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Globally Linked Pairs of Vertices in Minimally Rigid Graphs

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Abstract

A 2-dimensional *framework* (G, p) is a graph $G = (V, E)$ together with a map $p : V \to \mathbb{R}^2$. We view (G, p) as a straight line realization of G in $\mathbb{R}^2$. We shall only consider *generic* frameworks, in which the co-ordinates of all the vertices of G are algebraically independent. Two realizations of G are *equivalent* if the corresponding edges in the two frameworks have the same length. A pair of vertices $\{u, v\}$ is *globally linked* in G if the distance between the points corresponding to u and v is the same in all pairs of equivalent generic realizations of G. We extend the characterization of globally linked pairs of vertices given by Jackson, Jordán and the author [7] by characterizing globally linked pairs in minimally rigid graphs. In minimally rigid graphs, only those pairs of vertices are globally linked that are connected by an edge.

1 Introduction

We shall consider finite graphs without loops, multiple edges or isolated vertices. A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$.

The *edge function* of a graph $G = (V, E)$ is a map from the set of all realizations, $\mathbb{R}^{|V|d}$ to $\mathbb{R}^{|E|}$, given by

$$f_G(p) = (\dots, ||p(u) - p(v)||, \dots),$$

where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *equivalent*, if $f_G(p) = f_G(q)$, in other words, if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$.

A pair of vertices $\{u, v\}$ in a framework (G, p) is *globally linked* in (G, p) if, in all equivalent frameworks (G, q), we have $||p(u) - p(v)|| = ||q(u) - q(v)||$. A framework (G, p) is *globally rigid*, if all pairs of vertices in (G, p) are globally linked.

The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$ then (G, q) is congruent to (G, p). Intuitively, this means that if we think of a d-dimensional framework (G, p) as a collection of bars and joints where points correspond to joints and each edge to a rigid bar joining its end-points, then the framework is rigid if it has no non-trivial continuous deformations. A *flexing* of the framework (G, p) is a function $\pi : (-1, 1) \times V \to \mathbb{R}^d$, where $\pi(0) = p$ and the frameworks (G, p) and $(G, \pi(t))$ are equivalent for all $t \in (-1, 1)$. The flexing π is *trivial* if the frameworks (G, p) and $(G, \pi(t))$ are congruent for all $t \in (-1, 1)$. A framework is said to be *flexible* if it has a non-trivial continuous flexing. It is known [3, 1] that rigidity, flexibility and the existence of a non-trivial smooth flexing are all equivalent.

The first-order version of a flexing of the framework (G, p) is called an *infinitesimal motion*, which is an assignment of *infinitesimal velocities* to the vertices, $q : V \to \mathbb{R}^d$ satisfying

$$(p(u) - p(v))(q(u) - q(v)) = 0 \tag{1}$$

for all pairs u, v with $uv \in E$. If π is a smooth flexing of (G, p), then $\dot{\pi}$ is an infinitesimal motion of (G, p), where $\dot{\pi}(v)$ is defined as $\frac{d}{dt}\pi(t, v)|_{t=0}$. Let S be a $d \times d$ antisymmetric matrix and $t \in \mathbb{R}^d$. A *trivial infinitesimal motion* of (G, p) has the form $q(v) = Sp(v) + t$, for all $v \in V$. It is easy to see that these are indeed infinitesimal motions. A framework (G, p) is said to be *infinitesimally flexible* if it has a non-trivial infinitesimal motion, otherwise it is called *infinitesimally rigid*.

The set of infinitesimal motions of a framework (G, p) is a linear subspace of $\mathbb{R}^{|V|d}$, given by the $|E|$ linear equations of the form (1). The matrix of this system of linear equations is the *rigidity matrix* of (G, p) and it is denoted by $R(G, p)$. This is a matrix of size $|E| \times |V|d$, where, for each edge $uv \in E$, in the row corresponding to uv, the entries in the d columns corresponding to vertices u and v contain the d coordinates of $(p(u) - p(v))$ and $(p(v) - p(u))$, respectively, and the remaining entries are zeros. Note that the Jacobian of the edge function of G at a point $p \in \mathbb{R}^{|V|d}$ is $df_G|_p = 2R(G, p)$.

Gluck [3] has shown that if a framework (G, p) is infinitesimally rigid, then it is rigid. The converse of this is not true, however, if we exclude certain 'degenerate' configurations, rigidity and infinitesimal rigidity becomes equivalent. In order to establish this, let us recall some notions from differential topology. Given two smooth manifolds, M and N and a smooth map $f : M \to N$, we denote the derivative of f at some point $p \in M$ by $df|_p$, which is a linear map from T_pM – the tangent space of M at p – to $T_{f(p)}N$. Let k be the maximum rank of $df|_q$ over all $q \in M$. A point $p \in M$ is said to be a *regular point* of f, if $\operatorname{rank} df|_p = k$, and a *critical point*, if $\operatorname{rank} df|_p < k$. We say that a framework (G, p) is *regular*, if p is a regular point of f_G. Using the inverse function theorem, it can be shown (see e.g. [1, Proposition 2]) that if (G, p) is a regular framework, then there is an U_p neighbourhood of p, such that $f_G^{-1}(f_G(p)) \cap U_p$ is a manifold, whose tangent space at p is the kernel of df_p. This has the following corollary.

Theorem 1.1. *[1] Let (G, p) be a regular framework. If (G, p) is infinitesimally flexible, then it is flexible. Furthermore, if q is a non-trivial infinitesimal motion of (G, p), then there is a non-trivial smooth flexing π of (G, p) such that $\dot{\pi} = q$.*

Since the rank of the rigidity matrix for a given graph G is constant on the set of regular points of f_G and infinitesimal rigidity of a framework (G, p) depends only on

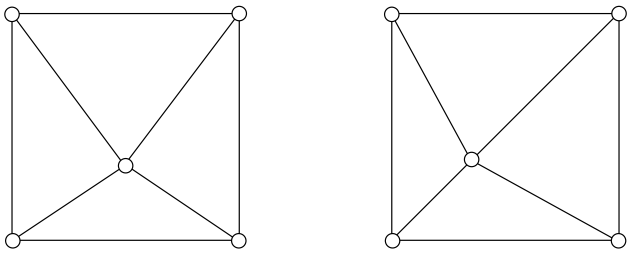

Figure 1: Two regular realizations of a graph G. The first one is globally rigid, but the second is not, since it can fold around the diagonal.

the rank of $R(G,p)$ it follows that if a regular framework (G,p) is infinitesimally rigid, then all other regular frameworks (G,q) are infinitesimally rigid as well. Let k be the maximum rank of the rigidity matrix $R(G,p)$ over all configurations. Then the set of critical points of f_G can be described by a polynomial equation, namely, if Q denotes the sum of the squares of the determinants of the $k \times k$ submatrices of $R(G,p)$ in terms of the coordinates of p, then $p \in \mathbb{R}^{|V|d}$ is a critical point if and only if $Q(p) = 0$. This means that the set of regular points of f_G is an open dense subset of $\mathbb{R}^{|V|d}$, and for almost all configurations $p \in \mathbb{R}^{|V|d}$ (with respect to the $|V|d$-dimensional Lebesgue-measure), the framework (G,p) is regular. So infinitesimal rigidity and rigidity are 'generic' properties in the sense that the infinitesimal rigidity or rigidity of a framework (G,p) depends only on the graph G for almost all configurations. Therefore we say that the graph G is *rigid in* $\mathbb{R}^d$, if every (or equivalently, if some) regular d-dimensional framework (G,p) is rigid (or equivalently, infinitesimally rigid).

The same is not true for global rigidity. Figure 1 shows two examples of regular frameworks for the same graph, where one of them is globally rigid, but the other is not. However, if we exclude a broader set of 'degenerate' realizations, those where the coordinates of the vertices satisfy a polynomial equation, then a similar situation holds. A framework (G,p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. Gortler, Healy and Thurston [4] showed that global rigidity is also a generic property of a graph G in the sense that if a generic framework (G,p) is globally rigid, then every other generic framework (G,q) is globally rigid as well. A graph G is *globally rigid* in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is globally rigid.

Unlike rigidity and global rigidity, however, 'global linkedness' is not a generic property in $\mathbb{R}^2$. Figures 2 and 3 give an example of a pair of vertices in a rigid graph G which is globally linked in one generic realization, but not in another. Note that if $d = 1$ then global linkedness is a generic property: $\{u,v\}$ is globally linked in G if and only if G has two openly disjoint uv-paths. We say that the pair $\{u,v\}$ is *globally linked* in G in $\mathbb{R}^d$ if it is globally linked in all d-dimensional generic frameworks (G,p).

In the rest of the paper we shall assume that $d = 2$, unless specified otherwise. In [6], the characterization of globally linked pairs of vertices were solved for M-connected graphs, an important family of rigid graphs. In this paper, we extend that characterization for minimally rigid graphs. A graph $G = (V,E)$ is *minimally rigid*,

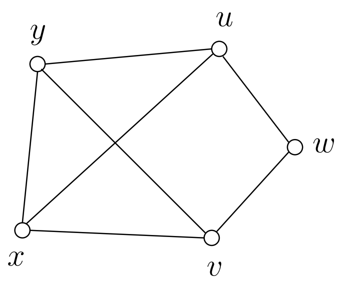

Figure 2: A realization (G, p) of a rigid graph G in $\mathbb{R}^2$. The pair $\{u, v\}$ is globally linked in (G, p).

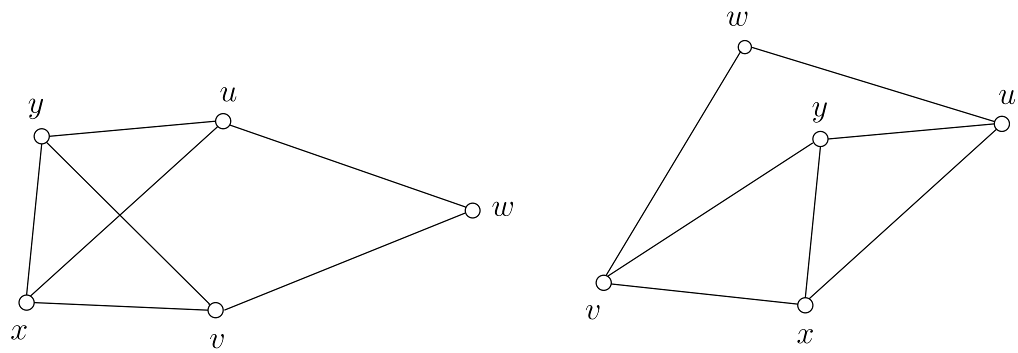

Figure 3: Two equivalent realizations of the rigid graph G of Figure 2, which show that the pair $\{u, v\}$ is not globally linked in G in $\mathbb{R}^2$.

if it is rigid, and $G - e$ is not rigid for all $e \in E$. As the main result of the paper, we shall prove that a pair of vertices $\{u, v\}$ is globally linked in a minimally rigid graph $G = (V, E)$ if and only if $uv \in E$. This result verifies [7, Conjecture 5.9] for minimally rigid graphs.

The rest of the paper is structured as follows. In Section 2 we review some preliminary results on minimally rigid graphs and globally linked pairs of vertices. In Section 3 we prove a key lemma about extending a pair of equivalent generic frameworks with a vertex of degree three without destroying the genericity of one of the frameworks. In Section 4 we show that the non-globally-linked relation of a pair of vertices is preserved by the Henneberg 1-extension operation if we apply it on a bridge, an edge whose removal destroys rigidity. In Section 5 we prove our main result about globally linked pairs of vertices in minimally rigid graphs.

2 Preliminary results

The characterization of minimally rigid graphs was first solved by Laman [8]. For a graph $G = (V, E)$ and $X \subseteq V$, let $E_G(X)$ denote the set, and $i_G(X)$ the number of edges in $G[X]$, that is, in the subgraph induced by X in G.

Theorem 2.1. *[8] A graph $G = (V, E)$ is minimally rigid if and only if $|E| = 2|V| - 3$, and $i_G(X) \leq 2|X| - 3$, for all $X \subseteq V$ with $|X| \geq 2$.*

An efficient combinatorial algorithm for testing this condition was given by Lovász and Yemini [9]. Laman's original proof of Theorem 2.1 used the following two simple graph operations that can construct minimally rigid graphs from an edge.

Given a graph $G = (V, E)$ and two distinct vertices $u, w \in V$, the Henneberg *0-extension* operation [5] adds a new vertex v and new edges vu and vw to G. The basic result about 0-extensions is the following.

Lemma 2.2. *[10] Let G be a graph and let H be obtained from G by a 0-extension. Then H is minimally rigid if and only if G is minimally rigid.*

The Henneberg *1-extension* operation [5] (on edge xy and vertex w) deletes an edge xy from a graph G and adds a new vertex z and new edges zx, zy, zw for some vertex $w \in V(G) - \{x, y\}$. It is known that the 1-extension operation preserves rigidity [10]. We shall need the following lemma about the inverse operation of 1-extension on minimally rigid graphs.

Lemma 2.3. *[10] Let $G = (V, E)$ be a minimally rigid graph and let $v \in V$ be a vertex with $d(v) = 3$. Then there are $u, w \in N_G(v)$, $uw \notin E$ such that the graph $H = G - v + uw$ is minimally rigid.*

By observing that a minimally rigid graph has a vertex of degree two or three, it follows that a graph is minimally rigid if and only if it can be constructed from an edge by a sequence of 0-extensions and 1-extensions.

Let $H = (V, E)$ be a graph and $x, y \in V$. We shall use $\kappa_H(x, y)$ to denote the maximum number of pairwise openly disjoint xy-paths in H. If $xy \notin E$ then, by Menger's theorem, $\kappa_H(x, y)$ is equal to the size of a smallest set $S \subseteq V(H) - \{x, y\}$ for which there is no xy-path in $H - S$. A simple necessary condition for a pair of vertices to be globally linked in a graph is the following.

Lemma 2.4. *[7] Let (G, p) be a generic framework, $x, y \in V(G)$, $xy \notin E(G)$, and suppose that $\kappa_G(x, y) \leq 2$. Then $\{x, y\}$ is not globally linked in (G, p).*

We shall also need the following lemma about 0-extensions and non-globally-linked pairs. In Section 4 we shall extend this result to 1-extensions.

Lemma 2.5. *[7] Let $H = (V, E)$ be a graph and $u, v \in V$. Suppose that $\{u, v\}$ is not globally linked in H and that G is a 0-extension of H. Then $\{u, v\}$ is not globally linked in G.*

3 Extending equivalent generic frameworks

Let $K \subseteq \mathbb{R}$ be a field. We call a point $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ *generic over K*, if the set $\{p_1, \ldots, p_n\}$ is algebraically independent over K. To prove the framework extension result of this section, we need the following lemma about polynomials where all roots are non-generic over a field. For a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$, we denote the set of points where f vanishes by $V(f)$.

Lemma 3.1. *Let K be a countable subfield of $\mathbb{R}$ and let $f \in \mathbb{R}[x_1, x_2]$ be an irreducible polynomial. If the cardinality of $V(f)$ is continuum, and all roots of f are non-generic over K, then there is $\lambda \neq 0$, such that $\lambda f \in K[x_1, x_2]$.*

Proof: For all $p \in \mathbb{R}^2$, which is a root of f, p is also the root of a non-zero irreducible polynomial in $K[x_1, x_2]$. Since the cardinality of $K[x_1, x_2]$ is countable, there is a non-zero irreducible polynomial $g \in K[x_1, x_2]$, such that the cardinality of $V(f) \cap V(g)$ is continuum. From this it follows[1] that f and g have a common factor, but since both f and g are irreducible, there is a $\lambda \neq 0$, such that $\lambda f = g$. •

Lemma 3.2. *Let $G = (V, E)$ be a graph and $v \in V$ with $N_G(v) = \{u, w, z\}$. Suppose that $(G - v, p)$ and $(G - v, q)$ are equivalent frameworks, where p is generic, $q(u)$, $q(w)$ and $q(z)$ are not collinear, and $||q(u) - q(w)|| \notin \overline{\mathbb{Q}(p)}$. Then there are equivalent frameworks (G, p^*) and (G, q^*) where p^* is generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = q$.*

Proof: Let $K = \overline{\mathbb{Q}(p)}$. The extension of the generic configuration p with a point $p_v \in \mathbb{R}^2$ is generic, if and only if p_v is a generic point over K. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry that takes $p(u)$ to the origin and $p(w)$ to a point on the first coordinate axis. This isometry has the form $Tz = Az + b$, where $A \in K^{2\times 2}$ and $b \in K^2$. With an argument similar to that in [7, Lemma 3.1], it can be shown that p_v is generic over K if and only if $T(p_v)$ is generic over K.

Let p' be a configuration that is congruent to p and $p'(u) = (0, 0)$, $p'(w) = (p_3, 0)$ and $p'(z) = (p_4, p_5)$. Similarly, let q' be a configuration that is congruent to q and $q'(u) = (0, 0)$, $q'(w) = (q_3, 0)$ and $q'(z) = (q_4, q_5)$. Since $q(u)$, $q(w)$ and $q(z)$ are not collinear, we have that $q_5 \neq 0$. Moreover $q_3 = ||q(u) - q(w)|| \notin K$. By reflecting the configuration q' on the first coordinate axis, if necessary, we may assume that $q_5 \neq p_5$.

We call a point $(p_1, p_2) \in \mathbb{R}^2$ *feasible*, if there exists a point $(q_1, q_2) \in \mathbb{R}^2$, such that the extended frameworks (G, p') and (G, q') are equivalent, where $p'(v) = (p_1, p_2)$ and $q'(v) = (q_1, q_2)$. To prove the lemma, we have to show that there exists a feasible point that is generic over K.

The set of feasible points can be described by the following equations:

$$q_1^2 + q_2^2 = p_1^2 + p_2^2 \tag{2}$$

$$(q_1 - q_3)^2 + q_2^2 = (p_1 - p_3)^2 + p_2^2 \tag{3}$$

$$(q_1 - q_4)^2 + (q_2 - q_5)^2 = (p_1 - p_4)^2 + (p_2 - p_5)^2 \tag{4}$$

From equation (2) and (3) we get that

$$q_1 = \frac{q_3^2 - p_3^2 + 2p_1p_3}{2q_3},$$

and using equation (4) we get that

$$q_2 = \frac{q_4^2 + q_5^2 - p_4^2 - p_5^2 + 2p_1p_4 + 2p_2p_5 - q_4\left(\frac{q_3^2 - p_3^2 + 2p_1p_3}{q_3}\right)}{2q_5}$$

[1] See e.g. [2, Section 1.6, Proposition 2], which states that if f, g are polynomials in $\mathbb{R}[x, y]$ with no common factors, then $V(f) \cap V(g)$ is finite.

And this (q_1, q_2) is a solution to the above equation system, if and only if equation (2) also holds, that is

$$4q_3^2q_5^2(q_1^2 + q_2^2 - p_1^2 - p_2^2) = a_{11}p_1^2 + a_{22}p_2^2 + a_{12}p_1p_2 + a_1p_1 + a_2p_2 + a_0 = 0.$$

This means, that there is an

$$f = a_{11}x_1^2 + a_{22}x_2^2 + a_{12}x_1x_2 + a_1x_1 + a_2x_2 + a_0 \in \mathbb{R}[x_1, x_2]$$

polynomial, such that (p_1, p_2) is feasible if and only if $f(p_1, p_2) = 0$. In other words, the feasible points are on a second-degree algebraic plane curve.

Let $r = q_4^2 - p_4^2$, and $s = q_5^2 - p_5^2$. The coefficients of f are the following:

$$\begin{aligned}
a_{11} &= 4q_5^2(p_3^2 - q_3^2) + 4(q_3p_4 - p_3q_4)^2 \\
a_{22} &= 4q_3^2(p_5^2 - q_5^2) \\
a_{12} &= 8p_5q_3(q_3p_4 - p_3q_4) \\
a_1 &= 4(q_3p_4 - p_3q_4)(q_3(r+s) - q_4(q_3^2 - p_3^2)) + 4p_3q_5^2(q_3^2 - p_3^2) \\
a_2 &= 4p_5q_3(q_3(r+s) - q_4(q_3^2 - p_3^2)) \\
a_0 &= (q_3(r+s) - q_4(q_3^2 - p_3^2))^2 + q_5^2(q_3^2 - p_3^2)^2
\end{aligned}$$

Since $q_3^2 \neq p_3^2$ and $q_5 \neq 0$ it follows that $a_0 > 0$.

Claim 3.3. *The algebraic plane curve defined by $f(x_1, x_2) = 0$ is not empty and it is not a single point.*

Proof: If $q_5^2 \neq p_5^2$, then the following two points $A, B \in \mathbb{R}^2$ are both on the curve.

$$A = \left(\frac{p_3 + q_3}{2}, \frac{r + s - (p_3 + q_3)(p_4 - q_4)}{2(p_5 - q_5)}\right)$$

$$B = \left(\frac{p_3 - q_3}{2}, \frac{r + s - (p_3 - q_3)(p_4 + q_4)}{2(p_5 + q_5)}\right)$$

Since $A \neq B$, in this case the proof is complete, so we may suppose that $q_5^2 = p_5^2$. Since $q_5 \neq p_5$, the point A belongs to the curve, so the curve is not empty, and since $a_{22} = 0$, it can not be a single point either. •

Let us suppose indirectly, that there is no point on the $f = 0$ curve that is generic over K. Since this curve is not empty, and is not a single point, it can be either an ellipse, a parabola, a hyperbola or the union of two lines. In either case the cardinality of the root-set of each irreducible component of f is continuum. Applying Lemma 3.1 to the irreducible components of f we get that there is $\lambda \neq 0$, such that $\lambda f \in K[x_1, x_2]$.

Claim 3.4. $q_5^2 = p_5^2$.

Proof: Consider the following two polynomials:

$$\begin{aligned}F(x) = {} & a_{12}^2(p_3^2a_{12}^2 - 4a_2^2)x^3 + \\ & 8p_5^2a_{22}\left[p_3^2a_{11}a_{12}^2 + 2a_2^2(a_{22} - a_{11}) - 2a_{12}^2a_0\right]x^2 + \\ & 16p_5^4a_{22}^2\left[4(a_{22} - a_{11})a_0 + p_3^2a_{11}^2 + a_2^2\right]x + \\ & 64p_5^6a_{22}^3a_0,\end{aligned}$$

$$G(x) = \left[4p_5^2q_5^2a_{22}^2 - 4sp_5^2a_{11}a_{22} - s^2a_{12}^2\right]x - 4p_3^2p_5^2q_5^2a_{22}^2.$$

If $a_{22} \neq 0$ then the constant term of both F and G are non-zero and substituting all coefficients with their appropriate expressions we get that $F(q_5^2 - p_5^2) = 0$ and $G(q_3^2) = 0$. Since $\lambda f \in K[x_1, x_2]$ it follows that $\lambda^4 F \in K[x]$, $q_5 \in K$, $\lambda^2 G \in K[x]$ and finally $q_3 \in K$, which is a contradiction. This means that a_{22} must be zero, and thus $q_5^2 = p_5^2$. •

Claim 3.5. *$a_{12} \neq 0$.*

Proof: Consider the polynomial

$$F(x) = p_4(p_3 - p_4)a_{11}x - p_3^2p_5a_2.$$

If $q_3p_4 - p_3p_4 = 0$, then $a_{11} = 4p_5^2(p_3^2 - q_3^2) \neq 0$ and $F(q_3^2) = 0$. Since $\lambda F \in K[x]$ and $F \neq 0$ we get that $q_3 \in K$, which is a contradiction, so $q_3p_4 - p_3q_4 \neq 0$ and thus $a_{12} \neq 0$. •

Claim 3.6. *Either $q_4 \in K$ or there is $\mu \in K$ such that $q_4 = \mu q_3$.*

Proof: Since

$$\left[2p_5(a_1 + p_3a_{11}) - p_4(2a_2 + p_3a_{12})\right]q_3 + p_3(2a_2 + p_3a_{12})q_4 = 0,$$

if $2a_2 + p_3a_{12} \neq 0$, then there is $\mu \in K$ such that $q_4 = \mu q_3$. If, on the other hand

$$2a_2 + p_3a_{23} = 8p_5q_3(q_4^2 - q_3q_4 - p_4^2 + p_3p_4) = 0,$$

then $q_4^2 - q_3q_4 = p_4^2 - p_3p_4$. In this case q_4^2 is the root of the following polynomial:

$$\begin{aligned}F(x) = {} & \left[((p_3 - p_4)^2 - p_5^2)a_{12} + 2p_5(p_3 - p_4)a_{11}\right]x^2 + \\ & \left[(p_4(p_3 - p_4)(p_4^2 - p_5^2 - 2p_3p_4) + p_3^2p_5^2)a_{12} + 2p_4p_5(p_3 - p_4)^2a_{11}\right]x + \\ & p_4^2(p_3 - p_4)^2((p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11}).\end{aligned}$$

Since $\lambda F \in K$, in order to show that $q_4 \in K$, we have to prove that $F \neq 0$. Let us suppose indirectly, that $F = 0$. In this case

$$\begin{aligned}((p_3 - p_4)^2 - p_5^2)a_{12} + 2p_5(p_3 - p_4)a_{11} &= 0,\\ (p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11} &= 0.\end{aligned}$$

But since $a_{12} \neq 0$, the determinant of this linear equation, which is a non-zero polynomial of p, is zero, which is a contradiction. $\bullet$

Now consider the following polynomial:

$$F(x,y) = \left[(p_4^2 - p_5^2)a_{12} - 2p_4p_5a_{11}\right] x^2 + 2p_3(p_5a_{11} - p_4a_{12})xy + p_3^2a_{12}y^2 + p_3^3p_5^2a_{12}.$$

Since $F(q_3, q_4) = 0$, if $q_4 \in K$, then q_3 is the root of $\lambda F(x, q_4) \in K[x]$, and the constant term of this polynomial is $p_3^2a_{12}(q_4^2 + p_5^2) \neq 0$. And if $q_3 = \mu q_4$ for some $\mu \in K$, then q_3 is the root of $\lambda F(x, \mu x) \in K[x]$, where the constant term is now $p_3^2p_5^2a_{12} \neq 0$. Either way we get that $q_3 \in K$, which is a contradiction that completes the proof. $\bullet$

4 1-extensions and globally linked pairs

We can use the framework extension result of the previous section to prove the converse of [7, Lemma 4.1].

Theorem 4.1. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose the $H - uw$ is not rigid. Then $\{u, w\}$ is not globally linked in G.*

Proof: Let (H, p) be a generic framework. Since (H, p) is infinitesimally rigid, but $(H-uw, p)$ is not infinitesimally rigid, there is an infinitesimal motion q of $(H-uw, p)$, such that

$$(p(u) - p(w))(q(u) - q(w)) \neq 0.$$

By Theorem 1.1 there is a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p)$, such that $\dot{\pi} = q$.

Suppose that G is the 1-extension of H with a new vertex v such that $N_G(v) = \{u, w, z\}$. Since p is generic, $p(u)$, $p(w)$ and $p(z)$ are not collinear, and since π is continuous, $\pi(t, u)$, $\pi(t, w)$ and $\pi(t, z)$ are not collinear for all $0 < t < \varepsilon$, where ε is sufficiently small. Let

$$d(t) = ||\pi(t, u) - \pi(t, w)||^2.$$

Since $d'(0) \neq 0$, there is $0 < \mu < \varepsilon$ such that $d(\mu) \notin \overline{\mathbb{Q}(p)}$. Particulary,

$$||\pi(\mu, u) - \pi(\mu, w)|| \neq ||p(u) - p(w)||.$$

Applying Lemma 3.2 to $(G-v, p)$ and $(G-v, \pi(\mu))$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is generic, $p^*|_{V-v} = p$ and $q^*|_{V-v} = \pi(\mu)$. From this we get that

$$||q^*(u) - q^*(w)|| = ||\pi(\mu, u) - \pi(\mu, w)|| \neq ||p(u) - p(w)|| = ||p^*(u) - p^*(w)||.$$

This means that $\{u, w\}$ is not globally linked in G. $\bullet$

We shall also need the following generalization of Lemma 2.5 to 1-extensions.

Theorem 4.2. *Let $H = (V, E)$ be a rigid graph and let G be a 1-extension of H on some edge $uw \in E$. Suppose that $H - uw$ is not rigid and that $\{x, y\}$ is not globally linked in H for some $x, y \in V$. Then $\{x, y\}$ is not globally linked in G.*

Proof: Since $\{x, y\}$ is not globally linked in H, there are equivalent frameworks (H, p_1) and (H, p_2) such that p_1 is generic and

$$||p_1(x) - p_1(y)|| \neq ||p_2(x) - p_2(y)||.$$

From [7, Corollary 3.7] we get that the framework (H, p_2) is quasi-generic and therefore regular. Since (H, p_2) is infinitesimally rigid, but $(H - uw, p_2)$ is not infinitesimally rigid, there is an infinitesimal motion q of $(H - uw, p_2)$, such that

$$(p_2(u) - p_2(w))(q(u) - q(w)) \neq 0.$$

By Theorem 1.1 there is a smooth flexing $\pi : [-1, 1] \times V \to \mathbb{R}^2$ of the framework $(H - uw, p_2)$, such that $\dot{\pi} = q$.

Suppose that G is the 1-extension of H with a new vertex v such that $N_G(v) = \{u, w, z\}$. Since p_2 is quasi-generic, $p_2(u)$, $p_2(w)$ and $p_2(z)$ are not collinear, and since π is continuous, $\pi(t, u)$, $\pi(t, w)$ and $\pi(t, z)$ are not collinear for all $0 < t < \varepsilon$, where ε is sufficiently small. Also, there is $\nu < \varepsilon$ such that

$$||\pi(t, x) - \pi(t, y)|| \neq ||p_1(x) - p_1(y)||$$

for all $0 < t < \nu$. Let

$$d(t) = ||\pi(t, u) - \pi(t, w)||^2.$$

Since $d'(0) \neq 0$, there is $0 < \mu < \nu$ such that $d(\mu) \notin \overline{\mathbb{Q}(p_1)}$. Applying Lemma 3.2 to $(G - v, p_1)$ and $(G - v, \pi(\mu))$ we can find equivalent frameworks (G, p^*) and (G, q^*) such that p^* is generic, $p^*|_{V-v} = p_1$ and $q^*|_{V-v} = \pi(\mu)$. Therefore

$$||q^*(x) - q^*(y)|| = ||\pi(\mu, x) - \pi(\mu, y)|| \neq ||p_1(x) - p_1(y)|| = ||p^*(x) - p^*(y)||.$$

This means that $\{x, y\}$ is not globally linked in G. •

5 Main result

Theorem 5.1. *Let $G = (V, E)$ be a minimally rigid graph and suppose that $xy \notin E$. Then $\{x, y\}$ is not globally linked.*

Proof: The proof is by induction on $|V|$. The theorem is trivially true for $|V| \leq 3$, so we may assume that $|V| \geq 4$ and that the theorem holds for all minimally rigid graphs with at most $|V| - 1$ vertices. From Theorem 2.1 it follows that G has either a vertex of degree two, or it has at least six vertices of degree three.

First suppose that G has a vertex v of degree two. If $v \in \{x, y\}$ then $\kappa_G(x, y) = 2$ and hence $\{x, y\}$ is not globally linked by Lemma 2.4. So suppose $v \neq x, y$ and consider $H = G - v$. By Lemma 2.2 H is also minimally rigid and $xy \notin E(H)$. By induction this implies that $\{x, y\}$ is not globally linked in H. Since G is a 0-extension of H, the theorem follows from Lemma 2.5.

If G has no vertex of degree two, then it has a vertex v of degree three, such that $v \neq x, y$. By Lemma 2.3 there are $u, w \in N_G(v)$, $uw \notin E$, such that $H = G - v + uw$ is also minimally rigid. If $xy \notin E(H)$, then we get by induction that $\{x, y\}$ is not globally linked in H. Since H is minimally rigid, $H - uw$ is not rigid and the theorem follows from Theorem 4.2. If $xy \in E(H)$, then $xy = uw$ and G is a 1-extension of H on xy. Since $H - xy$ is not rigid, the theorem follows from Theorem 4.1. •

Corollary 5.2. *Let G be a minimally rigid graph. Then $\{u, v\}$ is globally linked in G if and only if $uv \in E(G)$.*

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