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Abstract

A basic geometric question is to determine when a given framework $G(\mathbf{p})$ is globally rigid in Euclidean space $\mathbb{R}^d$, where G is a finite graph and $\mathbf{p}$ is a configuration of points corresponding to the vertices of G. $G(\mathbf{p})$ is *globally rigid in* $\mathbb{R}^d$ if for any other configuration $\mathbf{q}$ for G such that the edge lengths of $G(\mathbf{q})$ are the same as the corresponding edge lengths of $G(\mathbf{p})$, then $\mathbf{p}$ is congruent to $\mathbf{q}$. A framework $G(\mathbf{p})$ is *redundantly rigid*, if it is rigid and it remains rigid after the removal of any edge of G. Hendrickson [9] proved that redundant rigidity is a necessary condition for generic global rigidity, as is $(d+1)$-connectivity.

Recent results in [2] and [10] have shown that when the configuration $\mathbf{p}$ is generic and $d = 2$, redundant rigidity and 3-connectivity are also sufficient - a good combinatorial characterization that only depends on G and can be checked in polynomial time. It appears that a similar result for $d = 3$ is beyond the scope of present techniques and there are counterexamples to the sufficiency of Hendrickson's conditions.

However, there is a special class of generic frameworks that have polynomial time algorithms for their generic rigidity (and redundant rigidity) in $\mathbb{R}^d$ for any $d \geq 1$, as shown in [19], namely generic body-and-bar frameworks. Such frameworks are constructed from a finite number of rigid bodies that are connected by bars generically placed with respect to each body. We show that a body-and-bar framework is generically globally rigid in $\mathbb{R}^d$, for any $d \geq 1$, if it is redundantly rigid. As a consequence there is a deterministic polynomial time combinatorial algorithm on the graph to determine the generic global rigidity of body-and-bar frameworks in any dimension.

1 Introduction

Two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ are *equivalent in* $\mathbb{R}^d$ if corresponding edge lengths are the same, where $\mathbf{p}$ and $\mathbf{q}$ are configurations in $\mathbb{R}^d$ corresponding to the vertices

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of a finite graph G. We say that $G(\mathbf{p})$ is *globally rigid in* $\mathbb{R}^d$ if when $G(\mathbf{q})$ in $\mathbb{R}^d$ is equivalent to $G(\mathbf{p})$, $\mathbf{q}$ is congruent to $\mathbf{p}$. The configurations $\mathbf{p}$ and $\mathbf{q}$ are *congruent* if there is a rigid congruence of $\mathbb{R}^d$ that takes $\mathbf{p}$ to $\mathbf{q}$.

A framework $G(\mathbf{p})$ is *rigid in* $\mathbb{R}^d$ if there is a neighborhood $U_{\mathbf{p}}$ in the space of configurations in $\mathbb{R}^d$ such that if $G(\mathbf{q})$ is equivalent to $G(\mathbf{p})$ and $\mathbf{q} \in U_{\mathbf{p}}$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

If one is given a particular configuration $\mathbf{p}$, by [17], determining global rigidity for any $d \geq 1$ is infeasible, and even for rigidity for $d \geq 2$ it seems unrealistic. A natural way to address this difficulty is to consider the case when the configuration $\mathbf{p}$ is *generic*, which means that all the coordinates of all the points of the configuration $\mathbf{p}$ are algebraically independent over the rational numbers. In other words, the only polynomial with integer coefficients that is satisfied by these coordinates is the 0 polynomial. This is something of an overkill, especially in the case of rigidity, since a reasonable finite set of polynomial equations, given by certain determinants, can be used in many instances. In the case of global rigidity, the equations that would determine the "bad" cases for global rigidity are much harder to determine.

With the concept of generic in mind, we define a graph G to be *generically rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is rigid at all generic configurations $\mathbf{p}$, and *generically globally rigid in* $\mathbb{R}^d$ if $G(\mathbf{p})$ is globally rigid at all generic configurations $\mathbf{p}$ [3, 4]. It is not obvious that global rigidity is a generic property, but recent results in [4, 8] prove that indeed global rigidity is a generic property for graphs in each dimension.

Two natural necessary conditions, observed by Hendrickson [9], for generic global rigidity in $\mathbb{R}^d$ are that the graph G be vertex $(d+1)$-connected, and that, for a generic configuration $\mathbf{p}$, $G(\mathbf{p})$ be *redundantly rigid*, which means that $G(\mathbf{p})$ is rigid and remains rigid after the removal of any edge (Theorem 2.1 below).

For $d = 2$, Berg and Jordán [2] and Jackson and Jordán [10] confirm, using [4], that Hendrickson's necessary conditions are sufficient for generic global rigidity. For $d = 3$, Connelly [3] showed that the complete bipartite graph $K_{5,5}$ is generically redundantly rigid and vertex 5-connected, but not generically globally rigid, showing that Hendrickson's necessary conditions are not sufficient. Similar examples exist for all $d \geq 3$.

So it is natural to search for classes of graphs where generic global rigidity can be determined combinatorially in line with Hendrickson's necessary conditions, without recourse to matrix calculations for each graph, as in [4]. At a workshop at BIRS in 2008, two of the authors and Meera Sitharam conjectured that generic body-and-bar frameworks would be one such class. These consist of disjoint collections of vertices, grouped as *bodies*, where each body is joined to some of the other bodies by disjoint bars. Each body is assumed to be globally rigid in its own right, by insisting that each body have enough internal bars to ensure its own global rigidity. For a generic body-and-bar framework, all of the vertices of all of the bodies are generic. The connections between the bodies are recorded in a single multigraph H (without loops, but with multiple edges allowed), where each body is represented as a vertex in the multigraph. (During our inductions, we will allow H to have loops, but they will not be left when the induction is completed.) When we collect all the individual vertices of each body and their individual internal and external bars, we denote that graph by G_H. Note that any two bars joining a pair of bodies have disjoint vertices, making this a graph.

In [19, 20] it is shown that generic rigidity (and hence generic redundant rigidity) of body-and-bar frameworks in $\mathbb{R}^d$, for all $d \geq 1$, can be determined efficiently. The following is our main result. Theorem 5.2 is the same, but stated more in terms of the body-and-bar graph.

Theorem 1.1. *A body-and-bar framework is generically globally rigid in $\mathbb{R}^d$ if and only if it is generically redundantly rigid in $\mathbb{R}^d$.*

For the proofs of previous results [2, 10], and for our main theorem here, we rely on several key techniques. In [4], a sufficient condition is given in terms of the rank of a stress matrix (to be defined later), that combines with (infinitesimal) rigidity at a generic point to imply generic global rigidity in any specific dimension (see also [5]). To apply this result, certain key inductive constructions have been shown to preserve both the maximal rank of the corresponding stress matrix, and the infinitesimal rigidity. It is also necessary that these inductive constructions generate all members of the class from a generically globally rigid seed (a minimal complete graph).

These results have significant theoretical interest as steps towards a full theory of generic global rigidity of arbitrary frameworks. There are also a wide range of applications for the algorithms that detect global rigidity, such as localization in wireless sensor networks [1, 12], molecular conformation [27], and stability of molecules. We return to possible applications of our main theorem in §6.

We also note that by the results in [5], graphs G_H which are generically globally rigid in $\mathbb{R}^d$ are also generically globally rigid in spherical and hyperbolic d-space. $\mathbb{R}^d$ is the classical sample of a general class of metrics over which rigidity and generic global rigidity results are invariant.

2 Prior Results on Global Rigidity and Infinitesimal Rigidity

Hendrickson [9] proved two key necessary conditions for the global rigidity of a bar-and-joint framework at a generic configuration. This was conjectured by Whiteley in [24].

Theorem 2.1 (Hendrickson [9]). *Let $G(\mathbf{p})$ be a globally rigid generic bar-and-joint framework in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices or*

(i) the graph G is $(d+1)$-connected in a vertex sense;

(ii) the framework $G(\mathbf{p})$ is redundantly rigid in $\mathbb{R}^d$, in the sense that removing any one edge leaves a graph which is infinitesimally rigid.

Note that redundant rigidity is a generic property. Thus the conditions of Theorem 2.1 are necessary for generic global rigidity.

One critical technique used for proving global rigidity of frameworks uses stress matrices. This technique is at the core of the proof that global rigidity is a generic property, as well as some specific inductive techniques (below).

This stress matrix approach builds on the fact that any globally rigid generic framework is dependent (redundant), with an equilibrium stress ω which is non-zero on all edges. Let $G(\mathbf{p})$ be a framework with $G = (V, E)$. Recall that an *equilibrium stress* on $G(\mathbf{p})$ is an assignment of scalars ω_{ij} to the edges such that for each $i \in V$

$$\sum_{j|ij \in E} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = \mathbf{0}$$

This can also be visualized as a linear dependence of the rows of the rigidity matrix [25].

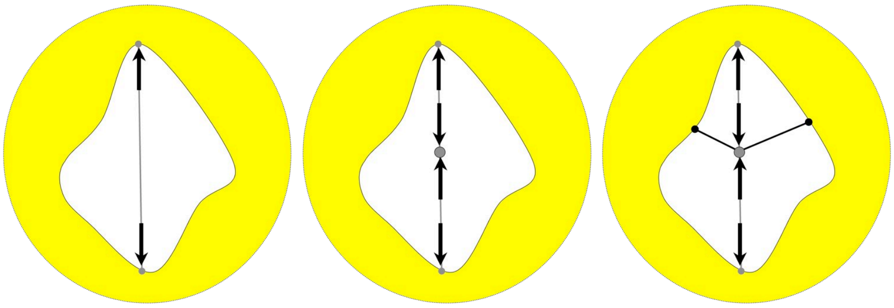
Given a stress, there is an associated $|V| \times |V|$ symmetric matrix Ω, the *stress matrix* such that for $i \neq j$, the i, j entry of Ω is $-\omega_{i,j}$, and the diagonal entries for i, i are $\sum_{j \neq i} \omega_{i,j}$. Note that all row and column sums are now zero. Connelly has developed a number of properties of these stress matrices [3, 4].

Theorem 2.2 (Connelly [4, 5])**.** *Let $G(\mathbf{p})$ be a framework in $\mathbb{R}^d$, where $\mathbf{p}$ is a generic configuration. If $G(\mathbf{p})$ has an equilibrium stress where the rank of the associated stress matrix Ω is $|V| - d - 1$ and $G(\mathbf{p})$ is infinitesimally rigid, then $G(\mathbf{p})$ is globally rigid in $\mathbb{R}^d$.*

Since the rank of Ω is determined by the non-vanishing of a polynomial in the entries of the configuration, the rank at one generic point is the rank at all generic points. Therefore, one generic configuration $\mathbf{p}$ making the framework $G(\mathbf{p})$ globally rigid makes $G(\mathbf{q})$ globally rigid whenever $\mathbf{q}$ is generic.

In order to understand this result and use it, it helps to interpret the rank condition on Ω. The kernel of Ω, $K(\Omega)$, always has the vector of all one's in it. Each additional vector in a basis for $K(\Omega)$ corresponds to an additional coordinate for a configuration in equilibrium. So the maximum dimension of the affine span d of a configuration $\mathbf{p}$ that is in equilibrium with respect to the stress ω, together with the vector of all one's, corresponds to a basis of $K(\Omega)$. Such a configuration is called *universal* with respect to the stress ω, and the rank of Ω is $|V| - (d + 1)$. When the emphasis is on a fixed configuration rather than a fixed equilibrium stress, we say Ω is of *full rank* if its rank is is $|V| - (d + 1)$.

A *subdivision* of a bar $\{i, j\}$ in a framework $G(\mathbf{p})$ is another framework, where the bar $\{i, j\}$ is removed and is replaced with two other bars $\{i, k\}$ and $\{k, j\}$, and the new vertex $\mathbf{p}_k$ is placed on the line through $\mathbf{p}_i$ and $\mathbf{p}_j \neq \mathbf{p}_i$, but not at $\mathbf{p}_i$ and $\mathbf{p}_j$ (Figure 1b). If there is an equilibrium stress on $G(\mathbf{p})$, it is easy to check that ω_{ij} can be replaced by $\omega_{ik} = \pm\omega_{ij}|\mathbf{p}_j - \mathbf{p}_i|/|\mathbf{p}_k - \mathbf{p}_i|$ and $\omega_{jk} = \pm\omega_{ij}|\mathbf{p}_i - \mathbf{p}_j|/|\mathbf{p}_k - \mathbf{p}_j|$, where the sign is chosen depending on whether $\mathbf{p}_k$ is between $\mathbf{p}_i$ and $\mathbf{p}_j$ or on one side, then the equilibrium condition will be preserved. Call the stress for the subdivided framework ω^*. When a subdivision is done on a bar $\{i, j\}$ with $\omega_{ij} \neq 0$ for a universal configuration for a given equilibrium stress ω, the resulting configuration is still universal with respect to ω^*, since the new vertex $\mathbf{p}_k$ has degree two and in



(a) (b) (c)

Figure 1: An edge with an equilibrium stress (a) can be subdivided with a modified stress on the parts (b) and $d-1$ edges added (c). The result is an *edge split* which preserves infinitesimal rigidity and the universality of the configuration.

order for any configuration to be in equilibrium the two new bars must be collinear, and the subdivision process can be reversed. This shows the following.

Proposition 2.3 (Connelly [4]). *Let $G(\mathbf{p})$ be a framework, whose vertices have affine span all of $\mathbb{R}^d$, and which is universal with respect to an equilibrium stress ω. If some of its bars with non-zero stress are subdivided, then the resulting framework is also universal with respect to ω^*.*

Note that for $d \geq 2$ a subdivided framework in $\mathbb{R}^d$ is never infinitesimally rigid. After subdivision it is necessary to add some additional bars to rigidify it infinitesimally (Figure 1c).

Recall that, given a graph G with edge $e = \{i, j\}$, and $d-1$ additional vertices $1, \ldots, d-1$, the *edge split on e in $\mathbb{R}^d$* is the addition of a new vertex k, the removal of e, and the insertion of $d+1$ new edges $\{k, i\}, \{k, j\}, \{k, 1\}, \ldots \{k, d-1\}$. The corresponding geometric operation on $G(\mathbf{p})$ subdivides the edge e and inserts $d-1$ new bars from the new vertex $\mathbf{p}_k$. Thus we can extend a stress ω on $\mathbf{p}$ to the edge split framework $G^*(\mathbf{p}^*)$ by using ω^* on the subdivided edge e, and making the stresses 0 on the edges $\{k, 1\}, \ldots, \{k, d-1\}$. We say that the edge split on $e = \{i, j\}$ is a *general position edge split on e* if the vectors $\mathbf{p}_k - \mathbf{p}_i, \mathbf{p}_k - \mathbf{p}_j, \mathbf{p}_k - \mathbf{p}_1, \ldots, \mathbf{p}_k - \mathbf{p}_d$ span $\mathbb{R}^d$ (Figure 1c). If $\mathbf{p}$ is generic, then the spanning condition will hold automatically. Even though some of the new edges have 0 stress, the following is true.

Proposition 2.4 (Tay and Whiteley [22]). *Let $G(\mathbf{p})$ be an infinitesimally rigid framework in $\mathbb{R}^d$ and let e be an edge of G. Then a general position edge split on e generates a new graph G^* and an extended configuration $G^*(\mathbf{p}^*)$ which is infinitesimally rigid in $\mathbb{R}^d$.*

We shall also use the well-known fact that if the edge split operation is applied to a graph G which is generically redundantly rigid in $\mathbb{R}^d$, then the resulting graph will also be redundantly rigid in $\mathbb{R}^d$.

Note also that since the new vertex $\mathbf{p}_k$ lies on the line between $\mathbf{p}_i$ and $\mathbf{p}_j$, the configuration $\mathbf{p}^*$ is not generic, even if $\mathbf{p}$ is. There is an elementary way to connect the rank calculations for particular configurations to the property of global rigidity of a framework $G(\mathbf{p})$ at a generic configuration $\mathbf{p}$. This is implicit in [4] and explicit in [5].

Proposition 2.5. *Suppose that $G(\mathbf{p})$ is an infinitesimally rigid framework in $\mathbb{R}^d$ and ω is an equilibrium stress on $G(\mathbf{p})$ with a stress matrix $\Omega(\mathbf{p})$ of full rank $|V|-(d+1)$. Then there is a neighborhood $U_\mathbf{p}$ in the space of configurations in $\mathbb{R}^d$ such that if $\mathbf{q} \in U_\mathbf{p}$, then $G(\mathbf{q})$ is infinitesimally rigid in $\mathbb{R}^d$ and has an equilibrium stress ω' with a stress matrix $\Omega(\mathbf{q})$ of full rank $|V|-(d+1)$. Furthermore, if G is generically redundantly rigid and $\mathbf{q} \in U_\mathbf{p}$ is a generic configuration then ω' can be chosen so that $w'_{ij} \neq 0$ on all edges ij of G.*

Putting all these propositions together we get the following, which was the original method to imply generic global rigidity for bar frameworks [4]: suppose that $G(\mathbf{p})$ is an infinitesimally rigid framework in $\mathbb{R}^d$, which is universal with respect to an equilibrium stress ω, and let e be an edge with non-zero stress. Then a general position edge split on e in $\mathbb{R}^d$ generates a new framework $G^*(\mathbf{p}^*)$ which is infinitesimally rigid and universal with respect to ω^*. After moving to a nearby generic point we may conclude that G^* is generically globally rigid in $\mathbb{R}^d$. (See Subsection 6.4 for an extension of this argument.)

We note that for the plane Jackson, Jordán and Szabadka have an alternative proof that edge-splitting preserves global rigidity [14]. This proof has recently been generalized to all dimensions.

3 An Inductive Construction of Redundantly Rigid Body-Bar Graphs

Let $H=(V,E)$ be a multigraph with minimum degree at least one. The *body-bar graph induced by* H, denoted by G_H, is the graph obtained from H by replacing each vertex $v \in V$ by a complete graph B_v (a 'body') on $d_H(v)$ vertices and replacing each edge uv by an edge (a 'bar') between B_u and B_v in such a way that the bars are pairwise disjoint. (We use $d_H(v)$ to denote the degree of vertex v in H. A loop on v contributes to $d_H(v)$ by two.)

We shall prove our main result by an inductive argument which relies on a combinatorial result of Frank and Szegő in [6]. Their result, stated as Theorem 3.1 below, provides an inductive construction for the multigraphs H that induce redundantly rigid body-bar graphs G_H in $\mathbb{R}^d$. By using the operations of the previous section we shall show how to construct an infinitesimally rigid framework $G_H(\hat{\mathbf{p}})$ with a full rank stress matrix, following the inductive construction of the underlying graph H. This will imply that G_H is generically globally rigid by Proposition 2.5 and Theorem 2.2.

Let $H=(V,E)$ be a multigraph. For a partition $\mathcal{P}$ of V let $E_H(\mathcal{P})$ denote the set, and let $e_H(\mathcal{P})$ be the number of edges of H connecting distinct members of $\mathcal{P}$. We

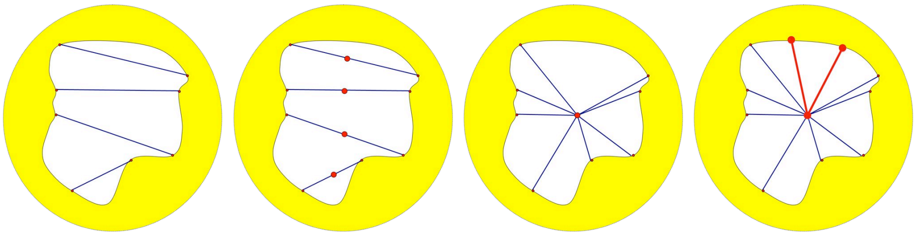


Figure 2: A combinatorial 6-split on 4 edges (a) with the pinch (b,c) and the addition of 2 edges (d).

say that H is *highly m-tree-connected* if

$$e_H(\mathcal{P}) \geq m(t-1)+1, \tag{1}$$

for all partitions $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$. Note that a theorem of Nash-Williams [16] and Tutte [23] implies that H satisfies (1) if and only if $H - e$ contains m edge-disjoint spanning trees for all $e \in E$.

The operation *pinching k edges (with vertex z)* subdivides k designated edges and then contracts the k subdividing vertices into a new vertex z.

Theorem 3.1 (Frank and Szegő [6]). *A multigraph H is highly m-tree-connected if and only if H can be obtained from a vertex by the following operations:*

(i) adding an edge (possibly a loop),

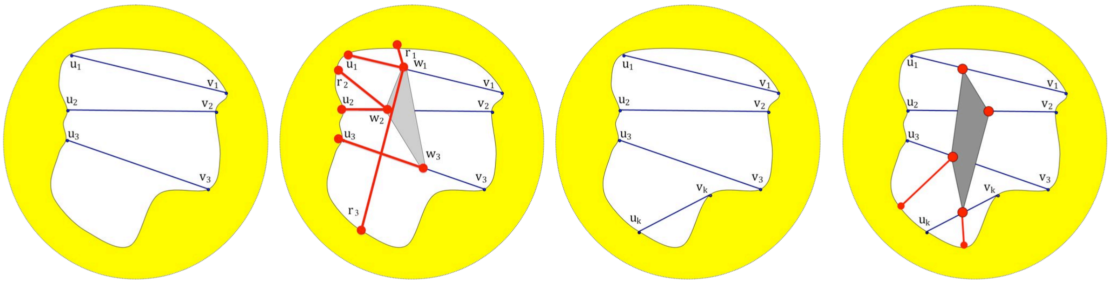
(ii) pinching k edges $(1 \leq k \leq m-1)$ with a new vertex z and adding $m-k$ new edges connecting z with existing vertices.

We call the combined operation of (ii), consisting of pinching and edge addition, an *m-split on k edges* (Figure 2). We shall prove (Lemma 4.6) that if a body-bar graph G_H induced by H is generically redundantly rigid in $\mathbb{R}^d$ then H is highly $\binom{d+1}{2}$-tree-connected. Thus we shall need Theorem 3.1 when $m = \binom{d+1}{2}$ for each $d \geq 1$.

4 Main Lemmas

Here we assemble a series of key lemmas needed for the proof of the main theorem.

We want to show that the combinatorial operation of a $\binom{d+1}{2}$-split on k edges in H gives rise to a geometric operation on the induced body-bar framework, which adds a new body and preserves global rigidity at generic configurations. To do this, we need steps that preserve infinitesimal rigidity and a stress matrix of full rank. We also need to preserve the structure of a body-bar framework and make sure the new body inserted is connected to distinct vertices in designated bodies, as prescribed by the split operation in H. These properties will be ensured by using the edge splitting operation



(a) (b) (c) (d)

Figure 3: Given a set of k edges to be split, we will insert a new body on k vertices with each vertex positioned on a bar, along with additional bars attaching this body to a set of designated vertices, preserving infinitesimal rigidity. (Illustrated for 3 edges in $d = 3$, and 4 edges in $d = 3$.)

on stressed edges to add new vertices. We shall use several different techniques to complete the geometric operation.

We also want to confirm that redundant rigidity implies high tree connectivity. With this in hand, the induction on H from Theorem 3.1 can be applied to generate the globally rigid generic realizations of G_H.

4.1 Body insertion lemmas $1 \leq k \leq d$

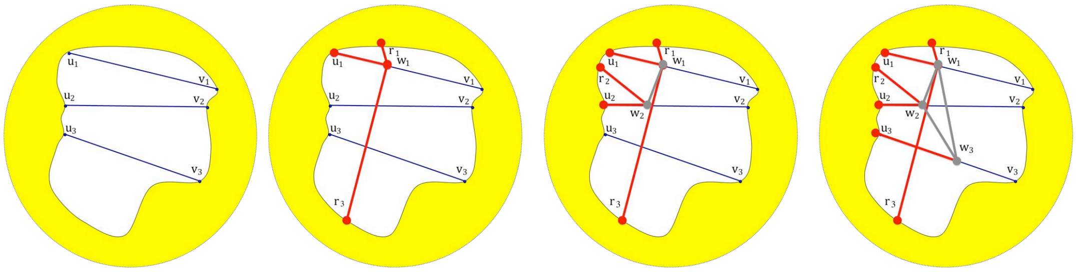
The techniques for $1 \leq k \leq d$ illustrate several basic principles. We present this in two stages. See Figure 3 for a schematic of the process. The central polygon on Figure 3(b) represents a new body, constructed on the vertices splitting the k edges, which will be infinitesimally rigidly attached to the prior framework. For these values of k this initial insertion can be done directly by a sequence of edge splits (Figure 4). This is our first lemma.

Lemma 4.1. *Let $G_H(\mathbf{p})$ be a generic framework in $\mathbb{R}^d$, where G_H is generically redundantly rigid in $\mathbb{R}^d$. Suppose that we have a set of designated edges u_iv_i, $1 \leq i \leq k$, and designated vertices r_j, $1 \leq j \leq dk - \binom{k+1}{2}$ in G_H, where $1 \leq k \leq d$ and these edges and vertices are pairwise disjoint. Then we can construct an extended graph G^*_H and an extended configuration $\hat{\mathbf{p}}$ by adding a complete graph on a new set $W = \{w_1, w_2, ..., w_k\}$ of vertices and adding one new edge from W to each of the designated vertices r_j, and specifying the positions of the vertices of W, such that*

(i) each w_i subdivides the designated edge u_iv_i, for $1 \leq i \leq k$,

(ii) $G^*_H(\hat{\mathbf{p}})$ is infinitesimally rigid, and

(iii) G^*_H is generically redundantly rigid.



(a) (b) (c) (d)

Figure 4: For placing vertices on up to d edges, we do a sequence of edge splits (illustrated for $d = 3$). The grey bars are forming the body, and the dark bars are attachments to the previous framework.

Proof. We construct $G_H^*(\hat{\mathbf{p}})$ by inserting $\mathbf{p}(w_i)$ along the line through $\mathbf{p}(u_i)$ and $\mathbf{p}(v_i)$, for $1 \le i \le k$, as follows. For $i = 1$, we simply add a vertex w_1 by an edge split on u_1v_1 which adds $d-1$ edges from w_1 to the designated vertices $r_1, \ldots, r_{d-1}$ (see Figure 4a). Since the points $\mathbf{p}(u_1), \mathbf{p}(v_1), \mathbf{p}(r_1), \ldots, \mathbf{p}(r_{d-1})$ are generic, the points $\mathbf{p}(w_1), \mathbf{p}(u_1), \mathbf{p}(r_1), \ldots, \mathbf{p}(r_{d-1})$ are in general position for all choices of $\mathbf{p}(w_1)$ along the line of $\mathbf{p}(u_1), \mathbf{p}(v_1)$, distinct from $\mathbf{p}(u_1)$. We make any such choice, distinct from both $\mathbf{p}(u_1)$ and $\mathbf{p}(v_1)$. Since $G_H(\mathbf{p})$ is infinitesimally rigid, the extended framework will also be infinitesimally rigid by Proposition 2.4. The edge split operation preserves the generic redundant rigidity of the graph as well. We will choose this point $\mathbf{p}(w_1)$ to also be in general position relative to all points $\mathbf{p}(r_j)$ and all $\mathbf{p}(v_l)$, $l > 1$. We say we have inserted $\mathbf{p}(w_1)$ along the line through $\mathbf{p}(u_1)$ and $\mathbf{p}(v_1)$.

Given the construction for i edges, $i < k$, we will position a new w_{i+1} along the line through $\mathbf{p}(u_{i+1}), \mathbf{p}(v_{i+1})$ by a related construction (see Figure 4b,c). The edge split connects w_{i+1} to u_{i+1}, v_{i+1} and to vertices $w_1, \ldots, w_i$ and the next (unused) vertices $r_{j+1}, \ldots, r_{j+d-i-1}$ (a total of $d-1$ edges beyond the split edge). Again, provided that $\mathbf{p}(w_{i+1})$ is in general position, relative to all the previous vertices as well as vertices $r_{j+1}, \ldots, r_{j+d-i-1}$, and positioned along the line of $\mathbf{p}(u_{i+1}), \mathbf{p}(v_{i+1})$, this maintains the infinitesimal rigidity. For the further insertions, we choose $\mathbf{p}(w_{i+1})$ to also be in general position relative to all of $\mathbf{p}(r_j)$ and any $\mathbf{p}(v_l)$, $l > i+1$. This way we have inserted $\mathbf{p}(w_{i+1})$ along the line through $\mathbf{p}(u_{i+1}), \mathbf{p}(v_{i+1})$.

The insertions can be completed, since we have $\sum_{i=1}^{k}(d-i) = dk - \binom{k+1}{2}$ designated vertices r_j. Also note that at each stage we have a complete graph among the new vertices w_i. Thus the extended graph and framework, after the k insertions, satisfies conditions (i),(ii), and (iii), as required. $\square$

At this point G_H^* is not a body-bar graph: each vertex w_i in the new body is connected to at least two vertices that belong to other bodies. Since in a body-bar graph each vertex is connected to precisely one other body, we refer to all but one of these connections at w_i as *bad*. The k insertions in Lemma 4.1 create $\sum_{i=1}^{k}(d+1-i)$ bad connections in total. The following lemma shows how to get rid of the bad connections and also confirms that the final body-bar structure is globally rigid for generic configurations.

Lemma 4.2. *Let G_H be a generically redundantly rigid body-bar graph in $\mathbb{R}^d$ and let $\mathbf{p}$ be a generic configuration for which $G(\mathbf{p})$ has an equilibrium stress ω with a stress matrix of full rank. Suppose that we have a set of designated edges u_iv_i, $1 \leq i \leq k$, and non-negative integers $s_1, s_2, ..., s_n$ assigned to the n bodies of G_H, where $1 \leq k \leq d$ and $\sum_{i=1}^{n} s_i = \binom{d+1}{2} - k$. Then we can construct an extended body-bar graph G_{H^*} and an extended generic configuration $\hat{\mathbf{p}}$ with one added body b^* on $\binom{d+1}{2} + k$ vertices, with s_i new vertices added to each existing body b_i, $1 \leq i \leq n$, and by replacing the k designated edges by $\binom{d+1}{2} + k$ disjoint edges connecting b^* to the added vertices and to vertices u_i, v_i, $1 \leq i \leq k$, such that*

(i) $G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid,

(ii) $G_{H^*}(\hat{\mathbf{p}})$ has a stress matrix of full rank, and

(iii) G_{H^*} is generically redundantly rigid.

Proof. By Proposition 2.5 we may suppose that ω is non-zero on all edges. We start the construction of G_{H^*} by extending each body b_i by s_i new vertices. We do this by a sequence of s_i edge splitting operations on stressed edges within b_i. This way we can maintain a stress matrix of full rank and, since $\mathbf{p}$ is generic, we can also preserve infinitesimal rigidity. These new vertices are labeled as r_j, $1 \leq j \leq \binom{d+1}{2} - k$.

Next we apply Lemma 4.1 to the resulting framework, by using the first $dk - \binom{k+1}{2} (\leq \binom{d+1}{2} - k)$ vertices r_j as designated vertices, to obtain an initial infinitesimally rigid framework $G_H^*(\hat{\mathbf{p}})$ with all new vertices placed on stressed edges. By Proposition 2.3 this implies that $G_H^*(\hat{\mathbf{p}})$ also has an equilibrium stress with a full rank stress matrix. Recall that the edge splitting operations preserve generic redundant rigidity. Moving to a generic configuration preserves all of these properties, and ensures that every edge has a non-zero stress by Proposition 2.5.

Now we can apply a sequence of edge splits on bad edges to separate the attachments and make sure that each vertex on the new body is connected to other bodies by precisely one bar. First we split $d - k$ bad edges incident with w_1 by new vertices w_{k+j}, $1 \leq j \leq d - k$, adding as many edges to other vertices in the growing body as possible and adding the other new edges to unused vertices r_j. As before, we perform general position edge splits on stressed bars, so by Propositions 2.3 and 2.4 infinitesimal rigidity is preserved and the resulting framework has a stress matrix of full rank.

In this phase vertex w_{k+j} will be connected to $k + j - 1$ vertices of the growing body and $d - k - j + 2$ vertices in other bodies, among which $d - k - j + 1$ will go to (unused) designated vertices. Furthermore, while one bad edge is eliminated by adding w_{k+j}, a new set of $d - k - j + 1$ bad edges arises. At this point we have used

$$dk - \binom{k+1}{2} + \sum_{j=1}^{d-k} (d - k - j + 1) = \binom{d+1}{2} - k$$

designated vertices and have

$$\sum_{i=1}^{k}(d+1-i) + \sum_{j=1}^{d-k}(d-k-j+1) - (d-k) = \binom{d+1}{2} - (d-k)$$

bad edges. Now we again move to a generic configuration to make all edges stressed.

Finally we separate the attachments and eliminate all remaining bad edges at vertices in the growing body by $\binom{d+1}{2} + k - d$ further edge splits on stressed edges. This step requires no new vertices r_j and creates no bad edges. At the end the new body has $\binom{d+1}{2} + k$ vertices. At this point we can also add any missing edges to make each body a complete graph on its vertices. Hence the constructed graph G_{H^*} is a body-bar graph which is induced by a multigraph H^*, obtained from H by a $\binom{d+1}{2}$-split on k edges.

At each stage the framework is infinitesimally rigid, and has an equilibrium stress with a stress matrix of full rank. Furthermore, the operations used (edge split and edge addition) preserve generic redundant rigidity. We conclude that the final framework $G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid with a full rank stress matrix and G_{H^*} is generically redundantly rigid, as required. □

The proofs of the previous lemmas, when restricted to the graphs of the frameworks, imply the following combinatorial statement.

Corollary 4.3. *Let H be a multigraph for which G_H is generically redundantly rigid in $\mathbb{R}^d$. Suppose that H^* is obtained from H by a $\binom{d+1}{2}$-split on k edges for some $1 \le k \le d$. Then G_{H^*} can be obtained from G_H by a sequence of edge splits, followed by a sequence of edge additions.*

4.2 Body insertion lemmas $d < k \le \binom{d+1}{2} - 1$

For splitting $d < k \le \binom{d+1}{2} - 1$ edges, we need an extended process which we introduce and verify here. In order to control the total number of edges in the final graph, we start from the output of Lemma 4.1. As we split some of the attachments which are not along edges previously split, we will place them onto the additional designated edges, creating additional edge splits in a new construction.

Lemma 4.4. *Let G_H be a generically redundantly rigid graph in $\mathbb{R}^d$ and let $\mathbf{p}$ be a generic configuration for which $G_H(\mathbf{p})$ has an equilibrium stress ω with a full rank stress matrix. Suppose that we have a set of designated edges u_iv_i, $1 \le i \le k$, and designated vertices r_j, $1 \le j \le \binom{d+1}{2} - k$, in G_H, where $d < k \le \binom{d+1}{2} - 1$ and these edges and vertices are pairwise disjoint. Then we can construct an extended graph G_H^* and an extended configuration $\hat{\mathbf{p}}$ by adding a complete graph on a new set $W = \{w_1, w_2, ..., w_k\}$ of vertices and adding one new edge from W to each of the designated vertices r_j, and specifying the positions of the vertices of W, such that*

(i) each w_i splits the designated edge u_iv_i, for $1 \le i \le k$, with non-zero stress on each of these split bars,

(ii) $G^*_H(\hat{\mathbf{p}})$ *is infinitesimally rigid,*

(iii) $G^*_H(\hat{\mathbf{p}})$ *has an equilibrium stress with a full rank stress matrix, and*

(iv) G^*_H *is generically redundantly rigid.*

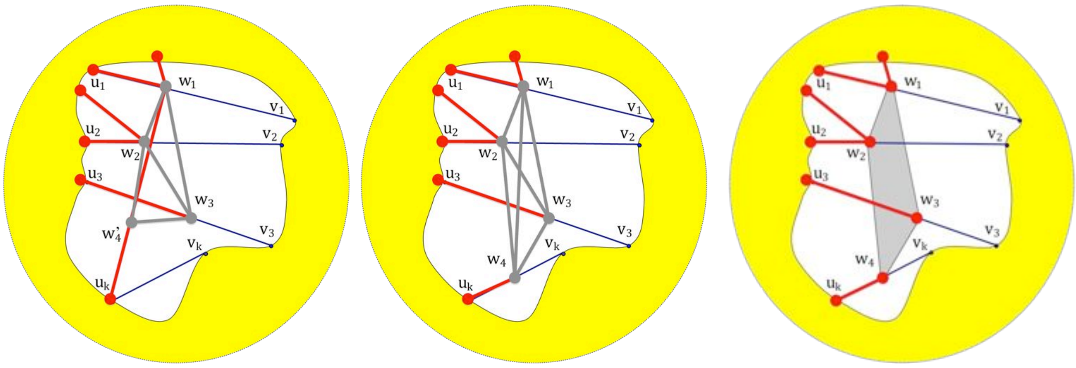
Proof. By Proposition 2.5 we may suppose that ω is non-zero on all edges. We follow Lemma 4.1 to place the first d vertices of the extended graph on the first d designated edges. We apply the lemma by choosing the required set of $\binom{d+1}{2} - d$ designated vertices to be equal to the union of $u_{d+1}, ..., u_k$ and the given designated vertices r_j, $1 \leq j \leq \binom{d+1}{2} - k$. After adding these d vertices the extended graph contains a complete graph on d vertices as the current new body and one edge from the body to each of u_i, v_i, $1 \leq i \leq d$, and to each of the designated vertices. Furthermore, the number of bad edges is $\binom{d+1}{2}$.

We will add the remaining $k - d$ new vertices by edge splits, carefully placing them on the further $k - d$ edges, without connecting the growing body to additional vertices in other bodies. Every subsequent edge split will be performed on a (bad) edge connecting some of the first d vertices of the new body to some vertex in the set $u_{d+1}, ..., u_k$. Note that the number $\binom{d+1}{2} - d$ of edges connecting the new body to designated vertices is greater than $k - d$, hence the number of such edges is indeed $k - d$.

Consider such an edge, say $w_1 u_{d+1}$. We perform an edge split on $w_1 u_{d+1}$, connecting the new vertex w_{d+1} to the $d - 1$ prior w_i's distinct from w_1. This will initially place w_{d+1} as $\hat{\mathbf{p}}(w_{d+1})$ along the bar $\mathbf{p}(w_1)\mathbf{p}(u_{d+1})$ (see Figure 5a). This is now infinitesimally rigid. Moreover, $\mathbf{p}(v_{d+1})$ is generic with respect to all of the vertices in the configuration (coordinates algebraically independent of all the initial vertices and the added w_i). In particular, it remains independent of the vertices $\mathbf{p}(u_i), \hat{\mathbf{p}}(w_i)$, $i \leq d$. So we could also place w_{d+1} at $\mathbf{p}(v_{d+1})$ and still have an infinitesimally rigid framework (see Figure 5b). Since infinitesimal rigidity is an open set property, we choose another distinct point $\hat{\mathbf{p}}(w_{d+1})$ along the line of $\mathbf{p}(u_{d+1})\mathbf{p}(v_{d+1})$ which preserves infinitesimal rigidity (see Figure 5c). Note that, at this point, there is no stress on any of the edges at $\hat{\mathbf{p}}(w_{d+1})$.

Next we do a 'triangle circuit exchange' with a stressed, collinear triangle on the three points $\mathbf{p}(u_{d+1}), \hat{\mathbf{p}}(w_{d+1}), \mathbf{p}(v_{d+1})$. This simple exchange cancels the stressed edge $\mathbf{p}(u_{d+1})\mathbf{p}(v_{d+1})$, and leaves two stressed edges $\mathbf{p}(u_{d+1})\hat{\mathbf{p}}(w_{d+1})$ and $\hat{\mathbf{p}}(w_{d+1})\mathbf{p}(v_{d+1})$. The net result is also infinitesimally rigid and has a non-zero stress through $\hat{\mathbf{p}}(w_{d+1})$ along only those two edges. By Proposition 2.3 this process also increases the rank of the corresponding stress matrix by 1. It is not difficult to show, by using the symmetry of our graphs (in particular, the fact that the endvertices u_i, v_i of the designated edges can be interchanged) that this triangle exchange operation preserves the generic redundant rigidity of the underlying graph.

We do essentially the same step to add w_i, for $d + 1 \leq i \leq k$, and position it on the next bar $u_i v_i$, preserving infinitesimal rigidity. As before, we split an edge $w_h u_i$ with w_i and, using the genericity of v_i relative to all other vertices used to this point, we can position $\hat{\mathbf{p}}(w_i)$ at some places along $\mathbf{p}(u_i)\mathbf{p}(v_i)$, so that the extended framework



(a) (b) (c)

Figure 5: For placing vertices on more than d edges, we do an edge split on a prior attachment (a), then reposition the new vertex on the desired additional bar (b), using the genericity of all the points. The grey bars are lumped as a body (c).

remains infinitesimally rigid. Again, an exchange with a collinear triangle completes a split of the bar $\mathbf{p}(u_i)\mathbf{p}(v_i)$, with a non-zero stress through $\hat{\mathbf{p}}(w_i)$ only on these two bars, and the rank of the stress matrix has been increased by 1. Finally we can add edges among the w_i's to make the body a complete graph.

By using Propositions 2.3, 2.4, and Lemma 4.1 we can deduce that when we are finished for $d < k \leq \binom{d+1}{2} - 1$, we have, as required: (i) each w_i splits the designated edge u_iv_i, for $1 \leq i \leq k$, with non-zero stress on each of these split bars, (ii) $G^*_H(\hat{\mathbf{p}})$ is infinitesimally rigid, and (iii) $G^*_H(\hat{\mathbf{p}})$ has an equilibrium stress with a full rank stress matrix. We can also deduce that (iv) G^*_H is generically redundantly rigid. □

It remains to show how to continue the geometric operation to obtain a body-bar structure with the required properties.

Lemma 4.5. *Let G_H be a generically redundantly rigid body-bar graph in $\mathbb{R}^d$ and let $\mathbf{p}$ be a generic configuration for which $G(\mathbf{p})$ has an equilibrium stress ω with a stress matrix of full rank. Suppose that we have a set of designated edges u_iv_i, $1 \leq i \leq k$, and non-negative integers $s_1, s_2, ..., s_n$ assigned to the n bodies of G_H, where $d < k \leq \binom{d+1}{2} - 1$ and $\sum_{i=1}^{n} s_i = \binom{d+1}{2} - k$. Then we can construct an extended body-bar graph G_{H^*} and an extended generic configuration $\hat{\mathbf{p}}$ with one added body b^* on $\binom{d+1}{2} + k$ vertices, with s_i new vertices added to each existing body b_i, $1 \leq i \leq n$, and by replacing the k designated edges by $\binom{d+1}{2} + k$ disjoint edges connecting b^* to the added vertices and to vertices u_i, v_i, $1 \leq i \leq k$, such that*

(i) $G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid,

(ii) $G_{H^*}(\hat{\mathbf{p}})$ has a stress matrix of full rank, and

(iii) G_{H^*} is generically redundantly rigid.

Proof. By Proposition 2.5 we may suppose that ω is non-zero on all edges. We start the construction of G_{H^*} by extending each body b_i by s_i new vertices. We do this by a sequence of s_i edge splitting operations on stressed edges within b_i. This way we can maintain a stress matrix of full rank and, since $\mathbf{p}$ is generic, we can also preserve infinitesimal rigidity. These new vertices are labeled as r_j, $1 \leq j \leq \binom{d+1}{2} - k$.

Next we apply Lemma 4.4 to obtain an initial infinitesimally rigid framework $G^*_H(\hat{\mathbf{p}})$ with an equilibrium stress with a full rank stress matrix, for which G^*_H is generically redundantly rigid. We may move to a generic configuration, preserving all of these properties, and ensure that every current edge has a non-zero stress by Proposition 2.5.

As in Lemma 4.2, we can apply a sequence of edge splits to separate the current attachments and eliminate the bad edges. Note that the first d edge splits have created $\binom{d+1}{2}$ bad edges, the next $k-d$ edge splits eliminated $k-d$ but also created $k-d$ bad edges. Thus we have $\binom{d+1}{2}$ bad edges and a body on k vertices when we start this phase. To eliminate all bad edges we need to add $\binom{d+1}{2}$ additional vertices to the new body, which gives $\binom{d+1}{2} + k$ in total, as required. At this point we can also add any missing edges to make each body a complete graph on its vertices. Hence the constructed graph G_{H^*} is a body-bar graph which is induced by a multigraph H^*, obtained from H by a $\binom{d+1}{2}$-split on k edges.

At each stage the framework is infinitesimally rigid, and has an equilibrium stress with a stress matrix of full rank by Propositions 2.3 and 2.4. The operations preserve generic redundant rigidity as well. We conclude that the final framework $G_{H^*}(\hat{\mathbf{p}})$ is infinitesimally rigid with a full rank stress matrix, and G_{H^*} is generically redundantly rigid, as required. $\square$

Remark We could also position new vertices on $\binom{d+1}{2}$ edges with exactly the same methods. The inductive construction of Frank and Szegő does not require this added case, but some other situations might make use of this extended result. This construction does not extend to the case when $k > \binom{d+1}{2}$ edges are split, if we want to retain the count on the attachments, and hence the condition that this extended framework has no new dependences.

4.3 Redundantly rigid implies highly tree-connected

Lemma 4.6. *Let $H = (V, E)$ be a multigraph with $|V| \geq 2$ and suppose that the body-bar graph G_H induced by H is generically redundantly rigid in $\mathbb{R}^d$. Then H is highly $\binom{d+1}{2}$-tree-connected.*

Proof. For a contradiction suppose that $e_H(\mathcal{P}) \leq \binom{d+1}{2}(t-1)$ for a partition $\mathcal{P} = \{X_1, X_2, ..., X_t\}$ of V with $t \geq 2$. Let $Y_i = \cup\{V(B_v) : v \in X_i\}$, for $1 \leq i \leq t$, and let $\mathcal{Q} = \{Y_1, Y_2, ..., Y_t\}$ be the corresponding partition of $V(G_H)$. The redundant rigidity of G_H implies that each vertex of G_H has degree at least $d+1$. Hence $|V(B_v)| \geq d+1$ and also $|Y_i| \geq d+1$ for all $v \in V$ and $1 \leq i \leq t$. Observe that $e_{G_H}(\mathcal{Q}) = e_H(\mathcal{P}) \neq \emptyset$.

Let $S \subseteq E(G_H)$ be a maximal set of independent edges in G_H, i.e. a base in the d-dimensional generic rigidity matroid of G_H. Since G_H is rigid and G_H has more than $d+1$ vertices, we have $|S| = d|V(G_H)| - \binom{d+1}{2}$. Thus, by using the fact that each subset $Y \subseteq V(G_H)$ with $|Y| \geq d+1$ induces at most $d|Y| - \binom{d+1}{2}$ edges of S, we obtain

$$d|V(G_H)| - \binom{d+1}{2} = |S| \leq \sum_{1}^{t}(d|Y_i| - \binom{d+1}{2}) + e_{G_H}(\mathcal{Q}) =$$

$$d|V(G_H)| - \binom{d+1}{2}t + e_H(\mathcal{P}) \leq d|V(G_H)| - \binom{d+1}{2}.$$

Thus we have equality everywhere. In particular, for all edges $e \in E_{G_H}(\mathcal{Q})$ and all bases S we must have $e \in S$. This implies that e is not redundant, and hence G_H is not redundantly rigid, a contradiction. Hence each partition of V satisfies (1) and the lemma follows. □

4.4 Highly tree-connected implies a redundantly rigid realization with a full rank stress matrix

Lemma 4.7. *Let $H = (V, E)$ be a highly $\binom{d+1}{2}$-tree-connected multigraph. Let $G_H = K_{d+2(l+1)}$, when $|V| = 1$ and E is a set of $l \geq 0$ loops, and otherwise let G_H be the body-bar graph induced by H. Then there exists a redundantly rigid generic realization $G_H(\mathbf{p})$ of G_H with an equilibrium stress ω for which the associated stress matrix Ω has rank $n-d-1$, where $n = |V(G_H)|$.*

Proof. The proof is by induction on $|V| + |E|$. In the base case, when $|V| = 1$ and $E = \emptyset$, G_H is a complete graph on $d+2$ vertices. This graph is generically redundantly rigid. In this case it is easy to construct an infinitesimally rigid realization $G_H(\mathbf{p})$ with an equilibrium stress ω for which the associated stress matrix has full rank. By using Proposition 2.5 we may suppose that the realization is generic.

Now consider a highly $\binom{d+1}{2}$-tree-connected multigraph $H = (V, E)$ and suppose that the lemma holds for all highly $\binom{d+1}{2}$-tree-connected multigraphs H' with $|V(H')|+|E(H')| < |V(H')|+|E(H')|$. By Theorem 3.1 H can be obtained from a smaller highly $\binom{d+1}{2}$-tree-connected multigraph H' by adding an edge or by a $\binom{d+1}{2}$-split on k edges, for some $1 \leq k \leq \binom{d+1}{2} - 1$. By induction, there exists a redundantly rigid generic realization $G_{H'}(\bar{\mathbf{p}})$ of $G_{H'}$ with an equilibrium stress ω for which the associated stress matrix Ω has rank $n'-d-1$, where $n' = |V(G_{H'})|$. By Proposition 2.5 we may suppose that ω is non-zero on all edges.

First suppose that H is obtained from H' by adding a new edge uv, possibly a loop. Then we may construct a realization $G_H(\mathbf{p})$ from $G_{H'}(\bar{\mathbf{p}})$ by performing two edge splits within B_u and B_v, respectively, which create two new vertices of degree $d+1$, followed by edge additions, which connect the new vertices and which make the two enlarged bodies complete. Note that the definition of $G_{H'}$ and the assumption on H' implies that each body in $G_{H'}$ has at least $d+1$ vertices. These operations preserve infinitesimal rigidity and the property of having a stress matrix of full rank by Propositions 2.3 and 2.4. They also preserve generic redundant rigidity. Thus the lemma follows by Proposition 2.5.

Next suppose that H is obtained from H' by a $\binom{d+1}{2}$-split on k edges. We have two cases. When $1 \leq k \leq d$ ($d < k \leq \binom{d+1}{2} - 1$) Lemma 4.2 (respectively, Lemma 4.5) confirms that there is an extended framework $G_H(\mathbf{p})$ which is infinitesimally rigid and has a stress matrix of full rank, and for which G_H is generically redundantly rigid. As above, the lemma now follows by Proposition 2.5. $\square$

5 Main Theorem

We can now assemble the pieces to give a full proof of the main theorem.

Theorem 5.1. *Let $H = (V, E)$ be a multigraph with $|V| \geq 2$ and let G_H be the body-bar graph induced by H. Let $d \geq 1$ be an integer. Then the following are equivalent:*
(a) there exists a redundantly rigid generic realization $G_H(\mathbf{p})$ of G_H in $\mathbb{R}^d$ with an equilibrium stress ω for which the associated stress matrix Ω has rank $n-d-1$, where $n = |V(G_H)|$,
(b) G_H is generically redundantly rigid in $\mathbb{R}^d$,
(c) H is highly $\binom{d+1}{2}$-tree-connected.

Proof. (a)→(b) is obvious.

(b)→(c) follows from Lemma 4.6.

(c)→(a) follows from Lemma 4.7. $\square$

We can now apply Theorem 2.2 to obtain the characterization of globally rigid body-bar graphs.

Theorem 5.2. *Let $H = (V, E)$ be a multigraph with $|V| \geq 2$ and $|E| \geq 2$ and let G_H be the body-bar graph induced by H. Let $d \geq 1$ be an integer. Then G_H is globally rigid in $\mathbb{R}^d$ if and only if G_H is redundantly rigid in $\mathbb{R}^d$.*

Proof. Since $|V| \geq 2$ and $|E| \geq 2$, it follows that G_H is not a complete graph. Thus the only if direction follows from Theorem 2.1. The if direction can be deduced from Theorem 5.1 and Theorem 2.2. $\square$

6 Further Remarks

6.1 Algorithmic implications

Theorems 5.1 and 5.2 give rise to a polynomial time algorithm to determine whether a body-bar graph is generically globally rigid in $\mathbb{R}^d$. This follows from the fact that, as we noted earlier, a multigraph H is highly m-tree-connected if and only if $H - e$ contains m edge-disjoint spanning trees for all $e \in E(H)$. Thus efficient tree-packing algorithms can be used to test whether a given multigraph is highly m-tree-connected. We refer the reader to [18, Chapter 51] for a complexity survey for tree packing algorithms.

By using similar techniques one can also compute the maximal highly m-tree-connected subgraphs of H. The vertex sets of these subgraphs form a partition of $V(H)$. See [13] for more details (where these subgraphs are called the *m-superbricks* of H).

6.2 Globally linked pairs

Given the characterization of globally rigid graphs in the plane, the methods have recently been extended to characterize *globally linked* pairs of vertices in some classes of graphs in $\mathbb{R}^2$ [14]. These are pairs of vertices whose distance is the same in all frameworks which are equivalent to any given generic framework of the graph. One can ask the analogous question for body-bar graphs G_H in $\mathbb{R}^d$. We *conjecture* that a pair of vertices is globally linked in G_H if and only if there is a globally rigid subgraph of G_H which contains both (or equivalently, if they are adjacent or the vertices of H corresponding to their bodies belong to the same $\binom{d+1}{2}$-superbrick of H). This conjecture is open even for $d = 2$, in which case it is consistent with the more general conjecture for the plane in [14].

6.3 Connectivity

We did not directly refer to Hendrickson's $(d + 1)$-connectivity condition of Theorem 2.1 in our proofs. This is due to the fact that high vertex-connectivity follows for 'free' for body-bar graphs G_H induced by highly $\binom{d+1}{2}$-tree-connected multigraphs. Another related observation is that if the multigraph H is $d(d + 1)$-edge connected, then the body-bar graph G_H is generically redundantly rigid in $\mathbb{R}^d$, see e.g. [25]. This now implies that G_H is globally rigid in $\mathbb{R}^d$. There are examples showing that the bound $d(d + 1)$ on the edge-connectivity of H cannot be improved.

In general, it has been conjectured that $d(d+1)$-vertex-connectivity is sufficient for generic rigidity for arbitrary bar-and-joint frameworks in $\mathbb{R}^d$. We can extend this and *conjecture* that $d(d + 1)$-vertex-connectivity is sufficient for global rigidity of general bar-and-joint frameworks.

6.4 Body insertion in one step

In Section 2 we deduced that edge splitting preserves generic global rigidity by showing that the corresponding geometric operation preserves infinitesimal rigidity and a stress matrix of full rank, when applied to a suitable realization of the graph.

However, we can be more ambitious. We do not have to subdivide edges one at a time and then add more edges to rigidify the framework infinitesimally. We can subdivide several edges at once and even one edge more than once, and then rigidify by adding the appropriate additional edges. This can be summarized as follows.

Corollary 6.1. *Let $G(\mathbf{p})$ be a framework which is infinitesimally rigid in $\mathbb{R}^d$ and is universal with respect to an equilibrium stress ω. Subdivide a subset of those edges of $G(\mathbf{p})$ which have non-zero stress, getting a new equilibrium stress ω^*. Add more edges* *so that the resulting framework* $G^*(\mathbf{p}^*)$ *is infinitesimally rigid in* $\mathbb{R}^d$*. Extend* ω^* *to be* 0 *on these additional edges. Then since* $\mathbf{p}^*$ *is universal with respect to* ω^* *as well,* G^* *is generically globally rigid in* $\mathbb{R}^d$*.*

By using Corollary 6.1 the proofs of some of our body insertion lemmas (in the case when $d < k$) can be simplified. The corollary may also be useful in other problems.

6.5 Examples of non-globally rigid frameworks

As mentioned in the introduction, in dimension three and greater there are examples in [3] of graphs that satisfy Hendrickson's necessary conditions for generic global rigidity in Theorem 2.1, but are not generically globally rigid. In the plane, those conditions are sufficient by the results in [10].

When one models globally rigid bodies as in this paper, it is possible to use a d-dimensional bar simplex as a base for each of those rigid bodies. Suppose it is true that when a bar-and-joint framework with graph G has such a simplex, namely a complete graph K_{d+1}, as a subgraph, and Hendrickson's conditions holds, then G is generically globally rigid. That supposed result would be much more general than our Theorem 5.2. Indeed, for the plane, in the inductive construction of Theorem 4 in [2], if such a simplex, a triangle, is present and there is a minimum number of bars ($2n-2$, where n is the number of vertices of the graph), then the edge splitting can avoid splitting any edge of the triangle. This allows the construction of configurations in the plane that are are explicitly infinitesimally rigid and globally rigid without passing to generic configurations. If such a construction were true in higher dimensions, it might provide the more general statement mentioned above.

However, such constructions are not always possible for higher dimensions. In [7] Frank and Jiang show, for $d \geq 5$, that there are graphs G containing a simplex K_{d+1} as a subgraph that are not generically globally rigid in $\mathbb{R}^d$, and yet they satisfy Hendrickson's necessary conditions in Theorem 2.1 as well.

6.6 Body-hinge and molecular frameworks

The results for body-bar frameworks have been generalized to body-hinge frameworks [19]. This suggests the following generalization of Theorem 5.2. For a graph G and integer k we use kG to denote the multigraph obtained from G by replacing each edge e of G by k parallel copies of e.

Conjecture 6.2 (Body-Hinge Global Rigidity Conjecture)**.** *A graph* G *is generically globally rigid in* $\mathbb{R}^d$ *as a body-hinge framework if the graph* $(\binom{d+1}{2}-1)G$ *is generically redundantly rigid as a body-bar framework in* $\mathbb{R}^d$*. Equivalently, a graph* G *is generically globally rigid in* $\mathbb{R}^d$ *as a body-hinge framework if the multigraph* $(\binom{d+1}{2}-1)G$ *is highly* $\binom{d+1}{2}$*-tree-connected.*

For generic rigidity, there is a further conjecture, which in its various forms is called the Molecular Conjecture [26]. A proof for this conjecture has recently been announced [15], so this 'Conjecture' may now be a Theorem.

Conjecture 6.3 (Molecular Global Rigidity Conjecture). *A graph G is generically globally rigid in $\mathbb{R}^d$ as a molecular-hinge framework if and only if the multigraph $(\binom{d+1}{2} - 1)G$ is highly $\binom{d+1}{2}$-tree-connected.*

In many contexts, including the study of infinitesimal rigidity, there is an equivalence between the molecular hinge structure on G and an associated bar-and-joint framework on the *square* G^2 of G in $\mathbb{R}^3$. However, for small cycles (of length 3,4,5) the shift between structures does not preserve equilibrium stresses (or redundance). Thus it may not preserve global rigidity, as the following example from [11] shows: consider two four-cycles with a common vertex. For this graph G we have that $5G$ is highly 6-tree-connected but G^2 is not even redundantly rigid in $\mathbb{R}^3$.

Conjecture 6.4. *Suppose that G has no cycles of length ≤ 5. Then G^2 is generically globally rigid in $\mathbb{R}^3$ as a bar-and-joint framework if the multi-graph $5G$ is highly 6-tree-connected.*

6.7 Body-bar frameworks with identifications

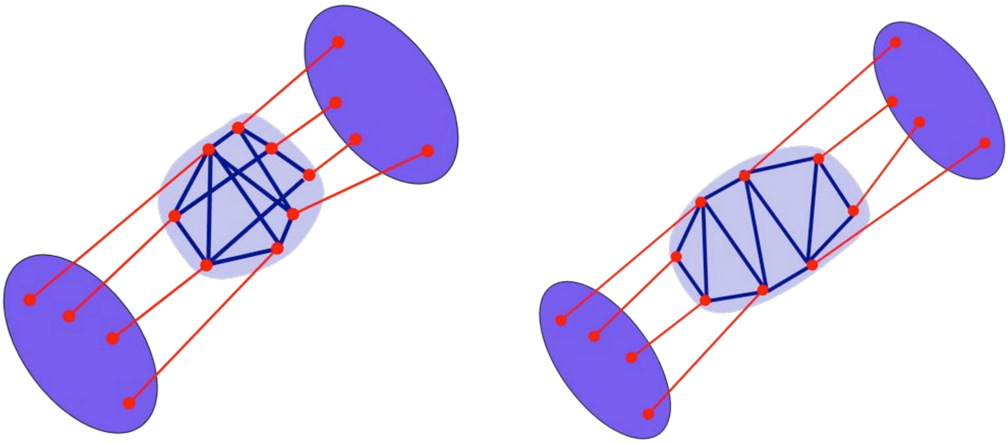
During the construction for the main theorem, we carefully did additional splits to separate the end-points of all bars, to give a simple body-bar framework. However, the framework was already globally rigid before doing these additional splits. So some identification of end-points will still preserve global rigidity. On the other hand, too much identification of the end points will destroy even first-order rigidity, as the 'double banana' can be cast as two bodies joined by six bars, where two triples of bars share endpoints.

An *identified body-bar framework* is a body-bar graph, with additional data for each body, which partitions the incident bars into classes which will share a vertex of attachment. It may be interesting to characterize which identifications preserve global rigidity (or even first-order rigidity) of a body-bar graph.

6.8 Isostatic frameworks for bodies.

It is not difficult to see, by rereading the proofs of the main lemmas, that if H is a highly $\binom{d+1}{2}$-tree-connected multigraph on at least two vertices then it is possible to replace each 'body' of the globally rigid body-bar graph G_H by *some* isostatic graph preserving global rigidity in $\mathbb{R}^d$. This follows by observing that the edge addition steps within the bodies are not necessary to ensure global rigidity, and that the other operations, when restricted to the individual bodies, build up isostatic graphs by edge splits and vertex additions.

For infinitesimal rigidity it is known that in all dimensions one can replace any isostatic subframework with any other isostatic subframework on the same vertices and preserve infinitesimal rigidity. However, the same general *isostatic replacement* does not necessarily preserve global rigidity. This issue most clearly arises in the steps of the insertion lemmas when we are separating the attachment points. While a careful separation (as used in our proof) does preserve global rigidity, a general replacement



(a) (b)

Figure 6: There are isostatic frameworks which generate a globally rigid framework (a), but others (b) which when substituted do not generate a globally rigid framework, due to failed connectivity.

can easily break down the simple necessary $(d+1)$-connectivity condition. See Figure 6, which, in the plane, breaks the required 3-connectivity.

We do not currently have a conjecture for which isostatic replacements for bodies would preserve global rigidity. So there is a residual puzzle about how to detect whether a bar and joint framework in which we have 'identified' bodies with isostatic subframeworks, and distinct edges joining them, has the required structure to apply this theorem and claim global rigidity.

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