d_{\mathcal{G}}(s)}{2} \geq 2 + \omega(\mathcal{G} - s)$ a contradiction that proves Lemma 4.6. □

Since $\mathcal{G}$ contains no obstacle, but by Lemma 4.6 and Corollary 4.4, G contains a simple $\mathcal{C}_4$-obstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, it follows that G contains a split edge.

Lemma 4.8. *For every split edge $e = xy$, G^e contains an obstacle. In particular, if $x \in A_i$ and $y \in A_{i+1}$ for some i, then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e, and if $x, y \in A_i$ for some i, then G^e contains a $\mathcal{C}_6$-obstacle in which $A_{i+1}, A_{i+2}, A_{i+3}$ are consecutive sets.*

Proof. First suppose that $x \in A_1$ and $y \in A_2$. Then $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle in G^e. By Claim 3.9 and Lemma 2.5 applied in G^e, the pairs sx, sa_3 and sy, sa_4 are admissible. If 3.6.3 does not hold in G^e, then one of them, say sx, sa_3 is a rainbow pair. In $G^e_{xa_3}$, each A_i is tight and s has no neighbor in A_3 so by Claims 3.10 and 3.8, no simple $\mathcal{C}_4$-obstacle exists, a contradiction by Corollary 4.4. Hence 3.6.3 holds and then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e.

Secondly, we may suppose, by 3.6.2, that $x, y \in A_1$. We may assume that (2) holds in G^e. Let d denote the degree function in G^e. Note that, by (2), $A_i = a_i$ for $i = 2, 3, 4$ because, by 3.6.1, these sets are tight in G^e. By possibly exchanging the role of x and y we can assume that sx, sa_3 is a rainbow pair in G^e by Claim 3.9 and Lemma 2.5 applied in G. By Lemma 4.4, $G^e_{xa_3}$ contains a simple $\mathcal{C}_4$-obstacle $\mathcal{A}' = \{A'_1, A'_2, A'_3, A'_4\}$. We may assume $x \in A'_1$. As above, $A'_i = a'_i$ for $i = 2, 3, 4$. If $a_3 \notin A'_1$, then $A'_1 = a'_1 = A_1$ and $k = d(A'_1) = d(A_1) = k + 2$, a contradiction. Thus $a_3 \in A'_1$. Then $d(A_1) = k + 2 = d(A'_1)$. Let $a_1 = x$ and $X = A_1 \cap A'_1$. Observe that $d(s, V - (A_1 \cup A'_1)) = d(s) - d(s, A_1) - d(s, A'_1) + d(s, A_1 \cap A'_1) = d(s, X)$. Since $x \in X$, $A_1 \cup A'_1 \neq V$. Moreover, $d(s, X) = 1$, otherwise $V - (A_1 \cup A'_1) = \{a_2, a_4\}$ and then, by (1) applied to X and $A'_1 - A_1$ and by 3.6.2 for $\mathcal{A}$, $k + 2 = d(A'_1) = d(X) + d(A'_1 - A_1) - d_1(X, A'_1 - A_1) \geq k + k - |\mathcal{F}| \geq k + 3$, a contradiction. Then we may assume $a_4 \notin A'_1$ and, by Claim 3.8 for $\mathcal{A}$, $d_0(a_3, a_4) \geq 1$ in $G^e_{xa_3}$. Since $a_3 \in A'_1$ and $a_4 \in \mathcal{A}'$, A'_1 and a_4 are consecutive in $\mathcal{A}'$ by 3.6.2. Hence we may assume that $a_4 = a'_4$. Then $a_2 \in A'_1$ and $a'_2, a'_3 \in A_1$. The following claim finishes the proof of Lemma 4.8.

Claim 4.9. *$\{a_1, a_2, a_3, a_4 = a'_4, a_5 = a'_3, a_6 = a'_2\}$ is a $\mathcal{C}_6$-obstacle in G^e.*

Proof. First we show 3.7.1-2: By (1) applied to A_1 and A'_1 and by $d(A_1 \cup A'_1) = d(V - (A_1 \cup A'_1)) + 4 = k + 4$, $a_1 = A_1 \cap A'_1$ is tight and $d_0(a_6, a_2) = 0$. By 3.6.2 for $\mathcal{A}$ and $\mathcal{A}'$, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}| \neq 1$ is odd and each hyperedge of $\mathcal{F}$ contains $V - a_1$. By Claim 3.8 applied to $\mathcal{A}$ and $\mathcal{A}'$, $d_0(a_i, a_{i+1}) = \frac{k-|\mathcal{F}|-1}{2}$ for $i = 2, \ldots, 5$. Since every a_i is tight and $d_0(a_6, a_2) = 0$, $d_0(a_i, a_{i+1}) = \frac{k-|\mathcal{F}|-1}{2}$ for $i = 6, 1$ and each hyperedge of $\mathcal{F}$ contains a_1. Then $d(a_i \cup a_{i+1}) = k+1$ for $i = 1, \ldots, 6$. Finally we show 3.7.3: Since $a_1 \cup a'_2$ and $a'_2 \cup a'_3$ are dangerous and sx, sy is admissible, $y = a'_3$. Since a_2 and a_4 are in consecutive sets of $\mathcal{A}'$, $c(sa_2) \neq c(sa_4)$. Thus, by 3.6.3 applied for $\mathcal{A}$, $c(sa_6) = c(sa'_2) = c(sa_3)$. Similarly, $c(sa_5) = c(sa_2)$. By Corollary 4.4, $\{a_1, a_2, a_3 \cup a_4 \cup a_5, a_6\}$ is a simple $\mathcal{C}_4$-obstacle in $G^e_{a_3a_5}$ and $c(sa_2) \neq c(sa_6)$ so $c(sa_1) = c(sa_4)$. □

Alternative proof of Lemma 4.8. The proof of the first case (i.e. when $x \in A_1$ and $y \in A_2$) is the same.

Secondly, we may suppose, by 3.6.2, that $x, y \in A_1$. We may assume that (2) holds in G^e. Let d denote the degree function in G^e. Note that, by (2), $A_i = a_i$ for $i = 2, 3, 4$ because, by 3.6.1, these sets are tight in G^e. First note that if sy, sa_2 is admissible then sx, sa_3 is again admissible in $G^e_{ya_2}$ by Claim 3.9 (applied in G), so such a sequence cannot be rainbow by the choice of G, in other words either $c(sy) = c(sa_2)$ or $c(sx) = c(sa_3)$. This implies that at least one of sx, sa_2 and sy, sa_2 is not admissible: if both of them were admissible then (by possibly switching x and y) we could achieve that $c(sy) \neq c(sa_2)$ and $c(sx) \neq c(sa_3)$ contradicting the conclusion of the previous sentence. Without loss of generality we may assume that sx, sa_2 is not admissible, i.e. there exists a dangerous set X with $x, a_2 \in X$ (observe that $y \notin X$ because sx, sy is admissible in G^e, and $X - A_1 = a_2$). Apply (1) for X and A_1 to get that $k+2+k+1 \geq d(A_1)+d(X) = d(A_1 \cap X)+d(A_1 \cup X)+d_0(A_1, X)+d_1(A_1, X) \geq k+k+3$, implying that $X \cap A_1$ is tight (consequently $X = \{x, a_2\}$ by (2)) and $d_0(A_1, X) = d_1(A_1, X) = 0$. By Claim 3.8 applied in G, $d_0(A_1, a_2) > 0$, implying that $d_0(x, a_2) > 0$. Now apply (1) for X and $V+s-A_1$ to get that $k+2+k+1 \geq d(V+s-A_1)+d(X) = d(A_1-X)+d(X-A_1)+d_0(V+s-A_1, X)+d_1(V+s-A_1, X)$, and since $d(x, s) = 1$ this implies that $d(A_1 - X) \leq k+1$ and $d_0(x, a_4) = 0$. This latter further implies that sx, sa_4 is admissible, because we have seen above that the existence of a dangerous set X' with $x, a_4 \in X'$ would imply $d_0(x, a_4) > 0$. However, sy, sa_4 cannot be admissible as shown above, thus there exists a dangerous set Y with $y, a_4 \in Y$, in fact $Y = \{y, a_4\}$. We claim that $\{A_1 - \{x, y\}, x, a_2, a_3, a_4, y\}$ is a $\mathcal{C}_6$-obstacle in G^e. Apply (1) for $A_1 - x$ and $A_1 - y$ to get that $2(k+1) \geq d(A_1 - x) + d(A_1 - y) = d(A_1) + d(A_1 - \{x, y\}) + d_0(A_1 - x, A_1 - y) + d_1(A_1 - x, A_1 - y) \geq k+2+k$, showing that $d(A_1 - x) = d(A_1 - y) = k+1$, $A_1 - \{x, y\} = a_1$ is tight and $d_0(A_1 - x, A_1 - y) = d_1(A_1 - x, A_1 - y) = 0$. This gives 3.7.1. From 3.6.2 applied in G, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}| \neq 1$ is odd and every hyperedge of $\mathcal{F}$ intersects A_1 and contains a_2, a_3, a_4. We claim that this hyperedge set actually satisfies 3.7.2 in G^e. Since $d_1(A_1, \{x, a_2\}) = 0$, every hyperedge of $\mathcal{F}$ contains x, and similarly every hyperedge of $\mathcal{F}$ contains y. Furthermore, $d_1(A_1 - x, A_1 - y) = 0$ implies that every hyperedge of $\mathcal{F}$ contains a_1, too. By Claim 3.8 applied in G, $d_0(a_2, x) = d_0(A_1, a_2) = \frac{k-|\mathcal{F}|-1}{2}$, since $d_1(A_1, \{x, a_2\}) = 0$: similarly $d_0(a_4, y) = \frac{k-|\mathcal{F}|-1}{2}$. Since $\delta(a_1) - (sa_1 + \mathcal{F})$ can only contain graph edges between a_1 and either x or y, and since $k = d(a_1) = d(x) = d(y)$, this gives that $d_0(a_1, x) = d_0(a_1, y) = \frac{k-|\mathcal{F}|-1}{2}$, finishing the proof of 3.7.2. To show 3.7.3, first observe that $c(sx) = c(sa_3)$ would imply that the sequence of splitting off sx, sa_4 followed by sy, sa_3 is rainbow, a contradiction, and similarly $c(sy) \neq c(sa_3)$ can also be deduced. Therefore $c(sa_4) = c(sx) \neq c(sy) = c(sa_2)$, giving 3.7.3. □

Lemma 4.10. *There exist two split edges e and f in G and a rainbow pair su, sv in $G^{e,f}$ such that $G' := G^{e,f}_{uv}$ contains no obstacle.*

Proof. By Lemma 4.8, we distinguish two cases.

Case 1: If every split edge in G connects consecutive members of $\mathcal{A}$ then $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle in $\mathcal{G}$. In G, let a_i denote the neighbor of s in A_i for every i. By Lemma 4.8, $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e for every split edge $e = xy$, hence one of x and y belongs to a dominating set in G^e, which is also a dominating set in G. If we had the same dominating set for all split edges, then $\mathcal{A}$ would be a $\mathcal{C}_4$-obstacle in $\mathcal{G}$ which is a contradiction. Thus there exist split edges $e = xy$ and $f = x'y'$ so that the different colors $\hat{P}$ of sx and $\hat{P}'$ of sx' are dominating in G and $sy, sy' \notin \hat{P} \cup \hat{P}'$. If $y \in A_j$, then $sa_j \in \hat{P}'$ and hence sx, sa_j is a rainbow pair. In $G^{e,f}_{xa_j}$, the sets of $\mathcal{A}$ are tight, so by Claim 3.10, no $\mathcal{C}_6$-obstacle exists. Moreover, there is no dominating color, so no $\mathcal{C}_4$-obstacle exists.

Case 2: If there exists a split edge e contained in A_i for some i then, by Lemma 4.8, G^e contains a $\mathcal{C}_6$-obstacle $\{B_1, \ldots, B_6\}$. In G^e, let b_i be the neighbor of s in B_i for every i. Since $\mathcal{G}$ contains no obstacle, G^e has a split edge $f = xy$. We may assume that $x \in B_1$. By 3.7.3, Claim 3.9 and Lemma 2.5, $\{sb_4, sb_2\}$ is a rainbow pair in G^e. By Claim 3.11, $\{B_2 \cup B_3 \cup B_4, B_5, B_6, B_1\}$ is a simple $\mathcal{C}_4$-obstacle in $G^e_{b_4 b_2}$. By Lemma 4.8, $G^{e,f}_{b_4 b_2}$ contains an obstacle $\mathcal{A}'$. If $y \notin B_1$, then we may assume, by 3.7.2, that $y \in B_2$. By Lemma 4.8, $\mathcal{A}'$ is a $\mathcal{C}_4$-obstacle. Then, by 3.6.3, $c(sb_3) = c(sy)$. The same argument applied to $\{sb_4, sb_6\}$ shows that $c(sb_5) = c(sy)$. The edge sy should be of two different colors, a contradiction. If $y \in B_1$, then by Lemma 4.8, $\mathcal{A}'$ must be a $\mathcal{C}_6$-obstacle in which $B_2 \cup B_3 \cup B_4, B_5, B_6$ are consecutive sets, but since $c(sb_3) = c(sb_6)$, this gives a contradiction. □

Lemma 4.11. *G' contains a rainbow pair sw, sz.*

Proof. First we show that G' contains an admissible pair. Otherwise, by Corollary 2.9, $5 = d_{G'}(s) - 1 \leq \omega(G' - s) \leq \omega(G - s) + 2$. However, since G contains an admissible pair, $\omega(G - s) \leq \frac{d_G(s)}{2} = 2$, a contradiction. Secondly, if there was no rainbow pair in G', then by Corollary 4.4, G' would contain a simple $\mathcal{C}_4$-obstacle, hence $4 = d_{G'}(s) = 6$, a contradiction. □

Since $d_{G'_{wz}}(s) = 4 = d_G(s)$, the hypergraph G'_{wz} is obtained from $\mathcal{G}$ by performing a longest sequence of rainbow splittings, so by Lemma 4.8, G' contains an obtacle contradicting Lemma 4.10. The theorem is proved. □

4.3 A new augmentation theorem

In this section we present our main theorem. It states that the lower bound Φ can be achieved except when the starting hypergraph contains a configuration. In this case, we need one more edge. Recall that Φ was defined in Section 3.

Theorem 4.12. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Then the minimum number of graph edges to be added between different members of $\mathcal{P}$ in order to make $\mathcal{H}$ k-edge-connected is Φ if $\mathcal{H}$ contains no configuration, and $\Phi + 1$ otherwise.*

Proof. The following lemma proves the theorem.

Lemma 4.13. $\Phi \leq OPT(\mathcal{H}, \mathcal{P}, k) \leq \Phi + 1$. *Moreover, $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$ if and only if $\mathcal{H}$ contains no configuration.*

Proof. By Lemma 3.2, we have the first inequality. By Theorem 3.4, there exists an optimal extension $(\hat{\mathcal{H}}, \hat{\mathcal{P}})$ of $(\mathcal{H}, \mathcal{P})$. We can add two edges incident to s without violating $|\hat{P}| \leq \frac{d(s)}{2}$ for all $\hat{P} \in \hat{\mathcal{P}}$ to get $(\mathcal{H}', \mathcal{P}')$. By 3.3.2, $d_{\mathcal{H}'}(s) = 2\Phi + 2$. The two additional edges do not enter tight sets in $\mathcal{H}'$ so no obstacle exists by 3.6.1 and 3.7.1. Theorem 4.5 applied to $\mathcal{H}'$ and $\mathcal{P}'$ provides a complete rainbow splitting off and we have the second inequality.

If $\mathcal{H}$ contains no configuration, then, by Lemma 3.17, there exists an optimal extension $(\hat{\mathcal{H}}, \hat{\mathcal{P}})$ of $(\mathcal{H}, \mathcal{P})$ that contains no obstacle. By 3.3.2, $d_{\hat{\mathcal{H}}}(s) = 2\Phi$. By Theorem

4.5 applied to $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$, there exists a complete rainbow splitting off in $\hat{\mathcal{H}}$, therefore $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$.

If $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$, then let F be an optimal solution, let $\mathcal{G}$ be obtained from $\mathcal{H} + F$ by adding a vertex s and by replacing every edge $uv \in F$ by the edges su and sv and let $\hat{\mathcal{P}}$ be defined as described in Definition 3.3. Since $|F| = \Phi$, $(\mathcal{G}, \hat{\mathcal{P}})$ is an optimal extension of $(\mathcal{H}, \mathcal{P})$. Since $\mathcal{H} + F$ is obtained from $\mathcal{G}$ by a complete rainbow splitting off, Lemma 3.13 implies that $\mathcal{G}$ contains no obstacle and hence, by Lemma 3.17, $\mathcal{H}$ contains no configuration. $\square$

We emphasize that Theorem 4.12 specialized to graphs provides Theorem 1.1 and specialized to the partition composed of singletons provides Theorem 1.2.

5 Algorithmic aspects

In this section, we explain why the proof of our main theorem yields a strongly polynomial algorithm that finds a set of edges, of the desired cardinality, respecting the partition constraints and whose addition makes the hypergraph k-edge-connected.

Our algorithm starts as the algorithm of [2]: find an optimal extension of the starting hypergraph. What follows is quite different: we first decide if this optimal extension contains an obstacle. If it does, then we modify the extension to get another one that contains no obstacle. Eventually, the new extension will contain two more edges incident to s. Then we find a complete rainbow splitting off that provides the set of edges of desired cardinality. We now explain why each of these steps is strongly polynomial.

Local edge-connectivity: Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and $x, y \in V$, we need a subroutine to compute the local edge-connectivity $\lambda(x, y)$ between x and y. By using a Max Flow-Min Cut algorithm in the capacitated bipartite incidence digraph of the hypergraph $\mathcal{H}$, this can be done in $O((n + m)^3)$, where $n = |V|$ and $m = |\mathcal{E}|$, for details see [1].

Deletion of edges incident to s: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges, one can compute in $O(n^2(n+m)^3)$ the maximum number of copies of sx that can be removed from the hypergraph without destroying the k-edge-connectivity in V, see [1].

Splitting off: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges and two neighbors x and y of s, one can compute the maximum number of admissible splittings sx, sy in $O(n(n + m)^3)$, see [1].

Tight sets: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges, if a vertex $u \in V$ belongs to a tight set, then the above results allow us to compute the minimal one in $O(n(n + m)^3)$, namely

$X_u = \{u\} \cup \{v \in V : \lambda(u,v) > k\}$. We can also compute the maximal one in $O(n^2(n+m)^3)$ as follows. For every vertex $v \in V$, contract s and v and find the minimal tight set $\bar{X}_v$ (if it exists) containing the resulting vertex. Then $V - \bar{X}_v$ is a maximal tight set in $\mathcal{G}$ that contains u but not v. Thus by n minimal tight set computations, we may find the maximal tight set containing u. We mention that if u is a neighbor of s, then the maximal tight set containing u is unique.

Optimal extension: The above facts imply that we may find an optimal extension of the starting hypergraph in $O(n^3(n+m)^3)$ using the algorithm given in the proof of Theorem 3.4: Steps 1. to 4. by results of [1], and Step 5. requires the computation of at most n minimal tight sets.

Obstacle: Deciding if V is partitioned into four or six maximal tight sets requires at most 6 computations of maximal tight sets. If it is the case, then checking if the partition is an obstacle is straightforward. Therefore, by Claim 3.10, deciding if the extension contains an obstacle is done in $O(n^2(n+m)^3)$.

Destroying obstacles: If the optimal extension contains an obstacle, then the proof of Lemma 3.17 gives the strongly polynomial algorithm that decides if the starting hypergraph contains a configuration. If there is no configuration, then it finds an optimal extension containing no obstacle by at most n computations of minimal tight sets. Otherwise, add two edges between s and V in order to ensure that no obstacle exists, as in the proof of Lemma 4.13. In both cases, by Theorem 4.5, there exists a complete rainbow splitting off in the resulting hypergraph.

Complete splitting off: To find a complete rainbow splitting off, we proceed in two steps. First, perform arbitrary rainbow splitting off as long as possible. The second step consists of unsplitting some split edges in order to find a longer sequence. This step considers two cases: either there are no admissible pairs, or there exists a simple $\mathcal{C}_4$-obstacle.

To start, perform an arbitrary sequence of rainbow splitting off until there are no rainbow pairs in the resulting hypergraph G. This can be done in $O(n^3(n+m)^3)$ because, for any two neigbors x and y of s, the maximum number of copies of sx, sy that can be split off can be computed in $O(n(n+m)^3)$. If the sequence is complete, then we are done. Otherwise, one of the following cases occurs.

(i) G contains no admissible pair. Then, by Corollary 2.9, there is a partition of V into tight set $\{V_1, \dots, V_\ell\}$ and a set $\mathcal{F}$ of $k-1$ hyperedges such that $\delta(V_i) = sv_i \cup \mathcal{F}$, where $v_i = V_i \cap \Gamma(s)$. In fact, each set V_i is a maximal tight set by Claim 2.6.2. As $d(s) \leq n$, we can compute $\{V_1, \dots, V_\ell\}$ and $\mathcal{F}$ in $O(n^3(n+m)^3)$. Repeating the following $(*)$ at most $\frac{d(s)}{2} \leq \frac{n}{2}$ times either completes the sequence of rainbow splitting off or puts us in Case (ii). By the proof of Lemma 4.6,

(∗) there exists a split edge $e = uv$, with $u, v \in V_i$ for some $i \in \{1, \ldots, \ell\}$, and a rainbow pair sx, sv_j in G^e, with $x \in \{u, v\}$ and $j \neq i$, such that $G' := G^e_{xv_j}$ contains an admissible pair. If G' contains no rainbow pair, then we are in Case (ii). If G' contains a rainbow pair, then split it. Note that this pair is sy, sv_k for some $k \neq i, j$, where $y = \{u, v\} - x$. Thereafter, no admissible pair exists and $\{V_i \cup V_j \cup V_k\} \cup \{V_r : r \neq i, j, k\}$ is the partition of V into maximal tight sets.

Note that, by Claim 4.7, finding e means finding a split edge that is not a cut edge in $G - \mathcal{F} - s$. Hence (∗) was done in $O(n^2(n+m)^3)$ as it simply required to find admissible pairs containing su and sv.

(ii) G contains an admissible pair. Then, by Theorem 4.1, it contains a simple $\mathcal{C}_4$-obstacle. As seen previously, finding the simple $\mathcal{C}_4$-obstacle can be done in $O(n^2(n+m)^3)$. Now, the proofs of Lemmas 4.10 and 4.11 find one or two edges to unsplit in order to find a complete rainbow splitting off, and then directly give the edges to be split off.

We have sketched why our algorithm finds a set of edges of the desired cardinality in strongly polynomial time, and its overall complexity is in $O(n^3(n+m)^3)$. It relies mostly on the strongly polynomiality of the subroutine — finding a minimum cut in a hypergraph — due to the flow techniques of [1]. We emphasize that our algorithm is quite different from the algorithm to make a graph k-edge-connected with partition constraints of [2] because we may decide if an obstacle exists before the splitting off step.

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